# Polynomial bounds for chromatic number II. Excluding a star-forest 

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#### Abstract

The Gyárfás-Sumner conjecture says that for every forest $H$, there is a function $f_{H}$ such that if $G$ is $H$-free then $\chi(G) \leq f_{H}(\omega(G))$ (where $\chi, \omega$ are the chromatic number and the clique number of $G)$. Louis Esperet conjectured that, whenever such a statement holds, $f_{H}$ can be chosen to be a polynomial. The Gyárfás-Sumner conjecture is only known to be true for a modest set of forests $H$, and Esperet's conjecture is known in to be true for almost no forests. For instance, it is not known when $H$ is a five-vertex path. Here we prove Esperet's conjecture when each component of $H$ is a star.


## 1 Introduction

The Gyárfás-Sumner conjecture [6, 20] asserts:
1.1 Conjecture: For every forest $H$, there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every $H$-free graph $G$.
(We use $\chi(G)$ and $\omega(G)$ to denote the chromatic number and the clique number of a graph $G$, and a graph is $H$-free if it has no induced subgraph isomorphic to $H$.) This remains open in general, though it has been proved for some very restricted families of trees (see, for example, [1, 7, 8, 9, 11, 13, 14]).

A class $\mathcal{C}$ of graphs is $\chi$-bounded if there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every graph $G$ that is an induced subgraph of a member of $\mathcal{C}$ (see [15] for a survey). Thus the Gyárfás-Sumner conjecture asserts that, for every forest $H$, the class of all $H$-free graphs is $\chi$-bounded. Esperet 5 conjectured that every $\chi$-bounded class is polynomially $\chi$-bounded, that is, $f$ can be chosen to be a polynomial. Neither conjecture has been settled in general. There is a survey by Schiermeyer and Randerath [19] on related material.

In particular, what happens to Esperet's conjecture when we exclude a forest? For which forests $H$ can we show the following?
1.2 Esperet's conjecture: There is a polynomial $f_{H}$ such that $\chi(G) \leq f_{H}(\omega(G))$ for every $H$-free graph $G$.

Not for very many forests $H$, far fewer than the forests that we know satisfy 1.1. For instance, 1.2 is not known when $H=P_{5}$, the five-vertex path. (This case is of great interest, because it would imply the Erdős-Hajnal conjecture [3, 4, 2] for $P_{5}$, and the latter is currently the smallest open case of the Erdős-Hajnal conjecture.)

We remark that, if in 1.2 we replace $\omega(G)$ by $\tau(G)$, defined to be the maximum $t$ such that $G$ contains $K_{t, t}$ as a subgraph, then all forests satisfy the modified 1.2. More exactly, the following is shown in [16]:
1.3 For every forest $H$, there is a polynomial $f_{H}$ such that $\chi(G) \leq f_{H}(\tau(G))$ for every $H$-free graph $G$.

One difficulty with 1.2 is that we cannot assume that $H$ is connected, or more exactly, knowing that each component of $H$ satisfies 1.2 does not seem to imply that $H$ itself satisfies 1.2, For instance, while $H=P_{4}$ satisfies 1.2, we do not know the same when $H$ is the disjoint union of two copies of $P_{4}$.

As far as we are aware, the only forests that were already known to satisfy 1.2 are those of the following three results, and their induced subgraphs (a star is a tree in which one vertex is adjacent to all the others):

### 1.4 The forest $H$ satisfies 1.2 if either:

- $H$ is the disjoint union of copies of $K_{2}$ (S. Wagon [21]); or
- $H$ is the disjoint union of $H^{\prime}$ and a copy of $K_{2}$, and $H^{\prime}$ satisfies 1.2 (I. Schiermeyer [18]); or
- H is obtained from a star by subdividing one edge once (X. Liu, J. Schroeder, Z. Wang and X. Yu [12]).

In the next paper of this series [17] we will show a strengthening of the third result of [1.4, that is, 1.2 is true when $H$ is a "double star", a tree with two internal vertices, the most general tree that does not contain a five-vertex path. Our main theorem in this paper is a strengthening of the second result of 1.4 :
1.5 If $H$ is the disjoint union of $H^{\prime}$ and a star, and $H^{\prime}$ satisfies 1.2, then $H$ satisfies 1.2.

A star-forest is a forest in which every component is a star. From 1.5 and the result of [17], we deduce
1.6 If $H^{\prime}$ is a double star, and $H$ is the disjoint union of $H^{\prime}$ and a star-forest, then $H$ satisfies 1.2.

As far as we know (although it seems unlikely), these might be all the forests that satisfy 1.2 ,

## 2 The proof

We will need the following well-known version of Ramsey's theorem:
2.1 For $k \geq 1$ an integer, if a graph $G$ has no stable subset of size $k$, then

$$
|V(G)| \leq \omega(G)^{k-1}+\omega(G)^{k-2}+\cdots+\omega(G) .
$$

Consequently $|V(G)|<\omega(G)^{k}$ if $\omega(G)>1$.
Proof. The claim holds for $k \leq 2$, so we assume that $k \geq 3$ and the result holds for $k-1$. Let $X$ be a clique of $G$ of cardinality $\omega(G)$, and for each $x \in X$ let $W_{x}$ be the set of vertices nonadjacent to $X$. From the inductive hypothesis, $\left|W_{x}\right| \leq \omega(G)^{k-2}+\cdots+\omega(G)$ for each $x$; but $V(G)$ is the union of the sets $W_{x} \cup\{x\}$ for $x \in X$, and the result follows by adding. This proves 2.1.

If $X \subseteq V(G)$, we denote the subgraph induced on $X$ by $G[X]$. When we are working with a graph $G$ and its induced subgraphs, it is convenient to write $\chi(X)$ for $\chi(G[X])$. Now we prove 1.5 , which we restate:
2.2 If $H^{\prime}$ satisfies 1.2, and $H$ is the disjoint union of $H^{\prime}$ and a star, then $H$ satisfies 1.2.

Proof. $H$ is the disjoint union of $H^{\prime}$ and some star $S$; let $S$ have $k+1$ vertices. Since $H^{\prime}$ satisfies 1.2, and $\chi(G)=\omega(G)$ for all graphs with $\omega(G) \leq 1$, there exists $c^{\prime}$ such that $\chi(G) \leq \omega(G)^{c^{\prime}}$ for every $H^{\prime}$-free graph $G$. Choose $c \geq k+2$ such that

$$
x^{c}-(x-1)^{c} \geq 1+x^{k+1}+x^{k(k+2)+c^{\prime}}
$$

for all $x \geq 2$ (it is easy to see that this is possible).
Let $G$ be an $H$-free graph, and write $\omega(G)=\omega$; we will show that $\chi(G) \leq \omega^{c}$ by induction on $\omega$. If $\omega=1$ then $\chi(G)=1$ as required, so we assume that $\omega \geq 2$. Let $n=\omega^{k+1}$. If every vertex of $G$ has degree less than $\omega^{c}$, then the result holds as we can colour greedily, so we assume that some vertex $v$ has degree at least $\omega^{c}$. Let $N$ be the set of all neighbours of $v$ in $G$. Let $X_{1}$ be the largest clique contained in $N$; let $X_{2}$ be the largest clique contained in $N \backslash X_{1}$; and in general, let $X_{i}$ be the largest clique contained in $N \backslash\left(X_{1} \cup \cdots \cup X_{i-1}\right)$. Since $|N| \geq \omega^{c} \geq n \omega$ (because $c \geq k+2$ ), it follows
that $X_{1}, \ldots, X_{n} \neq \emptyset$. Let $X=X_{1} \cup \cdots \cup X_{n}$, and $X_{0}=N \backslash X$, and $t=\left|X_{n}\right|$. Thus $1 \leq t \leq \omega-1$ (because $\omega(G[N])<\omega)$.
(1) $\chi(N \cup\{v\}) \leq t^{c}+n \omega$.

From the choice of $X_{n}$, it follows that the largest clique of $G\left[X_{0}\right]$ has cardinality at most $t<\omega$. From the inductive hypothesis, $\chi\left(X_{0}\right) \leq t^{c}$, and since $X \cup\{v\}$ has cardinality at most $n \omega$, it follows that $\chi(N \cup\{v\}) \leq t^{c}+n \omega$. This proves (1).

For each stable set $Y \subseteq X$ with $|Y|=k$, let $A_{Y}$ be the set of vertices in $V(G) \backslash(N \cup\{v\})$ that have no neighbour in $Y$. Let $A$ be the union of all the sets $A_{Y}$, and $B=V(G) \backslash(A \cup N \cup\{v\})$.
(2) $\chi(A) \leq(n \omega)^{k} \omega^{c^{\prime}}$.

For each choice of $Y, G\left[A_{Y}\right]$ is $H^{\prime}$-free (because $Y \cup\{v\}$ induces a copy of $S$ with no edges to $A_{Y}$, and so $\chi\left(A_{Y}\right) \leq \omega^{c^{\prime}}$. Since there are at most $|X|^{k} \leq(n \omega)^{k}$ choices of $Y$, it follows that the union $A$ of all the sets $A_{Y}$ has chromatic number at most $(n \omega)^{k} \omega^{c^{\prime}}$. This proves (2).
(3) For each $b \in B, b$ has fewer than $\omega^{k}$ non-neighbours in $X$.

Let $Z$ be the set of vertices in $X$ nonadjacent to $b$. Since $b \notin A, G[Z]$ has no stable set of cardinality $k$; and since it also has no clique of cardinality $\omega$, 2.1 implies that $|Z| \leq(\omega-1)^{k}<\omega^{k}$. This proves (3).
(4) $\chi(B) \leq(\omega-t)^{c}$.

Suppose that $C \subseteq B$ is a clique with $|C|=\omega-t+1$. For each $c \in C$, (3) implies that $c$ has a nonneighbour in fewer than $\omega^{k}$ of the cliques $X_{1}, \ldots, X_{n}$; and so fewer than $(\omega-t+1) \omega^{k}$ of the cliques $X, \ldots, X_{n}$ contain a vertex with a non-neighbour in $C$. Since $(\omega-t+1) \omega^{k} \leq \omega^{k+1}=n$, there exists $i \in\{1, \ldots, n\}$ such that every vertex in $X_{i}$ is adjacent to every vertex of $C$, and so $C \cup X_{i}$ is a clique. Since $\left|X_{i}\right| \geq\left|X_{n}\right|=t$, it follows that $\left|C \cup X_{i}\right|>\omega$, a contradiction. Thus there is no such clique $C$, and so $\omega(G[B]) \leq \omega-t$; and from the inductive hypothesis (since $t>0$ ) it follows that $\chi(B) \leq(\omega-t)^{c}$. This proves (4).

From (1), (2), (4) we deduce that

$$
\chi(G) \leq \chi(N \cup\{v\})+\chi(A)+\chi(B) \leq t^{c}+n \omega+(n \omega)^{k} \omega^{c^{\prime}}+(\omega-t)^{c} .
$$

Since $1 \leq t \leq \omega-1$ and $c \geq 1$, it follows that $t^{c}+(\omega-t)^{c} \leq 1+(\omega-1)^{c}$, and so

$$
\chi(G) \leq 1+\omega^{k+1}+(n \omega)^{k} \omega^{c^{\prime}}+(\omega-1)^{c} \leq \omega^{c}
$$

from the choice of $c$ and since $\omega \geq 2$. This proves 1.5,

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