

# Polynomial bounds for chromatic number. III. Excluding a double star

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### **Abstract**

A “double star” is a tree with two internal vertices. It is known that the Gyárfás-Sumner conjecture holds for double stars, that is, for every double star  $H$ , there is a function  $f_H$  such that if  $G$  does not contain  $H$  as an induced subgraph then  $\chi(G) \leq f_H(\omega(G))$  (where  $\chi, \omega$  are the chromatic number and the clique number of  $G$ ). Here we prove that  $f_H$  can be chosen to be a polynomial.

# 1 Introduction

A class of graphs is *hereditary* if it is closed under isomorphism and under taking induced subgraphs. A hereditary class of graphs  $\mathcal{C}$  is  $\chi$ -*bounded* if there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every graph  $G \in \mathcal{C}$ , where  $\chi(G)$  and  $\omega(G)$  denote the chromatic number and the clique number of  $G$ . There is a large literature addressing the question of which graph classes are  $\chi$ -bounded, and many open questions (see [16] for a survey).

Hereditary classes defined by excluding some fixed graph  $H$  are of particular interest. If  $G, H$  are graphs, we say  $G$  is  $H$ -*free* if no induced subgraph of  $G$  is isomorphic to  $H$ . It is easily seen that if the class of  $H$ -free graphs is  $\chi$ -bounded then  $H$  must be a forest, as Erdős [3] showed that there are graphs with arbitrarily large girth and chromatic number. The famous Gyárfás-Sumner conjecture [8, 20] asserts the converse:

**1.1 Conjecture:** *For every forest  $H$ , there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $H$ -free graph  $G$ .*

The Gyárfás-Sumner conjecture remains open in general, though it has been proved for some very restricted families of trees (see, for example, [2, 9, 10, 11, 12, 14, 15, 16]). In particular, it was proved by Kierstead and Penrice [11] for trees of radius two.

Louis Esperet [7] made the striking conjecture that, for every  $\chi$ -bounded class  $\mathcal{C}$ , the function  $f$  can be chosen to be a polynomial (see the survey by Schiermeyer and Randerath [19] for results on polynomial  $\chi$ -boundedness). In particular, this would imply a strengthening of the Gyárfás-Sumner conjecture, that the function  $f$  in 1.1 can always be chosen to be a polynomial. This is a bold conjecture, as frequently, when classes are known to be  $\chi$ -bounded, the best known function  $f$  grows quite rapidly, often because the proofs use multiple applications of Ramsey-type results. Nevertheless, Esperet's strengthening has been verified for some cases of the Gyárfás-Sumner conjecture, for instance when  $H$  is a star, or a four-vertex path, or a matching (see [19]); and recently it has been shown when  $H$  is obtained from a star by subdividing one edge once [13], and when  $H$  is a forest of stars [18].

A *double star* is a tree in which at most two vertices have degree more than one. Double stars have radius at most two, and so the result of Kierstead and Penrice [11] shows that the class of  $H$ -free graphs is  $\chi$ -bounded whenever  $H$  is a double star. In this paper, we prove a polynomial bound. Our main result is:

**1.2** *For every double star  $H$ , there is a polynomial  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $H$ -free graph  $G$ .*

This extends a theorem of Liu, Schroeder, Wang and Yu [13], who proved the same for double stars  $H$  that have at most one vertex of degree more than two.

Our result is partially motivated by the Erdős-Hajnal conjecture. In view of the recent result [4], the five-vertex path  $P_5$  is the smallest open case of this conjecture. It is known that  $P_5$  satisfies the Gyárfás-Sumner conjecture (in fact the Gyárfás-Sumner conjecture holds whenever  $H$  is a path), and if  $P_5$  satisfies Esperet's strengthening then  $P_5$  also satisfies the Erdős-Hajnal conjecture. Thus  $P_5$  appears likely to be a sticking point. We have not settled that; but this paper proves Esperet's strengthening for all trees that do not contain  $P_5$ .

We use standard notation throughout. When  $X \subseteq V(G)$ ,  $G[X]$  denotes the subgraph induced on  $X$ . We write  $\chi(X)$  for  $\chi(G[X])$ , and  $\omega$  for  $\omega(G)$ , when there is no ambiguity.

## 2 A degeneracy variant of defective colouring

A graph  $G$  is  $d$ -degenerate, or has *degeneracy at most  $d$* , if every non-null subgraph  $H$  has a vertex with degree (in  $H$ ) at most  $d$ . Every  $d$ -degenerate graph has chromatic number at most  $d + 1$ .

Let us say a  $(k, d)$ -colouring of a graph is a partition  $(A_1, \dots, A_k)$  of the vertex set  $V(G)$ , such that for  $1 \leq i \leq k$ , the subgraph induced on  $A_i$  has degeneracy at most  $d$ ; and we say that  $G$  or  $V(G)$  is  $(k, d)$ -colourable if there is such a partition. Thus, if a graph is  $(k, d)$ -colourable, its chromatic number is at most  $k(d+1)$ . We call this “degenerate colouring”; it is a relative of “defective colouring”, where we ask that the subgraph induced on each  $A_i$  has maximum degree at most  $d$ , but it is not exactly the same (see [21] for a survey of defective colouring). Let us explain why we need to use degenerate colourings.

A standard way to bound the chromatic number of a graph  $G$  is to partition  $V(G)$  into some number of parts  $V_1, V_2, \dots$ , and bound the chromatic numbers of the parts separately, and add to get a bound on  $\chi(G)$ . But we will be trying to prove that  $\chi(G)$  is at most  $\omega(G)^d$ , for some appropriately large constant  $d$ . So for this “addition” method to work when  $\omega$  is large, if the best bound we know for one of the parts is something like  $(\omega(G) - 1)^d$ , we would need much better bounds for all the other parts.

Fix a double star  $H$ , and choose a large constant  $d$ ; and suppose that we try to prove by induction on  $\omega(G)$  that every  $H$ -free graph  $G$  has chromatic number at most  $\omega(G)^d$ . The proof (by our method) does not work. There comes a stage where  $V(G)$  is partitioned into an unbounded number of parts  $V_1, V_2, \dots$ . We will know, from the induction on  $\omega(G)$ , that each part has chromatic number at most something like  $(\omega - 1)^d$  (where  $\omega = \omega(G)$ ), but we will not know a better bound for *any* of the parts. The “addition” method given above will therefore fail miserably. But we will know something about the edges between parts, which we might hope will save us (though in fact it will not). We will know that for each part  $V_i$ , each of its vertices has only a small number of neighbours in the union of the later parts  $V_{i+1} \cup \dots \cup V_n$ ; say at most  $\omega^r$  neighbours, where  $r$  is much less than  $d$ . Of course if there were no edges between the parts, all would be fine, and one might hope that similarly, because of the sparseness of the edges between parts, the effect of these edges could be fitted into the difference between  $\omega^d$  and  $(\omega - 1)^d$ . But we can’t do this (at least with no further information); the effect is multiplicative rather than additive. Even if for each  $i$ , each vertex of  $V_i$  has only at most one neighbour in  $V_{i+1} \cup \dots \cup V_n$ , the chromatic number of the union of the parts might be  $3/2$  times the maximum chromatic number of the individual parts, which is much too big.

Thus the inductive proof that every  $H$ -free graph  $G$  has chromatic number at most  $\omega(G)^d$  fails; and for that reason we will instead prove by induction a stronger statement, about degenerate colourings. We will prove by induction on  $\omega$  that if  $G$  is  $H$ -free, then  $G$  is  $(\omega^d, \omega^{r+1})$ -colourable. Then, when the situation above arises, we will know that each part admits an  $((\omega - 1)^d, (\omega - 1)^{r+1})$ -colouring and hence an  $(\omega^d, (\omega - 1)\omega^r)$ -colouring. The union of these colourings becomes an  $(\omega^d, \omega^{r+1})$ -colouring, which is what we want, and now it all works. Induction on  $\omega(G)$  will be used to prove the statement about degenerate colouring, and then we deduce the statement about normal colouring at the end.

Let us state formally the lemma we just mentioned:

**2.1** *Let  $k, d, d' \geq 0$  be integers. Let  $V(G)$  be partitioned into  $V_1, V_2, \dots, V_n$ , such that*

- *for  $1 \leq i \leq n$ ,  $G[V_i]$  admits a  $(k, d)$ -colouring; and*
- *for  $1 \leq i < n$ , every vertex of  $V_i$  has at most  $d'$  neighbours in  $V_{i+1} \cup \dots \cup V_n$ .*

Then  $G$  admits a  $(k, d + d')$ -colouring.

The proof is clear.

### 3 Templates

The paper by Kierstead and Penrice [11] uses the method of “templates”, an idea that was introduced in [10] and has been applied in several papers to prove special cases of 1.1. We will use the same idea, but substantially modified to keep the numbers polynomial. Fix an integer  $s$ . (Eventually  $s + 1$  will be the maximum degree in the double star we are excluding.) We say an  $s$ -template in a graph  $G$  is a sequence  $\mathcal{L}$  of pairwise disjoint subsets  $(L_0, L_1, \dots, L_k)$ , with  $k \geq 0$ , such that:

- $L_0$  is a clique of  $G$  (possibly empty), and every vertex in  $L_0$  is adjacent to every vertex in  $L_1 \cup \dots \cup L_k$ ;
- $\omega^{s+5} \leq |L_i| \leq 14\omega^{s+6}$  for  $1 \leq i \leq k$ ; and
- for all distinct  $i, j \in \{1, \dots, k\}$ , each vertex in  $L_i$  has at most  $\omega^{s+3}$  non-neighbours in  $L_j$ .

We say that  $k$  is the *length* of the  $s$ -template, and define its *value* to be

$$|L_1 \cup \dots \cup L_k| + 7\omega^{s+5}|L_0| + k\omega^{s+5}.$$

Let us define  $V(\mathcal{L}) = L_0 \cup L_1 \cup \dots \cup L_k$ , and  $N(\mathcal{L})$  to be the set of vertices in  $V(G) \setminus V(\mathcal{L})$  that have a neighbour in  $L_1 \cup \dots \cup L_k$ . (Note that we do not consider neighbours in  $L_0$ .)

The idea of the proof is as follows. Let  $G$  be an  $H$ -free graph (where  $H$  is a double star). Suppose that  $G$  has chromatic number at least some huge (but some fixed constant) power of  $\omega$ . It follows from a theorem of an earlier paper [17] that  $G$  has a subgraph which is a complete bipartite graph, in which both parts have cardinality  $14\omega^{s+6}$ . Consequently  $G$  contains an  $s$ -template of length two and value at least  $28\omega^{s+6}$ . Thus we may choose an  $s$ -template  $\mathcal{L}$  with maximum value, and its value will also be at least  $28\omega^{s+6}$ . We have tuned these values so that such  $s$ -templates have many useful properties. The bulk of the proof is to bound the chromatic number of the set  $N(\mathcal{L})$  (more exactly, to show it admits a certain degenerate colouring). Having done that, let  $Z$  consist of  $L_0$  together with all vertices in  $N(\mathcal{L})$  that have at least a few neighbours that are not in  $N(\mathcal{L})$  (a “few” means a constant power of  $\omega$ ). It is easy to show that  $|Z|$  is at most another constant power of  $\omega$ , and so every vertex of  $(V(\mathcal{L}) \cup N(\mathcal{L})) \setminus Z$  has only a few neighbours in the complement of this set. It remains to bound the chromatic number of the complementary set, that is of  $(V(G) \cup Z) \setminus (V(\mathcal{L}) \cup N(\mathcal{L}))$ ; and to do this, we choose an  $s$ -template in this graph with value as large as possible, and choose a third with no neighbours in the first or second, and so on. This is the situation we discussed in the previous section, which motivated us to use degenerate colouring, and this will allow us to produce a degenerate colouring of the whole of  $G$ .

We begin in this section by proving some properties of *optimal*  $s$ -templates, that is,  $s$ -templates chosen with maximum value.

**3.1** *Let  $(L_0, L_1, \dots, L_k)$  be an  $s$ -template in  $G$ . There is a clique of  $G$  with one vertex in each of  $L_1, \dots, L_k$ , and consequently  $k + |L_0| \leq \omega$ .*

**Proof.** Choose  $v_1 \in L_1$ ; and inductively for  $2 \leq i \leq k$ , having chosen  $v_1, \dots, v_{i-1}$ , choose  $v_i \in L_i$  as follows. There are at most  $\omega^{s+3}$  vertices in  $L_i$  nonadjacent to  $v_h$ , for  $1 \leq h < i$ ; and since  $\{v_1, \dots, v_{i-1}\}$  is a clique and therefore  $i-1 \leq \omega$ , it follows that  $\omega^{s+3}(i-1) \leq \omega^{s+4} < \omega^{s+5} \leq |L_i|$ . Consequently there exists  $v_i \in L_i$  adjacent to all of  $v_1, \dots, v_{i-1}$ . This completes the inductive definition of  $v_1, \dots, v_k$ . Hence  $L_0 \cup \{v_1, \dots, v_k\}$  is a clique, and so  $k + |L_0| \leq \omega$ . This proves 3.1.  $\blacksquare$

**3.2** Let  $(L_0, L_1, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . If  $\mathcal{L}$  has value at least  $28\omega^{s+6}$ , then  $k \geq 2$ , and  $|L_i| \geq 5\omega^{s+5}$  for  $1 \leq i \leq k$ .

**Proof.** If  $k = 0$ , the  $s$ -template has value  $7\omega^{s+5}|L_0| \leq 7\omega^{s+6} < 28\omega^{s+6}$ , a contradiction. If  $k = 1$ , then since  $|L_1| \leq 14\omega^{s+6}$ , and  $|L_0| + k \leq \omega$  by 3.1, the  $s$ -template has value at most

$$14\omega^{s+6} + 7\omega^{s+5}|L_0| + \omega^{s+5} \leq 14\omega^{s+6} + 7\omega^{s+5}(\omega - 1) + \omega^{s+5} \leq 21\omega^{s+6} < 28\omega^{s+6},$$

a contradiction. So  $k \geq 2$ .

By reordering  $L_1, \dots, L_k$  we may assume that  $|L_1|, \dots, |L_h| \geq 5\omega^{s+5}$  and  $|L_{h+1}|, \dots, |L_k| < 5\omega^{s+5}$ . We will show that  $h = k$ . For  $h+1 \leq i \leq k$ , choose  $v_i \in L_i$  such that  $\{v_{h+1}, \dots, v_k\}$  is a clique  $X$  (this is possible by 3.1). For  $1 \leq i \leq h$ , let  $L'_i$  be the set of all vertices in  $L_i$  that are adjacent to every vertex of  $X$ . Since each vertex of  $X$  has at most  $\omega^{s+3}$  non-neighbours in  $L_i$ , it follows that

$$|L'_i| \geq |L_i| - \omega^{s+4} \geq 5\omega^{s+5} - \omega^{s+5} \geq \omega^{s+5}$$

for  $1 \leq i \leq h$ . Consequently

$$(L_0 \cup \{x_{h+1}, \dots, x_k\}, L'_1, \dots, L'_h)$$

is an  $s$ -template in  $G$ . Its value is that of  $(L_0, L_1, \dots, L_k)$  plus

$$7\omega^{s+5}(k-h) - \omega^{s+5}(k-h) - (|L_1 \cup \dots \cup L_k| - |L'_1 \cup \dots \cup L'_h|),$$

and so this is at most zero, since  $(L_0, \dots, L_k)$  is optimal. But

$$|L_1 \cup \dots \cup L_k| - |L'_1 \cup \dots \cup L'_h| \leq \sum_{1 \leq i \leq h} (|L_i| - |L'_i|) + \sum_{h+1 \leq i \leq k} |L_i| \leq h\omega^{s+4} + (k-h)(5\omega^{s+5} - 1),$$

and consequently

$$7\omega^{s+5}(k-h) - \omega^{s+5}(k-h) \leq h\omega^{s+4} + (k-h)(5\omega^{s+5} - 1),$$

that is,

$$(\omega^{s+5} + 1)(k-h) \leq h\omega^{s+4}.$$

But  $\omega^{s+5} \geq h\omega^{s+4}$ , and so  $(\omega^{s+5} + 1)(k-h) \leq \omega^{s+5}$ , which implies that  $h = k$ . This proves 3.2.  $\blacksquare$

**3.3** Let  $(L_0, L_1, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . If its value is at least  $28\omega^{s+6}$ , then for  $1 \leq i \leq k$ , every vertex in  $L_i$  has at least  $4\omega^{s+5}$  non-neighbours in  $L_i$ .

**Proof.** Let  $i = 1$  say, and let  $v \in L_1$ . Let  $L'_1$  be the set of neighbours of  $v$  in  $L_1$ , and  $M = L_1 \setminus (L'_1 \cup \{v\})$ . We will show that  $|M| \geq 4\omega^{s+5}$ . If  $|L'_1| < \omega^{s+5}$ , then  $v$  has at least  $|L_1| - \omega^{s+5}$  non-neighbours in  $L_1$ ; and since  $|L_1| \geq 5\omega^{s+5}$  by 3.2, it follows that  $|M| \geq 4\omega^{s+5}$  as required. Thus we may assume that  $|L'_1| \geq \omega^{s+5}$ . For  $2 \leq i \leq k$  let  $L'_i$  be the set of vertices in  $L_i$  adjacent to  $v$ ; thus by 3.2,

$$|L'_i| \geq |L_i| - \omega^{s+3} \geq 5\omega^{s+5} - \omega^{s+3} \geq \omega^{s+5}$$

for  $2 \leq i \leq k$ . Consequently

$$(L_0 \cup \{v\}, L'_1, L'_2, \dots, L'_k)$$

is an  $s$ -template. From the optimality of  $(L_0, \dots, L_k)$ , it follows that  $7\omega^{s+5} - (|M| + 1) - (k-1)\omega^{s+3} \leq 0$ , and so

$$|M| \geq 7\omega^{s+5} - 1 - (k-1)\omega^{s+3} \geq 4\omega^{s+5}.$$

This proves 3.3. ■

**3.4** Let  $(L_0, L_1, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . Suppose that  $\omega \geq 4$ , and  $A, B$  are disjoint subsets of  $L_1$ , with  $|L_1 \setminus (A \cup B)| \leq \omega^{s+3}$ , such that every vertex in  $A$  has fewer than  $\omega^s$  non-neighbours in  $B$ . Then either  $|B| < 14\omega^{s+1}$  or  $A = \emptyset$ .

**Proof.** Suppose that  $|B| \geq 14\omega^{s+1}$ .

(1)  $|B| \geq 2\omega^{s+5}$ .

Suppose that  $|B| < 2\omega^{s+5}$ . Each vertex in  $B$  has at least  $4\omega^{s+5}$  non-neighbours in  $L_1$ , by 3.3, and only at most  $2\omega^{s+5} + \omega^{s+3}$  of them do not belong to  $A$ ; and since  $2\omega^{s+5} - \omega^{s+3} \geq \omega^{s+5}$ , there are at least  $\omega^{s+5}|B| \geq 14\omega^{2s+6}$  nonedges between  $B$  and  $A$ . Since  $|A| \leq |L_1| \leq 14\omega^{s+6}$ , some vertex in  $A$  has at least  $\omega^s$  non-neighbours in  $B$ , a contradiction. This proves (1).

(2)  $|A| < \omega^{s+5}$ .

Suppose that  $|A| \geq \omega^{s+5}$ . Let  $B'$  be the set of all vertices in  $B$  with at most  $\omega^{s+3}$  non-neighbours in  $A$ . Since there are only at most  $\omega^s|A|$  nonedges between  $A, B$ , there are at most  $|A|/\omega^3$  vertices in  $B$  that have more than  $\omega^{s+3}$  non-neighbours in  $A$ ; and so  $|B'| \geq |B| - |A|/\omega^3 \geq \omega^{s+5}$ , by (1) and since  $|A|/\omega^3 \leq 14\omega^{s+3} < \omega^{s+5}$  (the last because  $\omega \geq 4$ ). Hence

$$(L_0, A, B', L_2, \dots, L_k)$$

is an  $s$ -template, and the optimality of  $(L_0, \dots, L_k)$  implies that

$$|A| + |B'| + \omega^{s+5} \leq |L_1| \leq |A| + |B'| + \omega^{s+3} + |A|/\omega^3,$$

and so  $\omega^{s+5} \leq \omega^{s+3} + 14\omega^{s+3}$ , a contradiction. This proves (2).

Suppose that  $A \neq \emptyset$ , and choose  $v \in A$ . Since  $v$  has at least  $4\omega^{s+5}$  non-neighbours in  $L_1$  by 3.3, and at most  $\omega^{s+5}$  of them belong to  $A$  by (2), and at most  $\omega^{s+3}$  are not in  $A \cup B$ , it follows that  $v$  has at least  $3\omega^{s+5} - \omega^{s+3} \geq \omega^s$  non-neighbours in  $B$ , a contradiction. Thus  $A = \emptyset$ . This proves 3.4. ■

**3.5** Let  $\mathcal{L} = (L_0, L_1, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . There are fewer than  $14\omega^{s+6}$  vertices  $v \in N(\mathcal{L})$  such that for  $1 \leq i \leq k$ ,  $v$  has at most  $\omega^{s+2}/4$  non-neighbours in  $L_i$ .

**Proof.** Suppose that there is a set  $M \subseteq N(\mathcal{L})$  with  $|M| = 14\omega^{s+6}$ , such that for each  $v \in M$  and for  $1 \leq i \leq k$ ,  $v$  has at most  $\omega^{s+2}/4$  non-neighbours in  $L_i$ . For  $1 \leq i \leq k$ , there are at most  $|M|\omega^{s+2}/4 = 7\omega^{2s+8}/2$  nonedges between  $M$  and  $L_i$ ; and so there are at most  $7\omega^{s+5}/2$  vertices in  $L_i$  with at least  $\omega^{s+3}$  non-neighbours in  $M$ . Let  $L'_i$  be the set of vertices in  $L_i$  that have fewer than  $\omega^{s+3}$  non-neighbours in  $M$ . Hence  $|L'_i| \geq |L_i| - (7\omega^{s+5}/2) \geq \omega^{s+5}$ , since  $|L_i| \geq 5\omega^{s+5}$  by 3.2. It follows that

$$(\emptyset, L'_1, \dots, L'_k, M)$$

is an  $s$ -template. Its value is

$$|L'_1 \cup \dots \cup L'_k| + |M| + (k+1)\omega^{s+5} \geq |L_1 \cup \dots \cup L_k| - k(7\omega^{s+5}/2) + 14\omega^{s+6} + (k+1)\omega^{s+5}$$

and the optimality of  $(L_0, \dots, L_k)$  implies that

$$|L_1 \cup \dots \cup L_k| - k(7\omega^{s+5}/2) + 14\omega^{s+6} + (k+1)\omega^{s+5} \leq |L_1 \cup \dots \cup L_k| + 7\omega^{s+5}|L_0| + k\omega^{s+5},$$

that is,

$$2\omega + 1/7 \leq |L_0| + k/2,$$

contrary to 3.1. This proves 3.5. ■

## 4 Using the double star

We are concerned with graphs that do not contain some fixed double star, but so far we have not used that fact. For  $s \geq 1$ , let  $H_s$  be the double star with  $2s+2$  vertices, with two internal vertices both of degree  $s+1$ . Every double star is an induced subgraph of  $H_s$  for some  $s$ , so it suffices to prove the result for  $H_s$ -free graphs.

We will need the following result of [17]:

**4.1** Let  $H$  be a forest. Then there exists  $c > 0$  such that for every  $H$ -free graph  $G$  and every integer  $t \geq 0$ , either  $G$  contains the complete bipartite graph  $K_{t,t}$  as a subgraph, or  $G$  has degeneracy less than  $t^c$ , and hence has chromatic number at most  $t^c$ .

We will also need the following version of Ramsey's theorem (well-known, but proved for instance in [18]):

**4.2** If  $s \geq 0$  is an integer, then every graph  $G$  with no stable set of cardinality  $s$  has at most

$$\omega^{s-1} + \omega^{s-2} + \dots + \omega$$

vertices, and hence fewer than  $\omega^s$  vertices if  $\omega > 1$ .

Let  $\mathcal{L} = (L_0, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . With respect to this template, we say a vertex  $v \in N(\mathcal{L})$  is



- *pendant* if there exist distinct  $i, j \in \{1, \dots, k\}$ , and a vertex  $u \in L_j$ , and a stable set  $S$  of  $s + 1$  vertices in  $L_i$ , all adjacent to  $u$ , such that  $v$  is not adjacent to  $u$ , and  $v$  has exactly one neighbour in  $S$ ;
- *dense* if there exists  $j \in \{1, \dots, k\}$ , and  $u \in L_j$ , such that for all  $i \in \{1, \dots, k\} \setminus \{j\}$ , there are fewer than  $\omega^{s+2}/14$  vertices in  $L_i$  that are adjacent to  $u$  and not to  $v$ ;
- *pure* if there are least two values of  $i \in \{1, \dots, k\}$  such that  $v$  has no neighbour in  $L_i$ , and for  $1 \leq i \leq k$ , either  $v$  has no neighbour in  $L_i$  or  $v$  has at most  $\omega^{s+2}/7$  non-neighbours in  $L_i$ .

**4.3** Let  $s \geq 1$  be an integer, and let  $G$  be  $H_s$ -free, with  $\omega \geq 200$ . Let  $\mathcal{L} = (L_0, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . Then every vertex in  $N(\mathcal{L})$  is either pendant or dense or pure with respect to  $\mathcal{L}$ .

**Proof.** Let  $v \in N(\mathcal{L})$ , and suppose that  $v$  is neither dense nor pendant with respect to  $\mathcal{L}$ . We will prove that  $v$  is pure.

(1) For all distinct  $i, j \in \{1, \dots, k\}$ , if  $u \in L_j$  is nonadjacent to  $v$ , then either  $u, v$  have no common neighbour in  $L_i$ , or fewer than  $\omega^{s+2}/14$  neighbours of  $u$  in  $L_i$  are nonadjacent to  $v$ .

Let  $A$  be the set of all vertices in  $L_i$  adjacent to both  $u, v$ , and let  $B$  be the set of vertices in  $L_i$  adjacent to  $u$  and not to  $v$ . Suppose that  $A \neq \emptyset$ , and  $|B| \geq \omega^{s+2}/14$ . Since  $|L_i \setminus (A \cup B)| \leq \omega^{s+3}$  (because  $u$  has at most  $\omega^{s+3}$  non-neighbours in  $L_i$ ), and  $|B| \geq \omega^{s+2}/14 \geq 14\omega^{s+1}$  (because  $\omega \geq 200$ ), 3.4 implies that some vertex  $w \in A$  has at least  $\omega^s$  non-neighbours in  $B$ . By 4.2, this set of non-neighbours includes a stable set of size  $s$ , contradicting that  $v$  is not pendant. Thus either  $A = \emptyset$  or  $|B| < \omega^{s+2}/14$ . This proves (1).

(2) For all  $j \in \{1, \dots, k\}$ , if  $v$  has a non-neighbour in  $L_j$ , there exists  $i \in \{1, \dots, k\}$  with  $i \neq j$  such that  $v$  has at most  $\omega^{s+3}$  neighbours in  $L_i$ .

Choose  $u \in L_j$  nonadjacent to  $v$ . Since  $v$  is not dense, there exists  $i \in \{1, \dots, k\}$  with  $i \neq j$  such that there are at least  $\omega^{s+2}/14$  vertices in  $L_i$  adjacent to  $u$  and not to  $v$ . By (1),  $u, v$  have no common neighbour in  $L_i$ , and hence  $v$  has at most  $\omega^{s+3}$  neighbours in  $L_i$ . This proves (2).

(3) For  $1 \leq j \leq k$ , if  $v$  has at most  $\omega^{s+3}$  neighbours in  $L_j$ , then  $v$  has no neighbours in  $L_j$ .

By (2), there exists  $i \in \{1, \dots, k\}$  different from  $j$ , such that  $v$  has at most  $\omega^{s+3}$  neighbours in  $L_i$ . Suppose that  $v$  has a neighbour  $w \in L_j$ . Since  $w$  has at most  $\omega^{s+3}$  non-neighbours in  $L_i$ , and  $v$  has at most  $\omega^{s+3}$  neighbours in  $L_i$ , and  $|L_i| \geq \omega^{s+5} > 2\omega^{s+3}$ , there exists  $x \in L_i$  adjacent to  $w$  and not to  $v$ . Now  $w$  has at least  $4\omega^{s+5}$  non-neighbours in  $L_j$ , by 3.3; at most  $\omega^{s+3}$  of them are nonadjacent to  $x$ , and at most  $\omega^{s+3}$  of them are adjacent to  $v$ , and so at least  $4\omega^{s+5} - 2\omega^{s+3} \geq \omega^s$  of them are nonadjacent to  $v$  and adjacent to  $x$ . By 4.2, this set includes a stable set of size  $s$ , and so  $v$  is pendant, a contradiction. Thus  $v$  has no neighbour in  $L_j$ . This proves (3).

(4) For  $1 \leq i \leq k$ , either  $v$  has at most  $\omega^{s+2}/7$  non-neighbours in  $L_i$ , or  $v$  has no neighbours in  $L_i$ .

Let  $A_i, B_i$  be the sets of neighbours and non-neighbours respectively of  $v$  in  $L_i$ , and suppose that  $A_i \neq \emptyset$ , and  $|B_i| > \omega^{s+2}/7$ . By (2) and (3), there exists  $j \in \{1, \dots, k\}$  with  $j \neq i$  such that  $v$  has no neighbours in  $L_j$ . By (1), for each  $u \in L_j$ , either  $u$  has no neighbours in  $A_i$ , or  $u$  has at most  $\omega^{s+2}/14$  neighbours in  $B_i$ . Let  $X$  be the set of vertices in  $L_j$  with no neighbour in  $A_i$ , and  $Y = L_j \setminus X$ . Since  $A_i \neq \emptyset$ , and a vertex in  $A_i$  has at most  $\omega^{s+3}$  non-neighbours in  $L_j$ , it follows that  $|X| \leq \omega^{s+3}$ , and so  $|Y| \geq |L_j| - \omega^{s+3}$ . Every vertex in  $Y$  is adjacent to at most half the vertices in  $B_i$  (since  $|B_i| \geq \omega^{s+2}/7$ ), and so some vertex  $b \in B_i$  is adjacent to at most half the vertices in  $Y$ . Since  $b$  has at most  $\omega^{s+3}$  non-neighbours in  $L_j$ , it follows that  $|Y|/2 \leq \omega^{s+3}$ ; but  $|X| \leq \omega^{s+3}$ , and so  $|L_j| \leq 3\omega^{s+3}$ , a contradiction. This proves (4).

Since  $v$  is not dense, it has a non-neighbour in one of  $L_1, \dots, L_k$ ; and so by (2) and (3), it has no neighbours in some  $L_j$ . By (2),  $v$  has at most  $\omega^{s+3}$  neighbours in  $L_i$  for some  $i \neq j$ ; and so has no neighbours in  $L_i$  by (3). From (4) it follows that  $v$  is pure. This proves 4.3.  $\blacksquare$

**4.4** *Let  $s \geq 1$  be an integer, and let  $G$  be  $H_s$ -free. Let  $(L_0, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . There are at most  $14^{s+2}\omega^{s^2+9s+14}$  pendant vertices.*

**Proof.** Let  $i, j \in \{1, \dots, k\}$  be distinct, let  $u \in L_j$ , let  $S \subseteq L_i$  be a stable set of  $s+1$  neighbours of  $u$ , and let  $w \in S$ . Let  $X(i, j, u, S, w)$  be the set of all  $v \in V(G) \setminus (L_0 \cup \dots \cup L_k)$  such that  $v$  is adjacent to  $w$  and is nonadjacent to all other vertices in  $S \cup \{u\}$ . If  $X(i, j, u, S, w)$  includes a stable set of size  $s$ , say  $T$ , then the subgraph induces on  $S \cup T \cup \{u\}$  is isomorphic to  $H_s$ , a contradiction. Thus, 4.2 implies that  $|X(i, j, u, S, w)| \leq \omega^s$ . Since there are only  $k^2(14\omega^{s+6})^{s+2}$  choices for  $i, j, u, S, w$ , and every pendant vertex belongs to  $X(i, j, u, S, w)$  for some choice of  $i, j, u, S, w$ , it follows that there are at most  $k^2(14\omega^{s+6})^{s+2}\omega^s$  pendant vertices. Since  $k \leq \omega$  by 3.1, this proves 4.4.  $\blacksquare$

**4.5** *Let  $s \geq 1$  be an integer, and let  $G$  be  $H_s$ -free. Let  $\mathcal{L} = (L_0, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . Let  $c$  satisfy 4.1 when  $H = H_s$ . The chromatic number of the set of all dense vertices is at most  $(14\omega^{s+6})^{c+1}\omega$ .*

**Proof.** Let  $1 \leq j \leq k$ , let  $u \in L_j$ , and let  $X(j, u)$  be the set of all  $v \in N(\mathcal{L})$  such that for all  $i \in \{1, \dots, k\} \setminus \{j\}$ , there are fewer than  $\omega^{s+2}/14$  vertices in  $L_i$  that are adjacent to  $u$  and not to  $v$ .

Suppose that  $\chi(X(j, u)) > (14\omega^{s+6})^c$  for some  $u \in L_k$ . Then by 4.1,  $G[X(j, u)]$  contains a copy of  $K_{14\omega^{s+6}, 14\omega^{s+6}}$  as a subgraph; let  $M_1, M_2$  be disjoint subsets of  $X(j, u)$ , both of cardinality  $14\omega^{s+6}$ , such that every vertex of  $M_1$  is adjacent to every vertex of  $M_2$ . For  $1 \leq i \leq k-1$ , let  $L'_i$  be the set of vertices in  $L_i \cap N(u)$  that have at most  $\omega^{s+3}$  non-neighbours in  $M_1$  and at most  $\omega^{s+3}$  non-neighbours in  $M_2$ . There are at most  $(\omega^{s+2}/14)14\omega^{s+6}$  nonedges between  $M_1$  and  $L_i \cap N(u)$ , and it follows that at most  $\omega^{s+5}$  vertices in  $L_i \cap N(u)$  have more than  $\omega^{s+3}$  non-neighbours in  $M_1$ ; and the same for  $M_2$ . Consequently

$$|L'_i| \geq |L_i \cap N(u)| - 2\omega^{s+5} \geq 5\omega^{s+5} - \omega^{s+3} - 2\omega^{s+5} \geq \omega^{s+5},$$

and so

$$(\emptyset, L'_1, \dots, L'_{k-1}, M_1, M_2)$$

is an  $s$ -template. Moreover,  $|L'_i| \geq |L_i| - \omega^{s+3} - 2\omega^{s+5}$ , since  $u$  has at most  $\omega^{s+3}$  non-neighbours in  $L_i$ . From the optimality of  $\mathcal{L}$ , it follows that

$$28\omega^{s+6} \leq |L_k| + (\omega^{s+3} + 2\omega^{s+5})(k-1) + 7\omega^{s+5}|L_0| \leq 14\omega^{s+6} + 7\omega^{s+6},$$

a contradiction. Consequently  $\chi(X(k, u)) \leq (14\omega^{s+6})^c$ . The union of the sets  $X(k, u)$  over all  $u \in L_k$  thus has chromatic number at most  $(14\omega^{s+6})^{c+1}$ ; and from the symmetry between  $L_1, \dots, L_k$ , the set of all dense vertices has chromatic number at most  $k(14\omega^{s+6})^{c+1} \leq (14\omega^{s+6})^{c+1}\omega$ . This proves 4.5.  $\blacksquare$

## 5 Pure vertices

In view of 4.3, 4.4 and 4.5, in order to bound the chromatic number of  $N(\mathcal{L})$  it remains to bound the chromatic number of the set of pure vertices, and that is the topic of this section. But here we will need to use induction on  $\omega$ , and so as we discussed earlier, we will in fact work with degenerate colouring.

Throughout this section, let  $s \geq 1$  be an integer, let  $G$  be  $H_s$ -free, let  $\mathcal{L} = (L_0, \dots, L_k)$  be an optimal  $s$ -template in  $G$ . Let  $M$  denote the set of all vertices that are pure with respect to this template. For each  $v \in M$ , let  $I_v$  be the set of all  $i \in \{1, \dots, k\}$  such that  $v$  has a neighbour in  $L_i$  (and hence has at most  $\omega^{s+2}/14$  non-neighbours in  $L_i$ ). For each  $I \subseteq \{1, \dots, k\}$ , let  $M_I$  be the set of all  $v \in M$  with  $I_v = I$ .

We wish to find a degenerate colouring of the union of all the sets  $M_I$ . One problem is that the number of sets  $I$  with  $M_I$  nonempty may be superpolynomial; but we will show that there are only a linear number of sets  $M_I$  of large cardinality, and we can colour all the small ones simultaneously.

**5.1** *Let  $I \subseteq \{1, \dots, k\}$  and  $u \in M_I$  and  $j \in \{1, \dots, k\} \setminus I$ . For each  $i \in I$ , there are fewer than  $\omega^s$  vertices  $v$  adjacent to  $u$  such that  $v \in M_J$  for some  $J \subseteq \{1, \dots, k\} \setminus \{i, j\}$ . Hence there are fewer than  $\omega^{s+1}$  vertices  $v$  adjacent to  $u$  such that  $v \in M_J$  for some  $J \subseteq \{1, \dots, k\} \setminus \{j\}$  with  $I \not\subseteq J$ .*

**Proof.** Let  $i \in I$ , and let  $\mathcal{A}_i$  be the set of all  $J \subseteq \{1, \dots, k\} \setminus \{i, j\}$ . Let  $V_i$  be the union of all the sets  $M_J$  for  $J \in \mathcal{A}_i$ . Suppose that  $u$  has  $\omega^s$  neighbours in  $V_i$ . By 4.2, there is a stable set  $S \subseteq V_i$  of neighbours of  $u$  with  $|S| = s$ . Choose  $x \in L_i$  adjacent to  $u$ . Thus  $x$  has no neighbour in  $S$  since for each  $J \in \mathcal{A}_i$ , no vertex in  $M_J$  has a neighbour in  $L_i$ . Since  $x$  has at most  $\omega^{s+3}$  non-neighbours in  $L_j$ , and  $|L_j| \geq \omega^{s+5} \geq \omega^{s+3} + \omega^s$ , it follows that  $x$  has at least  $\omega^s$  neighbours in  $L_j$ , and hence by 4.2 there is a stable set  $T \subseteq L_j$  of neighbours of  $x$  with  $|T| = s$ . Since  $j \notin I \cup J$  for  $J \in \mathcal{A}_i$ , no vertex in  $S \cup \{u\}$  has a neighbour in  $T$ ; and so the subgraph induced on  $S \cup T \cup \{u, x\}$  is isomorphic to  $H_s$ , a contradiction.

Hence  $u$  has fewer than  $\omega^s$  neighbours in  $V_i$  for each choice of  $i$ . Since there are only at most  $|I| \leq \omega$  choices of  $i$ , this proves 5.1.  $\blacksquare$

For  $m \geq 0$ , we say  $I \subseteq \{1, \dots, k\}$  is  $m$ -small if  $|M_I| \leq m$ , and  $m$ -large if it is not  $m$ -small.

**5.2** *For each  $m \geq 0$ , the union of the sets  $M_I$  over all  $m$ -small  $I \subseteq \{1, \dots, k\}$  has chromatic number at most  $2\omega(m + \omega^{s+1})$ .*

**Proof.** For all  $j \in \{1, \dots, k\}$ , let  $\mathcal{A}_j$  be the set of all  $m$ -small  $I \subseteq \{1, \dots, k\} \setminus \{j\}$ , and let  $V_j$  be the union of the sets  $M_I$  for all  $I \in \mathcal{A}_j$ . If  $u, v \in V_j$  are adjacent, we will direct the edge  $uv$  as follows. Let  $u \in M_I$  and  $v \in M_J$  where  $I, J \in \mathcal{A}_j$ . If  $|I| > |J|$  we direct the edge  $uv$  from  $u$  to  $v$ . If  $|I| = |J|$  (and in particular, if  $I = J$ ) we direct  $uv$  arbitrarily. We claim every vertex  $u \in V_j$  has outdegree less than  $m + \omega^{s+1}$ . Let  $u \in M_I$ . Certainly  $u$  has at most  $m$  out-neighbours in  $M_I$  since  $|M_I| \leq m$ . If  $v$  is an out-neighbour of  $u$  and  $v \in M_J$  where  $J \neq I$ , then  $|J| \leq |I|$ , and it follows that  $I \not\subseteq J$ , and  $I \cup J \neq \{1, \dots, k\}$  since  $j \notin I \cup J$ ; and so there are fewer than  $\omega^{s+1}$  such vertices  $v$  by 5.1. This proves that every vertex  $u \in V_j$  has outdegree less than  $m + \omega^{s+1}$ . Hence every subgraph of  $G[V_j]$  with  $n$  vertices has at most  $n(m + \omega^{s+1} - 1)$  edges, and so (if  $n > 0$ ) has a vertex of degree at most  $2(m + \omega^{s+1} - 1)$ ; and hence  $G[V_j]$  has degeneracy at most  $2(m + \omega^{s+1} - 1)$ , and so has chromatic number less than  $2(m + \omega^{s+1})$ . Since there are at most  $\omega$  choices of  $j$ , this proves 5.2.  $\blacksquare$

It remains to handle the  $m$ -large sets. For  $t \geq 1$ , let us say two disjoint subsets  $A, B$  of  $V(G)$  are  $s$ -crowded if there is no stable set  $X$  with  $|X \cap A| = |X \cap B| = s$ .

**5.3** *Let  $m \geq (s+1)\omega^s$ , and let  $I, J \subseteq \{1, \dots, k\}$  be  $m$ -large, with  $I \neq J$ . If  $\omega^3 > \omega + s/7$ , then either:*

- $J \subseteq I$  and every vertex in  $M_I$  has fewer than  $\omega^s$  neighbours in  $M_J$ ; or
- $I \subseteq J$  and every vertex in  $M_J$  has fewer than  $\omega^s$  neighbours in  $M_I$ ; or
- $I \cup J = \{1, \dots, k\}$  and  $M_I, M_J$  are  $s$ -crowded.

Consequently there are at most  $k - 1$   $m$ -large sets.

**Proof.** If  $J \subseteq I$  then the first bullet holds by 5.1; so we may assume that  $J \not\subseteq I$  and similarly  $I \not\subseteq J$ . Choose  $i \in I \setminus J$  and  $j \in J \setminus I$ . Suppose that  $M_I, M_J$  are not  $s$ -crowded, and choose  $S \subseteq M_I$  and  $T \subseteq M_J$  with  $|S| = |T| = s$  such that  $S \cup T$  is stable. Since each vertex in  $S$  has at most  $\omega^{s+2}/7$  non-neighbours in  $i$ , and  $s\omega^{s+2}/7 < \omega^{s+5}$ , there exists  $u \in L_i$  adjacent to every vertex in  $S$ . Since  $u$  has at most  $\omega^{s+3}$  non-neighbours in  $L_j$ , and each vertex in  $T$  has at most  $\omega^{s+2}/7$  non-neighbours in  $L_j$ , and  $\omega^{s+3} + s\omega^{s+2}/7 < \omega^{s+5}$ , there exists  $v \in L_j$  adjacent to  $u$  and to each vertex in  $T$ . But then the subgraph induced on  $S \cup T \cup \{u, v\}$  is isomorphic to  $H_s$ . This proves that  $M_I, M_J$  are  $s$ -crowded.

Let  $S \subseteq M_I$  be stable with  $|S| = s$ . (This is possible by 4.2 since  $m \geq \omega^s$ .) Since  $M_I, M_J$  are  $s$ -crowded, 4.2 implies that there are fewer than  $\omega^s$  vertices in  $M_J$  with no neighbour in  $S$ ; and so some vertex in  $S$  has at least

$$(|M_J| - \omega^s)/s > (m - \omega^s)/s \geq \omega^s$$

neighbours in  $M_J$ . By 5.1 it follows that  $I \cup J = \{1, \dots, k\}$  and so the third bullet holds. This proves the first assertion of 5.3.

To show that there are at most  $k - 1$   $m$ -large sets, for each  $X \subseteq \{1, \dots, k\}$  with  $|X| \leq k - 1$ , let  $n(X)$  be the number of  $m$ -large sets that include  $X$ . We prove by induction on  $k - |X|$  that  $n(X) \leq k - |X| - 1$ . If  $|X| = k - 1$  then  $n(X) = 0$  and the claim holds, so we assume that  $|X| \leq k - 2$  and the result holds for all larger subsets of  $\{1, \dots, k\}$ . Since  $k - |X| - 1 \geq 1$ , we may assume that at least two  $m$ -large sets include  $X$ , and so at least one properly includes  $X$ . Choose an  $m$ -large set  $J$  minimal such that  $X \subseteq J$  and  $X \neq J$ . If  $I$  is an  $m$ -large set including  $X$ , then by 5.1 and the

minimality of  $J$ , either  $I = X$ , or  $J \subseteq I$ , or  $I \cup J = \{1, \dots, k\}$ . Moreover, if  $I \cup J = \{1, \dots, k\}$ , then  $I$  includes  $K$ , where  $K = \{1, \dots, k\} \setminus (J \setminus X)$ . Consequently  $n(X) \leq n(J) + n(K) + 1$ . Since  $J, K$  are both proper supersets of  $X$ , and not equal to  $\{1, \dots, k\}$ , the inductive hypothesis implies that  $n(J) \leq k - |J| - 1$  and  $n(K) \leq k - |K| - 1$ , and so

$$n(X) \leq (k - |J| - 1) + (k - |K| - 1) + 1 = k - |X| - 1.$$

This completes the proof that  $n(X) \leq k - |X| - 1$  for each  $X \subseteq \{1, \dots, k\}$  with  $|X| \leq k - 1$ . Setting  $X = \emptyset$ , it follows that there are only  $k - 1$   $m$ -large sets. This proves 5.3.  $\blacksquare$

**5.4** Suppose that  $d \geq 1$  and  $D \geq 0$  have the property that every  $H_s$ -free graph  $G'$  with  $\omega(G') < \omega(G)$  is  $(\omega(G')^d, D)$ -colourable, and let  $m = (s + 1)\omega^s$ . Assume that  $\omega^3 > \omega + s/7$  and  $\omega \geq 200$ . Then the union of the sets  $M_I$  over all  $m$ -large  $I$  is  $(K, D)$ -colourable where

$$K = (\omega - 1)^d + \omega^{s^2+4s+2} + \omega^{s+3} + 1.$$

**Proof.** Let  $M^*$  be the union of the sets  $M_I$  for all  $m$ -large  $I$ . We may assume that  $M^* \neq \emptyset$ .

(1) There is a partition  $\mathcal{A}, \mathcal{B}$  of the set of  $m$ -large sets, such that

- $M_I, M_J$  are  $s$ -crowded for each  $I \in \mathcal{A}$  and  $J \in \mathcal{B}$ ; and
- all the sets in  $\mathcal{A}$  have an element in common, and so do all the sets in  $\mathcal{B}$ .

There is an  $m$ -large set  $J$  since  $M^* \neq \emptyset$ ; choose  $J$  minimal. Let  $\mathcal{A}$  be the set of all  $m$ -large sets that include  $J$ , and let  $\mathcal{B}$  be the set of  $m$ -large sets that do not include  $J$ . Choose  $j \in J$  and  $j' \in \{1, \dots, k\} \setminus J$ . ( $J$  is nonempty since every pure vertex has a neighbour in  $L_1 \cup \dots \cup L_k$ , and  $J \neq \{1, \dots, k\}$  from the definition of pure.) Every set in  $\mathcal{A}$  includes  $J$ , and so contains  $j$ . We claim that every set in  $\mathcal{B}$  includes  $\{1, \dots, k\} \setminus J$  and so contains  $j'$ . To see this, let  $I' \in \mathcal{B}$ . Since  $I' \in \mathcal{B}$ ,  $I'$  does not include  $J$ . Also  $J \not\supseteq I'$  from the minimality of  $J$ , and so from 5.3,  $I' \cup J = \{1, \dots, k\}$ . This proves that every set in  $\mathcal{B}$  includes  $\{1, \dots, k\} \setminus J$  and so contains  $j'$ . Finally, we must show that  $M_I, M_{I'}$  are  $s$ -crowded for all  $I \in \mathcal{A}$  and  $I' \in \mathcal{B}$ . Since  $J \subseteq I$  and  $J \not\subseteq I'$ , it follows that  $I \not\subseteq I'$ ; and since  $\{1, \dots, k\} \setminus J \subseteq I'$  and  $\{1, \dots, k\} \setminus J \not\subseteq I$ , it follows that  $I' \not\subseteq I$ . By 5.3,  $M_I, M_{I'}$  are  $s$ -crowded. This proves (1).

Choose  $\mathcal{A}, \mathcal{B}$  as in (1). Let  $A$  be the union of all the sets  $M_I$  for  $I \in \mathcal{A}$ , and define  $B$  similarly. Thus  $M^* = A \cup B$ .

(2) Every clique included in  $A$  has cardinality at most  $\omega - 1$ , and the same for  $B$ .

Suppose that there is a clique  $X \subseteq A$  with  $|X| = \omega$ . Let  $j \in \{1, \dots, k\}$  belong to all the sets in  $\mathcal{A}$ . Since  $(\omega^{s+2}/7)\omega < \omega^{s+5} \leq |L_j|$ , there exists  $v \in L_j$  adjacent to every vertex of  $X$ , contradicting that  $X$  is a clique of  $G$  of maximum cardinality. The same holds for  $B$ . This proves (2).

Let  $n = \omega^{s+2}$ . If  $|A| \leq n\omega$ , then by (2) it follows that  $G[M^*]$  admits an  $(n\omega + (\omega - 1)^d, D)$ -colouring and the theorem holds. Thus we may assume that  $|A| > n\omega$ . Let  $X_1$  be the largest clique

contained in  $A$ , and inductively for  $i \geq 2$  let  $X_i$  be the largest clique contained in  $A \setminus (X_1 \cup \dots \cup X_{i-1})$ . Let  $|X_n| = t$ . Since  $|A| > n\omega$  it follows that  $t > 0$ . Let  $X = X_1 \cup \dots \cup X_n$ . Thus  $|X| \leq n\omega$ , and  $\omega(G[A \setminus X]) \leq t$ .

Let  $C$  be the set of all vertices  $v \in B$  such that for some  $I \in \mathcal{A}$ ,  $v$  has at least  $\omega^s$  non-neighbours in  $M_I \cap X$ .

$$(3) |C| \leq n^s \omega^{2s+2}.$$

For each  $v \in C$ , there exists  $I \in \mathcal{A}$  and a stable set  $S \subseteq M_I \cap X$  with  $|S| = s$  such that  $v$  has no neighbour in  $S$ , by 4.2. For each such  $I$  and  $S$ , and each  $I' \in \mathcal{B}$ , there are at most  $\omega^s$  vertices in  $M_{I'}$  that have no neighbours in  $S$ , by 4.2 and since  $M_I, M_{I'}$  are  $s$ -crowded (because  $I \not\subseteq I'$  and  $I' \not\subseteq I$  by (1), and by 5.3). Consequently for each choice of  $I, S$  there are at most  $k\omega^s$  vertices in  $C$  with no neighbour in  $S$ , since  $|\mathcal{B}| \leq k$  by 5.3. For each choice of  $I$  there are only  $|X|^s \leq n^s \omega^s$  choices of  $S$ , since  $|X| \leq n\omega$ ; and there are only  $k$  choices of  $I$  by 5.3. Thus  $|C| \leq (k\omega^s)(n^s \omega^s)k \leq n^s \omega^{2s+2}$ . This proves (3).

(4) Every clique in  $B \setminus C$  has cardinality at most  $\omega - t$ .

Let  $Y \subseteq B \setminus C$  be a clique. Since  $Y \cap C = \emptyset$ , every vertex in  $Y$  has at most  $k\omega^s$  non-neighbours in  $X$ , since it has at most  $\omega^s$  in each  $X \cap M_I$  and there are only  $k$  choices of  $M_I$ . Consequently at most  $k\omega^s|Y|$  vertices in  $X$  have a non-neighbour in  $Y$ . Since  $n > k\omega^s|Y|$  (since  $n = \omega^{s+2}$ , and  $k \leq \omega$  and  $|Y| < \omega$  by (2)) it follows that there exists  $i \in \{1, \dots, n\}$  such that every vertex in  $Y$  has no non-neighbour in  $X_i$ , and so  $X_i \cup Y$  is a clique. But  $|X| \geq t$  from the choice of  $t$ , and  $|X \cup Y| \leq \omega$ , and so  $|Y| \leq \omega - t$ . This proves (4).

Now  $\chi(M^*) \leq |X| + \chi(A \setminus X) + |C| + \chi(B \setminus C)$ . But  $|X| \leq n\omega$ ;  $A \setminus X$  is  $(t^d, D)$ -colourable, since  $\omega(G[A \setminus X]) \leq t$  and  $t < \omega$  (by (2));  $|C| \leq n^s \omega^{2s+2}$  by (3); and  $B \setminus C$  is  $((\omega - t)^d, D)$ -colourable by (4) and since  $\omega - t < \omega$  (because  $t > 0$ ). Thus  $M^*$  is  $(K_1, D)$ -colourable where

$$K_1 = n\omega + t^d + n^s \omega^{2s+2} + (\omega - t)^d.$$

Since  $1 \leq t \leq \omega - 1$ , it follows that  $t^d + (\omega - t)^d \leq (\omega - 1)^d + 1$  (since  $d \geq 1$ ), and so

$$K_1 = n\omega + t^d + n^s \omega^{2s+2} + (\omega - t)^d \leq n\omega + n^s \omega^{2s+2} + (\omega - 1)^d + 1 = \omega^{s+3} + \omega^{s^2+4s+2} + (\omega - 1)^d + 1.$$

Hence  $M^*$  is  $(K, D)$ -colourable where

$$K = \omega^{s+3} + \omega^{s^2+4s+2} + (\omega - 1)^d + 1.$$

This proves 5.4. ■

From 5.2 and 5.4, with  $m = (s + 1)\omega^s$ , we deduce:

**5.5** Suppose that  $d \geq 1$  and  $D \geq 0$  have the property that every  $H_s$ -free graph  $G'$  with  $\omega(G') < \omega(G)$  is  $(\omega(G')^d, D)$ -colourable; and  $c \geq 2s$  satisfies 4.1 when  $H = H_s$ . Assume that  $\omega^3 > \omega + s/7$  and  $\omega \geq 200$ . Then  $V(\mathcal{L}) \cup N(\mathcal{L})$  is  $(K, D)$ -colourable where

$$K = (\omega - 1)^d + \omega^{(c+1)(s+7)}.$$

**Proof.**  $V(\mathcal{L})$  has cardinality at most  $14\omega^{s+7}$  from the definition of an  $s$ -template and since  $k + |L_0| \leq \omega$  by 3.1. Every vertex in  $N(\mathcal{L})$  is pendant or dense or pure with respect to  $\mathcal{L}$ , by 4.3. At most  $14^{s+2}\omega^{s^2+9s+14}$  are pendant, by 4.4, and the chromatic number of the set of all dense vertices is at most  $(14\omega^{s+6})^{c+1}\omega$  by 4.5. By 5.2 with  $m = (s+1)\omega^s$ , and 5.4, the set of all pure vertices is  $(K_1, D)$ -colourable where

$$K_1 = (\omega - 1)^d + \omega^{s^2+4s+2} + \omega^{s+3} + 1 + (2s+2)\omega^{s+1} + 2\omega^{s+2}.$$

Adding, we deduce that  $V(\mathcal{L}) \cup N(\mathcal{L})$  is  $(K, D)$ -colourable where

$$K = 14\omega^{s+7} + 14^{s+2}\omega^{s^2+9s+14} + (14\omega^{s+6})^{c+1}\omega + (\omega - 1)^d + \omega^{s^2+4s+2} + \omega^{s+3} + 1 + (2s+2)\omega^{s+1} + 2\omega^{s+2}.$$

Since  $K \leq (\omega - 1)^d + \omega^{(c+1)(s+7)}$ , this proves 5.5. ■

## 6 Proof of the main theorem

In this section we prove 1.2. If  $\mathcal{L} = (L_0, \dots, L_k)$  is an optimal  $s$ -template in  $G$ , we define  $Z(\mathcal{L})$  to be the union of  $L_0$  and the set of all vertices  $v \in N(\mathcal{L})$  such that for  $1 \leq i \leq k$ ,  $v$  has at most  $\omega^{s+2}/4$  non-neighbours in  $L_i$ . Let  $Y(\mathcal{L}) = (V(\mathcal{L}) \cup N(\mathcal{L})) \setminus Z(\mathcal{L})$ . First we need:

**6.1** *Let  $s \geq 1$  be an integer, and let  $G$  be  $H_s$ -free, with  $\omega(G) \geq 15$  and  $\omega^2 > s+1$ . Let  $\mathcal{L}$  be an optimal  $s$ -template in  $G$ . Then every vertex in  $Y(\mathcal{L})$  has at most  $\omega^{s+7}$  neighbours in  $V(G) \setminus Y(\mathcal{L})$ .*

**Proof.** Let  $v \in Y(\mathcal{L})$ , and suppose that  $v$  has more than  $\omega^{s+7}$  neighbours in  $V(G) \setminus Y(\mathcal{L})$ . By 3.1 and 3.5,  $|Z(\mathcal{L})| \leq \omega + 14\omega^{s+6}$ , and so  $v$  has at least  $\omega^s$  neighbours in  $V(G) \setminus (V(\mathcal{L}) \cup N(\mathcal{L}))$ , since

$$\omega + 14\omega^{s+6} + \omega^s < \omega^{s+7}$$

(because  $\omega \geq 15$ ). By 4.2, there is a stable set  $S \subseteq V(G) \setminus (V(\mathcal{L}) \cup N(\mathcal{L}))$  of neighbours of  $v$ , with  $|S| = s$ . Since  $v \notin Z(\mathcal{L})$ , it follows that  $v \in N(\mathcal{L})$  and there exists  $i \in \{1, \dots, k\}$  such that  $v$  has more than  $\omega^{s+2}/4$  non-neighbours in  $L_i$ . Since  $k \geq 2$  by 3.2, and  $v$  has a neighbour in at least one of  $L_1, \dots, L_k$  because  $v \in N(\mathcal{L})$ , and each  $|L_j| \geq \omega^{s+5} \geq \omega^{s+2}/4$ , we may choose distinct  $i, j \in \{1, \dots, k\}$  such that  $v$  has a neighbour in  $L_j$  and  $v$  has at least  $\omega^{s+2}/4$  non-neighbours in  $L_i$ , and choose such a pair  $i, j$  such that  $v$  has as many non-neighbours in  $L_i$  as possible. Let  $B$  be the set of non-neighbours of  $v$  in  $L_i$ . Thus  $|B| \geq \omega^{s+2}/4$ .

(1)  *$v$  has at most  $\omega^s + \omega^{s+3}$  non-neighbours in  $L_j$ .*

Let  $u \in L_j$  be adjacent to  $v$ . If  $u$  has at least  $\omega^s$  neighbours in  $B$ , there is a stable set  $T$  of such neighbours with  $|T| = s$ , by 4.2, and then the subgraph induced on  $S \cup T \cup \{u, v\}$  is isomorphic to  $H_s$ , a contradiction. Thus  $u$  has fewer than  $\omega^s$  neighbours in  $B$ ; but it has at most  $\omega^{s+3}$  non-neighbours in  $L_i$ , and hence at most that many in  $B$ , and so  $|B| \leq \omega^s + \omega^{s+3}$ . Since  $|L_i| \geq \omega^{s+5}$ ,  $v$  has a neighbour in  $L_i$ ; and so from the choice of the pair  $i, j$ ,  $v$  has at most  $\omega^s + \omega^{s+3}$  non-neighbours in  $L_j$ . This proves (1).

Since  $|B| \geq \omega^{s+2}/4 \geq \omega^s$ , it includes a stable set  $T$  of cardinality  $s$ , by 4.2. Each vertex in  $S$  has at most  $\omega^{s+3}$  non-neighbours in  $L_j$ , and  $v$  has at most  $\omega^s + \omega^{s+3}$  non-neighbours in  $L_j$  by (1), and since

$$s\omega^{s+3} + \omega^s + \omega^{s+3} < \omega^{s+5} \leq |L_j|,$$

(because  $\omega^2 > s + 1$ ), it follows that some vertex  $u \in L_j$  is adjacent to every vertex in  $T \cup \{v\}$ . But then the subgraph induced on  $S \cup T \cup \{u, v\}$  is isomorphic to  $H_s$ , a contradiction. This proves 6.1.  $\blacksquare$

Let  $A \subseteq V(G)$ , and let  $\mathcal{L} = (L_0, \dots, L_k)$  be an  $s$ -template of  $G$  with  $V(\mathcal{L}) \subseteq A$ . We make the following definitions:

- $Z_A(\mathcal{L})$  is the union of  $L_0$  and the set of vertices in  $A \setminus V(\mathcal{L})$  that have at most  $\omega^{s+2}/4$  non-neighbours in  $L_i$  for each  $i \in \{1, \dots, k\}$ ;
- $N_A(\mathcal{L})$  is the set of vertices in  $A \setminus V(\mathcal{L})$  with a neighbour in  $L_1 \cup \dots \cup L_k$ ; and
- $Y_A(\mathcal{L}) = (V(\mathcal{L}) \cup N_A(\mathcal{L})) \setminus Z_A(\mathcal{L})$ .

**6.2** *Let  $s \geq 1$  be an integer, and let  $G$  be  $H_s$ -free. Let  $\omega = \omega(G)$ , and let  $A \subseteq V(G)$ , such that there is an  $s$ -template  $\mathcal{L}$  of  $G$  with  $V(\mathcal{L}) \subseteq A$  and with value at least  $28\omega^{s+6}$ . Let  $\mathcal{L} = (L_0, \dots, L_k)$  be such an  $s$ -template with maximum value.*

- *Suppose that  $d \geq 1$  and  $D \geq 0$  have the property that every  $H_s$ -free graph  $G'$  with  $\omega(G') < \omega$  is  $(\omega(G')^d, D)$ -colourable; and  $c \geq 2s$  satisfies 4.1 when  $H = H_s$ ; and  $\omega^3 > \omega + s/7$ ; and  $\omega \geq 200$ . Then  $V(\mathcal{L}) \cup N_A(\mathcal{L})$  is  $((\omega - 1)^d + \omega^{(c+1)(s+7)}, D)$ -colourable.*
- *If  $\omega \geq 15$  and  $\omega^2 > s + 1$ , then every vertex in  $Y_A(\mathcal{L})$  has at most  $\omega^{s+7}$  neighbours in  $A \setminus Y_A(\mathcal{L})$ .*

**Proof.**  $\mathcal{L}$  is not necessarily an optimal  $s$ -template of  $G$ , since it is constrained to have vertex set included in  $A$ ; and it is not necessarily an optimal  $s$ -template of  $G[A]$ , since perhaps  $\omega(G[A]) < \omega(G)$  and then the conditions that define an  $s$ -template of  $G$  are different from those that define an  $s$ -template of  $G[A]$ . Nevertheless, we can apply 5.5 and 6.1 to  $\mathcal{L}$  by the following trick. Let  $G'$  be the disjoint union of  $G[A]$  and a complete graph  $K_{\omega(G)}$  with vertex set  $B$  say. Then  $\omega(G') = \omega(G)$ , and since every optimal  $s$ -template of  $G'$  induces a connected subgraph with more than  $\omega(G)$  vertices (because its value is at least  $28\omega(G)^6$ ), it contains no vertex of  $B$  and so is an  $s$ -template of  $G[A]$ . This proves that  $\mathcal{L}$  is an optimal  $s$ -template of  $G'$ . The claims of the theorem follow by applying 5.5 and 6.1 to  $\mathcal{L}$  and  $G'$ . This proves 6.2.  $\blacksquare$

Now we prove 1.2, which we restate in a strengthened form:

**6.3** *For every integer  $s \geq 1$ , there exists  $d \geq 0$  such that if  $G$  is  $H_s$ -free, then  $G$  is  $(\omega^d, \omega^{s+8})$ -colourable, and hence has chromatic number at most  $\omega^d(\omega^{s+8} + 1)$ .*

**Proof.** Choose  $c \geq 2s$  satisfying 4.1 with  $H = H_s$ . It follows from the main theorem of [11] that there is a function  $f$  such that  $\chi(G) \leq f(\omega(G))$  for every  $H_s$ -free graph  $G$ ; and so by choosing  $d$  sufficiently large we may arrange that  $\chi(G) \leq \omega(G)^d$  for every  $H_s$ -graph  $G$  with  $\omega(G)$  at most  $\max(200, (s+1)^{1/2})$ . Let us also choose  $d$  so large that  $d \geq (c+1)(s+7) + 1$ . We claim that  $d$



satisfies 6.3. The proof is by induction on  $\omega = \omega(G)$ . If  $\omega \leq \max(200, (s+1)^{1/2})$  the claim is true, so we may assume that  $\omega > \max(200, (s+1)^{1/2})$ . Consequently  $\omega^3 > \omega + s/7$  and  $\omega^2 > s+1$ , and we can apply 6.2.

Let  $V_0 = V(G)$ , and choose  $n$  maximum such that there is a sequence  $\mathcal{L}_i$  ( $1 \leq i \leq n$ ) of  $s$ -templates of  $G$  and a sequence  $V_0 \supseteq V_1 \supseteq \dots \supseteq V_n$  of subsets of  $V(G)$ , with the following property. For  $1 \leq i \leq n$ ,  $\mathcal{L}_i$  is an  $s$ -template of  $G$  with value at least  $28\omega^{s+6}$ , with  $V(\mathcal{L}_i) \subseteq V_{i-1}$ , chosen with maximum value among all such  $s$ -templates; and  $V_i = V_{i-1} \setminus Y_{V_{i-1}}(\mathcal{L}_i)$ , in the notation of 6.2. For  $1 \leq i \leq n$ , let  $Y_i = Y_{V_{i-1}}(\mathcal{L}_i)$ , and let  $Y_{n+1} = V_n$ . We observe:

- The sets  $Y_1, \dots, Y_{n+1}$  are pairwise disjoint and have union  $V(G)$ .
- For  $1 \leq i \leq n$ ,  $G[Y_i]$  is  $(\omega^d, \omega^{s+7}(\omega-1))$ -colourable. To see this, observe that from the inductive hypothesis, every  $H_s$ -free graph  $G'$  with  $\omega(G') < \omega$  is  $(\omega(G')^d, \omega(G')^{s+8})$ -colourable, and hence  $(\omega(G')^d, D)$ -colourable where  $D = \omega^{s+7}(\omega-1)$ , since  $\omega(G') < \omega$ . From the first statement of 6.2, for  $1 \leq i \leq n$ ,  $Y_i$  is  $((\omega-1)^d + \omega^{(c+1)(s+7)}, \omega^{s+7}(\omega-1))$ -colourable and hence  $(\omega^d, \omega^{s+7}(\omega-1))$ -colourable, since  $(\omega-1)^d + \omega^{(c+1)(s+7)} \leq \omega^d$  (because  $d \geq (c+1)(s+7)+1$ ).
- $G[Y_{n+1}]$  is  $(\omega^d, \omega^{s+7}(\omega-1))$ -colourable. To see this, observe that the maximality of  $n$  implies that there is no  $s$ -template  $\mathcal{L}$  of  $G$  of value at least  $28\omega^{s+6}$  and with  $V(\mathcal{L}) \subseteq V_n$ , and so  $\chi(V_n) \leq (14\omega^{s+6})^c$  by 4.1. Consequently  $V_n = Y_{n+1}$  is  $(\omega^d, \omega^{s+7}(\omega-1))$ -colourable, since  $(14\omega^{s+6})^c \leq \omega^d$ .
- For  $1 \leq i \leq n$ , every vertex in  $Y_i$  has at most  $\omega^{s+7}$  neighbours in  $Y_{i+1} \cup \dots \cup Y_{n+1}$ . This follows from the second statement of 6.2.

By 2.1,  $G$  is  $(\omega^d, \omega^{s+7}(\omega-1) + \omega^{s+7})$ -colourable. This proves 6.3.

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