# A subquadratic algorithm for the simultaneous conjugacy problem

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#### Abstract

The d-Simultaneous Conjugacy problem in the symmetric group  $S_n$  asks whether there exists a permutation  $\tau \in S_n$  such that  $b_j = \tau^{-1}a_j\tau$  holds for all j = 1, 2, ..., d, where  $a_1, a_2, ..., a_d$  and  $b_1, b_2, ..., b_d$  are given sequences of permutations in  $S_n$ . The time complexity of existing algorithms for solving the problem is  $O(dn^2)$ . We show that for a given positive integer d the d-Simultaneous Conjugacy problem in  $S_n$  can be solved in  $o(n^2)$  time.

**Keywords:** canonical labeling, graph isomorphism, simultaneous conjugacy problem.

### 1 Introduction

The d-Simultaneous Conjugacy problem in the symmetric group  $S_n$  asks whether there exists a permutation in  $S_n$  which simultaneously conjugates two given d-tuples of permutations from  $S_n$ . More formally, given two ordered d-tuples  $a = (a_1, a_2, \ldots, a_d)$  and  $b = (b_1, b_2, \ldots, b_d)$  of permutations from  $S_n$ , is there a permutation  $\tau \in S_n$  such that  $b_j = \tau^{-1}a_j\tau$  holds for all indices  $j = 1, 2, \ldots, d$ ? To save words, we shall refer to this problem as d-SCP in  $S_n$  or even just as SCP.

This problem arises in many forms in various fields of mathematics and computer science, in particular, when deciding whether two objects from a given class are structurally equivalent. A brief list includes the following: in the theory of covering graphs, the problem of equivalence of covering projections [6], and moreover, the construction of

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all regular covering projections along which a given group of automorphisms lifts [8, 9]; in the theory of maps on surfaces, the question whether two oriented maps on a closed surface are combinatorially isomorphic [7]; in computational group theory, the problem whether the centralizer in the symmetric group of a given group is non-trivial [10]. Last but not least, in the design of efficient and fast interconnection networks for computer systems, the question of equivalence of permutation networks also reduces to the SCP [11, 12].

Because of its fundamental importance, the complexity of the d-SCP in  $S_n$  has been studied since mid seventies [4]. The problem can be viewed as a special case of the graph isomorphism problem. More precisely, let  $G_a$  be an arc-colored (multi)digraph on the vertex set  $[n] = \{1, 2, ..., n\}$  such that there is an arc from u to v colored j if and only if  $a_j$  maps u to v, for j = 1, 2, ..., d. The permutation digraph  $G_b$  is defined in a similar fashion. (See Section 2 for a more formal definition.) The permutation  $\tau \in S_n$  that simultaneously conjugates the two tuples is precisely a color and direction preserving isomorphism from  $G_a$  onto  $G_b$  (assuming that permutations in  $S_n$  are multiplied from left to right). The graph isomorphism problem is hard in general: no polynomial time algorithm is known, nor is the problem known to be NP-complete. However, there is a recent result due to Babai [1] presenting a quasipolynomial time algorithm.

In our context, things are fundamentally different. Namely, when considering the connected components of  $G_a$  and  $G_b$ , the additional structure imposed by colors is so strong that every color and direction preserving isomorphism is uniquely determined by the image of one arbitrary vertex. This implies that testing for the existence of such an isomorphism can be done in polynomial time. The first algorithm for the d-SCP in  $S_n$  was proposed in 1977 by Fontet running in time  $O(dn^2)$  [4]. Five years later, the algorithm was independently rediscovered by Hoffmann [5]. An important special case of the SCP occurs when the tuples a and b generate transitive permutation groups or, equivalently, when  $G_a$  and  $G_b$  are connected. This restricted problem, referred to as the transitive SCP, was considered by Sridhar in 1989 [11]. However, his  $O(dn \log(dn))$ -time algorithm does not work correctly as we recently showed in [2]. Moreover, in the same paper we also showed that the transitive SCP can be solved in subquadratic time in n at a given d; more precisely, we developed an algorithm with the running time  $O(n^2 \log d/\log n + dn \log n)$ .

A natural question arises whether the d-SCP in  $S_n$  can also be solved in subquadratic time in n at a given d. The following main result answers the question affirmatively.

**Theorem 1.1.** Given a positive integer d, the d-SCP in the symmetric group  $S_n$  can be solved in  $o(n^2)$  time.

The main idea behind our approach is as follows. First, we define two extreme cases depending on the number of connected components on the one hand, and on the size of individual components on the other hand. Second, a combination of solutions to these two extreme cases then yields the desired result in general.

As for the extreme cases, we say that a connected component is **large**, if it consists of  $\Theta(n)$  vertices, and **small** otherwise. The first extreme case is when a digraph consists of only large components (and consequently, there are O(1) of them). In the other extreme the digraph consists of only small components (and consequently, there are  $\omega(1)$  of them). In the first case, we simply consider each pair of connected components of the same size and test whether they are isomorphic by applying the above mentioned subquadratic algorithm from [2]. As for the other case, a different specially tailored approach is used. To this end, we present a canonical-labeling-based algorithm that takes  $O(dn^2)$  time;

however, when both digraphs consist only of small connected components its running time decreases to subquadratic in n at a given d.

The structure of the paper is the following. Section 2 contains the necessary notation and basic definitions to make the paper self-contained. In Section 3 we present a canonical-labeling-based algorithm for the SCP. The main theorem is proven in Section 4. We conclude the paper by discussing some open problems in Section 5.

# 2 Permutation digraphs and colour-isomorphism

We establish some notation and terminology used in the paper. For the concepts not defined here see [3].

For  $i \in [n]$  and  $g \in S_n$ , we write  $i^g$  for the image of g under the permutation g rather than by the more usual g(i). Let  $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d)$  be a d-tuple of permutations in  $S_n$ . The **permutation digraph** of  $\sigma$  is a pair  $G_{\sigma} = (V, A)$ , where  $V(G_{\sigma}) = V = [n]$  is the set of **vertices**, and  $A(G_{\sigma}) = A$  is the set of ordered pairs  $(i, \sigma_k)$ ,  $i \in [n]$ ,  $k \in [d]$ , called **arcs**. The **size** of  $G_{\sigma}$  is  $|V(G_{\sigma})|$ , while the **degree** of  $G_{\sigma}$  is  $|\sigma|$ . An arc  $e = (i, \sigma_k)$  has its initial vertex ini(e) = i, terminal vertex  $\text{ter}(e) = i^{\sigma_k}$ , and **color** c(e) = k; the vertex  $i^{\sigma_k}$  is also referred to as the **out-neighbour** of i coloured k. The vertices ini(e) and ter(e) are the end-vertices of e.

A walk from a vertex  $v_0$  to a vertex  $v_m$  in a permutation digraph  $G_{\sigma}$  is an alternating sequence  $W = v_0, e_1, v_1, e_2, \ldots, e_m, v_m$  of vertices and arcs in G such that for each  $i \in [m]$ , the vertices  $v_{i-1}$  and  $v_i$  are the end-vertices of the arc  $e_i$ . If for any two vertices u and v in  $G_a$  there is a walk from u to v, we say that  $G_{\sigma}$  is **connected**. Clearly,  $G_{\sigma}$  is connected if and only if the tuple a generates a transitive subgroup of  $S_n$ . A subdigraph H of  $G_{\sigma}$  consists of a subset  $V(H) \subseteq V(G_{\sigma})$  and a subset  $A(H) \subseteq A(G_{\sigma})$  such that every arc in A(H) has both end-vertices in V(H). A walk in a subdigraph H of  $G_{\sigma}$  is a walk in  $G_{\sigma}$  consisting only of arcs from A(H). If  $G_{\sigma}$  is not connected, its maximal connected subdigraphs are called the **connected components** of  $G_{\sigma}$ . Note that there are no arcs between connected components, and so the components are also permutation digraphs of degree d.

A **colour-isomorphism** between two permutation digraphs  $G_a$  and  $G_b$  is a pair  $(\phi_V, \phi_A)$  of bijections, where  $\phi_V \colon V(G_a) \to V(G_b)$  and  $\phi_A \colon A(G_a) \to A(G_b)$  such that  $\phi_V(\text{ini}(e)) = \text{ini}(\phi_A(e))$ ,  $\phi_V(\text{ter}(e)) = \text{ter}(\phi_A(e))$  and  $c(e) = c(\phi_A(e))$  for any arc  $e \in A(V_a)$ . If there is a colour-isomorphism between  $G_a$  and  $G_b$ , we say that  $G_a$  and  $G_b$  are **colour-isomorphic**, and we write  $G_a \cong G_b$ .

Let  $\mathcal{G}$  be the set of permutation digraphs of size n and degree d, and let L be a set of strings over some fixed-sized alphabet. A **labeling function** for  $\mathcal{G}$  is a function  $\mathcal{L} \colon \mathcal{G} \to L$ . Such a function  $\mathcal{L} \colon \mathcal{G} \to L$  is **canonical** whenever for all  $G_a, G_b \in \mathcal{G}$  a colour-isomorphism from  $G_a$  onto  $G_b$  exists if and only if  $\mathcal{L}(G_a) = \mathcal{L}(G_b)$ . In this case,  $\mathcal{L}(G)$  is the **canonical label** of G.

# 3 A canonical labeling algorithm

We present an algorithm for finding a canonical label of a permutation digraph  $G_a$  of size n and degree d based on publicly known techniques. It runs in  $O(dn^2)$  time in general, but in the case when  $G_a$  consists of only small connected components its running time decreases to subquadratic in n at a given d.

We first handle the case when  $G_a$  is connected. For a fixed  $v \in V(G_a)$  we relabel the vertices of  $G_a$  in a breadth-first-search order starting at v, see the algorithm Relabel  $(G_a, v)$ . The out-neighbours of a current vertex are visited in the ascending order of colours of the respective out-going arcs (lines 7-11 in Relabel). Let  $\gamma_v \colon V \to V$  be the resulting relabeling. The relabelled digraph induced by  $\gamma_v$  is  $G_{a^v}$ , where  $a^v = (\gamma_v^{-1} a_1 \gamma_v, \gamma_v^{-1} a_2 \gamma_v, \dots, \gamma_v^{-1} a_d \gamma_v)$  (line 12 in Relabel).

#### **Algorithm** Relabel $(G_a, v)$

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Input: Connected permutation digraph G_a of degree d on n vertices, v \in V(G_a).
    Output: The permutation digraph G_{a^v}.
 1: Initilize an empty queue Q;
 2: Visited = \{v\};
 3: v^{\gamma_v} = 1;
 4: Enqueue v into Q;
    while |Visited| \neq n do
         Dequeue Q into u;
         for k \leftarrow 1 to d do
 7:
             if u^{a_k} \notin \text{Visited then}
8:
                  Add u^{a_k} to Visited;
9:
                  (u^{a_k})^{\gamma_v} = |\text{Visited}|;
10:
                  Enqueue u^{a_k} into Q;
11:
12: Let a^v = (\gamma_v^{-1} a_1 \gamma_v, \gamma_v^{-1} a_2 \gamma_v, \dots, \gamma_v^{-1} a_d \gamma_v);
13: return The digraph G_{a^v};
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The **code** of a permutation digraph  $G_a$  is

$$C(G_a) = 1^{a_1} 2^{a_1} \cdots n^{a_1} 1^{a_2} 2^{a_2} \cdots n^{a_2} \cdots 1^{a_d} 2^{a_d} \cdots n^{a_d},$$

which is a string of length dn over [n] obtained by concatenating, in turn, the images of  $1, 2, \ldots, n$  under the permutations  $a_1, a_2, \ldots, a_d$ . For a connected digraph  $G_a$ , let  $\overline{C}(G_a)$  denote the lexicographically smallest string from among codes  $C(G_{a^v})$ ,  $v \in V(G_a)$ . We now prove that  $\overline{C}(G_a)$  is the canonical label of  $G_a$ .

**Proposition 3.1.** Let  $\mathcal{P}^c$  be the set of all connected permutation digraphs of size n and degree d, and let L be the set of all strings of length dn over [n]. Then the function  $\mathcal{L}^c \colon \mathcal{P}^c \to L$  defined by  $\mathcal{L}^c(G_a) = \overline{C}(G_a)$  is a canonical labeling function for  $\mathcal{P}^c$ .

Proof. We first show that if  $\overline{C}(G_a) = \overline{C}(G_b)$ , then  $G_a$  and  $G_b$  are colour-isomorphic. Let  $u \in V(G_a)$  be a vertex for which the permutation digraph  $G_{a^u}$  returned by Relabel $(G_a, u)$  has code  $C(G_{a^u}) = \overline{C}(G_a)$ . Similarly, let  $w \in V(G_b)$  be a vertex for which the permutation digraph  $G_{b^w}$  returned by Relabel $(G_b, w)$  has code  $C(G_{b^w}) = \overline{C}(G_b)$ . By assumption, it follows that  $C(G_{a^u}) = C(G_{b^w})$ . Hence  $G_{a^u} = G_{b^w}$ , and since  $G_{a^u} \cong G_a$  and  $G_{b^w} \cong G_b$  it follows that  $G_a \cong G_b$ .

Conversely, let f be a colour-isomorphism mapping  $G_a$  onto  $G_b$ , and let  $u \in V(G_a)$  be a vertex for which the permutation digraph  $G_{a^u}$  returned by Relabel $(G_a, u)$  has code  $C(G_{a^u}) = \overline{C}(G_a)$ . Next, let w = f(u) and consider the permutation digraph  $G_{b^w}$  returned by Relabel $(G_b, w)$ . Note that  $G_{a^u} = G_{b^w}$ , and hence  $C(G_{a^u}) = C(G_{b^w})$ . It remains to prove that  $C(G_{b^w}) = \overline{C}(G_b)$ . Suppose to the contrary that for some  $G_{b^z}$  returned by Relabel $(G_b, z)$ , the string  $C(G_{b^z})$  is lexicographically smaller than the string  $C(G_{b^w})$ .

Consider now the permutation digraph  $G_{af^{-1}(z)}$  returned by Relabel  $(G_a, f^{-1}(z))$ . Similarly as above,  $C(G_{b^z}) = C(G_{af^{-1}(z)})$ . Since  $C(G_{b^z})$  is lexicographically smaller than  $C(G_{a^u})$ . A contradiction.

Next, we bound the time complexity of computing  $\overline{C}(G_a)$ .

**Lemma 3.2.** The canonical label  $\overline{C}(G_a)$  of a connected permutation digraph  $G_a$  of size n and degree d can be computed in  $O(dn^2)$  time.

Proof. One call of Relabel  $(G_a, v)$  takes time O(dn) in order to construct  $G_{a^v}$ , while its code  $C(G_{a^v})$  can also be computed in linear time. Since this has to be repeated for each  $v \in V(G_a)$ , the total running time for constructing the codes is  $O(dn^2)$ . Clearly, choosing the lexicographically smallest code does not increase this time bound.

In the reminder of this section we consider the case when  $G_a$  is not connected. Let us denote the connected components of  $G_a$  by  $H_1, H_2, \ldots, H_k$ , and recall that each such component is a permutation digraph of degree d. Further, let us concatenate the respective canonical labels  $\overline{C}(H_1), \overline{C}(H_2), \ldots, \overline{C}(H_k)$  in such an order that the resulting string  $C^*(G_a)$  is lexicographically smallest. The following result shows that  $C^*(G_a)$  is the canonical label of  $G_a$ .

**Theorem 3.3.** Let  $\mathcal{P}$  be the set of all permutation digraphs of size n and degree d, and let L be the set of all strings of length dn over [n]. Then the function  $\mathcal{L} \colon \mathcal{P} \to L$  defined by  $\mathcal{L}(G_a) = C^*(G_a)$  is a canonical labeling function for  $\mathcal{P}$ .

*Proof.* The proof follows from the description of  $C^*(G_a)$  above and Proposition 3.1.

In the next section we will make use of the following result regarding permutation digraphs when all connected components have equal size.

Corollary 3.4. If a permutation digraph  $G_a$  of size n and degree d consists of precisely k equal-sized connected components, then its canonical label  $C^*(G_a)$  can be computed in  $O(dn^2/k)$  time.

Proof. Let  $H_1, H_2, \ldots, H_k$  be the connected components of  $G_a$ . Since each component  $H_i$  is of size n/k we can find its canonical label  $\overline{C}(H_i)$ , by Lemma 3.2, in  $O(d(n/k)^2)$  time. Consequently, the total running time for constructing the canonical labels of all components is  $O(dn^2/k)$ . To compute  $C^*(G_a)$ , all we need to do is to sort these labels. Using radix sort this can be done in time O(dn/k(n/k+k)), which obviously does not increase the time bound  $O(dn^2/k)$ .

## 4 Proof of main result

Recall that a tuple a is simultaneously conjugate to a tuple b if and only if the permutation digraph  $G_a$  is color-isomorphic to the permutation digraph  $G_b$ . Before considering the general case when the digraphs  $G_a$  and  $G_b$  have variable size components, we deal with two extreme cases, namely, when the digraphs have either only equal-sized small components or only equal-sized large components.

**Lemma 4.1.** Let  $G_a$  and  $G_b$  be permutation digraphs, each of size n and degree d, and let both  $G_a$  and  $G_b$  consist of only small equal-sized connected components. Then we can test whether  $G_a$  and  $G_b$  are colour-isomorphic in time  $o(n^2)$  at a given d.

*Proof.* Let k be the number of connected components. By Corollary 3.4 we can compute the canonical labels of both digraphs and hence perform the isomorphism test in time  $O(dn^2/k)$ . Since all components are small, we have  $k = \omega(1)$  and consequently  $O(dn^2/k) = o(n^2)$  at a given d.

**Lemma 4.2.** Let  $G_a$  and  $G_b$  be permutation digraphs, each of size n and degree d, and let both  $G_a$  and  $G_b$  consist of only large equal-sized connected components. Then we can test whether  $G_a$  and  $G_b$  are colour-isomorphic in  $O(n^2 \log d/\log n + dn \log n)$  time.

*Proof.* Let k be the number of connected components. Since all components are large, we have k = O(1). Obviously, at most  $k^2 = O(1)$  pairs of components, each of size n/k = O(n), need to be tested for isomorphism. By [2], this requires a total of  $O(n^2 \log d/\log n + dn \log n)$  time.

We are now ready to prove the main result.

Proof of Theorem 1.1. Finding the components of  $G_a$  and  $G_b$  requires O(dn) time. If  $G_a$  and  $G_b$  do not have an equal number of components of the same size, they are not isomorphic, which can be tested by sorting the sizes of the components in time  $o(n^2)$ . So, let  $G_a$  and  $G_b$  have  $p_i$  components of size  $n_i$ ,  $i \in [r]$ , where without loss of generality we may assume that components of sizes  $n_1, n_2, \ldots, n_j$  are large, and the remaining ones are small. Obviously, components of different sizes can be tested separately. By Lemma 4.2, we can test  $p_i = O(1)$  large components of size  $n_i$  for isomorphism in time  $O(n_i^2 \log d/\log(n_i) + dn_i \log n_i)$ . On the other hand, by Corollary 3.4, we can test  $p_i$  small components of size  $n_i$  in time  $O(dp_i n_i^2)$ . Finally, for a large enough constant c the total time is bounded from above by

$$c\left(\sum_{i=1}^{j} \frac{dn_i^2}{\log n_i} + \sum_{i=j+1}^{r} dp_i n_i^2\right) \le cd \max\left\{\frac{n_1}{\log(n_1)}, \dots, \frac{n_j}{\log(n_j)}, n_{j+1}, \dots, n_r\right\} \sum_{i=1}^{r} p_i n_i.$$

The max-term is o(n) since  $n_i/\log(n_i) = o(n)$  for each large component as well as  $n_i = o(n)$  for each small component. The final result follows as  $\sum_{i=1}^r p_i n_i = n$ .

# 5 Concluding remarks

It remains an open problem whether for a given positive integer d the d-SCP in the symmetric group  $S_n$  can be solved in a strongly subqudratic time in n, that is, in time  $O(n^{2-\epsilon})$  for some  $\epsilon > 0$ . Further, completely unanswered is the question of the problem's lower bound, except for the trivial one,  $\Omega(n)$ . The obvious question is whether it can be raised to  $\Omega(n \log n)$  reflecting erroneous Sridhar's upper bound, or even to a higher bound by proving conditional lower bounds based on conjectures of hardness for well-studied problems, as it was already done for a number of other problems.

### References

- [1] L. Babai, Graph isomorphism in quasipolynomial time [extended abstract]. In: *Proceedings of the Forty-eighth Annual ACM Symposium on Theory of Computing*, STOC 16, pages 684–697, New York, NY, USA, 2016.
- [2] A. Brodnik, A. Malnič, R. Požar, The simultaneous conjugacy problem in the symmetric group, arXiv:1907.07889v2.
- [3] R. Diestel, "Graph Theory", Springer-Verlag, New York, 2005.
- [4] M. Fontet, Calcul de Centralisateur d'un Grupe de Permutatations, Bull. Soc. Math. France Mem. 49-50 (1977), 53-63.
- [5] C. M Hoffmann, Subcomplete Generalization of Graph Isomorphism. J. of Comp. and Sys. Sci. 25 (1982), 332–359.
- [6] A. Malnič, R. Nedela, M. Škoviera, Lifting graph automorphisms by voltage assignments. *European J. Combin.* **21** (2000), 927–947.
- [7] A. Malnič, R. Nedela, M. Škoviera, Regular homomorphisms and regular maps. *European J. Combin.* **23** (2002), 449–461.
- [8] P. Potočnik, R. Požar, Smallest tetravalent half-arc-transitive graphs with the vertex-stabiliser isomorphic to the dihedral group of order 8, J. Combin. Theory Ser. A 145 (2017), 172–183.
- [9] R. Požar, Computing stable epimorphisms onto finite groups. *J. Symbolic Comput.* **92** (2019), 22–30.
- [10] Á. Seress, "Permutation group algorithms", Cambridge Tracts in Mathematics 152. Cambridge University Press, 2003.
- [11] M. A. Sridhar, A fast algorithm for testing isomorphism of permutation networks. *IEEE Trans. Computers (TC)* **38(6)** (1989), 903–909.
- [12] A. Yavuz Oruç, M. Yaman Oruç, On testing isomorphism of permutation networks. *IEEE Trans. Computers (TC)* **34** (1985), 958-962.