Complete minors and average degree – a short proof

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Abstract

We provide a short and self-contained proof of the classical result of Kostochka and of Thomason, ensuring that every graph of average degree d has a complete minor of order $\Omega(d/\sqrt{\log d})$.

Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$. How large a complete minor are we guaranteed to find in G? This classical question, closely related to the famed Hadwiger's conjecture, has been thoroughly studied over the last half a century. It is quite easy to see the answer is at least logarithmic in d. Mader [3] proved it is of order at least $d/\log d$. The right order of magnitude was established independently by Kostochka [1, 2] and by Thomason [4] to be $d/\sqrt{\log d}$, its tightness follows by considering random graphs. Finally, Thomason found in [5] the asymptotic value of this extremal function.

Here we provide a short and self-contained proof of the celebrated Kostochka–Thomason bound.

Theorem 1. *Let $G = (V, E)$ be a graph with $|E|/|V| \geq d$, where d is a sufficiently large integer. Then G contains a minor of the complete graph on at least $\frac{d}{10\sqrt{\ln d}}$ vertices.*

The constant $1/10$ in the above statement is inferior to the best constant $3.13\ldots$ found by Thomason [5] (yet is better than the constants in [1, 2]); we did not make any serious attempt to optimize it in our arguments. The main point here is to give a short proof of the tight $\Omega(d/\sqrt{\log d})$ bound for this classical extremal problem.

Throughout the proof we assume, whenever this is needed, that the parameters n and d are sufficiently large. To simplify the presentation we omit all floor and ceiling signs in several places. For a graph $G = (V, E)$, its minimum degree is denoted by $\delta(G)$, and for $v \in V$ we use $N_G(v)$ for the external neighborhood of v in G.

We need the following lemma proven by simple probabilistic arguments.

Lemma 2. *Let $H = (V, E)$ be a graph on at most n vertices with $\delta(H) \geq n/6$. Let $t \leq n/\sqrt{\ln n}$, and let $A_1, \ldots, A_t \subset V$ with $|A_j| \leq ne^{-\sqrt{\ln n}/3}$ for all $1 \leq j \leq t$. Then there is $B \subset V$ of size $|B| \leq 3.1\sqrt{\ln n}$*

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such that B dominates all but at most $ne^{-\sqrt{\ln n}/3}$ vertices of V, $B \setminus A_j \neq \emptyset$ for all $j = 1, \ldots, t$, and the induced subgraph $G[B]$ has at most six connected components.

Proof. Set $s = 3.1\sqrt{\ln n}$ and choose s vertices of V independently at random with repetitions. Let B be the set of chosen vertices. Observe that for every vertex $v \in V$,

$$Pr[N(v) \cap B = \emptyset] \leq \left(1 - \frac{d(v)}{n}\right)^s \leq e^{-\frac{sd(v)}{n}} \leq e^{-s/6}.$$

Hence the expected number of vertices not dominated by B is at most $ne^{-s/6} < ne^{-3.1\sqrt{\ln n}/6} < ne^{-\sqrt{\ln n}/2}$, and by Markov's inequality, it is at most $ne^{-\sqrt{\ln n}/3}$ with probability exceeding $1/2$ (with room to spare). Also, since $|V| > \delta(H) \geq n/6$, for every subset A_j,

$$Pr[B \subseteq A_j] = \left(\frac{|A_j|}{|V|}\right)^s < \left(\frac{6|A_j|}{n}\right)^s \leq 6^s e^{-s\sqrt{\ln n}/3} = 6^{\Theta(\sqrt{\log n})} e^{-3.1 \ln n/3} < \frac{1}{n}.$$

Therefore the probability that $B \setminus A_j \neq \emptyset$ for all j is at least $1 - t/n \geq 1 - 1/\sqrt{\ln n}$.

We now argue about the number of connected components in $G[B]$. Writing $B = (v_1 \ldots, v_s)$, for $1 \leq i \leq s$ let x_i be the random variable counting the number of indices $1 \leq j \neq i \leq s$ for which v_j is a neighbor of v_i. Conditioning on v_i, we see that x_i is distributed as a binomial random variable with parameters $s - 1$ and $d(v_i)/|V| > 1/6$. Hence invoking the standard Chernoff-type bound on the lower tail of the binomial distribution, we derive that $Pr[x_i < s/7] \leq e^{-\Theta(s)}$. Applying the union bound over all $1 \leq i \leq s$, we conclude that with probability $1 - o(1)$, we have $x_i \geq s/7$ for all i. Finally, observe that since $s \ll \sqrt{|V|}$, with probability $1 - o(1)$ there are no repetitions in B, and hence $d(v_i, B) = x_i \geq s/7$ for all $1 \leq i \leq s$. But then all connected components of $G[B]$ are of size exceeding $s/7$, and therefore $G[B]$ has at most six connected components.

Combining the above three estimates, the desired result follows. ◻

Proof of Theorem 1. Let $G' = (V', E')$ be a minor of G such that $|E'| \geq d|V'|$ and $|V'| + |E'|$ is minimal. If an edge e of G' is contained in t triangles then contracting e gives a minor of G with one vertex and $t + 1$ edges less. By the minimality of G' we have $t + 1 > d$, implying $t \geq d$. Hence for every edge $e = (u, v) \in E(G')$, the vertex u is connected by an edge of G' to at least d neighbors of v. The minimality of G' also implies $|E'| = d|V'|$, hence G' has a vertex v of degree at most $2d$. Let H be the subgraph of G' induced by $N_{G'}(v)$. Then H has at most $2d$ vertices and minimum degree at least d. Obviously a minor of H is a minor of G as well.

We now argue that H contains a $d/3$-connected subgraph H_1 with $\delta(H_1) \geq 2d/3$. If H itself is $d/3$-connected this holds for $H_1 = H$. Otherwise there is a partition $V(H) = A \cup B \cup S$, where $A, B \neq \emptyset$, $|S| < d/3$, and H has no edges between A and B. Assume without loss of generality $|A| \leq |B|$. Then $|A| \leq d$, and since $\delta(H) \geq d$, every vertex $v \in A$ has at least $2d/3$ neighbors in A, implying that every pair of vertices of A has at least $d/3$ common neighbors in A. Hence the induced subgraph $H_1 := H[A]$ is $d/3$-connected, has at most $2d$ vertices and satisfies $\delta(H_1) \geq 2d/3$.

Set $i = 1$ and repeat the following iteration $d/10\sqrt{\ln d}$ times. Let $H_i = (V_i, E_i) \subseteq H_1$ be the current graph, and suppose $A_1, \ldots, A_{i-1}$ are subsets of V_i of cardinalities $|A_j| \leq 2de^{-\sqrt{\ln(2d)}/3}$ (representing the non-neighbors of the previously found branch sets B_j in V_i). We assume (and justify it later) that H_i is connected and has $\delta(H_i) > d/3$. Then the diameter of H_i is at most 14, as on any shortest path $P = (v_0, v_1, \dots)$ in H_i the vertices at positions divisible by three have pairwise disjoint neighborhoods. Since $|V(H_i)|/\delta(H_i) < 6$, the number of such neighborhoods is at most 5, and therefore any shortest path has at most 15 vertices. Applying Lemma 2 with $H := H_i$, $n := 2d$, $t := i - 1$, and $A_1, \dots, A_{i-1}$ (for the initial step $i = 1$ there are no A_j's to plug into Lemma 2 — which of course does not hinder its application) we get a subset B_i of cardinality $|B_i| \le 3.1\sqrt{\ln(2d)}$ as promised by the lemma. We now turn B_i into a connected set by adding few vertices of H_i if necessary. Recall that $H_i[B_i]$ has at most six connected components. Connecting one of them by shortest paths in H_i to all others and recalling that H_i has diameter at most 14, we conclude that by appending to B_i all the vertices of these paths we make it connected by adding to it at most $13 \cdot 5 = 65$ vertices. Altogether we obtain a connected subset B_i of cardinality $|B_i| \le (3.1 + o(1))\sqrt{\ln(2d)}$, dominating all but at most $2de^{-\sqrt{\ln(2d)}/3}$ vertices of V_i and having a vertex outside every A_j (these properties are preserved under vertex addition when making B_i into a connected subset) — meaning connected to every previous B_j. We now update $V_{i+1} := V_i - B_i$, $A_i := V_{i+1} - N_{H_i}(B_i)$, and $A_j := A_j \cap V_{i+1}$, $j = 1, \dots, i-1$, and finally increment $i := i + 1$, set $H_i := H[V_i]$, and proceed to the next iteration. The total number of vertices deleted in all iterations satisfies:

$$\left| \cup_i B_i \right| \le \frac{d}{10\sqrt{\ln d}} \cdot (3.1 + o(1))\sqrt{\ln(2d)} < \frac{d}{3},$$

and since we started with the $d/3$-connected graph H_1 with $\delta(H_1) \ge 2d/3$, we indeed have that at each iteration the graph H_i is connected and has $\delta(H_i) > d/3$.

After having completed all $d/10\sqrt{\ln d}$ iterations, we get a family of $d/10\sqrt{\ln d}$ branch sets B_i, all connected, and with an edge of H_1 between every pair of branch sets. Hence they form a complete minor of order $d/10\sqrt{\ln d}$ as promised. □

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