# Subdivisions of maximal 3-degenerate graphs of order d + 1 in graphs of minimum degree d

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#### Abstract

We prove that every graph of minimum degree at least  $d \geq 1$  contains a subdivision of *some* maximal 3-degenerate graph of order d+1. This generalizes the classic results of Dirac (d=3) and Pelikán (d=4). We conjecture that for any planar maximal 3-degenerate graph H of order d+1 and any graph G of minimum degree at least d, G contains a subdivision of H. We verify this in the case H is  $P_6^3$  and  $P_7^3$ .

#### 1 Introduction

A classic result of Dirac [2] states that every graph of minimum degree at least 3 contains a subdivision of  $K_4$ . Pelikán [6] proved that every graph of minimum degree at least 4 contains a subdivision of  $K_5^-$ , the graph obtained by deleting an edge from  $K_5$ . In general, Mader [4] first showed that there exists a function f(k) such that every graph of minimum degree at least f(k) contains a subdivision of  $K_k$ . Bollobás and Thomason [1] showed that f(k) is  $O(k^2)$  and this is best possible.

We consider the question in the other direction. For a given integer  $d \geq 1$ , for what graphs H is it true that every graph of minimum degree at least d contains a subdivision of H? Clearly, any such graph

H can have at most d+1 vertices, since  $K_{d+1}$  has minimum degree d. We consider graphs H of order exactly d+1. We call a graph H good if every graph of minimum degree at least |H|-1 contains a subdivision of H. For  $1 \leq d \leq 3$ , it follows that  $K_{d+1}$  is good. Since there are planar graphs of minimum degree 4,  $K_5$  is not good, but Pelikán's theorem implies that  $K_5^-$  is good. We are interested in finding the maximal good graphs. Mader [5] showed that every graph of minimum degree at least  $d \geq 2$  contains a pair of adjacent vertices with d internally disjoint paths between them. This implies that the graph  $K_2 \vee \overline{K_{d-1}}$ , consisting of d-1 triangles that share a common edge, is good. However, this graph has only 2d-1 edges and is not a maximal good graph even for d=3. Turner [7] showed that the wheel  $W_d = C_d \vee K_1$  is good, for all  $d \geq 3$ , but again this has size 2d and is not a maximal good graph for d=4.

Our main result is that every graph of minimum degree at least  $d \geq 2$  contains a subdivision of *some* graph H of order d+1 and size 3d-3. For d=3,4 this implies the theorems of Dirac and Pelikán, respectively, since  $K_4$  and  $K_5^-$  are the only possible such graphs. Further, for d=5, this is the maximum possible number of edges in a good graph, since there exist planar graphs of minimum degree 5. We are unable to prove that any specific graph H of order d+1 and size 3d-3 is good, for general d, but we can say something more about the structure of the graph H. We show that H can be chosen to be 3-degenerate, that is, every subgraph of H contains a vertex of degree at most 3. We conjecture that every planar 3-degenerate graph of order d+1 and size 3d-3 is good. We prove this for two specific graphs  $P_6^3$  and  $P_7^3$ . A weaker conjecture would be that  $P_n^3$  is good for all  $n \geq 2$ .

#### 2 Notation

All graphs considered are undirected, finite and simple. The vertex set of a graph G is denoted by V(G) and the edge set by E(G). The order of a graph G is |V(G)| and |E(G)| is its size. The subset of vertices adjacent to a vertex  $v \in V(G)$  in a graph G is denoted by  $N_G(v)$  and  $d_G(v) = |N_G(v)|$  is the degree of the vertex v. If  $S \subset V(G) \cup E(G)$ , G-S is the graph obtained from G by deleting all vertices and edges in S and also edges incident with vertices in S. A graph F is a subgraph of a graph F if F if F if F is a subgraph of a graph F if F if

 $G - (V(G) \setminus S)$  is the subgraph of G induced by S. The union of two subgraphs  $H_1, H_2$  of a graph G is the subgraph with vertex set  $V(H_1) \cup V(H_2)$  and edge set  $E(H_1) \cup E(H_2)$ .

A path P in a graph G is a sequence of distinct vertices  $v_0, \ldots, v_l$  such that  $v_iv_{i+1}$  is an edge in G for  $0 \le i < l$ . We say P is a  $v_0-v_l$  path that joins  $v_0$  to  $v_l$ . The vertices  $v_0, v_l$  are the endpoints of P and  $\{v_1, \ldots, v_{l-1}\}$  are the internal vertices of P. The set of internal vertices of P is denoted I(P). We will also consider P to be a subgraph of G with vertex set  $\{v_0, \ldots, v_l\}$  and edge set  $\{v_iv_{i+1} \mid 0 \le i < l\}$ . A path P is said to be an A-B path in G, for  $A, B \subseteq V(G)$ , if P joins a vertex in A to a vertex in B and  $I(P) \cap (A \cup B) = \emptyset$ . An A-B path is also said to join A to B. A set of paths P in G is said to be internally disjoint if for any two distinct paths  $P, Q \in P$ ,  $I(P) \cap I(Q) = \emptyset$ . If P is a set of internally disjoint paths, let  $I(P) = \bigcup_{P \in P} I(P)$  be the set of internal vertices of P. If  $A \subset V(G)$  and  $u \in V(G) \setminus A$ , a u-A fan is a set of internally disjoint u-A paths having distinct endpoints in A.

A graph G is said to contain a subdivision of a graph H if there exists a subset  $B(H) \subseteq V(G)$  of vertices and a set  $\mathcal{P}$  of internally disjoint B(H)–B(H) paths in G such that:

- 1. There exist bijections  $f: V(H) \to B(H)$  and  $g: E(H) \to \mathcal{P}$ .
- 2. If  $uv \in E(H)$  then g(uv) is an f(u)-f(v) path in G.

We call the subgraph of G formed by the union of the paths in  $\mathcal{P}$  a subdivision of H and denote it  $\mathcal{T}(H)$ . The vertex  $f(v) \in V(\mathcal{T}(H))$  is said to correspond to the vertex  $v \in V(H)$ .

An ordered clique in a graph G is a complete subgraph of G together with a total ordering imposed on the vertices in the complete subgraph.

Let G be a graph and K an ordered clique in G. Let  $u_1, u_2, \ldots, u_t$  be a sequence of vertices in  $V(G) \setminus V(K)$  and  $n_1, n_2, \ldots, n_t$  a sequence of positive integers. We say  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K in G by a set of paths  $\mathcal{P}$  if the paths satisfy the following properties:

- 1. Every path in  $\mathcal{P}$  is a  $\{u_1,\ldots,u_t\}$ -V(K) path.
- 2.  $\mathcal{P}$  is a set of internally disjoint paths.
- 3. No two paths in  $\mathcal{P}$  have the same pair of endpoints.
- 4. Exactly  $n_i$  paths in  $\mathcal{P}$  have  $u_i$  as an endpoint, for all  $1 \leq i \leq t$ .

## 3 Unavoidable configurations

The main technique used in this paper is essentially the same as used by Mader in [5]. We consider ordered pairs of the form (G, K), where K is an ordered clique in the graph G. We define a reduction operation on such pairs.

Let G be a graph and K an ordered clique in G, such that  $V(K) \subset V(G)$ . Let  $v_1 < v_2 < \cdots < v_k$  be the ordering of the vertices in K.

- 1. Suppose there exists a vertex  $w \in V(G) \setminus V(K)$  that is adjacent to all vertices in V(K). Let K' be the ordered clique in G with  $V(K') = V(K) \cup \{w\}$  and the ordering  $w < v_1 < \cdots < v_k$  of V(K'). We say the pair (G, K') is obtained from the pair (G, K) by adding the vertex w.
- 2. Suppose every vertex in  $V(G) \setminus V(K)$  is not adjacent to at least one vertex in V(K). For every vertex  $u \in N_G(v_1) \setminus V(K)$  let f(u) be the smallest index such that  $v_{f(u)} \notin N_G(u)$ . Let G' be the graph obtained from G by deleting the vertex  $v_1$  and adding the edge  $uv_{f(u)}$ , for all vertices  $u \in N_G(v_1) \setminus V(K)$ . Let K' be the ordered clique in G' with  $V(K') = V(K) \setminus \{v_1\}$  and the ordering  $v_2 < \cdots < v_k$  of V(K'). We say the pair (G', K') is obtained from (G, K) by deleting the vertex  $v_1$ .

Note that for any pair (G,K) with  $V(K) \subset V(G)$ , exactly one of the two operations can be applied. We say a pair (G',K') can be derived from the pair (G,K), denoted  $(G,K) \to (G',K')$ , if (G',K') can be obtained from (G,K) by a sequence of vertex deletion or addition operations.

**Lemma 1** If  $(G, K) \to (G', K')$  then the following properties hold.

- 1.  $V(G') \setminus V(K') \subseteq V(G) \setminus V(K)$ .
- 2. For every vertex  $u \in V(G') \setminus V(K')$ ,  $d_{G'}(u) = d_G(u)$ .
- 3. For any subset  $S \subseteq V(G' V(K')) \cup E(G' V(K'))$ ,  $(G S, K) \rightarrow (G' S, K')$ .
- 4.  $|V(G')| + |V(G') \setminus V(K')| < |V(G)| + |V(G) \setminus V(K)|$ .

**Proof:** The proof follows by induction on the number of reduction operations used to derive (G', K') from (G, K). It is easy to check that each operation satisfies the required properties.

**Lemma 2** Suppose (G', K') is obtained from (G, K) by deleting a vertex. If  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K' in G', then it is also  $(n_1, \ldots, n_t)$ -joined to K in G.

**Proof:** Let K be the ordered clique  $v_1 < v_2 < \cdots < v_k$  and suppose K' is obtained from K by deleting  $v_1$ , keeping the order of the remaining vertices the same. Let  $\mathcal{P}$  be the set of paths that  $(n_1, \ldots, n_t)$ -joins  $(u_1, \ldots, u_t)$  to K' in G'. By the definition of the reduction operation, the only edges in G' that are not in G are edges of the form  $uv_{f(u)}$  for every vertex  $u \in N_G(v_1) \setminus V(K)$ . We call such edges bad edges. Note that there is at most one bad edge incident with any vertex  $u \in V(G') \setminus V(K')$ , and it must have one endpoint in V(K'). Also, if  $uv_{f(u)}$  is a bad edge, by the definition of f(u),  $uv_i$  is an edge in G, for all  $1 \leq i < f(u)$ .

Let  $\mathcal{P}_i$  be the set of  $n_i$  paths in  $\mathcal{P}$  that form a  $u_i$ –V(K') fan. If none of these paths contain a bad edge, these form a  $u_i$ –V(K) fan in G. If any of these paths contains a bad edge, it must be the last edge in the path. Let  $w_m v_{j_m}$ , for  $1 \leq m \leq l$ , be the bad edges contained in the paths in  $\mathcal{P}_i$ , where  $1 = j_0 < j_1 < j_2 < \cdots < j_l$ , and  $1 \leq l \leq n_i$ . Then, replacing the bad edge  $w_m v_{j_m}$  by the edge  $w_m v_{j_{m-1}}$ , for  $1 \leq m \leq l$ , gives a set of  $n_i$  paths that form a  $u_i$ –V(K) fan in G. These paths have the same set of internal vertices as the paths in  $\mathcal{P}_i$ . Since the paths in  $\mathcal{P}_i$  and  $\mathcal{P}_j$  can only have vertices in V(K') in common for  $i \neq j$ , this step can be done independently for each  $u_i$ . Thus we get a set of paths that  $(n_1, \ldots, n_t)$ -joins  $(u_1, \ldots, u_t)$  to K in G.

**Lemma 3** Suppose (G, K') is obtained from (G, K) by adding a vertex w. Suppose  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K' in G by a set of paths  $\mathcal{P}$ . If the sequence  $n_1, \ldots, n_t$  does not have a unique maximum, and at most one path in  $\mathcal{P}$  has w as an endpoint, then  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K in G.

**Proof:** If none of the paths in  $\mathcal{P}$  has w as an endpoint, then  $\mathcal{P}$  is a set of paths that  $(n_1, \ldots, n_t)$ -joins  $(u_1, \ldots, u_t)$  to K = K' - w in G. Suppose  $\mathcal{P}$  contains exactly one path terminating in w. Without loss of generality, we may assume  $u_1$  is its other endpoint. Since the sequence  $n_1, \ldots, n_t$  does not have a unique maximum, there exists an i > 1 such that  $n_i \geq n_1$ . Without loss of generality, assume i = 2. Since  $\mathcal{P}$  contains  $n_2 u_2 - V(K')$  paths having distinct endpoints in  $V(K') \setminus \{w\}$ , we must have  $|V(K')| > n_2 \geq n_1$ . This implies  $|V(K)| \geq n_1$ . Since  $\mathcal{P}$  contains  $n_1 u_1 - V(K')$  paths, one of which terminates in w, there

exists a vertex  $v \in V(K') \setminus \{w\}$ , such that there is no  $u_1$ -v path in  $\mathcal{P}$ . Since w is adjacent to all vertices in K, adding the edge wv to the  $u_1$ -w path in  $\mathcal{P}$ , together with all other paths in  $\mathcal{P}$ , gives a set of paths that  $(n_1, \ldots, n_t)$ -joins  $(u_1, \ldots, u_t)$  to K in G.

In view of Lemma 2, we will henceforth only need to consider cases where (G, K') is obtained from (G, K) by adding a vertex w. Suppose  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K' in G by a set of paths  $\mathcal{P}$ . In any such case, we will denote by  $\mathcal{P}' \subseteq \mathcal{P}$  the subset of paths that terminate in w, and by  $U' \subseteq \{u_1, \ldots, u_t\}$  the endpoints of paths in  $\mathcal{P}'$  other than w. We will only consider cases where the sequence  $n_1, \ldots, n_t$  does not have a unique maximum, and hence by Lemma 3, we only need to consider cases where  $|\mathcal{P}'| = |U'| \geq 2$ .

**Lemma 4** Suppose (G, K') is obtained from (G, K) by adding a vertex w. Suppose  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K' in G by a set of paths  $\mathcal{P}$  and suppose  $|\mathcal{P}'| \geq 2$ . Then  $(u_1, u_2, \ldots, u_t, w)$  is  $(n'_1, \ldots, n'_t, m)$ -joined to K in  $G - I(\mathcal{P}')$ , where  $n'_i = n_i - 1$  if  $u_i \in U'$  else  $n'_i = n_i$ , and  $m = \max_{1 \leq i \leq t} n'_i$ .

**Proof:** Since the paths in  $\mathcal{P}$  are internally disjoint,  $\mathcal{P} \setminus \mathcal{P}'$  is a set of paths that  $(n'_1, \ldots, n'_t)$ -joins  $(u_1, \ldots, u_t)$  to K in  $G - I(\mathcal{P}')$ . Since any two paths in  $\mathcal{P}$  can have at most one endpoint in common,  $|V(K)| \geq m = \max_{1 \leq i \leq t} n'_i$ . Since w is adjacent to every vertex in V(K), adding m edges joining w to V(K) to the set of paths  $\mathcal{P} \setminus \mathcal{P}'$  gives the required set of paths that  $(n'_1, \ldots, n'_t, m)$ -joins  $(u_1, \ldots, u_t, w)$  to K in  $G - I(\mathcal{P}')$ .

Let  $\mathcal{C}$  be a set of graphs such that  $A = \{a_1, \ldots, a_t\} \subseteq V(H)$ , for all graphs  $H \in \mathcal{C}$ . Suppose each vertex  $a_i \in A$  is assigned a positive integer weight  $n_i$ , for  $1 \leq i \leq t$ . We call such a set of graphs  $\mathcal{C}$  a configuration with terminal vertices  $(a_1, \ldots, a_t)$  having weights  $(n_1, \ldots, n_t)$ .

Let  $\mathcal{C}$  be a configuration with terminal vertices  $(a_1, \ldots, a_t)$  having weights  $(n_1, \ldots, n_t)$ . We say that  $\mathcal{C}$  is unavoidable if for every graph G and (G', K') such that  $(G, \emptyset) \to (G', K')$ , the following property holds.

• If  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K' in G', then G contains a subdivision of some graph  $H \in \mathcal{C}$  such that the vertex  $u_i$  in G corresponds to the vertex  $a_i$  in H, for  $1 \leq i \leq t$ .

The basic idea to prove that a configuration C is unavoidable is to use induction on the length of the sequence of reductions  $(G, \emptyset)$ 

 $(G_0, K_0), (G_1, K_1), \ldots, (G_l, K_l)$  such that  $(G_{i+1}, K_{i+1})$  is obtained from  $(G_i, K_i)$  by addition or deletion of vertices. If  $\mathcal{C}$  has t terminals  $(a_1, \ldots, a_t)$  of weights  $(n_1, \ldots, n_t)$ , we assume  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to  $K_l$  in  $G_l$ . In all configurations that we consider, the sequence of weights does not have a unique maximum. Lemmas 2 and 3 then imply that if  $(G_l, K_l)$  is obtained from  $(G_{l-1}, K_{l-1})$  by deleting a vertex or if  $|\mathcal{P}'| = 1$ , we can apply induction. If  $|\mathcal{P}'| \geq 2$ , we use Lemma 4 and an appropriate configuration  $\mathcal{C}'$  that is either known or assumed to be unavoidable as part of the induction hypothesis, and apply induction. This may require that several configurations are proved unavoidable simultaneously.

The following lemma gives a starting point for applying this argument to a graph of minimum degree at least d.

**Lemma 5** Let G be a graph of minimum degree at least  $d \geq 2$ . Then there exists a pair (G', K') such that  $(G, \emptyset) \rightarrow (G', K')$  and G' - V(K') contains an edge  $u_1u_2$  such that  $(u_1, u_2)$  is (d-1, d-1)-joined to K' in G'.

**Proof:** Let  $(G,\emptyset) = (G_0,K_0)$ ,  $(G_1,K_1)$ , ...,  $(G_l,K_l)$  be a maximal sequence of pairs such that  $(G_i,K_i)$  is obtained from  $(G_{i-1},K_{i-1})$  by either deleting or adding a vertex, for  $1 \leq i \leq l$ . Such a sequence exists since  $|V(G_{i+1})| + |V(G_{i+1})| \setminus V(K_{i+1})| < |V(G_i)| + |V(G_i)| \setminus V(K_i)|$ . Then we must have  $V(G_l) = V(K_l)$ , otherwise we can add one more pair to the sequence. Let i < l be the largest index such that  $G_i - V(K_i)$  contains an edge  $u_1u_2$ . The choice of i implies that  $(G_{i+1}, K_{i+1})$  is obtained from  $(G_i, K_i)$  be adding either the vertex  $u_1$  or  $u_2$  to  $K_i$ , otherwise  $u_1u_2$  is an edge in  $G_{i+1} - V(K_{i+1})$ . Without loss of generality,  $V(K_{i+1}) = V(K_i) \cup \{u_1\}$ . Then  $u_2$  cannot be adjacent to any vertex other than  $u_1$  in  $G_i - V(K_i)$ . Since G has minimum degree at least d,  $u_2$  has at least d-1 neighbors in  $V(K_i)$  and thus  $|V(K_i)| \geq d-1$ . Since  $u_1$  is adjacent to every vertex in  $V(K_i)$ , it has at least d-1 neighbors in  $V(K_i)$ . Thus  $(u_1,u_2)$  is (d-1,d-1)-joined to  $K_i$  in  $G_i$ , and  $(G_i,K_i)$  is the required pair.

Let H be any graph of order d+1 and  $\mathcal{C}(H)$  the configuration containing all possible graphs  $H-a_1a_2$ , for every edge  $a_1a_2 \in E(H)$ , with terminal vertices  $(a_1, a_2)$  having weights (d-1, d-1). If this configuration is unavoidable, Lemma 5 implies that H is good.

We illustrate the method by restating the proof of Mader's theorem in terms of unavoidable configurations. Let C(d) be the configuration containing the single graph  $K_{2,d}$ , with the two vertices in the part of

size 2 being the terminal vertices having weight d. We claim that for all  $d \geq 1$ , the configuration C(d) is unavoidable.

Applying the general strategy, we may assume  $(G_l, K_l)$  is obtained from  $(G_{l-1}, K_{l-1})$  by adding a vertex w and  $U' = \{u_1, u_2\}$ . Then the union of the two paths in  $\mathcal{P}'$  is a  $u_1-u_2$  path P in  $G_{l-1}-V(K_{l-1})$  that contains w. If d=1, this gives a subdivision of  $K_{2,1}$  in G, otherwise by Lemma 4,  $(u_1, u_2)$  is (d-1, d-1)-joined to  $K_{l-1}$  in  $G_{l-1}-I(P)$ . By induction, G-I(P) contains a subdivision of  $K_{2,d-1}$  with vertices  $u_1, u_2$  corresponding to the two terminals in  $K_{2,d-1}$ . The union of this with the path P gives the required subdivision of  $K_{2,d}$ . The unavoidability of  $\mathcal{C}(d)$  and Lemma 5 proves Mader's theorem for  $d \geq 2$ .

Turner's theorem for wheels can be proved in a similar way. In this case, we consider the configuration  $W_d - a_1 a_2$ , where  $a_1 a_2$  is a spoke and  $a_1$  the center of the wheel. Both  $a_1, a_2$  have weight d-1. We also need another configuration  $W_d - \{a_1 a_2, a_1 a_3, a_2 a_3\}$ , where  $\{a_1, a_2, a_3\}$  induce a triangle in  $W_d$  with  $a_1$  being the center of the wheel. If d=3,  $a_1, a_2, a_3$  all have weight 1, while for  $d \geq 4$ ,  $a_1, a_2$  have weight d-2 and  $a_3$  has weight d-3. It can be argued in a similar way that both these configurations are unavoidable for all  $d \geq 3$ .

## 4 Maximal 3-degenerate graphs

A maximal 3-degenerate graph of order  $n \geq 3$  is a graph whose vertices can be ordered  $v_1, \ldots, v_n$  such that  $\{v_1, v_2, v_3\}$  induce a  $K_3$  and  $v_i$  is adjacent to exactly 3 vertices in  $\{v_1, \ldots, v_{i-1}\}$ , for  $4 \leq i \leq n$ .

**Theorem 1** Every graph of minimum degree at least  $d \geq 2$  contains a subdivision of some maximal 3-degenerate graph of order d + 1.

**Proof:** The proof follows the same general strategy. We define a set of configurations and show that they are unavoidable. The theorem then follows by applying Lemma 5.

Consider the following configurations.

1.  $C_1(d)$  for  $d \geq 1$ , contains all graphs of order d+2 with two terminal vertices  $a_1, a_2$ , and d other vertices  $b_1, \ldots, b_d$ , ordered so that  $b_1$  is adjacent to  $a_1$  and  $a_2$ , and  $b_i$  is adjacent to exactly 3 vertices in  $\{a_1, a_2, b_1, \ldots, b_{i-1}\}$ , for  $2 \leq i \leq d$ . The two terminal vertices  $a_1, a_2$  have weight d.

- 2.  $C_2(d)$  for  $d \ge 1$ , contains all graphs of order d+3 with 3 terminal vertices  $a_1, a_2, a_3$ , and d other vertices  $b_1, b_2, \ldots, b_d$ , ordered so that  $b_i$  is adjacent to exactly 3 vertices in  $\{a_1, a_2, a_3, b_1, \ldots, b_{i-1}\}$ , for  $1 \le i \le d$ . The 3 terminal vertices have weight d each.
- 3.  $C_3(d)$  for  $d \ge 1$ , contains all graphs of order d+3 with 3 terminal vertices  $a_1, a_2, a_3$ , such that  $a_1$  adjacent to  $a_2$ , and d other vertices  $b_1, \ldots, b_d$ , ordered so that  $b_i$  is adjacent to exactly 3 vertices in  $\{a_1, a_2, a_3, b_1, \ldots, b_{i-1}\}$  for  $1 \le i \le d$ . The vertices  $a_1, a_2$  have weight d+1, while  $a_3$  has weight d.
- 4.  $C_4(d,t)$  for  $d \geq 0$ ,  $t \geq 3$ , contains all graphs with t terminal vertices  $a_1, a_2, \ldots, a_t$  such that  $a_1$  is adjacent to  $a_i$  for  $1 \leq i \leq t$ , and i0 other vertices i1, ..., i2, i3 such that i3 is adjacent to exactly 3 vertices in  $\{a_1, \ldots, a_t, b_1, \ldots, b_{i-1}\}$  for  $1 \leq i \leq d$ . The weight of i3 is i4 to i4, and the weight of i5 is i6 to i7.

We show that the configurations  $C_1$ ,  $C_2$ ,  $C_3$ ,  $C_4$  are unavoidable. We consider each of the 4 configurations.

Case 1. Consider the configuration  $C_1$ . If d=1, this just contains the graph  $K_{2,1}$  with 2 terminal vertices of weight 1. This is unavoidable, as argued in the proof of Mader's theorem. Suppose  $d \geq 2$ . We may assume  $\mathcal{P}'$  contains exactly 2 paths. Lemma 4 implies  $(u_1, u_2, w)$  is (d-1, d-1, d-1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of some graph in  $C_2(d-1)$ , with vertices  $u_1, u_2, w$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of a graph in  $C_1(d)$ .

Case 2. Consider the configuration  $C_2$ . If d=1, this contains the graph  $K_{3,1}$  with 3 terminal vertices of weight 1. If  $\mathcal{P}'$  contains 3 paths, this gives a subdivision of  $K_{3,1}$  in  $G_{l-1} - V(K_{l-1})$ , with  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. Suppose  $|\mathcal{P}'| = 2$ , and assume without loss of generality  $U' = \{u_1, u_2\}$ . Lemma 4 implies  $(u_3, w)$  is (1,1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . This implies  $G - I(\mathcal{P}')$  contains a  $u_3$ —w path. Adding this to the paths in  $\mathcal{P}'$  gives a subdivision of  $K_{3,1}$  in which  $u_1, u_2, u_3$  correspond to  $a_1, a_2, a_3$ , respectively.

A similar argument holds if  $d \geq 2$ . If  $\mathcal{P}'$  contains 3 paths, by Lemma 4,  $(u_1, u_2, u_3)$  is (d-1, d-1, d-1)-joined to  $K_{l-1}$  in  $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$ . By induction,  $G - (I(\mathcal{P}') \cup \{w\})$  contains a subdivision of some graph in  $\mathcal{C}_2(d-1)$ , with vertices  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the vertex w and the paths in  $\mathcal{P}'$  to this, gives a subdivision of a graph in  $\mathcal{C}_2(d)$ . Suppose  $|\mathcal{P}'| = 2$  and assume without loss of generality,  $U' = \{u_1, u_2\}$ . Lemma 4 implies

 $(u_3, w, u_1)$  is (d, d, d - 1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . Therefore  $G - I(\mathcal{P}')$  contains a subdivision of some graph in  $\mathcal{C}_3(d-1)$ , with vertices  $u_3, w, u_1$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of a graph in  $\mathcal{C}_2(d)$ .

Case 3. Consider the configuration  $C_3$ . Suppose  $|\mathcal{P}'| = 3$ . If d = 1, then  $(u_1, u_2)$  is (1, 1)-joined to  $K_{l-1}$  in  $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$ . This implies  $G - (I(\mathcal{P}') \cup \{w\})$  contains a  $u_1$ - $u_2$  path. Adding w and the paths in  $\mathcal{P}'$  to this, gives the required subdivision of the graph in  $C_3(1)$ . If  $d \geq 2$ , then  $(u_1, u_2, u_3)$  is (d, d, d-1)-joined to  $K_{l-1}$  in  $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$ . By induction,  $G - (I(\mathcal{P}') \cup \{w\})$  contains a subdivision of some graph in  $C_3(d-1)$ , with vertices  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the vertex w to this along with the paths in  $\mathcal{P}'$ , gives the required subdivision of a graph in  $C_3(d)$ .

Suppose  $|\mathcal{P}'| = 2$  and  $U' = \{u_1, u_2\}$ . The union of the two paths in  $\mathcal{P}'$  is a  $u_1$ - $u_2$  path P in  $G_{l-1} - V(K_{l-1})$  that contains w. Lemma 4 implies  $(u_1, u_2, u_3)$  is (d, d, d)-joined to  $K_{l-1}$  in  $G_{l-1} - I(P)$ . By induction, G - I(P) contains a subdivision of some graph in  $\mathcal{C}_2(d)$ , with vertices  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the path P to this gives a subdivision of some graph in  $\mathcal{C}_3(d)$ .

Suppose  $|\mathcal{P}'| = 2$  and  $U' = \{u_2, u_3\}$ . The case when  $U' = \{u_1, u_3\}$  is symmetric. Then  $(u_1, u_2, w)$  is (d + 1, d, d + 1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of some graph in  $\mathcal{C}_4(d-1,3)$ , with vertices  $u_1, u_2, w$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to it, gives a subdivision of a graph in  $\mathcal{C}_3(d)$ .

Case 4. Consider the configuration  $C_4$ .

Case 4.1 Suppose d=0. The only graph in  $C_4(0,t)$  has  $t\geq 3$  terminals  $a_1,\ldots,a_t$  with edges  $a_1a_i$ , for  $2\leq i\leq t$ . The weight of  $a_1$  is t-1 and that of  $a_i$  is i-1 for  $2\leq i\leq t$ . In this case, we need to show that there exist t-1 paths in G that form a  $u_1-\{u_2,\ldots,u_t\}$  fan.

Suppose  $u_1 \in U'$ . Let i be the smallest index greater than 1 such that  $u_i \in U'$ . The union of the  $u_1$ -w and  $u_i$ -w paths in  $\mathcal{P}'$  is a  $u_1$ - $u_i$  path P in  $G_{l-1}-V(K_{l-1})$ . If t>3, then  $(u_1,u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_t)$  is  $(t-2,1,\ldots,i-2,i-1,\ldots,t-2)$ -joined to  $K_{l-1}$  in  $G_{l-1}-(I(\mathcal{P}')\cup\{w\})$ . By induction,  $G-(I(\mathcal{P}')\cup\{w\})$  contains t-2 internally disjoint paths that form a  $u_1$ - $\{u_2,\ldots,u_{i-1},u_{i+1},\ldots,u_t\}$  fan. Adding the path P to this gives the required set of t-1 paths. If t=3, then  $(u_1,u_{5-i})$  is (1,1)-joined to  $K_{l-1}$  in  $G_{l-1}-(I(\mathcal{P}')\cup\{w\})$ . Thus  $G-(I(\mathcal{P}')\cup\{w\})$  contains a  $u_1$ - $u_{5-i}$  path. Adding the path P to this gives the required paths that form a  $u_1$ - $\{u_2,u_3\}$  fan.

Suppose  $u_1 \notin U'$ . Again, let i be the smallest index such that  $u_i \in U'$ . Then  $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t, w)$  is  $(t-1, 1, 2, \ldots, t-2, t-1)$ -joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains t-1 internally disjoint paths that form a  $u_1 - \{u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t, w\}$  fan. The union of the  $u_1 - w$  path in this set with the  $u_i - w$  path in  $\mathcal{P}'$  is a  $u_1 - u_i$  path in G that is internally disjoint from the other paths in the set. Replacing the  $u_1 - w$  path in the fan by this gives t-1 paths that form a  $u_1 - \{u_2, \ldots, u_t\}$  fan.

Case 4.2 Suppose d > 0. If  $|\mathcal{P}'| \geq 3$ , then  $(u_1, \ldots, u_t)$  is  $(d + t - 2, d, \ldots, d+t-2)$ -joined to  $K_{l-1}$  in  $G_{l-1} - (I(\mathcal{P}') \cup \{w\})$ . By induction,  $G - (I(\mathcal{P}') \cup \{w\})$  contains a subdivision of some graph in  $\mathcal{C}_4(d-1,t)$  with vertex  $u_i$  corresponding to  $a_i$ , for  $1 \leq i \leq t$ . Adding the vertex w and any 3 paths in  $\mathcal{P}'$ , we get a subdivision of graph in  $\mathcal{C}_4(d,t)$  that is contained in G.

Suppose  $|\mathcal{P}'| = 2$  and  $u_1 \in U'$ . Let  $u_i, i > 1$  be the other vertex in U'. Then the union of the two paths in  $\mathcal{P}'$  is a  $u_1-u_i$  path P in  $G_{l-1} - V(K_{l-1})$ . If t = 3, then  $(u_1, u_{5-i}, u_i)$  is (d+1, d+1, d)-joined to  $K_{l-1}$  in  $G_{l-1} - I(P)$ . By induction, G - I(P) contains a subdivision of some graph in  $\mathcal{C}_3(d)$  with vertices  $u_1, u_{5-i}, u_i$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the  $u_1-u_i$  path P to this gives a subdivision of a graph in  $\mathcal{C}_4(d,3)$ . If t > 3, then  $(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t)$  is  $(d+t-2, d+1, \ldots, d+i-2, d+i-1, \ldots, d+t-2)$ -joined to  $K_{l-1}$  in  $G_{l-1} - I(P)$ . By induction, G - I(P) contains a subdivision of some graph in  $\mathcal{C}_4(d, t-1)$  with vertices  $u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_t$  corresponding to  $a_1, \ldots, a_{t-1}$ , respectively. Adding the path P to this gives a subdivision of a graph in  $\mathcal{C}_4(d, t)$ .

Finally, suppose  $|\mathcal{P}'| = 2$  and  $u_1 \notin U'$ . Then  $(u_1, u_2, \ldots, u_t, w)$  is  $(d+t-1, d, \ldots, d+t-2, d+t-1)$ -joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G-I(\mathcal{P}')$  contains a subdivision of some graph in  $\mathcal{C}_4(d-1, t+1)$ , with vertices  $u_1, \ldots, u_t, w$  corresponding to  $a_1, \ldots, a_{t+1}$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of a graph in  $\mathcal{C}_4(d, t)$ .

This completes all cases and we conclude that all the 4 configurations are unavoidable. The theorem then follows from Lemma 5 and the fact that the configuration  $C_1(d-1)$  is unavoidable. Note that for any graph in  $C_1(d-1)$ , adding an edge between the two terminal vertices  $a_1, a_2$  gives a maximal 3-degenerate graph of order d+1.  $\square$ 

#### 5 Planar Maximal 3-degenerate Graphs

Since there exist non-planar 3-degenerate graphs of order 6, not every maximal 3-degenerate graph is good. However, we do not know of any planar graph that is not good. This suggests the following problem.

Conjecture 1 Every planar maximal 3-degenerate graph is good.

A specific family of planar maximal 3-degenerate graphs is  $P_n^3$  with vertices  $v_1, \ldots, v_n$  and  $v_i$  adjacent to  $v_j$  iff  $1 \leq |j-i| \leq 3$ . We verify Conjecture 1 for two graphs  $P_6^3$  and  $P_7^3$ . Note that  $P_4^3$  is  $K_4$ ,  $P_5^3$  is  $K_5^-$  and  $P_6^3$  is the only planar maximal 3-degenerate graph of order 6.

**Theorem 2** Every graph of minimum degree at least 5 contains a subdivision of  $P_6^3$ .

**Proof:** The proof is again based on the same technique, using more restricted configurations than those used in Theorem 1. Consider the following set of configurations.

- 1.  $C_5$  contains a subset of the graphs in the configuration  $C_1(4)$ . The graphs have 6 vertices  $\{a_1, a_2, b_1, b_2, b_3, b_4\}$ , where  $a_1, a_2$  are terminal vertices of weight 4. The edge sets of the 3 graphs are
  - (a)  $\{a_1b_1, a_2b_1, a_1b_2, a_2b_2, b_1b_2, a_1b_3, a_2b_3, b_2b_3, a_1b_4, b_2b_4, b_3b_4\}.$
  - (b)  $\{a_1b_1, a_2b_1, a_1b_2, a_2b_2, b_1b_2, a_1b_3, b_1b_3, b_2b_3, b_1b_4, b_2b_4, b_3b_4\}.$
  - (c)  $\{a_1b_1, a_2b_1, a_1b_2, a_2b_2, b_1b_2, a_2b_3, b_1b_3, b_2b_3, a_2b_4, b_2b_4, b_3b_4\}.$
- 2.  $C_6$  contains a subset of the graphs in the configuration  $C_2(3)$ . The graphs have 6 vertices  $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ , where  $a_1, a_2, a_3$  are terminal vertices of weight 3. The edge sets of the 3 graphs are
  - (a)  $\{a_1b_1, a_2b_1, a_3b_1, a_1b_2, a_2b_2, b_1b_2, a_1b_3, b_1b_3, b_2b_3\}.$
  - (b)  $\{a_1b_1, a_2b_1, a_3b_1, a_1b_2, a_3b_2, b_1b_2, a_3b_3, b_1b_3, b_2b_3\}.$
  - (c)  $\{a_1b_1, a_2b_1, a_3b_1, a_2b_2, a_3b_2, b_1b_2, a_2b_3, b_1b_3, b_2b_3\}.$
- 3.  $C_7$  contains only one graph from the configuration  $C_2(2)$ . This graph has 5 vertices  $\{a_1, a_2, a_3, b_1, b_2\}$ , where  $a_1, a_2, a_3$  are terminal vertices of weight 2. The edges in the graph are  $\{a_1b_1, a_2b_1, a_3b_1, a_1b_2, a_2b_2, b_1b_2\}$ .
- 4.  $C_8$  contains only one graph from the configuration  $C_3(2)$ . This graph has 5 vertices  $\{a_1, a_2, a_3, b_1, b_2\}$ , where  $a_1, a_2, a_3$  are terminal vertices,  $a_1, a_2$  have weight 3 and  $a_3$  has weight 2. The edge set of the graph is  $\{a_1a_2, a_1b_1, a_2b_1, a_3b_1, a_1b_2, a_2b_2, b_1b_2\}$ .

- 5.  $C_9$  contains two graphs with 5 vertices  $\{a_1, a_2, a_3, a_4, b_1\}$ , where  $a_1, a_2, a_3, a_4$  are terminals,  $a_1, a_2$  have weight 1 and  $a_3, a_4$  have weight 2. The edge sets of the two graphs are
  - (a)  $\{a_1b_1, a_2b_1, a_3b_1, a_4b_1\}.$
  - (b)  $\{a_1b_1, a_2b_1, a_3a_4, a_3b_1\}.$

We show that  $C_5$ ,  $C_6$ ,  $C_7$ ,  $C_8$  and  $C_9$  are unavoidable.

Case 1. Consider the configuration  $C_5$ . The only case to be considered here is if  $|\mathcal{P}'| = 2$ . Lemma 4 implies that  $(u_1, u_2, w)$  is (3, 3, 3)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of one of the graphs (a), (b) or (c) in  $C_6$ , with vertices  $u_1, u_2, w$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of the corresponding graph (a), (b) or (c) in  $C_5$  with  $u_1, u_2$  corresponding to  $a_1, a_2$  and w corresponding to  $b_1$ .

Case 2. Consider the configuration  $C_6$ . If  $|\mathcal{P}'| = 3$ , then  $(u_1, w, u_2)$  is (2, 2, 2)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of the graph in  $C_7$ , with vertices  $u_1, w, u_2$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of the graph (a) in  $C_6$ .

Suppose  $U' = \{u_1, u_2\}$ . The other cases can be argued symmetrically. Then  $(u_3, w, u_1)$  is (3, 3, 2)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of the graph in  $\mathcal{C}_8$ , with vertices  $u_3, w, u_1$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of the graph (b) in  $\mathcal{C}_6$ .

Case 3. Consider the configuration  $C_7$ . If  $|\mathcal{P}'| = 3$ , then  $(u_1, u_2, w)$  is (1, 1, 1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . Since the configuration  $C_2(1)$  is unavoidable,  $G - I(\mathcal{P}')$  contains a subdivision of  $K_{3,1}$  in which the vertices  $u_1, u_2, w$  correspond to the vertices in the part of size 3. Adding the paths in  $\mathcal{P}'$  to this, gives a subdivision of the graph in  $C_7$ .

Suppose  $U' = \{u_1, u_2\}$ . Then  $(u_1, u_2, w, u_3)$  is (1, 1, 2, 2)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$  and  $G - I(\mathcal{P}')$  contains a subdivision of one of the two graphs in  $\mathcal{C}_9$ , with vertices  $u_1, u_2, w, u_3$  corresponding to  $a_1, a_2, a_3, a_4$ , respectively. In either case, adding the paths in  $\mathcal{P}'$  to this gives a subdivision of the graph in  $\mathcal{C}_7$ .

Suppose  $U' = \{u_2, u_3\}$ . Then  $(u_1, w, u_2)$  is (2, 2, 1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$  and  $G - I(\mathcal{P}')$  contains a subdivision of the graph in  $\mathcal{C}_3(1)$ , with vertices  $u_1, w, u_2$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of the graph in  $\mathcal{C}_7$ . The case when  $U' = \{u_1, u_3\}$  can be argued symmetrically.

Case 4. Consider the configuration  $C_8$ . If  $|\mathcal{P}'| = 3$  then  $(u_1, u_2, w)$ 

is (2,2,1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of the graph in  $\mathcal{C}_3(1)$ , with vertices  $u_1, u_2, w$  corresponding to  $a_1, a_2, a_3$ , respectively. Together with the paths in  $\mathcal{P}'$ , this gives a subdivision of the graph in  $\mathcal{C}_8$ .

Suppose  $U' = \{u_1, u_2\}$ . The union of the two paths in  $\mathcal{P}'$  is a  $u_1$ – $u_2$  path P in  $G_{l-1} - V(K_{l-1})$  that contains w. Since  $(u_1, u_2, u_3)$  is (2, 2, 2)-joined to  $K_{l-1}$  in  $G_{l-1} - I(P)$ , G - I(P) contains a subdivision of the graph in  $\mathcal{C}_7$ , with vertices  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the path P to this gives a subdivision of the graph in  $\mathcal{C}_8$ .

Suppose  $U' = \{u_2, u_3\}$ . Then  $(u_1, u_2, w)$  is (3, 2, 3)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of the graph in  $\mathcal{C}_4(1,3)$ , with vertices  $u_1, u_2, w$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  to this gives a subdivision of the graph in  $\mathcal{C}_8$ . The case when  $U' = \{u_1, u_3\}$  is similar.

Case 5. Consider the configuration  $C_9$ . If  $|\mathcal{P}'| = 4$ , then  $G_{l-1} - V(K_{l-1})$  contains a subdivision of the graph (a) in  $C_9$ , with vertices  $u_1, u_2, u_3, u_4, w$  corresponding to  $a_1, a_2, a_3, a_4, b_1$ , respectively.

If  $|\mathcal{P}'| = 3$ , then  $(u_i, w)$  is (1, 1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ , where  $u_i \notin U'$ . Then  $G - I(\mathcal{P}')$  contains a  $u_i$ -w path, which together with the paths in  $\mathcal{P}'$  gives a subdivision of the graph (a) in  $\mathcal{C}_9$ .

Suppose  $|\mathcal{P}'| = 2$  and  $U' \neq \{u_3, u_4\}$ . Then  $(w, u_i, u_j)$  is (2, 1, 2)-joined to  $K_{l-1}$  in  $G_{l-1} - I(\mathcal{P}')$ , where  $u_i, u_j \notin U'$  and  $1 \leq i < j \leq 4$ . By induction,  $G - I(\mathcal{P}')$  contains a subdivision of the graph in  $\mathcal{C}_4(0, 3)$  with vertices  $w, u_i, u_j$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the paths in  $\mathcal{P}'$  gives a subdivision of the graph (a) in  $\mathcal{C}_9$ .

The only other possibility is that  $U' = \{u_3, u_4\}$ . The union of the 2 paths in  $\mathcal{P}'$  is a  $u_3$ - $u_4$  path P in  $G_{l-1} - V(K_{l-1})$ . Since  $(u_1, u_2, u_3)$  is (1, 1, 1)-joined to  $K_{l-1}$  in  $G_{l-1} - I(P)$ , G - I(P) contains a subdivision of the graph in  $\mathcal{C}_2(1)$ , with vertices  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. Adding the path P to this gives a subdivision of the graph (b) in  $\mathcal{C}_9$ .

This completes all cases and shows that the configurations  $C_5$ ,  $C_6$ ,  $C_7$ ,  $C_8$  and  $C_9$  are unavoidable. Theorem 2 then follows from Lemma 5, since adding the edge  $a_1a_2$  to any graph in  $C_5$  gives the graph  $P_6^3$ .  $\square$ 

We next consider planar maximal 3-degenerate graphs of order 7. There are 3 different such graphs, but we consider only the graph  $P_7^3$ . While it is possible to use the same technique, the number of configurations required appears to be large. We can reduce the number of configurations required by starting with an initial graph other than

an edge.

Let  $\mathcal{C}$  be a configuration with terminal vertices  $a_1, \ldots, a_t$  of weights  $n_1, \ldots, n_t$ , respectively. We say a pair (G, K) contains the configuration  $\mathcal{C}$  if G - V(K) contains a subdivision  $\mathcal{T}(H)$  of some graph  $H \in \mathcal{C}$ , such that vertices  $u_1, \ldots, u_t$  correspond to  $a_1, \ldots, a_t$ , respectively, and  $(u_1, \ldots, u_t)$  is  $(n_1, \ldots, n_t)$ -joined to K in  $G - V(\mathcal{T}(H)) \setminus \{u_1, \ldots, u_t\}$ . Consider the following set of configurations.

- 1.  $C_{10}(d)$  for  $d \geq 1$  is the configuration containing only the graph  $K_2$  with 2 terminal vertices of weight d.
- 2.  $C_{11}(d)$  for  $d \geq 1$  is the configuration containing only the graph  $K_3$  with 3 terminal vertices of weight d.
- 3.  $C_{12}(d)$  for  $d \geq 1$  is the configuration containing only the graph  $K_4^-$ , obtained by deleting an edge from  $K_4$ . There are 3 terminal vertices  $a_1, a_2, a_3$  with  $a_1, a_3$  of weight d + 1 and  $a_2$  of weight d. The missing edge is  $a_1a_3$ .
- 4.  $C_{13}(d)$  for  $d \geq 1$  is the configuration containing only the graph  $K_4$  with 3 terminal vertices  $a_1, a_2, a_3$  of weight d.

**Lemma 6** Let G be a graph of minimum degree at least  $d \geq 4$ . Then there exists a pair (G', K') such that  $(G, \emptyset) \rightarrow (G', K')$  and (G', K') contains the configuration  $C_{13}(d-3)$ .

**Proof:** Let  $(G,\emptyset) = (G_0,K_0)$ ,  $(G_1,K_1)$ ,..., $(G_l,K_l)$  be a maximal sequence of pairs such that  $(G_{i+1},K_{i+1})$  is obtained from  $(G_i,K_i)$  by adding or deleting a vertex, for  $0 \le i < l$ . Let i be the smallest index such that  $(G_i,K_i)$  contains the configuration  $\mathcal{C}_{10}(d-1)$ . Lemma 5 implies there exists such an index i. Since  $d \ge 4$ , we have i > 0. Then  $(G_i,K_i)$  must be obtained from  $(G_{i-1},K_{i-1})$  by adding a vertex w, and  $U' = \{u_1,u_2\}$ . This implies  $(G_{i-1},K_{i-1})$  contains the configuration  $\mathcal{C}_{11}(d-2)$  with vertices  $u_1,u_2,w$  corresponding to  $a_1,a_2,a_3$ , respectively.

Let j be the smallest index such that  $(G_j, K_j)$  contains  $\mathcal{C}_{11}(d-2)$ . Since  $d \geq 4$ , we have j > 0. Again,  $(G_j, K_j)$  must be obtained from  $(G_{j-1}, K_{j-1})$  by adding a vertex w. If  $|\mathcal{P}'| = 3$ , then  $(G_{j-1}, K_{j-1})$  contains  $\mathcal{C}_{13}(d-3)$ , with vertices  $u_1, u_2, u_3$  corresponding to the vertices  $a_1, a_2, a_3$ , respectively. Then  $(G_{j-1}, K_{j-1})$  is the required pair.

Suppose  $|\mathcal{P}'| = 2$ , and without loss of generality,  $U' = \{u_2, u_3\}$ . Then  $(G_{j-1}, K_{j-1})$  contains  $\mathcal{C}_{12}(d-3)$  with vertices  $u_1, u_2, w$  corresponding to  $a_1, a_2, a_3$ , respectively. Let m be the smallest index such that  $(G_m, K_m)$  contains  $\mathcal{C}_{12}(d-3)$ . Since  $d \geq 4$ , we have m > 0. Again,  $(G_m, K_m)$  must be obtained from  $(G_{m-1}, K_{m-1})$  by adding a vertex w. If  $|\mathcal{P}'| = 3$ , then  $(G_{m-1}, K_{m-1})$  contains the configuration  $\mathcal{C}_{13}(d-3)$  with vertices  $u_1, u_3, w$  corresponding to  $a_1, a_2, a_3$ , respectively. If  $U' = \{u_1, u_3\}$  then  $(G_{m-1}, K_{m-1})$  contains  $\mathcal{C}_{13}(d-3)$  with vertices  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. If  $U' = \{u_2, u_3\}$  then  $(G_{m-1}, K_{m-1})$  contains  $\mathcal{C}_{12}(d-3)$ , with vertices  $u_1, u_3, w$  corresponding to  $a_1, a_2, a_3$ , respectively. This contradicts the choice of m. Similarly, if  $U' = \{u_1, u_2\}$ , then  $(G_{m-1}, K_{m-1})$  contains  $\mathcal{C}_{12}(d-3)$ , with vertices  $u_3, u_1, w$  corresponding to  $a_1, a_2, a_3$ , respectively. Again, this contradicts the choice of m. Therefore  $(G_{m-1}, K_{m-1})$  must contain  $\mathcal{C}_{13}(d-3)$ .

**Theorem 3** Every graph of minimum degree at least 6 contains a subdivision of  $P_7^3$ .

**Proof:** Let G be a graph of minimum degree at least 6. Lemma 6 implies there exists a pair (G', K') such that  $(G, \emptyset) \to (G', K')$  and (G', K') contains the configuration  $\mathcal{C}_{13}(3)$ . Thus G' - V(K') contains a subdivision H of  $K_4$ , with vertices  $u_1, u_2, u_3$  corresponding to the vertices  $a_1, a_2, a_3$ , respectively, such that  $(u_1, u_2, u_3)$  is (3, 3, 3)-joined to K' in  $G' - (V(H) \setminus \{u_1, u_2, u_3\})$ . Since  $\mathcal{C}_6$  is unavoidable,  $G - (V(H) \setminus \{u_1, u_2, u_3\})$  contains a subdivision of one of the graphs (a), (b), or (c) in  $\mathcal{C}_6$ , with vertices  $u_1, u_2, u_3$  corresponding to  $a_1, a_2, a_3$ , respectively. In all cases, the union of this graph with H gives a subdivision of  $P_7^3$  in G.

#### 6 Remarks

We have verified Conjecture 1 for the other two planar maximal 3-degenerate graphs of order 7. Although the method is the same, the number of configurations required is larger, and we omit the details. A planar maximal 3-degenerate graph is also a maximal planar graph. An interesting question is whether all maximal planar graphs are good? The smallest case to consider is the octahedron, obtained by deleting a perfect matching from  $K_6$ . While we do not know a graph of minimum degree 5 that does not contain a subdivision of this, the technique used in this paper cannot be applied since the required configuration is avoidable. It would be interesting to see if there is any characterization of unavoidable configurations. Perhaps

the first question to answer would be to find the maximum number of edges in a good graph of order d+1. For  $2 \le d \le 5$ , this is exactly 3d-3. Does this hold in general? An even simpler question would be to find the largest number m such that every graph of minimum degree d contains a subdivision of some graph of order d+1 and size m. Theorem 1 shows that  $m \ge 3d-3$  and the bound is tight for  $2 \le d \le 5$ . Does this hold for all d? Finally, it would be interesting to consider non-separating versions of these results. Kriesell [3] generalized Dirac's theorem to show that every connected graph G with minimum degree at least 4 contains a subdivision H of  $K_4$  such that G-V(H) is connected. Can the results in this paper be extended in a similar way, by increasing the minimum degree bound by one?

#### References

- [1] B. Bollobás and A. G. Thomason, Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs, Eur. J. Comb. 19 (1998) 883–887.
- [2] G. A. Dirac, In abstrakten Graphen vorhandene vollständige 4-Graphen und ihre Unterteilungen, Math. Nachr. 22 (1960) 61–85.
- [3] M. Kriesell, Nonseparating  $K_4$ -subdivisions in graphs of minimum degree at least 4, J. Graph Theory 89 (2018) 194–213.
- [4] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Ann. 174 (1967) 265–268.
- [5] W. Mader, Existenz gewisser Konfigurationen in *n*-gesättigten Graphen und in Graphen genügend großer Kantendichte, Math. Ann. 194 (1971) 295–312.
- [6] J. Pelikán, Valency conditions for the existence of certain subgraphs, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp 251-258.
- [7] G. E. Turner III, A generalization of Dirac's theorem: Subdivisions of wheels, Discrete Math. 297 (2005) 202–205.