Counting spanning trees in a complete bipartite graph which contain a given spanning forest

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Abstract

In this article, we extend Moon's classic formula for counting spanning trees in complete graphs containing a fixed spanning forest to complete bipartite graphs. Let (X, Y) be the bipartition of the complete bipartite graph $K_{m,n}$ with |X| = m and |Y| = n. We prove that for any given spanning forest F of $K_{m,n}$ with components T_1, T_2, \ldots, T_k , the number of spanning trees in $K_{m,n}$ which contain all edges in F is equal to

$$\frac{1}{mn}\left(\prod_{i=1}^{k}(m_{i}n+n_{i}m)\right)\left(1-\sum_{i=1}^{k}\frac{m_{i}n_{i}}{m_{i}n+n_{i}m}\right),$$

where $m_i = |V(T_i) \cap X|$ and $n_i = |V(T_i) \cap Y|$ for $i = 1, 2, \ldots, k$.

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1 Introduction

In this paper, we assume that all graphs are loopless, while parallel edges are allowed. For any graph G, let V(G) and E(G) be the vertex set and edge set of G. For any edge set $F \subseteq E(G)$, let G/F be the graph obtained from G by contracting all edges in F, and removing all loops. Let $\mathcal{T}(G)$ denote the set of spanning trees of G. For any positive integer k, let [k] denote the set $\{1, 2, \dots, k\}$.

Suppose G is a weighted graph with weight function $\omega : E(G) \to \mathbb{R}$. For any $F \subseteq E(G)$ and any subgraph H of G, define $\omega(F) = \prod_{e \in F} \omega(e)$ and $\omega(H) = \omega(E(H))$. Let $\tau(G, \omega) = \sum_{T \in \mathcal{T}(G)} \omega(T)$. Sometimes we use $\tau(G)$ instead of $\tau(G, \omega)$ when there is no confusion. It is obvious that for an unweighted graph G (that is to say, a weighted graph with unit weight on each edge), $\tau(G) = |\mathcal{T}(G)|$, i.e., the number of spanning trees of G. Throughout this paper, any graph is assumed to be unweighted, unless it is claimed.

Counting spanning trees in graphs is a very old topic in graph theory having modern connections with many other fields in mathematics, statistical physics and theoretical computer science, such as

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random walks, the Ising model and Potts model, network reliability, parking functions, knot/link determinants. See [2–4,7,10] for some recent work on counting spanning trees.

For a subgraph H of G, let $\mathcal{T}_H(G)$ denote the set of spanning trees $T \in \mathcal{T}(G)$ with $E(H) \subseteq E(T)$, and let $\tau_H(G) = \sum_{T \in \mathcal{T}_H(G)} \omega(T)$. For unweighted graph G, $\tau_H(G) = |\mathcal{T}_H(G)|$, i.e., the number of spanning trees of G containing all edges in H. Note that usually the graph H here is a forest or a tree, because otherwise $\mathcal{T}_H(G) = \emptyset$ and $\tau_H(G) = 0$.

The celebrated Cayley's formula [1] states that $\tau(K_n) = n^{n-2}$. In 1964, Moon generalized Cayley's formula by obtaining a nice expression of $\tau_F(K_n)$ for any spanning forest F of K_n .

Theorem 1 ([9], also see Problem 4.4 in [8]). For any spanning forest F of K_n , if c is the number of components of F and n_1, n_2, \ldots, n_c are the orders of those components, then

$$\tau_F(K_n) = n^{c-2} \prod_{i=1}^c n_i.$$

It is easy to see that Cayley's formula is the special case that F is an empty graph. It is also well known that $\tau(K_{m,n}) = m^{n-1}n^{m-1}$ for any complete bipartite graph $K_{m,n}$ by Fiedler and Sedláček [5]. So there is a natural question: is there a bipartite analogue of Moon's formula (Theorem 1)? That is to say, for any given spanning forest F in $K_{m,n}$, what is the explicit expression of $\tau_F(K_{m,n})$?

It turns out that this question is much harder than the case of complete graphs. In [6], this question was partially answered for two special cases: F is a matching or a tree plus several possible isolated vertices.

Theorem 2 ([6]). For any matching M of size k in $K_{m,n}$,

$$\tau_M(K_{m,n}) = (m+n)^{k-1}(m+n-k)m^{n-k-1}n^{m-k-1}.$$

Theorem 3 ([6]). For any tree T of $K_{m,n}$,

$$\tau_T(K_{m,n}) = (sn + tm - st)m^{n-t-1}n^{m-s-1},$$

where $s = |V(T) \cap X|$, $t = |V(T) \cap Y|$, and (X, Y) is the bipartition of $K_{m,n}$ with |X| = m and |Y| = n.

In this paper, we obtain an explicit expression for $\tau_F(K_{m,n})$ for an arbitrary spanning forest F of $K_{m,n}$.

Theorem 4. Let (X, Y) be the bipartition of $K_{m,n}$ with |X| = m and |Y| = n. For any spanning forest F of $K_{m,n}$ with components T_1, T_2, \ldots, T_k ,

$$\tau_F(K_{m,n}) = \frac{1}{mn} \left(\prod_{i=1}^k (m_i n + n_i m) \right) \left(1 - \sum_{i=1}^k \frac{m_i n_i}{m_i n + n_i m} \right),$$
(1)

where $m_i = |X \cap V(T_i)|$ and $n_i = |Y \cap V(T_i)|$ for all $i \in [\![k]\!]$.

2 Preliminary results

This section provides some results which will be applied in the next section for proving a key identity.

Lemma 1. For any set of k pairs of real numbers $\{a_i, b_i : i \in [\![k]]\!\}$, where $k \ge 1$, if $a_i B + b_i A \ne 0$ for all $i \in [\![k]\!]$ where $A = a_1 + a_2 + \cdots + a_k$ and $B = b_1 + b_2 + \cdots + b_k$, then

$$\left(1 - \sum_{i=1}^{k} \frac{a_i b_i}{a_i B + b_i A}\right)^2 = \left(\sum_{i=1}^{k} \frac{a_i^2}{a_i B + b_i A}\right) \cdot \left(\sum_{i=1}^{k} \frac{b_i^2}{a_i B + b_i A}\right).$$
(2)

Proof. Since $a_i B + b_i A \neq 0$, A and B cannot both be 0.

When A = 0 and $B \neq 0$, equality (2) holds because

$$\left(1 - \sum_{i=1}^{k} \frac{a_i b_i}{a_i B + b_i A}\right)^2 = \left(1 - \sum_{i=1}^{k} \frac{b_i}{B}\right)^2 = 0,$$

and

$$\left(\sum_{i=1}^k \frac{a_i^2}{a_i B + b_i A}\right) \cdot \left(\sum_{i=1}^k \frac{b_i^2}{a_i B + b_i A}\right) = \frac{A}{B} \cdot \sum_{i=1}^k \frac{b_i^2}{a_i B} = 0.$$

Similarly, equality (2) holds when B = 0 and $A \neq 0$. Let $W_i = a_i B + b_i A$ for $i \in [[k]]$. When $A \neq 0$ and $B \neq 0$, observe that

$$A\sum_{i=1}^{k} \frac{a_i b_i}{W_i} + B\sum_{i=1}^{k} \frac{a_i^2}{W_i} = \sum_{i=1}^{k} \frac{a_i (Ab_i + Ba_i)}{W_i} = \sum_{i=1}^{k} a_i = A,$$
(3)

implying that

$$\sum_{i=1}^{k} \frac{a_i^2}{W_i} = \frac{A}{B} \left(1 - \sum_{i=1}^{k} \frac{a_i b_i}{W_i} \right).$$
(4)

Similarly,

$$\sum_{i=1}^{k} \frac{b_i^2}{W_i} = \frac{B}{A} \left(1 - \sum_{i=1}^{k} \frac{a_i b_i}{W_i} \right).$$
(5)

Clearly, equality (2) follows from (4) and (5).

For a set A of real numbers, in the following, if A is the empty set, we set

$$\prod_{a \in A} a = 1 \quad \text{and} \quad \sum_{a \in A} a = 0.$$
(6)

Lemma 2. Let S be a set of positive integers. For any set of 2|S| real numbers $\{a_i, b_i : i \in S\}$,

$$\sum_{\emptyset \neq I \subseteq S} \left(\left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \prod_{i \in S} (a_i + b_i) - \prod_{i \in S} b_i.$$
(7)

Proof. The identity follows from the following fact:

$$\sum_{\emptyset \subseteq I \subseteq S} \left(\left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \prod_{i \in S} (a_i + b_i).$$
(8)

Lemma 3. Let S be a set of positive integers. For any set of 3|S| real numbers $\{a_i, b_i, c_i : i \in S\}$,

$$\sum_{\emptyset \neq I \subseteq S} \left(\left(\sum_{i \in I} c_i \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \left(\prod_{j \in S} (a_j + b_j) \right) \cdot \sum_{i \in S} \frac{c_i a_i}{a_i + b_i}.$$
(9)

Proof. The identity follows from the following fact:

$$\sum_{\emptyset \neq I \subseteq S} \left(\left(\sum_{i \in I} c_i \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \sum_{i \in S} \left(c_i a_i \prod_{j \in S \setminus \{i\}} (a_j + b_j) \right).$$
(10)

By Lemmas 2 and 3, for an arbitrary real number c, we have

$$\sum_{\emptyset \neq I \subseteq S} \left(\left(c + \sum_{i \in I} c_i \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \left(\prod_{j \in S} (a_j + b_j) \right) \cdot \left(c + \sum_{i \in S} \frac{c_i a_i}{a_i + b_i} \right) - c \prod_{j \in S} b_j.$$
(11)

Lemma 4. Let S be a set of positive integers. For any set of 3|S| real numbers $\{a_i, b_i, d_i : i \in S\}$,

$$\sum_{\emptyset \neq I \subseteq S} \left(\left(\sum_{i \in S \setminus I} d_i \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \left(\prod_{j \in S} (a_j + b_j) \right) \cdot \left(\sum_{i \in S} \frac{d_i b_i}{a_i + b_i} \right) - \left(\prod_{r \in S} b_r \right) \cdot \left(\sum_{i \in S} d_i \right).$$
(12)

Proof. It follows from Lemma 3 and the following fact:

$$\sum_{\emptyset \neq I \subseteq S} \left(\left(\sum_{i \in S \setminus I} d_i \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \sum_{\emptyset \neq I \subseteq S} \left(\left(\sum_{i \in I} d_i \right) \prod_{j \in S \setminus I} a_j \prod_{r \in I} b_r \right) - \left(\prod_{r \in S} b_r \right) \cdot \left(\sum_{i \in S} d_i \right)$$
(13)

.

Lemma 5. Let S be a set of positive integers. For any set of 4|S| real numbers $\{a_i, b_i, c_i, d_i : i \in S\}$,

$$\sum_{\emptyset \neq I \subsetneq S} \left(\left(\sum_{i \in I} c_i \right) \left(\sum_{q \in S \setminus I} d_q \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \left(\prod_{r \in S} (a_r + b_r) \right) \cdot \sum_{\substack{i, j \in S \\ i \neq j}} \frac{c_i a_i d_j b_j}{(a_i + b_i)(a_j + b_j)}.$$
(14)

Proof. The result follows from the following fact:

$$\sum_{\emptyset \neq I \subsetneq S} \left(\left(\sum_{i \in I} c_i \right) \left(\sum_{q \in S \setminus I} d_q \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right) = \sum_{\substack{i, j \in S \\ i \neq j}} \left(c_i a_i d_j b_j \prod_{r \in S \setminus \{i, j\}} (a_r + b_r) \right).$$
(15)

By Lemmas 4 and 5, for an arbitrary real number c, we have

$$\sum_{\emptyset \neq I \subseteq S} \left(\left(c + \sum_{i \in I} c_i \right) \left(\sum_{q \in S \setminus I} d_q \right) \left(\prod_{j \in I} a_j \right) \left(\prod_{r \in S \setminus I} b_r \right) \right)$$
$$= \left(\prod_{r \in S} (a_r + b_r) \right) \cdot \left(\sum_{\substack{i,j \in S \\ i \neq j}} \frac{c_i a_i d_j b_j}{(a_i + b_i)(a_j + b_j)} + c \sum_{i \in S} \frac{d_i b_i}{a_i + b_i} \right) - c \left(\prod_{r \in S} b_r \right) \cdot \left(\sum_{i \in S} d_i \right).$$
(16)

3 An identity

Define a function ϕ on 2k variables x_1, x_2, \ldots, x_k and y_1, y_2, \ldots, y_k , where $k \ge 1$, as follows:

$$\phi(x_1, y_1, x_2, y_2, \dots, x_k, y_k) = \frac{1}{XY} \left(\prod_{i=1}^k (x_i Y + y_i X) \right) \left(1 - \sum_{i=1}^k \frac{x_i y_i}{x_i Y + y_i X} \right), \tag{17}$$

where $X = x_1 + x_2 + \cdots + x_k$ and $Y = y_1 + y_2 + \cdots + y_k$. Observe that

$$\phi(x_1, y_1, x_2, y_2, \dots, x_k, y_k) = \frac{1}{XY} \left(\left(\prod_{i=1}^k (x_i Y + y_i X) \right) - \sum_{i=1}^k \left(x_i y_i \prod_{\substack{1 \le j \le k \\ j \ne i}} (x_j Y + y_j X) \right) \right).$$

In the expansion of $\left(\prod_{i=1}^{k} (x_i Y + y_i X)\right) - \sum_{i=1}^{k} \left(x_i y_i \prod_{\substack{1 \le j \le k \\ j \ne i}} (x_j Y + y_j X)\right)$, the expression consisting of all monomials not divisible by XY is identically 0, as shown below:

$$Y^{k} \prod_{i=1}^{k} x_{i} + X^{k} \prod_{i=1}^{k} y_{i} - \sum_{i=1}^{k} \left(y_{i} Y^{k-1} \prod_{j=1}^{k} x_{j} \right) - \sum_{i=1}^{k} \left(x_{i} X^{k-1} \prod_{j=1}^{k} y_{j} \right)$$

$$= Y^{k} \prod_{j=1}^{k} x_{j} + X^{k} \prod_{j=1}^{k} y_{j} - Y^{k-1} \prod_{j=1}^{k} x_{j} \cdot \left(\sum_{i=1}^{k} y_{i} \right) - X^{k-1} \prod_{j=1}^{k} y_{j} \cdot \left(\sum_{i=1}^{k} x_{i} \right)$$

$$= 0.$$
(18)

It follows that $\phi(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$ is a polynomial on 2k variables x_1, x_2, \dots, x_k and y_1, y_2, \dots, y_k .

For $k \leq 3$,

$$\begin{cases} \phi(x_1, y_1) = 1; \\ \phi(x_1, y_1, x_2, y_2) = z_{1,2}; \\ \phi(x_1, y_1, x_2, y_2, x_3, y_3) = z_{1,2} z_{1,3} + z_{1,2} z_{2,3} + z_{1,3} z_{2,3}, \end{cases}$$
(19)

where $z_{i,j} = x_i y_j + x_j y_i$ for all $1 \le i < j \le 3$. For any $I \subseteq \llbracket k \rrbracket \setminus \{1\}$, let

$$x_I = x_1 + \sum_{i \in I} x_i, \quad y_I = y_1 + \sum_{i \in I} y_i.$$
 (20)

In this section, we shall establish the following identity, which will be applied to prove the main result in the article.

Theorem 5. For any 2k real numbers x_1, x_2, \ldots, x_k and y_1, y_2, \ldots, y_k , where $k \ge 2$,

$$\phi(x_1, y_1, x_2, y_2, \dots, x_k, y_k) = \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left((-1)^{|I|-1} \left(\prod_{j \in I} (x_1 y_j + x_j y_1) \right) \phi(x_I, y_I, \underbrace{x_s, y_s}_{\forall s \in \llbracket k \rrbracket \setminus (I \cup \{1\})} \right), \tag{21}$$

where $\phi(x_I, y_I, \underbrace{x_s, y_s}_{\forall s \in \llbracket k \rrbracket \setminus (I \cup \{1\})}) = \phi(x_I, y_I, x_{i_1}, y_{i_2}, \dots, x_{i_r}, y_{i_r}), \ \{i_1, i_2, \dots, i_r\} = \llbracket k \rrbracket \setminus (I \cup \{1\}) \ and \ r = k - 1 - |I|.$

Proof. Let I be a non-empty subset of $\llbracket k \rrbracket \setminus \{1\}$. Then

$$x_{I} + \sum_{s \in [\![k]\!] \setminus (I \cup \{1\})} x_{s} = \sum_{i=1}^{k} x_{i} = X$$
(22)

and

$$y_I + \sum_{s \in [[k]] \setminus (I \cup \{1\})} y_s = \sum_{i=1}^k y_i = Y.$$
(23)

In the remainder of the proof of Theorem 5, let $W_I = x_I Y + y_I X$, and for each $i \in [k]$, let

$$W_i = x_i Y + y_i X$$
 and $w_i = x_i y_1 + y_i x_1$.

By the definition of the function ϕ ,

$$\phi(x_I, y_I, \underbrace{x_s, y_s}_{\forall s \in \llbracket k \rrbracket \setminus (I \cup \{1\})}) = \frac{W_I}{XY} \cdot \left(\prod_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} W_i\right) \left(1 - \frac{x_I y_I}{W_I} - \sum_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} \frac{x_i y_i}{W_i}\right).$$
(24)

Thus, the right-hand side of (21) can be expressed as

$$\frac{1}{XY}\left(\Gamma_1 - \Gamma_2 - \Gamma_3\right),\tag{25}$$

where

$$\Gamma_{1} = \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left((-1)^{|I|-1} W_{I} \left(\prod_{j \in I} w_{j} \right) \left(\prod_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} W_{i} \right) \right),$$
(26)

$$\Gamma_2 = \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left((-1)^{|I|-1} x_I y_I \left(\prod_{j \in I} w_j \right) \left(\prod_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} W_i \right) \right), \tag{27}$$

$$\Gamma_3 = \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left((-1)^{|I|-1} W_I \left(\prod_{j \in I} w_j \right) \left(\prod_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} W_i \right) \left(\sum_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} \frac{x_i y_i}{W_i} \right) \right).$$
(28)

In the following, we shall apply Lemmas 1– 5 to simplify Γ_1, Γ_2 and Γ_3 in order to show that

$$\Gamma_1 - \Gamma_2 - \Gamma_3 = \left(\prod_{i=1}^k (x_i Y + y_i X)\right) \left(1 - \sum_{i=1}^k \frac{x_i y_i}{x_i Y + y_i X}\right).$$
(29)

Let $X' = X - x_1$ and $Y' = Y - y_1$. Also let

$$\begin{cases} Z = \prod_{i=2}^{k} (x_i Y' + y_i X'), \\ Z_1 = \sum_{i=2}^{k} \frac{x_i y_i}{x_i Y' + y_i X'}, \quad Z_2 = \sum_{i=2}^{k} \frac{x_i^2}{x_i Y' + y_i X'}, \quad Z_3 = \sum_{i=2}^{k} \frac{y_i^2}{x_i Y' + y_i X'}. \end{cases}$$
(30)

Note that for any non-empty subset I of $\llbracket k \rrbracket \setminus \{1\}$,

$$y_I X + x_I Y - x_I y_I = \left(y_1 + \sum_{i \in I} y_i \right) X + \left(x_1 + \sum_{i \in I} x_i \right) \left(\sum_{i \in [[k]] \setminus (I \cup \{1\})} y_i \right).$$
(31)

In the remainder of this section, let W'_i denote $x_iY' + y_iX'$ for each $i \in [k]$. By applying identities (11), (16) and (31),

$$\Gamma_{1} - \Gamma_{2}$$

$$= \sum_{\substack{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}}} \left((-1)^{|I|-1} (W_{I} - x_{I} y_{I}) \left(\prod_{j \in I} w_{j}\right) \left(\prod_{i \in \llbracket k \rrbracket \setminus \{I \cup \{1\}\}} W_{i}\right) \right)$$

$$\stackrel{by (31)}{=} X \sum_{\substack{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}}} \left((-1)^{|I|-1} \left(y_{1} + \sum_{i \in I} y_{i}\right) \left(\prod_{j \in I} w_{j}\right) \left(\prod_{i \in \llbracket k \rrbracket \setminus \{I \cup \{1\}\}} W_{i}\right) \right)$$

$$+ \sum_{\emptyset \neq I \subseteq [[k]] \setminus \{1\}} \left((-1)^{|I|-1} \left(x_1 + \sum_{i \in I} x_i \right) \left(\sum_{i \in [[k]] \setminus (I \cup \{1\})} y_i \right) \left(\prod_{j \in I} w_j \right) \left(\prod_{i \in [[k]] \setminus (I \cup \{1\})} W_i \right) \right)$$

$$= -X \sum_{\emptyset \neq I \subseteq [[k]] \setminus \{1\}} \left(\left(y_1 + \sum_{i \in I} y_i \right) \left(\prod_{j \in I} (-w_j) \right) \left(\prod_{i \in [[k]] \setminus (I \cup \{1\})} W_i \right) \right) \right)$$

$$- \sum_{\emptyset \neq I \subseteq [[k]] \setminus \{1\}} \left(\left(x_1 + \sum_{i \in I} x_i \right) \left(\sum_{i \in [[k]] \setminus (I \cup \{1\})} y_i \right) \left(\prod_{i \in [[k]] \setminus (I \cup \{1\})} W_i \right) \right) \right)$$

$$by (11).(16) \quad Xy_1 \prod_{i=2}^k W_i - X \prod_{j=2}^k W_j' \left(y_1 - \sum_{i=2}^k \frac{y_i w_i}{W_i'} \right) + x_1 \left(\prod_{i=2}^k W_i \right) \left(\sum_{j=2}^k y_j \right)$$

$$+ \left(\prod_{r=2}^k W_r' \right) \left(\sum_{\substack{2 \le i, j \le k \\ i \neq j}} \frac{x_i y_j w_i W_j}{W_i' W_j'} - x_1 \sum_{i=2}^k \frac{y_i W_i}{W_i'} \right) \right)$$

$$= \prod_{i=1}^k W_i - x_1 y_1 \prod_{i=2}^k W_i - XZ \left(y_1 - x_1 Z_3 - y_1 Z_1 \right) + Z \left(-x_1 (Y Z_1 + X Z_3) + \sum_{\substack{2 \le i, j \le k \\ i \neq j}} \frac{x_i y_j w_i W_j}{W_i' W_j'} \right)$$

$$= \prod_{i=1}^k W_i - x_1 y_1 \prod_{i=2}^k W_i - Z \left(y_1 X - y_1 X Z_1 + x_1 Y Z_1 \right) + Z \sum_{\substack{2 \le i, j \le k \\ i \neq j}} \frac{x_i y_j w_i W_j}{W_i' W_j'},$$

$$(32)$$

where in the second last equality, we combine $Xy_1 \prod_{i=2}^k W_i$ and $x_1 \prod_{i=2}^k W_i \left(\sum_{j=2}^k y_j\right)$ to obtain $\prod_{i=1}^k W_i - \sum_{j=2}^k W_j$ $x_1y_1\prod_{i=2}^k W_i.$ By applying identity (16), we have

$$\Gamma_{3} = -\sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left(\left(W_{1} + \sum_{i \in I} W_{i} \right) \left(\sum_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} \frac{x_{i}y_{i}}{W_{i}} \right) \left(\prod_{j \in I} (-w_{j}) \right) \left(\prod_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} W_{i} \right) \right)$$

$$\stackrel{by (16)}{=} W_{1} \left(\prod_{i=2}^{k} W_{i} \right) \left(\sum_{i=2}^{k} \frac{x_{i}y_{i}}{W_{i}} \right) + \left(\prod_{i=2}^{k} W_{i}' \right) \left(-W_{1} \sum_{i=2}^{k} \frac{x_{i}y_{i}}{W_{i}'} + \sum_{\substack{2 \leq i, j \leq k \\ i \neq j}} \frac{w_{i}W_{i}x_{j}y_{j}}{W_{i}'W_{j}'} \right)$$

$$= \left(\prod_{i=1}^{k} W_{i} \right) \left(\sum_{i=2}^{k} \frac{x_{i}y_{i}}{W_{i}} \right) - W_{1}ZZ_{1} + Z \sum_{\substack{2 \leq i, j \leq k \\ i \neq j}} \frac{w_{i}W_{i}x_{j}y_{j}}{W_{i}'W_{j}'}. \tag{33}$$

Note that for any set S of positive integers, and real numbers a_i, b_i for $i \in S$, we have

$$\sum_{\substack{i,j\in S\\i\neq j}} (a_i b_j) = \left(\sum_{i\in S} a_i\right) \cdot \left(\sum_{i\in S} b_i\right) - \sum_{i\in S} (a_i b_i).$$

Thus,

$$\sum_{\substack{2 \le i,j \le k \\ i \ne j}} \frac{x_i y_j w_i W_j}{W'_i W'_j} - \sum_{\substack{2 \le i,j \le k \\ i \ne j}} \frac{w_i W_i x_j y_j}{W'_i W'_j}$$

$$= \left(\sum_{i=2}^k \frac{x_i w_i}{W'_i}\right) \cdot \left(\sum_{i=2}^k \frac{y_i W_i}{W'_i}\right) - \sum_{i=2}^k \frac{x_i y_i w_i W_i}{(W'_i)^2} - \left(\left(\sum_{i=2}^k \frac{x_i y_i}{W'_i}\right) \cdot \left(\sum_{i=2}^k \frac{w_i W_i}{W'_i}\right) - \sum_{i=2}^k \frac{x_i y_i w_i W_i}{(W'_i)^2}\right)$$

$$= (x_1 Z_1 + y_1 Z_2)(Y' + y_1 Z_1 + x_1 Z_3) - Z_1(x_1 Y' + y_1 X' + x_1^2 Z_3 + y_1^2 Z_2 + 2x_1 y_1 Z_1)$$

$$= y_1(Y' Z_2 - X' Z_1) + x_1 y_1(Z_2 Z_3 - Z_1^2)$$

$$= y_1(Y'Z_2 - X'Z_1) + x_1y_1(1 - 2Z_1)$$

= $y_1(Y'Z_2 + x_1 - X'Z_1 - 2x_1Z_1),$ (34)

where the second last equality follows from the fact that $(Z_1 - 1)^2 = Z_2 Z_3$ by Lemma 1. Thus, by (32), (33) and (34),

$$\Gamma_{1} - \Gamma_{2} - \Gamma_{3} - \left(\prod_{i=1}^{k} (x_{i}Y + y_{i}X)\right) \left(1 - \sum_{i=1}^{k} \frac{x_{i}y_{i}}{x_{i}Y + y_{i}X}\right)$$

$$= -Z \left(y_{1}X - y_{1}XZ_{1} + x_{1}YZ_{1}\right) + ZZ_{1}(x_{1}Y + y_{1}X) + Zy_{1}(Y'Z_{2} + x_{1} - X'Z_{1} - 2x_{1}Z_{1})$$

$$= y_{1}Z(X'Z_{1} + Y'Z_{2} - X')$$

$$= y_{1}Z \left(-X' + \sum_{i=2}^{k} \frac{x_{i}(y_{i}X' + x_{i}Y')}{x_{i}Y' + y_{i}X'}\right)$$

$$= 0.$$
(35)

Thus, (21) follows from (25), (35) and the definition of $\phi(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$ in (17).

4 Counting spanning trees in a special type of multigraphs

Let $V = \{v_i : i \in [k]\}$ and $E = \{v_i v_j : i, j \in [k]\}$ and $i \neq j\}$ be the vertex set and edge set of the complete graph K_k , where $k \geq 1$. Let ω be a weight function on E. If $\omega(v_i v_j)$ is a nonnegative integer for all i, j with $1 \leq i < j \leq k$, then $\tau(K_k, \omega)$ is the number of spanning trees of the multigraph with vertex set $\{u_1, u_2, \ldots, u_k\}$ which contains exactly $\omega(v_i v_j)$ parallel edges joining u_i and u_j for all i, j with $1 \leq i < j \leq k$.

For any non-empty subset I of $[\![k]\!] \setminus \{1\}$, let G_I denote the complete graph of order k - |I| with vertex set $\{v_I\} \cup \{v_i : i \in [\![k]\!] \setminus (I \cup \{1\})\}$ and weight function ω_I on the edge set of G_I defined as follows:

$$\begin{cases} \omega_I(v_i v_j) = \omega(v_i v_j), & \forall i, j \in \llbracket k \rrbracket \setminus (I \cup \{1\}), i \neq j; \\ \omega_I(v_I v_j) = \sum_{r \in I \cup \{1\}} \omega(v_r v_j), & \forall j \in \llbracket k \rrbracket \setminus (I \cup \{1\}). \end{cases}$$
(36)

Note that G_I is actually the graph obtained from K_k by identifying all vertices in $\{v_i : i \in I \cup \{1\}\}$ as one vertex.

By the inclusion-exclusion principle, the following recursive relation on $\tau(K_k, \omega)$ can be obtained.

Lemma 6. For any weight function ω on the edge set E of K_k ,

$$\tau(K_k,\omega) = \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left((-1)^{|I|-1} \tau(G_I,\omega_I) \prod_{i \in I} \omega(v_1 v_i) \right).$$
(37)

Proof. For any $i \in [[k]] \setminus \{1\}$, let A_i denote the set of members T in $\mathcal{T}(K_k)$ with $v_1 v_i \in E(T)$. Clearly,

$$\mathcal{T}(K_k) = \bigcup_{i=2}^k A_i.$$
(38)

For any $T \in \mathcal{T}(K_k)$, by the inclusion-exclusion principle,

$$\sum_{\emptyset \neq I \subseteq [\![k]\!] \setminus \{1\}} \left((-1)^{|I|-1} | \{T\} \cap \bigcap_{i \in I} A_i | \right) = 1.$$
(39)

Thus,

$$\tau(K_k,\omega) = \sum_{T \in \bigcup_{2 \le i \le k} A_i} \omega(T) = \sum_{\emptyset \ne I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left((-1)^{|I|-1} \sum_{T \in \bigcap_{i \in I} A_i} \omega(T) \right).$$
(40)

Let I be any non-empty subset of $\llbracket k \rrbracket \setminus \{1\}$ and let H_I denote the multiple graph obtained from K_k by identifying all vertices in the set $\{v_i : i \in I \cup \{1\}\}$ and removing all loops produced. The vertex set of H_I is $(V(K_k) \setminus \{v_i : i \in I \cup \{1\}\}) \cup v_I$. Clearly, H_I includes each edge $v_i v_j$, where $i, j \in \llbracket k \rrbracket \setminus (I \cup \{1\})$, while each edge $v_i v_j$ in K_k , where $i \in I \cup \{1\}$ and $j \in \llbracket k \rrbracket \setminus (I \cup \{1\})$, is changed to an edge of H_I joining v_I and v_j . There are exactly 1 + |I| parallel edges in H_I joining v_I and v_j for each $j \in \llbracket k \rrbracket \setminus (I \cup \{1\})$. The weight function on $E(H_I)$ is the restriction of ω to $E(H_I)$ and parallel edges in H_I may have different weights.

For each $T \in \bigcap_{i \in I} A_i$, let T_I be the tree obtained from T by identifying all vertices in the set $\{v_i : i \in I \cup \{1\}\}$. Clearly $\omega(T)$ and $\omega(T_I)$ have the following relation:

$$\omega(T) = \prod_{i \in I} \omega(v_1 v_i) \cdot \omega(T_I).$$
(41)

Moreover, $T \to T_I$ is a bijection from $\bigcap_{i \in I} A_i$ to $\mathcal{T}(H_I)$, implying that

$$\sum_{T \in \bigcap_{i \in I} A_i} \omega(T) = \prod_{i \in I} \omega(v_1 v_i) \cdot \sum_{T \in \bigcap_{i \in I} A_i} \omega(T_I) = \tau(H_I, \omega) \cdot \prod_{i \in I} \omega(v_1 v_i).$$
(42)

Note that G_T can be obtained from H_I by merging all parallel edges with ends v_I and v_j into one for each $j \in [\![k]\!] \setminus (I \cup \{1\})$. By the definition of ω_I , $\tau(H_I, \omega) = \tau(G_I, \omega_I)$. Thus, (37) follows from (40) and (42).

Recall the function ϕ defined in the previous section. In the following, we shall show that $\tau(K_k, \omega)$ can be expressed in terms of ϕ when ω satisfies certain conditions.

Theorem 6. Let $V = \{v_1, v_2, ..., v_k\}$ be the vertex set of the complete graph K_k , where $k \ge 1$, and ω be a weight function on the edge set E of K_k . If there exist 2k real numbers $x_1, x_2, ..., x_k$ and $y_1, y_2, ..., y_k$ such that $\omega(v_i v_j) = x_i y_j + x_j y_i$ holds for every pair i and j with $1 \le i < j \le k$, then,

$$\tau(K_k, \omega) = \phi(x_1, y_1, x_2, y_2, \dots, x_k, y_k).$$
(43)

Proof. Note that for $1 \le k \le 3$,

$$\tau(K_k,\omega) = \begin{cases} 1, & k = 1; \\ \omega(v_1v_2), & k = 2; \\ \omega(v_1v_2)\omega(v_1v_3) + \omega(v_1v_2)\omega(v_2v_3) + \omega(v_1v_3)\omega(v_2v_3), & k = 3. \end{cases}$$
(44)

As $\omega(v_i v_j) = x_i y_j + x_j y_i$, (43) follows from (19) and (44) when $k \leq 3$.

Assume that the result holds for $k \leq N$, where $N \geq 3$. In the following, we assume that k = N + 1and show that it holds in this case by induction.

Recall that for any non-empty subset of I of $[\![k]\!] \setminus \{1\}$, G_I is the complete graph with vertex set $\{v_I\} \cup \{v_j : j \in [\![k]\!] \setminus (I \cup \{1\})\}$ and weight function ω_I on its edge set defined in (36), i.e.,

$$\begin{cases} \omega_I(v_iv_j) = \omega(v_iv_j) = x_iy_j + x_jy_i, & \forall i, j \in \llbracket k \rrbracket \setminus (I \cup \{1\}), i \neq j; \\ \omega_I(v_Iv_j) = \sum_{r \in I \cup \{1\}} (x_ry_j + x_jy_r) = x_Iy_j + y_Ix_j, & \forall j \in \llbracket k \rrbracket \setminus (I \cup \{1\}). \end{cases}$$

$$\tag{45}$$

where $x_I = x_1 + \sum_{r \in I} x_r$ and $y_I = y_1 + \sum_{r \in I} y_r$. As G_I is a complete graph of order k - |I| < k with a weight function ω_I satisfying conditions in (45), by inductive assumption, (43) holds for G_I , i.e.,

$$\tau(G_I, \omega_I) = \phi(x_I, y_I, \underbrace{x_i, y_i}_{\forall i \in \llbracket k \rrbracket \setminus \{I \cup \{1\}\}}).$$
(46)

By Lemma 6, (46) and Theorem 5,

$$\tau(K_{k},\omega) = \sum_{\substack{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}}} (-1)^{|I|-1} \tau(G_{I},\omega_{I}) \prod_{i \in I} \omega(v_{1}v_{i})$$

$$= \sum_{\substack{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}}} \left((-1)^{|I|-1} \tau(G_{I},\omega_{I}) \prod_{i \in I} (x_{i}y_{1}+y_{i}x_{1}) \right)$$

$$= \sum_{\substack{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}}} \left((-1)^{|I|-1} \prod_{i \in I} (x_{i}y_{1}+y_{i}x_{1}) \cdot \phi(x_{I},y_{I},\underbrace{x_{i},y_{i}}_{\forall i \in \llbracket k \rrbracket \setminus \{I \in I\}}) \right)$$

$$= \phi(x_{1},y_{1},x_{2},y_{2},\cdots,x_{k},y_{k}).$$
(47)

Hence the result holds.

Spanning trees in $K_{m,n}$ containing a spanning forest F5

Now we are ready to prove the main result.

Proof of Theorem 4. For any spanning forest F of $K_{m,n}$ with components T_1, T_2, \ldots, T_k , observe that

$$\tau_F(K_{m,n}) = \tau(K_{m,n}/F), \tag{48}$$

where $K_{m,n}$ is unweighted and $K_{m,n}/F$ is the multigraph obtained from $K_{m,n}$ by contracting all edges in F. Note that $K_{m,n}/F$ is a multigraph of order k whose vertices correspond to components of F, as $K_{m,n}/F$ can also be obtained from $K_{m,n}$ by identifying all vertices in T_i for all $i \in [k]$, and removing all loops. Thus, we may assume that $K_{m,n}/F$ has vertices v_1, v_2, \ldots, v_k such that the number of parallel edges joining v_i and v_j is equal to the number of edges in $K_{m,n}$ with one end in T_i and the other end in T_i .

As $|X \cap V(T_s)| = m_s$ and $|Y \cap V(T_s)| = n_s$ for all $s = 1, 2, \ldots, k, K_{m,n}/F$ contains exactly $m_i n_j + m_j n_i$ parallel edges joining v_i and v_j for all $1 \le i < j \le k$. By Theorem 6,

$$\tau(K_{m,n}/F) = \phi(m_1, n_1, m_2, n_2, \dots, m_k, n_k).$$
(49)

Thus, by (48) and the definition of $\phi(x_1, y_1, x_2, y_2, \dots, x_k, y_k)$ in (17), the result holds.

Remarks 6

Another approach for proving the main result is to establish results analogue to Lemma 6 and Theorem 5. The following identity analogue to Lemma 6 can be obtained easily:

$$\tau(K_k,\omega) = \sum_{\emptyset \neq I \subseteq \llbracket k \rrbracket \setminus \{1\}} \left(\tau(G_I,\omega_I') \prod_{i \in I} \omega(v_1 v_i) \right),$$
(50)

where ω'_I is different from ω_I defined in (36), as for any $j \in [\![k]\!] \setminus (I \cup \{1\})$,

$$\omega_I'(v_I v_j) = \sum_{r \in I} \omega(v_r v_j) = \omega_I(v_I v_j) - \omega(v_1 v_j), \tag{51}$$

although $\omega'_I(v_iv_j) = \omega_I(v_iv_j)$ for all $i, j \in \llbracket k \rrbracket \setminus (I \cup \{1\})$ with $i \neq j$.

By Theorem 4 and (50), the following identity analogue to Theorem 5 holds:

$$\phi(x_1, y_1, x_2, y_2, \cdots, x_k, y_k) = \sum_{\emptyset \neq I \subseteq [\![k]\!] \setminus \{1\}} \left(\prod_{j \in I} (x_1 y_j + x_j y_1) \phi(x'_I, y'_I, \underbrace{x_s, y_s}_{\forall s \in [\![k]\!] \setminus (I \cup \{1\})} \right),$$
(52)

where $x'_I = \sum_{i \in I} x_i = x_I - x_1$ and $y'_I = \sum_{i \in I} y_i = y_I - y_1$. However, it is quite challenging to prove (52) directly. Note that for any I with $\emptyset \neq I \subseteq [\![k]\!] \setminus \{1\}$,

$$\begin{cases} x'_{I} + \sum_{i \in I} x_{i} = x_{2} + x_{3} + \dots + x_{k} = X - x_{1}, \\ y'_{I} + \sum_{i \in I} y_{i} = y_{2} + y_{3} + \dots + y_{k} = Y - x_{1}. \end{cases}$$
(53)

By the definition of the function ϕ ,

$$\phi(x'_{I}, y'_{I}, \underbrace{x_{s}, y_{s}}_{\forall s \in \llbracket k \rrbracket \setminus (I \cup \{1\})}) = \frac{x'_{I}Y' + y'_{I}X'}{X'Y'} \cdot \left(\prod_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} (x_{i}Y' + y_{i}X')\right) \\ \cdot \left(1 - \frac{x'_{I}y'_{I}}{x'_{I}Y' + y'_{I}X'} - \sum_{i \in \llbracket k \rrbracket \setminus (I \cup \{1\})} \frac{x_{i}y_{i}}{x_{i}Y' + y_{i}X'}\right),$$
(54)

where $X' = X - x_1$ and $Y' = Y - y_1$. Observe that the left-hand of (52) has a denominator XY, while its right-hand side has a denominator X'Y'.

Clearly, the main result (i.e., Theorem 4) also follows from (50) and (52).

In the end, we propose some problems.

Problem 1. Find a bijective proof for Theorem 4.

Another problem is to extend Theorem 4 to complete k-partite graphs, where $k \geq 3$.

Problem 2. Let K_{n_1,n_2,\dots,n_k} be a complete k-partite graph and F be a spanning forest in K_{n_1,n_2,\dots,n_k} , where $k \geq 3$. Find a formula for counting the number of spanning trees in K_{n_1,n_2,\ldots,n_k} which contain all edges in F.

For k = 3, we propose the following conjecture for a lower bound of $\tau_F(K_{n_1,n_2,n_3})$.

Conjecture 1. Let X_1, X_2 and X_3 be the partite sets of the complete tripartite graph K_{n_1,n_2,n_3} , where $|X_i| = n_i$ for $i \in [3]$. For any spanning forest F in K_{n_1,n_2,n_3} with k components T_1, T_2, \cdots, T_k ,

$$\tau_F(K_{n_1,n_2,n_3}) \geq \frac{1}{n_1n_2 + n_1n_3 + n_2n_3} \left(\prod_{i=1}^k \left((n-n_1)n_{1,i} + (n-n_2)n_{2,i} + (n-n_3)n_{3,i} \right) \right) \\ \cdot \left(1 - \sum_{i=1}^k \frac{n_{1,i}n_{2,i} + n_{1,i}n_{3,i} + n_{2,i}n_{3,i}}{(n-n_1)n_{1,i} + (n-n_2)n_{2,i} + (n-n_3)n_{3,i}} \right),$$
(55)

where $n = n_1 + n_2 + n_3$ and $n_{s,i} = |X_s \cap V(T_i)|$ for s = 1, 2, 3 and $i \in [k]$.

It is trivial to verify that the equality of (55) holds for $k \leq 2$.

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