# Counting spanning trees in a complete bipartite graph which contain a given spanning forest 

Fengming Dong ${ }^{1 *}$ and Jun $\mathrm{Ge}^{2 \dagger}$<br>${ }^{1}$ National Institute of Education, Nanyang Technological University, Singapore<br>${ }^{2}$ School of Mathematical Sciences \& Laurent Mathematics Center, Sichuan Normal University, China


#### Abstract

In this article, we extend Moon's classic formula for counting spanning trees in complete graphs containing a fixed spanning forest to complete bipartite graphs. Let $(X, Y)$ be the bipartition of the complete bipartite graph $K_{m, n}$ with $|X|=m$ and $|Y|=n$. We prove that for any given spanning forest $F$ of $K_{m, n}$ with components $T_{1}, T_{2}, \ldots, T_{k}$, the number of spanning trees in $K_{m, n}$ which contain all edges in $F$ is equal to


$$
\frac{1}{m n}\left(\prod_{i=1}^{k}\left(m_{i} n+n_{i} m\right)\right)\left(1-\sum_{i=1}^{k} \frac{m_{i} n_{i}}{m_{i} n+n_{i} m}\right)
$$

where $m_{i}=\left|V\left(T_{i}\right) \cap X\right|$ and $n_{i}=\left|V\left(T_{i}\right) \cap Y\right|$ for $i=1,2, \ldots, k$.
Keywords: spanning tree; multigraph; weighted graph; complete bipartite graph
Mathematics Subject Classification (2010): 05C30, 05C05

## 1 Introduction

In this paper, we assume that all graphs are loopless, while parallel edges are allowed. For any graph $G$, let $V(G)$ and $E(G)$ be the vertex set and edge set of $G$. For any edge set $F \subseteq E(G)$, let $G / F$ be the graph obtained from $G$ by contracting all edges in $F$, and removing all loops. Let $\mathcal{T}(G)$ denote the set of spanning trees of $G$. For any positive integer $k$, let $\llbracket k \rrbracket$ denote the set $\{1,2, \cdots, k\}$.

Suppose $G$ is a weighted graph with weight function $\omega: E(G) \rightarrow \mathbb{R}$. For any $F \subseteq E(G)$ and any subgraph $H$ of $G$, define $\omega(F)=\prod_{e \in F} \omega(e)$ and $\omega(H)=\omega(E(H))$. Let $\tau(G, \omega)=\sum_{T \in \mathcal{T}(G)} \omega(T)$. Sometimes we use $\tau(G)$ instead of $\tau(G, \omega)$ when there is no confusion. It is obvious that for an unweighted graph $G$ (that is to say, a weighted graph with unit weight on each edge), $\tau(G)=|\mathcal{T}(G)|$, i.e., the number of spanning trees of $G$. Throughout this paper, any graph is assumed to be unweighted, unless it is claimed.

Counting spanning trees in graphs is a very old topic in graph theory having modern connections with many other fields in mathematics, statistical physics and theoretical computer science, such as

[^0]random walks, the Ising model and Potts model, network reliability, parking functions, knot/link determinants. See [2,4, 7, 10 for some recent work on counting spanning trees.

For a subgraph $H$ of $G$, let $\mathcal{T}_{H}(G)$ denote the set of spanning trees $T \in \mathcal{T}(G)$ with $E(H) \subseteq E(T)$, and let $\tau_{H}(G)=\sum_{T \in \mathcal{T}_{H}(G)} \omega(T)$. For unweighted graph $G, \tau_{H}(G)=\left|\mathcal{T}_{H}(G)\right|$, i.e., the number of spanning trees of $G$ containing all edges in $H$. Note that usually the graph $H$ here is a forest or a tree, because otherwise $\mathcal{T}_{H}(G)=\emptyset$ and $\tau_{H}(G)=0$.

The celebrated Cayley's formula [1] states that $\tau\left(K_{n}\right)=n^{n-2}$. In 1964, Moon generalized Cayley's formula by obtaining a nice expression of $\tau_{F}\left(K_{n}\right)$ for any spanning forest $F$ of $K_{n}$.

Theorem 1 ( [9], also see Problem 4.4 in [8]). For any spanning forest $F$ of $K_{n}$, if $c$ is the number of components of $F$ and $n_{1}, n_{2}, \ldots, n_{c}$ are the orders of those components, then

$$
\tau_{F}\left(K_{n}\right)=n^{c-2} \prod_{i=1}^{c} n_{i} .
$$

It is easy to see that Cayley's formula is the special case that $F$ is an empty graph. It is also well known that $\tau\left(K_{m, n}\right)=m^{n-1} n^{m-1}$ for any complete bipartite graph $K_{m, n}$ by Fiedler and Sedláček [5]. So there is a natural question: is there a bipartite analogue of Moon's formula (Theorem 1)? That is to say, for any given spanning forest $F$ in $K_{m, n}$, what is the explicit expression of $\tau_{F}\left(K_{m, n}\right)$ ?

It turns out that this question is much harder than the case of complete graphs. In 6], this question was partially answered for two special cases: $F$ is a matching or a tree plus several possible isolated vertices.

Theorem 2 ( [6]). For any matching $M$ of size $k$ in $K_{m, n}$,

$$
\tau_{M}\left(K_{m, n}\right)=(m+n)^{k-1}(m+n-k) m^{n-k-1} n^{m-k-1}
$$

Theorem 3 ( [6]). For any tree $T$ of $K_{m, n}$,

$$
\tau_{T}\left(K_{m, n}\right)=(s n+t m-s t) m^{n-t-1} n^{m-s-1}
$$

where $s=|V(T) \cap X|, t=|V(T) \cap Y|$, and $(X, Y)$ is the bipartition of $K_{m, n}$ with $|X|=m$ and $|Y|=n$.

In this paper, we obtain an explicit expression for $\tau_{F}\left(K_{m, n}\right)$ for an arbitrary spanning forest $F$ of $K_{m, n}$.

Theorem 4. Let $(X, Y)$ be the bipartition of $K_{m, n}$ with $|X|=m$ and $|Y|=n$. For any spanning forest $F$ of $K_{m, n}$ with components $T_{1}, T_{2}, \ldots, T_{k}$,

$$
\begin{equation*}
\tau_{F}\left(K_{m, n}\right)=\frac{1}{m n}\left(\prod_{i=1}^{k}\left(m_{i} n+n_{i} m\right)\right)\left(1-\sum_{i=1}^{k} \frac{m_{i} n_{i}}{m_{i} n+n_{i} m}\right) \tag{1}
\end{equation*}
$$

where $m_{i}=\left|X \cap V\left(T_{i}\right)\right|$ and $n_{i}=\left|Y \cap V\left(T_{i}\right)\right|$ for all $i \in \llbracket k \rrbracket$.

## 2 Preliminary results

This section provides some results which will be applied in the next section for proving a key identity.

Lemma 1. For any set of $k$ pairs of real numbers $\left\{a_{i}, b_{i}: i \in \llbracket k \rrbracket\right\}$, where $k \geq 1$, if $a_{i} B+b_{i} A \neq 0$ for all $i \in \llbracket k \rrbracket$ where $A=a_{1}+a_{2}+\cdots+a_{k}$ and $B=b_{1}+b_{2}+\cdots+b_{k}$, then

$$
\begin{equation*}
\left(1-\sum_{i=1}^{k} \frac{a_{i} b_{i}}{a_{i} B+b_{i} A}\right)^{2}=\left(\sum_{i=1}^{k} \frac{a_{i}^{2}}{a_{i} B+b_{i} A}\right) \cdot\left(\sum_{i=1}^{k} \frac{b_{i}^{2}}{a_{i} B+b_{i} A}\right) . \tag{2}
\end{equation*}
$$

Proof. Since $a_{i} B+b_{i} A \neq 0, A$ and $B$ cannot both be 0 .
When $A=0$ and $B \neq 0$, equality (2) holds because

$$
\left(1-\sum_{i=1}^{k} \frac{a_{i} b_{i}}{a_{i} B+b_{i} A}\right)^{2}=\left(1-\sum_{i=1}^{k} \frac{b_{i}}{B}\right)^{2}=0
$$

and

$$
\left(\sum_{i=1}^{k} \frac{a_{i}^{2}}{a_{i} B+b_{i} A}\right) \cdot\left(\sum_{i=1}^{k} \frac{b_{i}^{2}}{a_{i} B+b_{i} A}\right)=\frac{A}{B} \cdot \sum_{i=1}^{k} \frac{b_{i}^{2}}{a_{i} B}=0 .
$$

Similarly, equality (22) holds when $B=0$ and $A \neq 0$. Let $W_{i}=a_{i} B+b_{i} A$ for $i \in \llbracket k \rrbracket$. When $A \neq 0$ and $B \neq 0$, observe that

$$
\begin{equation*}
A \sum_{i=1}^{k} \frac{a_{i} b_{i}}{W_{i}}+B \sum_{i=1}^{k} \frac{a_{i}^{2}}{W_{i}}=\sum_{i=1}^{k} \frac{a_{i}\left(A b_{i}+B a_{i}\right)}{W_{i}}=\sum_{i=1}^{k} a_{i}=A, \tag{3}
\end{equation*}
$$

implying that

$$
\begin{equation*}
\sum_{i=1}^{k} \frac{a_{i}^{2}}{W_{i}}=\frac{A}{B}\left(1-\sum_{i=1}^{k} \frac{a_{i} b_{i}}{W_{i}}\right) \tag{4}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\sum_{i=1}^{k} \frac{b_{i}^{2}}{W_{i}}=\frac{B}{A}\left(1-\sum_{i=1}^{k} \frac{a_{i} b_{i}}{W_{i}}\right) \tag{5}
\end{equation*}
$$

Clearly, equality (2) follows from (4) and (5).
For a set $A$ of real numbers, in the following, if $A$ is the empty set, we set

$$
\begin{equation*}
\prod_{a \in A} a=1 \quad \text { and } \quad \sum_{a \in A} a=0 \tag{6}
\end{equation*}
$$

Lemma 2. Let $S$ be a set of positive integers. For any set of $2|S|$ real numbers $\left\{a_{i}, b_{i}: i \in S\right\}$,

$$
\begin{equation*}
\sum_{\emptyset \neq I \subseteq S}\left(\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\prod_{i \in S}\left(a_{i}+b_{i}\right)-\prod_{i \in S} b_{i} . \tag{7}
\end{equation*}
$$

Proof. The identity follows from the following fact:

$$
\begin{equation*}
\sum_{\emptyset \subseteq I \subseteq S}\left(\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\prod_{i \in S}\left(a_{i}+b_{i}\right) \tag{8}
\end{equation*}
$$

Lemma 3. Let $S$ be a set of positive integers. For any set of $3|S|$ real numbers $\left\{a_{i}, b_{i}, c_{i}: i \in S\right\}$,

$$
\begin{equation*}
\sum_{\emptyset \neq I \subseteq S}\left(\left(\sum_{i \in I} c_{i}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\left(\prod_{j \in S}\left(a_{j}+b_{j}\right)\right) \cdot \sum_{i \in S} \frac{c_{i} a_{i}}{a_{i}+b_{i}} \tag{9}
\end{equation*}
$$

Proof. The identity follows from the following fact:

$$
\begin{equation*}
\sum_{\emptyset \neq I \subseteq S}\left(\left(\sum_{i \in I} c_{i}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\sum_{i \in S}\left(c_{i} a_{i} \prod_{j \in S \backslash\{i\}}\left(a_{j}+b_{j}\right)\right) . \tag{10}
\end{equation*}
$$

By Lemmas 2 and 3, for an arbitrary real number $c$, we have

$$
\begin{equation*}
\sum_{\emptyset \neq I \subseteq S}\left(\left(c+\sum_{i \in I} c_{i}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\left(\prod_{j \in S}\left(a_{j}+b_{j}\right)\right) \cdot\left(c+\sum_{i \in S} \frac{c_{i} a_{i}}{a_{i}+b_{i}}\right)-c \prod_{j \in S} b_{j} \tag{11}
\end{equation*}
$$

Lemma 4. Let $S$ be a set of positive integers. For any set of $3|S|$ real numbers $\left\{a_{i}, b_{i}, d_{i}: i \in S\right\}$,

$$
\begin{equation*}
\sum_{\emptyset \neq I \subseteq S}\left(\left(\sum_{i \in S \backslash I} d_{i}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\left(\prod_{j \in S}\left(a_{j}+b_{j}\right)\right) \cdot\left(\sum_{i \in S} \frac{d_{i} b_{i}}{a_{i}+b_{i}}\right)-\left(\prod_{r \in S} b_{r}\right) \cdot\left(\sum_{i \in S} d_{i}\right) \tag{12}
\end{equation*}
$$

Proof. It follows from Lemma 3 and the following fact:

$$
\begin{equation*}
\sum_{\emptyset \neq I \subseteq S}\left(\left(\sum_{i \in S \backslash I} d_{i}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\sum_{\emptyset \neq I \subseteq S}\left(\left(\sum_{i \in I} d_{i}\right) \prod_{j \in S \backslash I} a_{j} \prod_{r \in I} b_{r}\right)-\left(\prod_{r \in S} b_{r}\right) \cdot\left(\sum_{i \in S} d_{i}\right) . \tag{13}
\end{equation*}
$$

Lemma 5. Let $S$ be a set of positive integers. For any set of $4|S|$ real numbers $\left\{a_{i}, b_{i}, c_{i}, d_{i}: i \in S\right\}$,

$$
\begin{equation*}
\sum_{\emptyset \neq I \subsetneq S}\left(\left(\sum_{i \in I} c_{i}\right)\left(\sum_{q \in S \backslash I} d_{q}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\left(\prod_{r \in S}\left(a_{r}+b_{r}\right)\right) \cdot \sum_{\substack{i, j \in S \\ i \neq j}} \frac{c_{i} a_{i} d_{j} b_{j}}{\left(a_{i}+b_{i}\right)\left(a_{j}+b_{j}\right)} \tag{14}
\end{equation*}
$$

Proof. The result follows from the following fact:

$$
\begin{equation*}
\sum_{\emptyset \neq I \subsetneq S}\left(\left(\sum_{i \in I} c_{i}\right)\left(\sum_{q \in S \backslash I} d_{q}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right)=\sum_{\substack{i, j \in S \\ i \neq j}}\left(c_{i} a_{i} d_{j} b_{j} \prod_{\substack{r \in S \backslash\{i, j\}}}\left(a_{r}+b_{r}\right)\right) \tag{15}
\end{equation*}
$$

By Lemmas 4 and 5, for an arbitrary real number $c$, we have

$$
\begin{align*}
& \sum_{\emptyset \neq I \subseteq S}\left(\left(c+\sum_{i \in I} c_{i}\right)\left(\sum_{q \in S \backslash I} d_{q}\right)\left(\prod_{j \in I} a_{j}\right)\left(\prod_{r \in S \backslash I} b_{r}\right)\right) \\
= & \left(\prod_{r \in S}\left(a_{r}+b_{r}\right)\right) \cdot\left(\sum_{\substack{i, j \in S \\
i \neq j}} \frac{c_{i} a_{i} d_{j} b_{j}}{\left(a_{i}+b_{i}\right)\left(a_{j}+b_{j}\right)}+c \sum_{i \in S} \frac{d_{i} b_{i}}{a_{i}+b_{i}}\right)-c\left(\prod_{r \in S} b_{r}\right) \cdot\left(\sum_{i \in S} d_{i}\right) . \tag{16}
\end{align*}
$$

## 3 An identity

Define a function $\phi$ on $2 k$ variables $x_{1}, x_{2}, \ldots, x_{k}$ and $y_{1}, y_{2}, \ldots, y_{k}$, where $k \geq 1$, as follows:

$$
\begin{equation*}
\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{k}, y_{k}\right)=\frac{1}{X Y}\left(\prod_{i=1}^{k}\left(x_{i} Y+y_{i} X\right)\right)\left(1-\sum_{i=1}^{k} \frac{x_{i} y_{i}}{x_{i} Y+y_{i} X}\right) \tag{17}
\end{equation*}
$$

where $X=x_{1}+x_{2}+\cdots+x_{k}$ and $Y=y_{1}+y_{2}+\cdots+y_{k}$. Observe that

$$
\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{k}, y_{k}\right)=\frac{1}{X Y}\left(\left(\prod_{i=1}^{k}\left(x_{i} Y+y_{i} X\right)\right)-\sum_{i=1}^{k}\left(x_{i} y_{i} \prod_{\substack{1 \leq j \leq k \\ j \neq i}}\left(x_{j} Y+y_{j} X\right)\right)\right)
$$

In the expansion of $\left(\prod_{i=1}^{k}\left(x_{i} Y+y_{i} X\right)\right)-\sum_{i=1}^{k}\left(x_{i} y_{i} \prod_{\substack{1 \leq j \leq k \\ j \neq i}}\left(x_{j} Y+y_{j} X\right)\right)$, the expression consisting of all monomials not divisible by $X Y$ is identically 0 , as shown below:

$$
\begin{align*}
& Y^{k} \prod_{i=1}^{k} x_{i}+X^{k} \prod_{i=1}^{k} y_{i}-\sum_{i=1}^{k}\left(y_{i} Y^{k-1} \prod_{j=1}^{k} x_{j}\right)-\sum_{i=1}^{k}\left(x_{i} X^{k-1} \prod_{j=1}^{k} y_{j}\right) \\
= & Y^{k} \prod_{j=1}^{k} x_{j}+X^{k} \prod_{j=1}^{k} y_{j}-Y^{k-1} \prod_{j=1}^{k} x_{j} \cdot\left(\sum_{i=1}^{k} y_{i}\right)-X^{k-1} \prod_{j=1}^{k} y_{j} \cdot\left(\sum_{i=1}^{k} x_{i}\right) \\
= & 0 . \tag{18}
\end{align*}
$$

It follows that $\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{k}, y_{k}\right)$ is a polynomial on $2 k$ variables $x_{1}, x_{2}, \ldots, x_{k}$ and $y_{1}, y_{2}, \ldots, y_{k}$.
For $k \leq 3$,

$$
\left\{\begin{array}{l}
\phi\left(x_{1}, y_{1}\right)=1  \tag{19}\\
\phi\left(x_{1}, y_{1}, x_{2}, y_{2}\right)=z_{1,2} \\
\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, x_{3}, y_{3}\right)=z_{1,2} z_{1,3}+z_{1,2} z_{2,3}+z_{1,3} z_{2,3}
\end{array}\right.
$$

where $z_{i, j}=x_{i} y_{j}+x_{j} y_{i}$ for all $1 \leq i<j \leq 3$. For any $I \subseteq \llbracket k \rrbracket \backslash\{1\}$, let

$$
\begin{equation*}
x_{I}=x_{1}+\sum_{i \in I} x_{i}, \quad y_{I}=y_{1}+\sum_{i \in I} y_{i} . \tag{20}
\end{equation*}
$$

In this section, we shall establish the following identity, which will be applied to prove the main result in the article.

Theorem 5. For any $2 k$ real numbers $x_{1}, x_{2}, \ldots, x_{k}$ and $y_{1}, y_{2}, \ldots, y_{k}$, where $k \geq 2$,

$$
\begin{equation*}
\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{k}, y_{k}\right)=\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}((-1)^{|I|-1}\left(\prod_{j \in I}\left(x_{1} y_{j}+x_{j} y_{1}\right)\right) \phi(x_{I}, y_{I}, \underbrace{x_{s}, y_{s}}_{\forall s \in \llbracket k \rrbracket \backslash(I \cup\{1\})})), \tag{21}
\end{equation*}
$$

where $\phi(x_{I}, y_{I}, \underbrace{x_{s}, y_{s}}_{\forall s \in \llbracket \rrbracket \backslash(I \cup\{1\})})=\phi\left(x_{I}, y_{I}, x_{i_{1}}, y_{i_{2}}, \ldots, x_{i_{r}}, y_{i_{r}}\right),\left\{i_{1}, i_{2}, \ldots, i_{r}\right\}=\llbracket k \rrbracket \backslash(I \cup\{1\})$ and $r=k-1-|I|$.

Proof. Let $I$ be a non-empty subset of $\llbracket k \rrbracket \backslash\{1\}$. Then

$$
\begin{equation*}
x_{I}+\sum_{s \in \llbracket k \rrbracket \backslash(I \cup\{1\})} x_{s}=\sum_{i=1}^{k} x_{i}=X \tag{22}
\end{equation*}
$$

and

$$
\begin{equation*}
y_{I}+\sum_{s \in \llbracket k] \backslash(I \cup\{1\})} y_{s}=\sum_{i=1}^{k} y_{i}=Y . \tag{23}
\end{equation*}
$$

In the remainder of the proof of Theorem 5 , let $W_{I}=x_{I} Y+y_{I} X$, and for each $i \in \llbracket k \rrbracket$, let

$$
W_{i}=x_{i} Y+y_{i} X \text { and } w_{i}=x_{i} y_{1}+y_{i} x_{1} .
$$

By the definition of the function $\phi$,

$$
\begin{equation*}
\phi(x_{I}, y_{I}, \underbrace{x_{s}, y_{s}}_{\forall s \in \llbracket k \rrbracket \backslash(I \cup\{1\})})=\frac{W_{I}}{X Y} \cdot\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} W_{i}\right)\left(1-\frac{x_{I} y_{I}}{W_{I}}-\sum_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} \frac{x_{i} y_{i}}{W_{i}}\right) . \tag{24}
\end{equation*}
$$

Thus, the right-hand side of (21) can be expressed as

$$
\begin{equation*}
\frac{1}{X Y}\left(\Gamma_{1}-\Gamma_{2}-\Gamma_{3}\right), \tag{25}
\end{equation*}
$$

where

$$
\begin{align*}
& \Gamma_{1}=\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1} W_{I}\left(\prod_{j \in I} w_{j}\right)\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} W_{i}\right)\right)  \tag{26}\\
& \Gamma_{2}= \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1} x_{I} y_{I}\left(\prod_{j \in I} w_{j}\right)\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} W_{i}\right)\right)  \tag{27}\\
& \Gamma_{3}=\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1} W_{I}\left(\prod_{j \in I} w_{j}\right)\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} W_{i}\right)\left(\sum_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} \frac{x_{i} y_{i}}{W_{i}}\right)\right) . \tag{28}
\end{align*}
$$

In the following, we shall apply Lemmas 1-5 to simplify $\Gamma_{1}, \Gamma_{2}$ and $\Gamma_{3}$ in order to show that

$$
\begin{equation*}
\Gamma_{1}-\Gamma_{2}-\Gamma_{3}=\left(\prod_{i=1}^{k}\left(x_{i} Y+y_{i} X\right)\right)\left(1-\sum_{i=1}^{k} \frac{x_{i} y_{i}}{x_{i} Y+y_{i} X}\right) \tag{29}
\end{equation*}
$$

Let $X^{\prime}=X-x_{1}$ and $Y^{\prime}=Y-y_{1}$. Also let

$$
\left\{\begin{array}{l}
Z=\prod_{i=2}^{k}\left(x_{i} Y^{\prime}+y_{i} X^{\prime}\right)  \tag{30}\\
Z_{1}=\sum_{i=2}^{k} \frac{x_{i} y_{i}}{x_{i} Y^{\prime}+y_{i} X^{\prime}}, \quad Z_{2}=\sum_{i=2}^{k} \frac{x_{i}^{2}}{x_{i} Y^{\prime}+y_{i} X^{\prime}}, \quad Z_{3}=\sum_{i=2}^{k} \frac{y_{i}^{2}}{x_{i} Y^{\prime}+y_{i} X^{\prime}}
\end{array}\right.
$$

Note that for any non-empty subset $I$ of $\llbracket k \rrbracket \backslash\{1\}$,

$$
\begin{equation*}
y_{I} X+x_{I} Y-x_{I} y_{I}=\left(y_{1}+\sum_{i \in I} y_{i}\right) X+\left(x_{1}+\sum_{i \in I} x_{i}\right)\left(\sum_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} y_{i}\right) . \tag{31}
\end{equation*}
$$

In the remainder of this section, let $W_{i}^{\prime}$ denote $x_{i} Y^{\prime}+y_{i} X^{\prime}$ for each $i \in \llbracket k \rrbracket$. By applying identities (11), (16) and (31),

$$
\begin{array}{cc} 
& \Gamma_{1}-\Gamma_{2} \\
= & \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1}\left(W_{I}-x_{I} y_{I}\right)\left(\prod_{j \in I} w_{j}\right)\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} W_{i}\right)\right) \\
b y \xlongequal{b 3} \quad & X \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1}\left(y_{1}+\sum_{i \in I} y_{i}\right)\left(\prod_{j \in I} w_{j}\right)\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} W_{i}\right)\right)
\end{array}
$$

$$
\begin{align*}
& +\sum_{\emptyset \neq I \subseteq \llbracket k \backslash \backslash\{1\}}\left((-1)^{|I|-1}\left(x_{1}+\sum_{i \in I} x_{i}\right)\left(\sum_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} y_{i}\right)\left(\prod_{j \in I} w_{j}\right)\left(\prod_{i \in \llbracket[\rrbracket \backslash(I U\{1\})} W_{i}\right)\right) \\
& =\quad-X \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left(\left(y_{1}+\sum_{i \in I} y_{i}\right)\left(\prod_{j \in I}\left(-w_{j}\right)\right)\left(\prod_{i \in \llbracket k\rfloor \backslash(I \cup\{1\})} W_{i}\right)\right) \\
& -\sum_{\emptyset \neq I \subseteq \llbracket k] \backslash\{1\}}\left(\left(x_{1}+\sum_{i \in I} x_{i}\right)\left(\sum_{i \in \llbracket k] \backslash(I \cup\{1\})} y_{i}\right)\left(\prod_{j \in I}\left(-w_{j}\right)\right)\left(\prod_{i \in \llbracket k] \backslash(I \cup\{1\})} W_{i}\right)\right) \\
& \text { by [11), (16) } X y_{1} \prod_{i=2}^{k} W_{i}-X \prod_{j=2}^{k} W_{j}^{\prime}\left(y_{1}-\sum_{i=2}^{k} \frac{y_{i} w_{i}}{W_{i}^{\prime}}\right)+x_{1}\left(\prod_{i=2}^{k} W_{i}\right)\left(\sum_{j=2}^{k} y_{j}\right) \\
& +\left(\prod_{r=2}^{k} W_{r}^{\prime}\right)\left(\sum_{\substack{2 \leq i, j \leq k \\
i \neq j}} \frac{x_{i} y_{j} w_{i} W_{j}}{W_{i}^{\prime} W_{j}^{\prime}}-x_{1} \sum_{i=2}^{k} \frac{y_{i} W_{i}}{W_{i}^{\prime}}\right) \\
& =\quad \prod_{i=1}^{k} W_{i}-x_{1} y_{1} \prod_{i=2}^{k} W_{i}-X Z\left(y_{1}-x_{1} Z_{3}-y_{1} Z_{1}\right)+Z\left(-x_{1}\left(Y Z_{1}+X Z_{3}\right)+\sum_{\substack{\leq \leq, j, j \leq k \\
i \neq j}} \frac{x_{i} y_{j} w_{i} W_{j}}{W_{i}^{\prime} W_{j}^{\prime}}\right) \\
& =\quad \prod_{i=1}^{k} W_{i}-x_{1} y_{1} \prod_{i=2}^{k} W_{i}-Z\left(y_{1} X-y_{1} X Z_{1}+x_{1} Y Z_{1}\right)+Z \sum_{\substack{\leq \leq i, j \leq k \\
i \neq j}} \frac{x_{i} y_{j} w_{i} W_{j}}{W_{i}^{\prime} W_{j}^{\prime}}, \tag{32}
\end{align*}
$$

where in the second last equality, we combine $X y_{1} \prod_{i=2}^{k} W_{i}$ and $x_{1} \prod_{i=2}^{k} W_{i}\left(\sum_{j=2}^{k} y_{j}\right)$ to obtain $\prod_{i=1}^{k} W_{i}-$ $x_{1} y_{1} \prod_{i=2}^{k} W_{i}$.

By applying identity (16), we have

$$
\begin{align*}
\Gamma_{3} & =-\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left(\left(W_{1}+\sum_{i \in I} W_{i}\right)\left(\sum_{i \in \llbracket k \backslash \backslash(I \cup\{1\})} \frac{x_{i} y_{i}}{W_{i}}\right)\left(\prod_{j \in I}\left(-w_{j}\right)\right)\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} W_{i}\right)\right) \\
& \stackrel{b y}{ } \stackrel{\lfloor 16}{=} W_{1}\left(\prod_{i=2}^{k} W_{i}\right)\left(\sum_{i=2}^{k} \frac{x_{i} y_{i}}{W_{i}}\right)+\left(\prod_{i=2}^{k} W_{i}^{\prime}\right)\left(-W_{1} \sum_{i=2}^{k} \frac{x_{i} y_{i}}{W_{i}^{\prime}}+\sum_{\substack{2 \leq i, j \leq k \\
i \neq j}} \frac{w_{i} W_{i} x_{j} y_{j}}{W_{i}^{\prime} W_{j}^{\prime}}\right) \\
& =\left(\prod_{i=1}^{k} W_{i}\right)\left(\sum_{i=2}^{k} \frac{x_{i} y_{i}}{W_{i}}\right)-W_{1} Z Z_{1}+Z \sum_{\substack{2 \leq i, i \leq \leq k \\
i \neq j}} \frac{w_{i} W_{i} x_{j} y_{j}}{W_{i}^{\prime} W_{j}^{\prime}} . \tag{33}
\end{align*}
$$

Note that for any set $S$ of positive integers, and real numbers $a_{i}, b_{i}$ for $i \in S$, we have

$$
\sum_{\substack{i, j \in S \\ i \neq j}}\left(a_{i} b_{j}\right)=\left(\sum_{i \in S} a_{i}\right) \cdot\left(\sum_{i \in S} b_{i}\right)-\sum_{i \in S}\left(a_{i} b_{i}\right) .
$$

Thus,

$$
\begin{aligned}
& \sum_{\substack{2 \leq i, i \leq k \\
i \neq j}} \frac{x_{i} y_{j} w_{i} W_{j}}{W_{i}^{\prime} W_{j}^{\prime}}-\sum_{\substack{2 \leq i, i \leq j \\
i \neq j}} \frac{w_{i} W_{i} x_{j} y_{j}}{W_{i}^{\prime} W_{j}^{\prime}} \\
= & \left(\sum_{i=2}^{k} \frac{x_{i} w_{i}}{W_{i}^{\prime}}\right) \cdot\left(\sum_{i=2}^{k} \frac{y_{i} W_{i}}{W_{i}^{\prime}}\right)-\sum_{i=2}^{k} \frac{x_{i} y_{i} w_{i} W_{i}}{\left(W_{i}^{\prime}\right)^{2}}-\left(\left(\sum_{i=2}^{k} \frac{x_{i} y_{i}}{W_{i}^{\prime}}\right) \cdot\left(\sum_{i=2}^{k} \frac{w_{i} W_{i}}{W_{i}^{\prime}}\right)-\sum_{i=2}^{k} \frac{x_{i} y_{i} w_{i} W_{i}}{\left(W_{i}^{\prime}\right)^{2}}\right) \\
= & \left(x_{1} Z_{1}+y_{1} Z_{2}\right)\left(Y^{\prime}+y_{1} Z_{1}+x_{1} Z_{3}\right)-Z_{1}\left(x_{1} Y^{\prime}+y_{1} X^{\prime}+x_{1}^{2} Z_{3}+y_{1}^{2} Z_{2}+2 x_{1} y_{1} Z_{1}\right) \\
= & y_{1}\left(Y^{\prime} Z_{2}-X^{\prime} Z_{1}\right)+x_{1} y_{1}\left(Z_{2} Z_{3}-Z_{1}^{2}\right)
\end{aligned}
$$

$$
\begin{align*}
& =y_{1}\left(Y^{\prime} Z_{2}-X^{\prime} Z_{1}\right)+x_{1} y_{1}\left(1-2 Z_{1}\right) \\
& =y_{1}\left(Y^{\prime} Z_{2}+x_{1}-X^{\prime} Z_{1}-2 x_{1} Z_{1}\right) \tag{34}
\end{align*}
$$

where the second last equality follows from the fact that $\left(Z_{1}-1\right)^{2}=Z_{2} Z_{3}$ by Lemma Thus, by (321), (33) and (34),

$$
\begin{align*}
& \Gamma_{1}-\Gamma_{2}-\Gamma_{3}-\left(\prod_{i=1}^{k}\left(x_{i} Y+y_{i} X\right)\right)\left(1-\sum_{i=1}^{k} \frac{x_{i} y_{i}}{x_{i} Y+y_{i} X}\right) \\
= & -Z\left(y_{1} X-y_{1} X Z_{1}+x_{1} Y Z_{1}\right)+Z Z_{1}\left(x_{1} Y+y_{1} X\right)+Z y_{1}\left(Y^{\prime} Z_{2}+x_{1}-X^{\prime} Z_{1}-2 x_{1} Z_{1}\right) \\
= & y_{1} Z\left(X^{\prime} Z_{1}+Y^{\prime} Z_{2}-X^{\prime}\right) \\
= & y_{1} Z\left(-X^{\prime}+\sum_{i=2}^{k} \frac{x_{i}\left(y_{i} X^{\prime}+x_{i} Y^{\prime}\right)}{x_{i} Y^{\prime}+y_{i} X^{\prime}}\right) \\
= & 0 . \tag{35}
\end{align*}
$$

Thus, (21) follows from (25), (35) and the definition of $\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{k}, y_{k}\right)$ in (17).

## 4 Counting spanning trees in a special type of multigraphs

Let $V=\left\{v_{i}: i \in \llbracket k \rrbracket\right\}$ and $E=\left\{v_{i} v_{j}: i, j \in \llbracket k \rrbracket\right.$ and $\left.i \neq j\right\}$ be the vertex set and edge set of the complete graph $K_{k}$, where $k \geq 1$. Let $\omega$ be a weight function on $E$. If $\omega\left(v_{i} v_{j}\right)$ is a nonnegative integer for all $i, j$ with $1 \leq i<j \leq k$, then $\tau\left(K_{k}, \omega\right)$ is the number of spanning trees of the multigraph with vertex set $\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}$ which contains exactly $\omega\left(v_{i} v_{j}\right)$ parallel edges joining $u_{i}$ and $u_{j}$ for all $i, j$ with $1 \leq i<j \leq k$.

For any non-empty subset $I$ of $\llbracket k \rrbracket \backslash\{1\}$, let $G_{I}$ denote the complete graph of order $k-|I|$ with vertex set $\left\{v_{I}\right\} \cup\left\{v_{i}: i \in \llbracket k \rrbracket \backslash(I \cup\{1\})\right\}$ and weight function $\omega_{I}$ on the edge set of $G_{I}$ defined as follows:

$$
\begin{cases}\omega_{I}\left(v_{i} v_{j}\right)=\omega\left(v_{i} v_{j}\right), & \forall i, j \in \llbracket k \rrbracket \backslash(I \cup\{1\}), i \neq j ;  \tag{36}\\ \omega_{I}\left(v_{I} v_{j}\right)=\sum_{r \in I \cup\{1\}} \omega\left(v_{r} v_{j}\right), & \forall j \in \llbracket k \rrbracket \backslash(I \cup\{1\}) .\end{cases}
$$

Note that $G_{I}$ is actually the graph obtained from $K_{k}$ by identifying all vertices in $\left\{v_{i}: i \in I \cup\{1\}\right\}$ as one vertex.

By the inclusion-exclusion principle, the following recursive relation on $\tau\left(K_{k}, \omega\right)$ can be obtained.
Lemma 6. For any weight function $\omega$ on the edge set $E$ of $K_{k}$,

$$
\begin{equation*}
\tau\left(K_{k}, \omega\right)=\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1} \tau\left(G_{I}, \omega_{I}\right) \prod_{i \in I} \omega\left(v_{1} v_{i}\right)\right) . \tag{37}
\end{equation*}
$$

Proof. For any $i \in \llbracket k \rrbracket \backslash\{1\}$, let $A_{i}$ denote the set of members $T$ in $\mathcal{T}\left(K_{k}\right)$ with $v_{1} v_{i} \in E(T)$. Clearly,

$$
\begin{equation*}
\mathcal{T}\left(K_{k}\right)=\bigcup_{i=2}^{k} A_{i} \tag{38}
\end{equation*}
$$

For any $T \in \mathcal{T}\left(K_{k}\right)$, by the inclusion-exclusion principle,

$$
\begin{equation*}
\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1}\left|\{T\} \cap \bigcap_{i \in I} A_{i}\right|\right)=1 . \tag{39}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\tau\left(K_{k}, \omega\right)=\sum_{T \in \bigcup_{2 \leq i \leq k} A_{i}} \omega(T)=\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1} \sum_{T \in \bigcap_{i \in I} A_{i}} \omega(T)\right) . \tag{40}
\end{equation*}
$$

Let $I$ be any non-empty subset of $\llbracket k \rrbracket \backslash\{1\}$ and let $H_{I}$ denote the multiple graph obtained from $K_{k}$ by identifying all vertices in the set $\left\{v_{i}: i \in I \cup\{1\}\right\}$ and removing all loops produced. The vertex set of $H_{I}$ is $\left(V\left(K_{k}\right) \backslash\left\{v_{i}: i \in I \cup\{1\}\right\}\right) \cup v_{I}$. Clearly, $H_{I}$ includes each edge $v_{i} v_{j}$, where $i, j \in \llbracket k \rrbracket \backslash(I \cup\{1\})$, while each edge $v_{i} v_{j}$ in $K_{k}$, where $i \in I \cup\{1\}$ and $j \in \llbracket k \rrbracket \backslash(I \cup\{1\})$, is changed to an edge of $H_{I}$ joining $v_{I}$ and $v_{j}$. There are exactly $1+|I|$ parallel edges in $H_{I}$ joining $v_{I}$ and $v_{j}$ for each $j \in \llbracket k \rrbracket \backslash(I \cup\{1\})$. The weight function on $E\left(H_{I}\right)$ is the restriction of $\omega$ to $E\left(H_{I}\right)$ and parallel edges in $H_{I}$ may have different weights.

For each $T \in \bigcap_{i \in I} A_{i}$, let $T_{I}$ be the tree obtained from $T$ by identifying all vertices in the set $\left\{v_{i}: i \in I \cup\{1\}\right\}$. Clearly $\omega(T)$ and $\omega\left(T_{I}\right)$ have the following relation:

$$
\begin{equation*}
\omega(T)=\prod_{i \in I} \omega\left(v_{1} v_{i}\right) \cdot \omega\left(T_{I}\right) \tag{41}
\end{equation*}
$$

Moreover, $T \rightarrow T_{I}$ is a bijection from $\bigcap_{i \in I} A_{i}$ to $\mathcal{T}\left(H_{I}\right)$, implying that

$$
\begin{equation*}
\sum_{T \in \bigcap_{i \in I} A_{i}} \omega(T)=\prod_{i \in I} \omega\left(v_{1} v_{i}\right) \cdot \sum_{T \in \bigcap_{i \in I} A_{i}} \omega\left(T_{I}\right)=\tau\left(H_{I}, \omega\right) \cdot \prod_{i \in I} \omega\left(v_{1} v_{i}\right) . \tag{42}
\end{equation*}
$$

Note that $G_{T}$ can be obtained from $H_{I}$ by merging all parallel edges with ends $v_{I}$ and $v_{j}$ into one for each $j \in \llbracket k \rrbracket \backslash(I \cup\{1\})$. By the definition of $\omega_{I}, \tau\left(H_{I}, \omega\right)=\tau\left(G_{I}, \omega_{I}\right)$. Thus, (37) follows from (40) and (42).

Recall the function $\phi$ defined in the previous section. In the following, we shall show that $\tau\left(K_{k}, \omega\right)$ can be expressed in terms of $\phi$ when $\omega$ satisfies certain conditions.

Theorem 6. Let $V=\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}$ be the the vertex set of the complete graph $K_{k}$, where $k \geq 1$, and $\omega$ be a weight function on the edge set $E$ of $K_{k}$. If there exist $2 k$ real numbers $x_{1}, x_{2}, \ldots, x_{k}$ and $y_{1}, y_{2}, \ldots, y_{k}$ such that $\omega\left(v_{i} v_{j}\right)=x_{i} y_{j}+x_{j} y_{i}$ holds for every pair $i$ and $j$ with $1 \leq i<j \leq k$, then,

$$
\begin{equation*}
\tau\left(K_{k}, \omega\right)=\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{k}, y_{k}\right) \tag{43}
\end{equation*}
$$

Proof. Note that for $1 \leq k \leq 3$,

$$
\tau\left(K_{k}, \omega\right)= \begin{cases}1, & k=1  \tag{44}\\ \omega\left(v_{1} v_{2}\right), & k=2 \\ \omega\left(v_{1} v_{2}\right) \omega\left(v_{1} v_{3}\right)+\omega\left(v_{1} v_{2}\right) \omega\left(v_{2} v_{3}\right)+\omega\left(v_{1} v_{3}\right) \omega\left(v_{2} v_{3}\right), & k=3\end{cases}
$$

As $\omega\left(v_{i} v_{j}\right)=x_{i} y_{j}+x_{j} y_{i}$, (43) follows from (19) and (44) when $k \leq 3$.
Assume that the result holds for $k \leq N$, where $N \geq 3$. In the following, we assume that $k=N+1$ and show that it holds in this case by induction.

Recall that for any non-empty subset of $I$ of $\llbracket k \rrbracket \backslash\{1\}, G_{I}$ is the complete graph with vertex set $\left\{v_{I}\right\} \cup\left\{v_{j}: j \in \llbracket k \rrbracket \backslash(I \cup\{1\})\right\}$ and weight function $\omega_{I}$ on its edge set defined in (36), i.e.,

$$
\begin{cases}\omega_{I}\left(v_{i} v_{j}\right)=\omega\left(v_{i} v_{j}\right)=x_{i} y_{j}+x_{j} y_{i}, & \forall i, j \in \llbracket k \rrbracket \backslash(I \cup\{1\}), i \neq j ;  \tag{45}\\ \omega_{I}\left(v_{I} v_{j}\right)=\sum_{r \in I \cup\{1\}}\left(x_{r} y_{j}+x_{j} y_{r}\right)=x_{I} y_{j}+y_{I} x_{j}, & \forall j \in \llbracket k \rrbracket \backslash(I \cup\{1\})\end{cases}
$$

where $x_{I}=x_{1}+\sum_{r \in I} x_{r}$ and $y_{I}=y_{1}+\sum_{r \in I} y_{r}$.
As $G_{I}$ is a complete graph of order $k-|I|<k$ with a weight function $\omega_{I}$ satisfying conditions in (45), by inductive assumption, (43) holds for $G_{I}$, i.e.,

$$
\begin{equation*}
\tau\left(G_{I}, \omega_{I}\right)=\phi(x_{I}, y_{I}, \underbrace{x_{i}, y_{i}}_{\forall i \in \llbracket k \rrbracket \backslash(I \cup\{1\})}) . \tag{46}
\end{equation*}
$$

By Lemma 6, (46) and Theorem 5 ,

$$
\begin{align*}
\tau\left(K_{k}, \omega\right) & =\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}(-1)^{|I|-1} \tau\left(G_{I}, \omega_{I}\right) \prod_{i \in I} \omega\left(v_{1} v_{i}\right) \\
& =\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left((-1)^{|I|-1} \tau\left(G_{I}, \omega_{I}\right) \prod_{i \in I}\left(x_{i} y_{1}+y_{i} x_{1}\right)\right) \\
& =\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}((-1)^{|I|-1} \prod_{i \in I}\left(x_{i} y_{1}+y_{i} x_{1}\right) \cdot \phi(x_{I}, y_{I}, \underbrace{x_{i}, y_{i}}_{\forall i \in \llbracket k \rrbracket \backslash(I \cup\{1\})})) \\
& =\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \cdots, x_{k}, y_{k}\right) . \tag{47}
\end{align*}
$$

Hence the result holds.

## 5 Spanning trees in $K_{m, n}$ containing a spanning forest $F$

Now we are ready to prove the main result.
Proof of Theorem 4. For any spanning forest $F$ of $K_{m, n}$ with components $T_{1}, T_{2}, \ldots, T_{k}$, observe that

$$
\begin{equation*}
\tau_{F}\left(K_{m, n}\right)=\tau\left(K_{m, n} / F\right) \tag{48}
\end{equation*}
$$

where $K_{m, n}$ is unweighted and $K_{m, n} / F$ is the multigraph obtained from $K_{m, n}$ by contracting all edges in $F$. Note that $K_{m, n} / F$ is a multigraph of order $k$ whose vertices correspond to components of $F$, as $K_{m, n} / F$ can also be obtained from $K_{m, n}$ by identifying all vertices in $T_{i}$ for all $i \in \llbracket k \rrbracket$, and removing all loops. Thus, we may assume that $K_{m, n} / F$ has vertices $v_{1}, v_{2}, \ldots, v_{k}$ such that the number of parallel edges joining $v_{i}$ and $v_{j}$ is equal to the number of edges in $K_{m, n}$ with one end in $T_{i}$ and the other end in $T_{j}$.

As $\left|X \cap V\left(T_{s}\right)\right|=m_{s}$ and $\left|Y \cap V\left(T_{s}\right)\right|=n_{s}$ for all $s=1,2, \ldots, k, K_{m, n} / F$ contains exactly $m_{i} n_{j}+m_{j} n_{i}$ parallel edges joining $v_{i}$ and $v_{j}$ for all $1 \leq i<j \leq k$. By Theorem6,

$$
\begin{equation*}
\tau\left(K_{m, n} / F\right)=\phi\left(m_{1}, n_{1}, m_{2}, n_{2}, \ldots, m_{k}, n_{k}\right) \tag{49}
\end{equation*}
$$

Thus, by (48) and the definition of $\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \ldots, x_{k}, y_{k}\right)$ in (17), the result holds.

## 6 Remarks

Another approach for proving the main result is to establish results analogue to Lemma 6 and Theorem 5. The following identity analogue to Lemma 6 can be obtained easily:

$$
\begin{equation*}
\tau\left(K_{k}, \omega\right)=\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}\left(\tau\left(G_{I}, \omega_{I}^{\prime}\right) \prod_{i \in I} \omega\left(v_{1} v_{i}\right)\right), \tag{50}
\end{equation*}
$$

where $\omega_{I}^{\prime}$ is different from $\omega_{I}$ defined in (36), as for any $j \in \llbracket k \rrbracket \backslash(I \cup\{1\})$,

$$
\begin{equation*}
\omega_{I}^{\prime}\left(v_{I} v_{j}\right)=\sum_{r \in I} \omega\left(v_{r} v_{j}\right)=\omega_{I}\left(v_{I} v_{j}\right)-\omega\left(v_{1} v_{j}\right) \tag{51}
\end{equation*}
$$

although $\omega_{I}^{\prime}\left(v_{i} v_{j}\right)=\omega_{I}\left(v_{i} v_{j}\right)$ for all $i, j \in \llbracket k \rrbracket \backslash(I \cup\{1\})$ with $i \neq j$.
By Theorem 4 and (50), the following identity analogue to Theorem 5 holds:

$$
\begin{equation*}
\phi\left(x_{1}, y_{1}, x_{2}, y_{2}, \cdots, x_{k}, y_{k}\right)=\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}}(\prod_{j \in I}\left(x_{1} y_{j}+x_{j} y_{1}\right) \phi(x_{I}^{\prime}, y_{I}^{\prime}, \underbrace{x_{s}, y_{s}}_{\forall s \in \llbracket k \rrbracket \backslash(I \cup\{1\})})) \tag{52}
\end{equation*}
$$

where $x_{I}^{\prime}=\sum_{i \in I} x_{i}=x_{I}-x_{1}$ and $y_{I}^{\prime}=\sum_{i \in I} y_{i}=y_{I}-y_{1}$.
However, it is quite challenging to prove (52) directly. Note that for any $I$ with $\emptyset \neq I \subseteq \llbracket k \rrbracket \backslash\{1\}$,

$$
\left\{\begin{array}{l}
x_{I}^{\prime}+\sum_{i \in I} x_{i}=x_{2}+x_{3}+\cdots+x_{k}=X-x_{1}  \tag{53}\\
y_{I}^{\prime}+\sum_{i \in I} y_{i}=y_{2}+y_{3}+\cdots+y_{k}=Y-x_{1}
\end{array}\right.
$$

By the definition of the function $\phi$,

$$
\begin{align*}
\phi(x_{I}^{\prime}, y_{I}^{\prime}, \underbrace{x_{s}, y_{s}}_{\forall s \in \llbracket k \rrbracket \backslash(I \cup\{1\})}) & =\frac{x_{I}^{\prime} Y^{\prime}+y_{I}^{\prime} X^{\prime}}{X^{\prime} Y^{\prime}} \cdot\left(\prod_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})}\left(x_{i} Y^{\prime}+y_{i} X^{\prime}\right)\right) \\
& \cdot\left(1-\frac{x_{I}^{\prime} y_{I}^{\prime}}{x_{I}^{\prime} Y^{\prime}+y_{I}^{\prime} X^{\prime}}-\sum_{i \in \llbracket k \rrbracket \backslash(I \cup\{1\})} \frac{x_{i} y_{i}}{x_{i} Y^{\prime}+y_{i} X^{\prime}}\right), \tag{54}
\end{align*}
$$

where $X^{\prime}=X-x_{1}$ and $Y^{\prime}=Y-y_{1}$. Observe that the left-hand of (52) has a denominator $X Y$, while its right-hand side has a denominator $X^{\prime} Y^{\prime}$.

Clearly, the main result (i.e., Theorem (4) also follows from (50) and (52).
In the end, we propose some problems.
Problem 1. Find a bijective proof for Theorem 4 .
Another problem is to extend Theorem 4 to complete $k$-partite graphs, where $k \geq 3$.
Problem 2. Let $K_{n_{1}, n_{2}, \cdots, n_{k}}$ be a complete $k$-partite graph and $F$ be a spanning forest in $K_{n_{1}, n_{2}, \ldots, n_{k}}$, where $k \geq 3$. Find a formula for counting the number of spanning trees in $K_{n_{1}, n_{2}, \ldots, n_{k}}$ which contain all edges in $F$.

For $k=3$, we propose the following conjecture for a lower bound of $\tau_{F}\left(K_{n_{1}, n_{2}, n_{3}}\right)$.
Conjecture 1. Let $X_{1}, X_{2}$ and $X_{3}$ be the partite sets of the complete tripartite graph $K_{n_{1}, n_{2}, n_{3}}$, where $\left|X_{i}\right|=n_{i}$ for $i \in \llbracket 3 \rrbracket$. For any spanning forest $F$ in $K_{n_{1}, n_{2}, n_{3}}$ with $k$ components $T_{1}, T_{2}, \cdots, T_{k}$,

$$
\begin{align*}
\tau_{F}\left(K_{n_{1}, n_{2}, n_{3}}\right) \geq & \frac{1}{n_{1} n_{2}+n_{1} n_{3}+n_{2} n_{3}}\left(\prod_{i=1}^{k}\left(\left(n-n_{1}\right) n_{1, i}+\left(n-n_{2}\right) n_{2, i}+\left(n-n_{3}\right) n_{3, i}\right)\right) \\
& \cdot\left(1-\sum_{i=1}^{k} \frac{n_{1, i} n_{2, i}+n_{1, i} n_{3, i}+n_{2, i} n_{3, i}}{\left(n-n_{1}\right) n_{1, i}+\left(n-n_{2}\right) n_{2, i}+\left(n-n_{3}\right) n_{3, i}}\right), \tag{55}
\end{align*}
$$

where $n=n_{1}+n_{2}+n_{3}$ and $n_{s, i}=\left|X_{s} \cap V\left(T_{i}\right)\right|$ for $s=1,2,3$ and $i \in \llbracket k \rrbracket$.
It is trivial to verify that the equality of (55) holds for $k \leq 2$.

## Acknowledgements

Jun Ge is supported by NSFC (No. 11701401) and the joint research project of Laurent Mathematics Center of Sichuan Normal University and National-Local Joint Engineering Laboratory of System Credibility Automatic Verification. The authors are grateful to two anonymous referees for their careful examination and constructive comments.

## References

[1] A. Cayley, A theorem on trees, Quart. J. Pure Appl. Math. 23 (1889), 376-378.
[2] F. Dong, J. Ge and Z. Ouyang, Express the number of spanning trees in term of degrees, Appl. Math. Comput. 415 (2022), 126697.
[3] F. Dong and W. Yan, Expression for the number of spanning trees of line graphs of arbitrary connected graphs, J. Graph Theory 85 (2017), 74-93.
[4] R. Ehrenborg, The number of spanning trees of the Bruhat graph, Adv. in Appl. Math. 125 (2021), 102150.
[5] M. Fiedler and J. Sedláček, Über Wurzelbasen von gerichteten Graphen, Časopis Pěst. Mat. $8 \mathbf{3}$ (1958), 214-225.
[6] J. Ge and F. Dong, Spanning trees in complete bipartite graphs and resistance distance in nearly complete bipartite graphs, Discrete Appl. Math. 283 (2020), 542-554.
[7] H. Gong and X. Jin, A simple formula for the number of spanning trees of line graphs, J. Graph Theory 88 (2018), 294-301.
[8] L. Lovász, Combinatorial problems and exercises, second edition, North-Holland Publishing Co., Amsterdam, 1993.
[9] J. W. Moon, The second moment of the complexity of a graph, Mathematika 11 (1964), 95-98.
[10] W. Yan, On the number of spanning trees of some irregular line graphs, J. Combin. Theory Ser. A 120 (2013), 1642-1648.


[^0]:    *Email: fengming.dong@nie.edu.sg and donggraph@163.com.
    ${ }^{\dagger}$ Corresponding author. Email: mathsgejun@163.com.

