arXiv:2203.07704v1 [math.CO] 15 Mar 2022

DP color functions versus chromatic polynomials (II)

Meiqiao Zhang[*], Fengming Dong[†]
National Institute of Education, Nanyang Technological University, Singapore

Abstract

For any connected graph G, let $P(G,m)$ and $P_{DP}(G,m)$ denote the chromatic polynomial and DP color function of G, respectively. It is known that $P_{DP}(G,m) \le P(G,m)$ holds for every positive integer m. Let $DP_{\approx}$ (resp. $DP_{<}$) be the set of graphs G for which there exists an integer M such that $P_{DP}(G,m) = P(G,m)$ (resp. $P_{DP}(G,m) < P(G,m)$) holds for all integers $m \ge M$. Determining the sets $DP_{\approx}$ and $DP_{<}$ is a key problem on the study of the DP color function. For any edge set E_0 of G, let $\ell_G(E_0)$ be the length of a shortest cycle C in G such that $|E(C) \cap E_0|$ is odd whenever such a cycle exists, and $\ell_G(E_0) = \infty$ otherwise. Write $\ell_G(E_0)$ as $\ell_G(e)$ if $E_0 = \{e\}$.

In this paper, we prove that if G has a spanning tree T such that $\ell_G(e)$ is odd for each $e \in E(G) \setminus E(T)$, the edges in $E(G) \setminus E(T)$ can be labeled as $e_1, e_2, \cdots, e_q$ with $\ell_G(e_i) \le \ell_G(e_{i+1})$ for all $1 \le i \le q-1$ and each edge e_i is contained in a cycle C_i of length $\ell_G(e_i)$ with $E(C_i) \subseteq E(T) \cup \{e_j : 1 \le j \le i\}$, then G is a graph in $DP_{\approx}$. As a direct application of this conclusion, all plane near-triangulations and complete multipartite graphs with at least three partite sets belong to $DP_{\approx}$. We also show that if E^* is an edge set of G such that $\ell_G(E^*)$ is even and E^* satisfies certain conditions, then G belongs to $DP_{<}$. In particular, if $\ell_G(E^*) = 4$, where E^* is a set of edges between two disjoint vertex subsets of G, then G belongs to $DP_{<}$. Both results extend known ones in [DP color functions versus chromatic polynomials, *Advances in Applied Mathematics* **134** (2022), article 102301].

1 Introduction

1.1 Proper coloring, list coloring and DP coloring

In this article, we consider simple graphs only. For any graph G, let $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. For any two disjoint subsets V_1 and V_2 of $V(G)$, let $E_G(V_1, V_2)$ be the set of edges $uv \in E(G)$, where $u \in V_1$ and $v \in V_2$. For any non-empty subset V_0 of $V(G)$, let $G[V_0]$ denote the subgraph of G induced by V_0. For any $A \subseteq E(G)$, let $V(A)$ be the

[*]Email: nie21.zm@e.ntu.edu.sg and meiqiaozhang95@163.com.
[†]Corresponding author. Email: fengming.dong@nie.edu.sg and donggraph@163.com.

set of vertices in G which are incident to some edges in A, and let $G[A]$ be the subgraph of G with vertex set $V(A)$ and edge set A when $A \neq \emptyset$. Let $G\langle A\rangle$ be the spanning subgraph of G with edge set A and $G - A = G\langle E(G) \setminus A\rangle$, and denote by $c(A)$ the number of components of $G\langle A\rangle$. For any $u \in V(G)$, let $N_G(u)$ (or simply $N(u)$) be the set of the neighbors of u in G.

Denote the set of positive integers by $\mathbb{N}$. For any $m \in \mathbb{N}$, let $[m] = \{1, \cdots, m\}$. For any graph G, a *proper coloring* of G is a mapping $c : V(G) \to \mathbb{N}$, such that $c(u) \neq c(v)$ for all $uv \in E(G)$. For any positive integer m, a *proper m-coloring* of G is a proper coloring c with $c(v) \in [m]$ for all $v \in V(G)$. The *chromatic polynomial* $P(G, m)$ of G is a function which counts the number of proper m-colorings of G for each $m \in \mathbb{N}$. The chromatic polynomial was originally designed as a tool to attack the Four Color Conjecture [1], but later gained unique research significance because of its elegant properties, see [2, 3, 11, 12] for reference.

To generalize proper coloring, Vizing [14] and Erdős, Rubin and Taylor [7] independently introduced the notion of list coloring. For any graph G, a *list assignment* L of G is a mapping from $V(G)$ to the power set of $\mathbb{N}$, and an *L-coloring* of G is a proper coloring c with $c(v) \in L(v)$ for all $v \in V(G)$. Denote the number of L-colorings of G by $P(G, L)$.

L is an *m-list assignment* of G if $|L(v)| = m$ holds for all $v \in V(G)$. Then the *list color function* $P_l(G, m)$ of G counts the minimum value of $P(G, L)$ among all m-list assignments L for each $m \in \mathbb{N}$. Obviously, $P_l(G, m) \leq P(G, m)$ holds for each $m \in \mathbb{N}$. And surprisingly, $P_l(G, m) = P(G, m)$ holds whenever $m > \frac{|E(G)|-1}{ln(1+\sqrt{2})}$ (see [15]). While this implies that the list color function of some graph might not be a polynomial [5], the list color function $P_l(G, m)$ of any graph G inherits all the nice properties of its chromatic polynomial when m is sufficiently large. See [13] for some open problems of list color functions.

To make breakthroughs in list coloring, Dvořák and Postle [6] recently defined the *correspondence coloring*, or *DP-coloring*. The formal definition is as follows.

For any graph G, a *cover* of G is an ordered pair $\mathcal{H} = (L, H)$, where H is a graph and L is a mapping from $V(G)$ to the power set of $V(H)$ satisfying the conditions below:

- the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$,
- for every $u \in V(G)$, $H[L(u)]$ is a complete graph,
- if u and v are not adjacent in G, then $E_H(L(u), L(v)) = \emptyset$, and
- for each edge $uv \in E(G)$, $E_H(L(u), L(v))$ is a matching.

For any cover $\mathcal{H} = (L, H)$ of G, $\mathcal{H}$ is *m-fold* if $|L(v)| = m$ for all $v \in V(G)$, and $\mathcal{H}$ is *full* if for each edge $uv \in E(G)$, $E_H(L(u), L(v))$ is a perfect matching. An *$\mathcal{H}$-coloring* of G is an independent set I in H with $|I| = |V(G)|$. Obviously, any $\mathcal{H}$-coloring I of G has the property that $|I \cap L(v)| = 1$ for each $v \in V(G)$. Denote the number of $\mathcal{H}$-colorings of G by $P_{DP}(G, \mathcal{H})$.

The *DP color function* $P_{DP}(G,m)$ of G, introduced by Kaul and Mudrock [9] in 2019, counts the minimum value of $P_{DP}(G,\mathcal{H})$ among all m-fold covers $\mathcal{H}$ of G for each $m \in \mathbb{N}$. Note that $P_{DP}(G,m) \le P_l(G,m)$ holds for each $m \in \mathbb{N}$. Therefore, for each $m \in \mathbb{N}$,

$$P_{DP}(G,m) \le P_l(G,m) \le P(G,m). \tag{1.1}$$

It is known that all the equalities in (1.1) can hold simultaneously. For example, the authors of [9] proved that $P_{DP}(G,m) = P(G,m)$ holds for all $m \in \mathbb{N}$ when G is a chordal graph. However, different from list color functions, not the DP color functions of all graphs tend to be the same as their chromatic polynomials. In [9], it is shown that for any graph G with even girth, there exists an $N \in \mathbb{N}$, such that $P_{DP}(G,m) < P(G,m)$ for all integers $m \ge N$. Therefore, how to characterize the two classes of graphs $DP_{\approx}$ and $DP_{<}$ becomes a research focus in the study of DP color functions, where

- $DP_{\approx}$ is the set of graphs G for which there exists an integer M such that $P_{DP}(G,m) = P(G,m)$ holds for all integers $m \ge M$, and
- $DP_{<}$ is the set of graphs G for which there exists an integer M such that $P_{DP}(G,m) < P(G,m)$ holds for all integers $m \ge M$.

So far it is still unknown if there exists a graph G such that $G \notin DP_{\approx}$ and $G \notin DP_{<}$. Thus, a characterization of the graphs in $DP_{\approx}$ or $DP_{<}$ does not necessarily guarantee a characterization of the graphs in the other class.

In this paper, we shall introduce our new findings on determining $DP_{\approx}$ and $DP_{<}$.

1.2 Known results

Throughout this paper, we need only to consider connected graphs because for disconnected graph G with components $G_1, \cdots, G_k$,

$$P_{DP}(G,m) = \prod_{i=1}^{k} P_{DP}(G_i,m). \tag{1.2}$$

In this subsection, we introduce the known graphs contained in sets $DP_{\approx}$ and $DP_{<}$ respectively.

Let $\mathcal{H}_{G,m} = (L_{G,m}, H_{G,m})$ denote the special full m-fold cover such that $L_{G,m}(u) = \{(u,i) : i \in [m]\}$ for each vertex $u \in V(G)$ and $E_{H_{G,m}}(L_{G,m}(u), L_{G,m}(v)) = \{(u,i)(v,i) : i \in [m]\}$ for each edge $uv \in E(G)$. Obviously, $P_{DP}(G,\mathcal{H}_{G,m}) = P(G,m)$ for all $m \in \mathbb{N}$. Let DP^* denote the set of graphs G for which there exists $M \in \mathbb{N}$ such that for every m-fold cover $\mathcal{H} = (L,H)$ of G, if $H \not\cong H_{G,m}$, then $P_{DP}(G,\mathcal{H}) > P(G,m)$ holds for all integers $m \ge M$. Apparently, $DP^* \subseteq DP_{\approx}$, but whether $DP^* = DP_{\approx}$ or not is currently unknown.

On one hand, Mudrock and Thomason [10] showed that each graph with a dominating vertex belongs to $DP_{\approx}$. Actually they proved that each graph with a dominating vertex belongs to DP^*. Dong and Yang [4] then extended their conclusion to a large set of connected graphs (see Theorem 1.1).

Let $\mathcal{C}_G(e)$ be the set of cycles in G containing e with the minimum order. Note that $\mathcal{C}_G(e) = \emptyset$ if e is a bridge. Denote by $\ell_G(e)$ the *girth of edge e in G*, which is the order of any $C \in \mathcal{C}_G(e)$ if $\mathcal{C}_G(e) \neq \emptyset$; otherwise, $\ell_G(e) = \infty$.

Theorem 1.1 ([4]) *Let G be a graph with a spanning tree T. If for each edge e in $E(G) \setminus E(T)$, $\ell_G(e)$ is odd and there exists $C \in \mathcal{C}_G(e)$ such that $\ell_G(e') < \ell_G(e)$ for each $e' \in E(C) \setminus (E(T) \cup \{e\})$, then $G \in DP^*$ and hence $G \in DP_{\approx}$.*

On the other hand, some families of graphs belonging to $DP_<$ were found. Kaul and Mudrock [9] discovered the fact that for any graph G with an edge e, if $P(G - e, m) < mP(G, m)/(m - 1)$, then $P_{DP}(G, m) < P(G, m)$ holds, and showed that every graph with an even girth belongs to $DP_<$. The latter conclusion was extended to the following one.

Theorem 1.2 ([4]) *Graph G belongs to $DP_<$ if G contains an edge of even girth.*

An *edge gluing* of vertex disjoint graphs G_1 and G_2 is a graph obtained by identifying an edge in G_1 and an edge in G_2 as a same one. Then, it is easy to check [4, 8] that G belongs to $DP_<$ if $G_1 \in DP_<$ and either G_1 is a block of G or G is an edge-gluing of G_1 and some other graph. Therefore, as shown in [4], Theorem 1.2 cannot be a characterization of all the graphs in $DP_<$ because by edge gluing any graph G in $DP_<$ with a number of 3-cycles, infinitely many graphs G' in $DP_<$ can be obtained in which $\ell_{G'}(e) = 3$ holds for all $e \in E(G')$.

1.3 New results

In this article, we will further extend Theorems 1.1 and 1.2. We first give the definition of a family of graphs.

A graph G is called *DP-good* if G has a spanning tree T and a labeling $e_1, \cdots, e_q$ of the edges in $E(G) \setminus E(T)$, where $q = |E(G)| - |E(T)|$, such that $\ell_G(e_1) \le \cdots \le \ell_G(e_q)$ and for each $i \in [q]$, $\ell_G(e_i)$ is odd and $E(C_i) \subseteq E(T) \cup \{e_1, \cdots, e_i\}$ holds for some $C_i \in \mathcal{C}_G(e_i)$. Obviously, the q cycles $C_1, \cdots, C_q$ are pairwise distinct.

It is clear that any graph satisfying the condition in Theorem 1.1 is DP-good. But the graph shown in Figure 1 is a DP-good graph which doesn't satisfy the requirement in Theorem 1.1. The following theorem shows that each DP-good graph belongs to $DP_{\approx}$.

Theorem 1.3 *Every DP-good graph is in DP^*.*

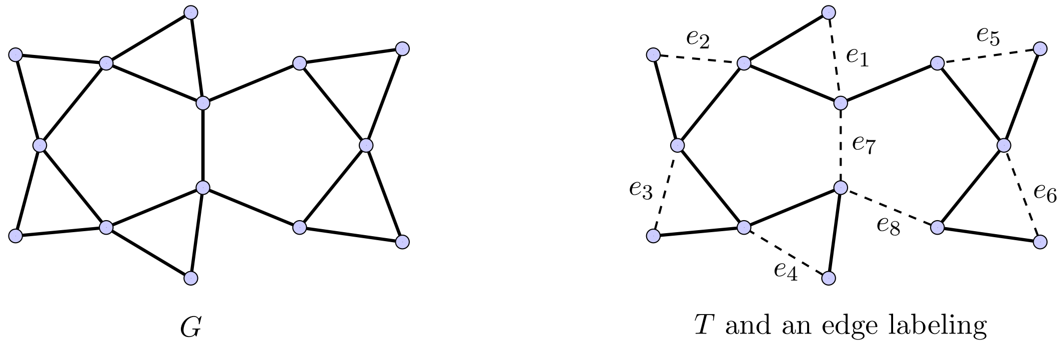


Figure 1: A DP-good graph G with a spanning tree T and an edge labeling of the edges in $E(G) \setminus E(T)$

As an immediate consequence of Theorem 1.3, Corollary 1.4 below suggests that many special classes of graphs are DP-good and therefore contained in DP^*, such as chordal graphs, complete multipartite graphs with at least three partite sets, and plane near-triangulations.

Corollary 1.4 *Let G be a graph with vertex set $\{v_i : i = 0, 1, 2, \cdots, n\}$. If for each $i \in [n]$, the set $N(v_i) \cap \{v_j : 0 \le j \le i-1\}$ is not empty and the subgraph of G induced by this vertex set is connected, then G is DP-good.*

On the other hand, in order to extend Theorem 1.2, we shall first generalize the definition of the girth of an edge to the girth of an edge set. Given any subset E_0 of $E(G)$, let $\mathcal{C}'_G(E_0)$ be the set of the shortest cycles C in G such that $|E(C) \cap E_0|$ is odd (i.e., $|E(C) \cap E_0|$ is odd and $|E(C) \le |E(C')|$ holds for each cycle C' in G whenever $|E(C') \cap E_0|$ is odd), and *the girth of* E_0, denoted by $\ell_G(E_0)$, is defined to be the length of any cycle in $\mathcal{C}'_G(E_0)$ if this set is non-empty, and $\ell_G(E_0) = \infty$ otherwise. Obviously, $\ell_G(E_0) < \infty$ if and only if G contains a cycle C such that $|E(C) \cap E_0|$ is odd, and if $E_0 = \{e\}$, then $\mathcal{C}'_G(\{e\}) = \mathcal{C}_G(e)$ and $\ell_G(\{e\}) = \ell_G(e)$.

Let E^* be a set of edges in G. Assume that each edge e in E^* is assigned a direction $\overrightarrow{e}$ and $\overrightarrow{E^*}$ is the set of directed edges $\overrightarrow{e}$ for all $e \in E^*$. In graph G, only edges in E^* are assigned directions. For any cycle C in G, we say the directed edges in $\overrightarrow{E^*}$ are *balanced on* C if $|E(C) \cap E^*|$ is even and exactly half of the edges in $E(C) \cap E^*$ are oriented clockwise along C, and *unbalanced* otherwise. Obviously, the directed edges of $\overrightarrow{E^*}$ are balanced on C when $E(C) \cap E^* = \emptyset$, and unbalanced on C if $|E(C) \cap E^*|$ is odd. Examples of cycles on which directed edges of $\overrightarrow{E^*}$ are balanced or unbalanced are shown in Figure 2 (a) and (b), respectively, where $E(C) \cap E^* = \{e_1, e_2, e_3, e_4\}$.

We are now going to introduce the second main result in this article.

Theorem 1.5 *Let G be a connected graph and E^* be a set of edges in G. If the following conditions are satisfied, then G belongs to $DP_<$:*

(a) Balanced directed edges on C (b) Unbalanced directed edges on C

Figure 2: $E(C) \cap E^* = \{e_1, e_2, e_3, e_4\}$

(i) *$r_0 = \ell_G(E^*)$ is even; and*

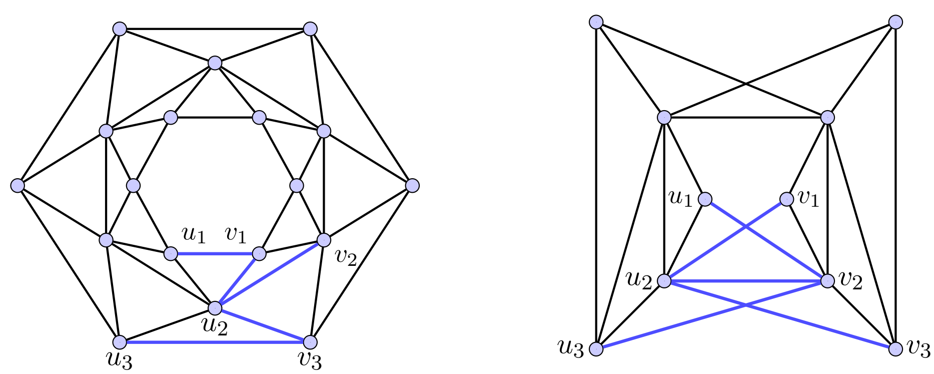
(ii) *there exists a way to assign an orientation $\overrightarrow{e}$ for each edge $e \in E^*$ such that the directed edges in $\overrightarrow{E^*} = \{\overrightarrow{e} : e \in E^*\}$ are balanced on each cycle C of G with $|E(C)| < r_0$.*

The following Corollary 1.6 of Theorem 1.5 introduces a family of graphs in $DP_<$, including the graphs determined by Theorem 1.2.

Corollary 1.6 *Let G be any graph and let $E^* \subseteq E_G(V_1, V_2)$, where V_1 and V_2 are disjoint vertex subsets of $V(G)$ with $V_1 \cup V_2 \neq V(G)$. If the following conditions are satisfied, then $G \in DP_<$:*

(i) *$r_0 = \ell_G(E^*)$ is even; and*

(ii) *for each cycle C in G such that $|E(C) \cap E^*|$ is positive, either $|E(C)| \geq r_0$ or no component of the subgraph $C - (E^* \cap E(C))$ is a (v_1, v_2)-path for some $v_1 \in V_1$ and $v_2 \in V_2$.*



(a) (b)

Figure 3: Two graphs in $DP_<$

It is easy to verify that both the graphs in Figure 3 satisfy the conditions in Corollary 1.6 by taking $V_1 = \{u_i : i = 1, 2, 3\}$, $V_2 = \{v_i : i = 1, 2, 3\}$ and $E^* = E_G(V_1, V_2)$. Note that the graph in Figure 3 (b) belongs to a family of graphs stated in the following corollary, which follows from Corollary 1.6 directly.

Corollary 1.7 *Let G be any graph and let $E^* \subseteq E_G(V_1, V_2)$, where V_1 and V_2 are disjoint vertex subsets of $V(G)$. If $\ell_G(E^*) = 4$, then $G \in DP_<$.*

We will introduce some notations and fundamental results on an m-fold cover of a graph in Section 2. We will then prove Theorem 1.3 and Corollary 1.4 in Section 3, and Theorem 1.5 and Corollary 1.6 in Section 4. Finally, in Section 5, we will apply Theorem 1.5 to determine some families of plane graphs belonging to $DP_<$.

2 Notations and preliminary facts on an m-fold cover

In this section, we introduce some notations and preliminary facts on an m-fold cover which will be applied in the proofs of Theorems 1.3 and 1.5.

Let G be a graph. By the definition of $P_{DP}(G, m)$, $P_{DP}(G, m)$ is actually equal to the minimum value of $P_{DP}(G, \mathcal{H})$'s over all those full m-fold covers $\mathcal{H} = (L, H)$ of G with $L(u) = \{(u, i) : i = 1, \cdots, m\}$ for every $u \in V(G)$. Now we assume that $\mathcal{H} = (L, H)$ is any full m-fold cover of G with $L(u) = \{(u, i) : i = 1, \cdots, m\}$ for every $u \in V(G)$.

For any edge $e = uv$ in $E(G)$, let

$$X_e(G, \mathcal{H}) = E_H(L(u), L(v)) \setminus \{(u, i)(v, i) : i = 1, \cdots, m\}, \text{ and}$$

$$Y_e(G, \mathcal{H}) = \{i \in [m] : (u, i)(v, j) \in X_e(G, \mathcal{H})\}.$$

Then $|X_e(G, \mathcal{H})| = |Y_e(G, \mathcal{H})|$, and if $(u, i)(v, j) \in X_e(G, \mathcal{H})$, j is also included in set $Y_e(G, \mathcal{H})$ as $(u, j)(v, s) \in X_e(G, \mathcal{H})$ for some $s \neq j$. We say an edge e in G is *horizontal* with respect to $\mathcal{H}$ if $X_e(G, \mathcal{H}) = \emptyset$; *sloping* otherwise. Denote the set of sloping edges in G with respect to $\mathcal{H}$ by $\mathcal{S}_G(\mathcal{H})$. For a given spanning tree T of G, it is common to further assume that each edge in T is horizontal with respect to $\mathcal{H}$ because we can rename the vertices in $L(u)$ for every vertex $u \in V(G)$ to guarantee that $E_H(L(u), L(v)) = \{(u, i)(v, i) : i = 1, \cdots, m\}$ holds whenever $uv \in E(T)$, during which the structure of graph H remains unchanged.

Let $\mathscr{S}(\mathcal{H})$ (simply $\mathscr{S}$) be the set of subsets S of $V(H)$ with $|S \cap L(v)| = 1$ for each $v \in V(G)$. Clearly, $|S| = |V(G)|$ for each $S \in \mathscr{S}$. For each $U \subseteq V(G)$, let $\mathscr{S}|_U$ be the set of subsets S of $V(H)$ such that $|S \cap L(v)| = 1$ for each $v \in U$ and $S \cap L(v) = \emptyset$ for each $v \in V(G) \setminus U$. Clearly, $|S| = |U|$ for each $S \in \mathscr{S}|_U$, and $\mathscr{S} = \mathscr{S}|_U$ when $U = V(G)$.

For any subgraph G_0 of G, let H_{G_0} be the subgraph of H with vertex set $\cup_{u \in V(G_0)} L(u)$ and edge set $\cup_{uv \in E(G_0)} E_H(L(u), L(v))$. Let $\mathcal{G}_{\mathcal{H}}(G_0)$ be the set of graphs $H_{G_0}[S]$ (i.e., the subgraph of

H_{G_0} induced by S), where $S \in \mathscr{S}|_{V(G_0)}$, such that $H_{G_0}[S] \cong G_0$. Note that $H_{G_0}[S]$ is the induced subgraph $H[S]$ whenever G_0 is an induced subgraph of G. For each $j \in [m]$, let $S_j(G_0) = \{(v, j) : v \in V(G_0)\}$ and write $H_{G_0}[S_j(G_0)]$ as $H_j[G_0]$.

For each edge $e = uv \in E(G)$, let $\mathscr{S}_e$ be the set of $S \in \mathscr{S}$ such that the two vertices in $S \cap (L(u) \cup L(v))$ are adjacent in H. For each $A \subseteq E(G)$, let $\mathscr{S}_A = \cap_{e \in A} \mathscr{S}_e$. Then, by the inclusion-exclusion principle,

$$P_{DP}(G, \mathcal{H}) = \sum_{A \subseteq E(G)} (-1)^{|A|} |\mathscr{S}_A|, \tag{2.1}$$

which generalizes a well known property of the chromatic polynomial that

$$P(G, m) = \sum_{A \subseteq E(G)} (-1)^{|A|} m^{c(A)}. \tag{2.2}$$

For any graph F, let $\mathscr{B}(F)$ be the set of bridges (i.e., cut-edges) in F, and let $\bar{\mathscr{B}}(F) = E(F) \setminus \mathscr{B}(F)$. Write $\bar{\mathscr{B}}(G\langle A \rangle)$ as $\bar{\mathscr{B}}(A)$ for any $A \subseteq E(G)$. The following properties hold, as proved in [4].

(i) For any $A \subseteq E(G)$, if $G_1, G_2, \cdots G_{c(A)}$ are the components of $G\langle A \rangle$, then

$$|\mathscr{S}_A| = \prod_{i=1}^{c(A)} |\mathcal{G}_{\mathcal{H}}(G_i)|. \tag{2.3}$$

(ii) For any connected subgraph G_0 of G, we have $|\mathcal{G}_{\mathcal{H}}(G_0)| \leq m$, where the equality holds if $\bar{\mathscr{B}}(G_0) \cap \mathcal{S}_G(\mathcal{H}) = \emptyset$ (i.e., $\bar{\mathscr{B}}(G_0)$ does not contain sloping edges with respect to $\mathcal{H}$).

(iii) By Facts (i) and (ii), for each $A \subseteq E(G)$, we have $|\mathscr{S}_A| \leq m^{c(A)}$, where the equality holds if $\bar{\mathscr{B}}(A) \cap \mathcal{S}_G(\mathcal{H}) = \emptyset$.

(iv) Let $\mathscr{E}(\mathcal{H})$ (or simply $\mathscr{E}$) be the set of subsets A of $E(G)$ such that $\bar{\mathscr{B}}(A)$ contains at least one sloping edge with respect to $\mathcal{H}$. Then Fact (iii) implies that

$$P_{DP}(G, \mathcal{H}) - P(G, m) = \sum_{A \in \mathscr{E}} (-1)^{|A|} (|\mathscr{S}_A| - m^{c(A)}). \tag{2.4}$$

(v) Fact (iii) also implies that for any $k \in [n]$,

$$\sum_{\substack{A \in \mathscr{E} \\ c(A) = k}} (-1)^{|A|} (|\mathscr{S}_A| - m^{c(A)}) \geq \sum_{\substack{A \in \mathscr{E},\ c(A) = k \\ |A| \text{ is even}}} (|\mathscr{S}_A| - m^k) \tag{2.5}$$

and

$$\sum_{\substack{A\in\mathscr{E}\\ c(A)=k}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) \le \sum_{\substack{A\in\mathscr{E},\ c(A)=k\\ |A| \text{ is odd}}} (m^k - |\mathscr{S}_A|). \tag{2.6}$$

(vi) For any $A \in \mathscr{E}$ and any sloping edge e in $\bar{\mathscr{B}}(A)$, let G_1 be the component of $G\langle A\rangle$ containing e. Then $|V(G_1)| \ge \ell_G(e)$ and $c(A) \le |V(G)| - \ell_G(e) + 1$, and $|A| = \ell_G(e)$ whenever $c(A) = |V(G)| - \ell_G(e) + 1$.

3 Proof of Theorem 1.3.

Now we give the proof of Theorem 1.3.

Proof of Theorem 1.3. We need only to prove that there exists an $M \in \mathbb{N}$, such that whenever $m \ge M$, $P_{DP}(G, \mathcal{H}) > P(G, m)$ holds for every full m-fold cover $\mathcal{H} = (L, H)$ of G with $H \not\cong H_{G,m}$.

Suppose $n = |V(G)|$. As G is DP-good, G has a spanning tree T and an edge labeling $e_1, \cdots, e_q$ of the edges in $E(G) \setminus E(T)$, such that $\ell_G(e_1) \le \cdots \le \ell_G(e_q)$ and for all $i \in [q]$, $\ell_G(e_i)$ is odd and $E(C_i) \subseteq E(T) \cup \{e_1, \cdots, e_i\}$ for some $C_i \in \mathcal{C}_G(e_i)$.

Let $\mathcal{H} = (L, H)$ be a full m-fold cover of G with $L(u) = \{(u, i) : i = 1, \cdots, m\}$ for every $u \in V(G)$ and $H \not\cong H_{G,m}$. We can further assume that $\mathcal{S}_G(\mathcal{H}) \ne \emptyset$ and all the edges in $E(T)$ are horizontal with respect to $\mathcal{H}$. Then, every sloping edge e in G with respect to $\mathcal{H}$ is of odd girth as $\mathcal{S}_G(\mathcal{H}) \subseteq E(G) \setminus E(T)$.

In the following, write $X_e(G, \mathcal{H})$ and $Y_e(G, \mathcal{H})$ simply as X_e and Y_e for any edge $e \in E(G)$. Let $r = \min\{\ell_G(e) : e \in \mathcal{S}_G(\mathcal{H})\}$, and let $E_0 = \{e_{k_1}, \cdots, e_{k_t}\}$ be the set of sloping edges in G with $\ell_G(e_{k_i}) = r$, where $k_1 < k_2 < \cdots < k_t$. Then r is odd, $r \ge 3$ and $1 \le t \le q$. Let $\mathscr{X}_r = \cup_{i=1}^t X_{e_{k_i}}$. Then $|\mathscr{X}_r| = \sum_{i=1}^t |X_{e_{k_i}}| \ge 1$.

Recall that the cycles $C_1, \cdots, C_q$ are pairwise distinct. We first prove the following three claims.

Claim 1 $\sum\limits_{i=1}^{t} |\mathcal{G}_{\mathcal{H}}(C_{k_i})| \le mt - \left|\bigcup\limits_{i=1}^{t} Y_{e_{k_i}}\right|$, *i.e.,* $\sum\limits_{i=1}^{t}(m - |\mathcal{G}_{\mathcal{H}}(C_{k_i})|) \ge \left|\bigcup\limits_{i=1}^{t} Y_{e_{k_i}}\right|$.

Proof. It suffices to prove the two facts below:

(i) $|\mathcal{G}_{\mathcal{H}}(C_{k_1})| = m - |Y_{e_{k_1}}|$; and

(ii) for any integer $p \in [t-1]$, $|\mathcal{G}_{\mathcal{H}}(C_{k_{p+1}})| \le m - |Y_{e_{k_{p+1}}} \setminus (\cup_{i=1}^p Y_{e_{k_i}})|$.

Since $E(C_{k_1}) \subseteq E(T) \cup \{e_1, \cdots, e_{k_1}\}$ and $\ell_G(e_1) \le \cdots \le \ell_G(e_{k_1}) = r$, C_{k_1} contains exactly one sloping edge e_{k_1}. Thus for any $j \in [m]$, $H_j[C_{k_1} - \{e_{k_1}\}] \cong C_{k_1} - \{e_{k_1}\}$, and $H_j[C_{k_1}] \cong C_{k_1}$ if and only if $j \notin Y_{e_{k_1}}$. Hence Fact (i) holds.

Similarly, for $p \in [t-1]$, all the edges in $E(C_{k_{p+1}}) \setminus \{e_{k_1}, e_{k_2}, \cdots e_{k_{p+1}}\}$ are horizontal as $E(C_{k_{p+1}}) \subseteq E(T) \cup \{e_1, e_2, \cdots, e_{k_{p+1}}\}$ and $\ell_G(e_1) \leq \cdots \leq \ell_G(e_{k_{p+1}}) = r$. Let $j \in Y_{e_{k_{p+1}}} \setminus (\cup_{i=1}^{p} Y_{e_{k_i}})$. Then, $H_j[C_{k_{p+1}} - \{e_{k_{p+1}}\}] \cong C_{k_{p+1}} - \{e_{k_{p+1}}\}$ but $H_j[C_{k_{p+1}}] \not\cong C_{k_{p+1}}$. Hence Fact (ii) holds and Claim 1 follows. $\natural$

Claim 2 *The following inequality holds:*

$$\sum_{i=1}^{t} \left(m^{n-r+1} - |\mathscr{S}_{E(C_{k_i})}| \right) \geq \frac{|\mathscr{X}_r|}{q} m^{n-r}. \tag{3.1}$$

<ins>Proof</ins>. Since $|Y_e| = |X_e|$ for every edge $e \in E(G)$, we have

$$|\bigcup_{i=1}^{t} Y_{e_{k_i}}| \geq \max_{i \in [t]} |X_{e_{k_i}}| \geq \frac{1}{t} \sum_{i=1}^{t} |X_{e_{k_i}}| = \frac{1}{t} |\mathscr{X}_r| \geq \frac{1}{q} |\mathscr{X}_r|. \tag{3.2}$$

Then, by (2.3) and Claim 1,

$$\begin{aligned} \sum_{i=1}^{t} (m^{n-r+1} - |\mathscr{S}_{E(C_{k_i})}|) &= \sum_{i=1}^{t} (m^{n-r+1} - m^{n-r} |\mathcal{G}_{\mathcal{H}}(C_{k_i})|) \\ &\geq |\bigcup_{i=1}^{t} Y_{e_{k_i}}| m^{n-r} \\ &\geq \frac{|\mathscr{X}_r|}{q} m^{n-r}. \end{aligned} \tag{3.3}$$

$\natural$

Claim 3 *The following inequality holds:*

$$\sum_{\substack{A \in \mathscr{E} \\ c(A) = n-r+1}} (-1)^{|A|} (|\mathscr{S}_A| - m^{c(A)}) \geq \frac{|\mathscr{X}_r|}{q} m^{n-r}. \tag{3.4}$$

<ins>Proof</ins>. Recall that for any $A \in \mathscr{E}$, $\bar{\mathscr{B}}(A)$ contains a sloping edge e, where $\ell_G(e) \geq r$. Thus, by (vi) in Section 2, $\ell_G(e) = r = |A|$ holds whenever $c(A) = n - r + 1$. Therefore,

$$\begin{aligned} \sum_{\substack{A \in \mathscr{E} \\ c(A) = n-r+1}} (-1)^{|A|} (|\mathscr{S}_A| - m^{c(A)}) &= \sum_{\substack{A \in \mathscr{E},\ |A| = r \\ c(A) = n-r+1}} (-1)^{r} (|\mathscr{S}_A| - m^{c(A)}) \\ &= \sum_{\substack{A \in \mathscr{E},\ |A| = r \\ c(A) = n-r+1}} (m^{c(A)} - |\mathscr{S}_A|), \end{aligned} \tag{3.5}$$

where the last equality holds as r is odd.

By (iii) in Section 2, $m^{c(A)} \geq |\mathscr{S}_A|$ for any $A \subseteq E(G)$, hence

$$\begin{aligned}\sum_{\substack{A\in\mathscr{E}\\ c(A)=n-r+1}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) &\geq \sum_{i=1}^{t}(m^{n-r+1} - |\mathscr{S}_{E(C_{k_i})}|)\\ &\geq \frac{|\mathscr{X}_r|}{q} m^{n-r}, \end{aligned} \tag{3.6}$$

where the last inequality follows from Claim 2. ♮

The rest of the proof is basically the same as in the proof of Theorem 1.1 that are given in [4]. For completeness, we restate the proofs of Claims 4-7 here with slight changes.

Claim 4 *For any subgraph G_0 of G, if $\ell_G(e) \leq r$ for each sloping edge e in G_0, then $|\mathcal{G}_{\mathcal{H}}(G_0)| \geq m - |\mathscr{X}_r|$.*

Proof. Since $\ell_G(e) \leq r$ for each sloping edge e in G_0, each sloping edge in G_0 belongs to $E_0 = \{e_{k_1}, \cdots, e_{k_t}\}$. Thus, for every $j \in [m] \setminus (\cup_{i=1}^{t} Y_{e_{k_i}})$, $H_j[G_0] \cong G_0$ holds, implying that

$$|\mathcal{G}_{\mathcal{H}}(G_0)| \geq m - \left|\bigcup_{i=1}^{t} Y_{e_{k_i}}\right| \geq m - \sum_{i=1}^{t}|Y_{e_{k_i}}| = m - \sum_{i=1}^{t}|X_{e_{k_i}}| = m - |\mathscr{X}_r|. \tag{3.7}$$

Hence Claim 4 holds. ♮

Claim 5 *For any $A \in \mathscr{E}$ with $c(A) = n - r$, we have $|\mathscr{S}_A| \geq (m - |\mathscr{X}_r|)m^{n-r-1}$.*

Proof. Since $A \in \mathscr{E}$, $\bar{\mathscr{B}}(A)$ contains a sloping edge e with $\ell_G(e) \geq r$. Thus, by (vi) in Section 2, $G\langle A\rangle$ has a component G_0 with $e \in E(G_0)$ and $|V(G_0)| \geq r$. Moreover, as $c(A) = n - r$, $|V(G_0)| \leq r + 1$ holds, and for any other component G' of $G\langle A\rangle$, G' is either an isolated vertex or an edge, and thus $|\mathcal{G}_{\mathcal{H}}(G')| = m$. Hence by (2.3), it suffices to prove that $|\mathcal{G}_{\mathcal{H}}(G_0)| \geq m - |\mathscr{X}_r|$.

If G_0 is 2-connected, then for every edge $e \in E(G_0)$, $\ell_G(e) \leq |V(G_0)| \leq r + 1$. Moreover, for each sloping edge e in G_0, $\ell_G(e) \leq r$ as $r + 1$ is even and $\ell_G(e)$ is odd. Hence $|\mathcal{G}_{\mathcal{H}}(G_0)| \geq m - |\mathscr{X}_r|$ holds by Claim 4.

Otherwise, G_0 has exaxctly two blocks as G_0 contains a cycle C with $|V(C)| \geq r$. Then, it is clear that the two blocks of G_0 are $G_0[V(C)]$ and an edge f, where $|V(C)| = r$ and $|\mathcal{G}_{\mathcal{H}}(G_0[f])| = m$. As $G_0[V(C)]$ is 2-connected, for every edge $e \in E(G_0[V(C)])$, $\ell_G(e) \leq |V(C)| \leq r$ holds, and thus $|\mathcal{G}_{\mathcal{H}}(G_0[V(C)])| \geq m - |\mathscr{X}_r|$ follows from Claim 4. Consequently, $|\mathcal{G}_{\mathcal{H}}(G_0)| \geq m - |\mathscr{X}_r|$ and Claim 5 holds. ♮

For any $s \in \mathbb{N}$ with $s \leq n - r$, let ϕ_s be the number of subsets $A \subseteq E(G)$ such that $c(A) = s$, $G\langle A\rangle$ is not a forest and $|A|$ is even.

Claim 6 *The following inequality holds:*

$$\sum_{\substack{A\in\mathscr{E}\\ c(A)=n-r}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) \geq -\phi_{n-r}|\mathscr{X}_r|m^{n-r-1}. \tag{3.8}$$

Proof. By (2.5) and Claim 5,

$$\begin{aligned}\sum_{\substack{A\in\mathscr{E}\\ c(A)=n-r}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) &\geq \sum_{\substack{A\in\mathscr{E},\ c(A)=n-r\\ |A| \text{ is even}}} (|\mathscr{S}_A| - m^{n-r}) \\ &\geq \sum_{\substack{A\in\mathscr{E},\ c(A)=n-r\\ |A| \text{ is even}}} (-|\mathscr{X}_r|m^{n-r-1}) \\ &\geq -\phi_{n-r}|\mathscr{X}_r|m^{n-r-1}. \end{aligned} \tag{3.9}$$

♮

Claim 7 *For each $s \in [n-r-1]$, we have*

$$\sum_{\substack{A\in\mathscr{E}\\ c(A)=s}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) \geq -\phi_s m^s. \tag{3.10}$$

Proof. By (2.5),

$$\begin{aligned}\sum_{\substack{A\in\mathscr{E}\\ c(A)=s}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) &\geq \sum_{\substack{A\in\mathscr{E},\ c(A)=s\\ |A| \text{ is even}}} (|\mathscr{S}_A| - m^{s}) \\ &\geq \sum_{\substack{A\in\mathscr{E},\ c(A)=s\\ |A| \text{ is even}}} (-m^{s}) \\ &\geq -\phi_s m^s. \end{aligned} \tag{3.11}$$

♮

Now we are going to prove the main result by recalling (2.4) that

$$P_{DP}(G,\mathcal{H}) - P(G,m) = \sum_{A\in\mathscr{E}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}).$$

By (vi) in Section 2 and Claims 3, 6, 7, we have

$$P_{DP}(G,\mathcal{H}) - P(G,m) \;=\; \sum_{s=1}^{n-r+1} \sum_{\substack{A\in\mathscr{E}\\ c(A)=s}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)})$$

$$
\begin{aligned}
&\geq \frac{|\mathscr{X}_r|}{q}m^{n-r} - \phi_{n-r}|\mathscr{X}_r|m^{n-r-1} - \sum_{s=1}^{n-r-1}\phi_s m^s \\
&\geq \frac{1}{q}m^{n-r} - \phi_{n-r}m^{n-r-1} - \sum_{s=1}^{n-r-1}\phi_s m^s, \qquad (3.12)
\end{aligned}
$$

where the last inequality holds when $m \geq q\phi_{n-r}$. As $q, \phi_1, \cdots, \phi_{n-r}$ are independent of the value of m, there exists $M_r \in \mathbb{N}$, such that $P_{DP}(G, \mathcal{H}) - P(G, m) > 0$ for all $m \geq M_r$. Let $M = \max\{M_r : 3 \leq r \leq n, r \text{ is odd}\}$. Then the result is proven. □

The proof of Corollary 1.4 is given below.

Proof of Corollary 1.4. For $i = 0, 1, \cdots, n$, let $V_i = \{v_j : 0 \leq j \leq i\}$ and $G_i = G[V_i]$. Obviously, $G_n = G$.

By Theorem 1.3, it suffices to show that for any $i \geq 1$, G_i has a spanning tree T_i and the edges in $E(G_i) \setminus E(T_i)$ can be labeled as $e_1, e_2, \cdots, e_{s_i}$ such that for all $j = 1, 2, \cdots, s_i$, $|V(C_j)| = 3$ and $E(C_j) \subseteq E(T_i) \cup \{e_t : 1 \leq t \leq j\}$ hold for some $C_j \in \mathcal{C}_G(e_j)$.

The above conclusion is obvious for $i = 1$, as $G_1 \cong K_2$ by the given conditions. Now assume that the above conclusion holds for $1 \leq i < n$.

Since $G[V_i \cap N(v_{i+1})]$ is connected, the vertices in $V_i \cap N(v_{i+1})$ can be labeled as $v_{q_0}, v_{q_1}, \cdots, v_{q_l}$, where $l = |N(v_{i+1}) \cap V_i| - 1$, such that for any $1 \leq j \leq l$, $N(v_{q_j}) \cap \{v_{q_0}, \cdots, v_{q_{j-1}}\} \neq \emptyset$. Now, let T_{i+1} be the spanning tree of G_{i+1} obtained from T_i by adding edge $v_{i+1}v_{q_0}$. Let $s_{i+1} = s_i + l$, and for any $1 \leq j \leq l$, let e_{s_i+j} denote the edge $v_{i+1}v_{q_j}$. Then, it is obvious that for any $1 \leq j \leq l$, e_{s_i+j} is contained in a cycle C_{s_i+j} of length 3 such that $E(C_{s_i+j}) \subseteq E(T_{i+1}) \cup \{e_t : 1 \leq t \leq s_i + j\}$.

Hence the above conclusion holds for $i + 1$. Therefore G_n is DP-good. □

By Corollary 1.4, chordal graphs, complete k-partite graphs, where $k \geq 3$, and plane near-triangulations are DP-good.

4 Proof of Theorem 1.5

We shall prove Theorem 1.5 in this section.

Proof of Theorem 1.5. Assume $|V(G)| = n$ and $E^* = \{e_1, \cdots, e_k\}$, where $k \geq 1$.

If $k = 1$, then $\ell_G(e_1)$ is even, and the result follows from Theorem 1.2 directly.

In the following, we assume that $k \geq 2$. For each $i \in [k]$, let e_i be the edge u_iv_i for $u_i, v_i \in V(G)$, and let $\overrightarrow{e_i}$ be the directed edge (u_i, v_i) with tail u_i. By condition (ii) in Theorem 1.5, the directed edges in $\overrightarrow{E^*} = \{\overrightarrow{e_i} : i \in [k]\}$ are balanced on every cycle C in G with $E(C) < r_0$.

For any positive integer m, let $\mathcal{H} = (L, H)$ be the m-fold cover of G defined below:

- $L(x) = \{(x, i) : i = 1, \cdots, m\}$ for all $x \in V(G)$;
- $E_H(L(x), L(y)) = \{(x, i)(y, i) : i = 1, \cdots, m\}$ for every edge $xy \in E(G) \setminus E^*$; and
- $E_H(L(u_i), L(v_i)) = \{(u_i, q)(v_i, q+1) : q = 1, \cdots, m-1\} \cup \{(u_i, m)(v_i, 1)\}$ for every edge $e_i = u_i v_i \in E^*$.

Clearly, $\mathcal{S}_G(\mathcal{H}) = E^*$ (i.e., only edges in E^* are sloping in G with respect to $\mathcal{H}$).

An *induced* cycle of G is a cycle in G which is induced by some subset of $V(G)$. We first analyze the structure of connected subgraphs G_0 of G with $|V(G_0)| \le r_0$ by several claims.

Claim 1 *Let C be a cycle in G. If $|V(C)| \le r_0$ and $|E(C) \cap E^*|$ is odd, then C is an induced cycle of G with $|V(C)| = r_0$.*

Proof. By Condition (i) in Theorem 1.5, $|V(C)| = r_0$ trivially holds.

Suppose that there exists $e \in E(G) \setminus E(C)$ such that e joins two vertices in $V(C)$. Then, G contains a cycle C' such that $V(C') \subseteq V(C)$, $|V(C')| < |V(C)| = r_0$ and $|E(C') \cap E^*|$ is odd, a contradiction to the definition of r_0.

Hence $G[V(C)] = C$ and Claim 1 holds. $\natural$

Claim 2 *Let $G_0 = (V_0, E_0)$ be a 2-connected subgraph of G. If $|V_0| \le r_0$ and $|E^* \cap E_0| = 1$, then $|V_0| = r_0$ and G_0 is an induced cycle of G.*

Proof. Since G_0 is 2-connected, G_0 contains a cycle C with $|E(C) \cap E^*| = 1$, where $|V(C)| \le |V_0| \le r_0$. By Claim 1, C is an induced cycle of G with $|V(C)| = r_0$, which implies that $|V_0| = r_0$, $V_0 = V(C)$ and G_0 is C. Hence Claim 2 holds. $\natural$

Claim 3 *Let $G_0 = (V_0, E_0)$ be a connected subgraph of G with $|V_0| \le r_0$. If $|E^* \cap E_0| \ge 2$, then $G_0 - (E^* \cap E_0)$ is disconnected.*

Proof. Suppose that $G_0 - (E^* \cap E_0)$ is connected. Let $e', e'' \in E^* \cap E_0$, and let P be a path in $G_0 - (E^* \cap E_0)$ connecting the two end-vertices of e'. Consequently, the edge set $E(P) \cup \{e'\}$ forms a cycle C in G_0 with $|V(C)| \le r_0$ and $E(C) \cap E^* = \{e'\}$. By Claim 1, C is an induced cycle with $|V(C)| = r_0$, implying that G_0 is C, a contradiction to the fact that $e'' \in E_0$. Hence Claim 3 holds. $\natural$

Claim 4 *Let $G_0 = (V_0, E_0)$ be a connected subgraph of G with $|V_0| \le r_0$. If $|E^* \cap E_0| \ge 2$, then no edge e in $E^* \cap E_0$ joins two vertices in any component G' of $G_0 - (E^* \cap E_0)$ (i.e., each component G' of $G_0 - (E^* \cap E_0)$ is an induced subgraph of G_0).*

Proof. Assume that G' is a component of $G_0 - (E^* \cap E_0)$ and e is an edge in $E^* \cap E_0$ which joins two vertices in G'.

Then, $G' + e$ has a block, say G_1, which contains e. By Claim 3, $|V(G')| < |V(G_0)|$. Thus, $|V(G_1)| \le |V(G')| < |V_0| \le r_0$. But, as $|E(G_1) \cap E^*| = 1$, Claim 2 implies that $|V(G_1)| = r_0$, a contradiction. $\natural$

Claim 5 *Let $G_0 = (V_0, E_0)$ be a connected subgraph of G with $|V_0| \le r_0$. Assume that $\{U_1, U_2\}$ is a partition of V_0 such that $E_0 \cap E^* = E_{G_0}(U_1, U_2)$ and $G_0[U_i]$ is connected for both $i = 1, 2$. If G_0 is not a cycle of length r_0, then for all the edges $e_{i_1}, \cdots, e_{i_t}$ in $E_{G_0}(U_1, U_2)$, the vertices $u_{i_1}, \cdots, u_{i_t}$ must be in the same set U_s for some $s \in \{1, 2\}$.*

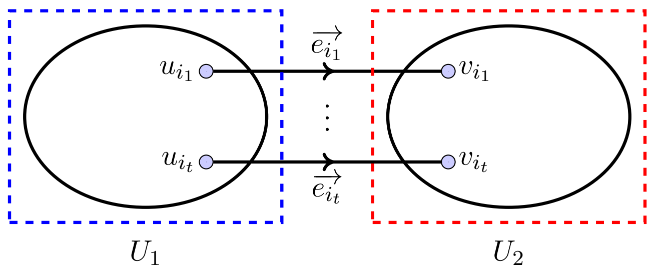


Figure 4: Graph G_0 with $E_{G_0}(U_1, U_2) = \{e_{i_1}, \cdots, e_{i_t}\}$ and $u_{i_1}, \cdots, u_{i_t} \in U_1$

Proof. Let $E' = E_{G_0}(U_1, U_2)$. If $|E'| = 1$, then the result trivially holds.

In the following, assume that $|E'| \ge 2$. We need only to prove the two facts below on any two edges e_{i_p}, e_{i_q} in E':

(i) if there is a cycle C in G_0 shorter than r_0 with $|E(C) \cap E^*| = \{e_{i_p}, e_{i_q}\}$, then u_{i_p} and u_{i_q} are contained in the same set U_s for some $s \in \{1, 2\}$;

(ii) otherwise, there exists $e_{i_j} \in E' \setminus \{e_{i_p}, e_{i_q}\}$, such that there is a cycle C_1 in G_0 shorter than r_0 with $E(C_1) \cap E^* = \{e_{i_p}, e_{i_j}\}$ and a cycle C_2 in G_0 shorter than r_0 with $E(C_2) \cap E^* = \{e_{i_j}, e_{i_q}\}$.

Since $G_0[U_i]$ is connected for both $i = 1, 2$, G_0 has a cycle C with $|E(C) \cap E^*| = \{e_{i_p}, e_{i_q}\}$. If $|V(C)| < r_0$, Condition (ii) in Theorem 1.5 indicates that $\overrightarrow{e_{i_p}} = (u_{i_p}, v_{i_p})$ and $\overrightarrow{e_{i_q}} = (u_{i_q}, v_{i_q})$ are balanced on C, implying that u_{i_p} and u_{i_q} must be in the same set U_s for some $s \in \{1, 2\}$. Fact (i) holds.

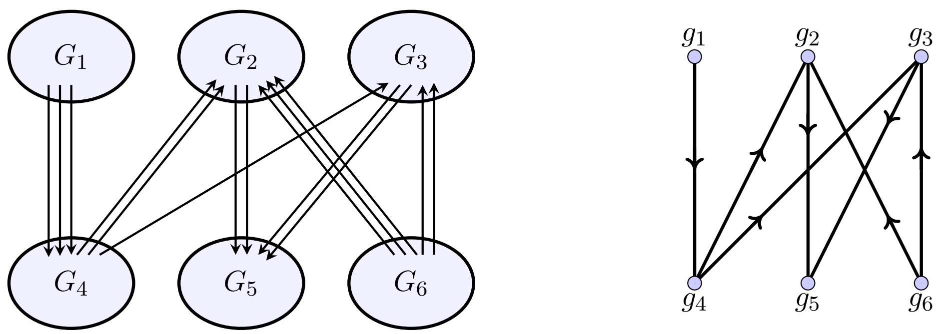
Now suppose that G_0 does not have a cycle C shorter than r_0 with $|E(C) \cap E^*| = \{e_{i_p}, e_{i_q}\}$. Thus, $|V(C)| = r_0$, implying that $V(C) = V(G_0)$. As G_0 is not a cycle of length r_0, there is an edge $e \in E_0 \setminus E(C)$. Obviously, $e \notin E(G_0[U_1]) \cup E(G_0[U_2])$. Otherwise, G_0 has a cycle C' shorter than r_0 with $|E(C') \cap E^*| = \{e_{i_p}, e_{i_q}\}$, a contradiction. Thus, $e \in E' = E_{G_0}(U_1, U_2)$. Assume that

$e = e_{i_j} \in E' \setminus \{e_{i_p}, e_{i_q}\}$. Then, there are cycles C_1 and C_2 in $C+e$ such that $|E(C_1) \cap E^*| = \{e_{i_p}, e_{i_j}\}$ and $|E(C_2) \cap E^*| = \{e_{i_j}, e_{i_q}\}$. Note that both C_1 and C_2 are shorter than r_0. Fact (ii) holds and Claim 5 follows. $\natural$

Claim 6 *Let $G_0 = (V_0, E_0)$ be a connected subgraph of G with $|V_0| \le r_0$ and $|E_0 \cap E^*| \ge 2$. If G_0 is not a cycle of length r_0, then $|\mathcal{G}_{\mathcal{H}}(G_0)| = m$.*

Proof. By Claim 3, we can assume that $G_0 - (E_0 \cap E^*)$ has s (≥ 2) components $G_1, G_2, \cdots, G_s$, where $G_i = (V_i, E_i)$ for $i = 1, 2, \cdots, s$. Then, Claim 4 implies that each G_i is an induced subgraph of G_0, i.e., $E_0 \cap E^* = \bigcup_{1 \le i < j \le s} E_{G_0}(V_i, V_j)$.

Let G' be the graph with vertex set $V(G') = \{g_1, \cdots, g_s\}$ in which $g_i g_j$ is an edge if and only if $E_{G_0}(V_i, V_j) \ne \emptyset$. Let $\overrightarrow{G'}$ be the digraph obtained from G' by converting each edge $g_i g_j$ in G' into a directed edge whose tail is g_i if and only if $u_q \in V_i$ for some edge $e_q = u_q v_q$ in $E_{G_0}(V_i, V_j)$. Note that the orientation of directed edges in $\overrightarrow{G'}$ is well-defined due to the result in Claim 5. An example is shown in Figure 5 (b).



(a) G_0 (b) $\overrightarrow{G'}$

Figure 5: An example of $\overrightarrow{G'}$

As G_0 is connected, G' is also connected. Let T be a spanning tree of G' with as many leaves as possible. Thus, T has at least $\Delta(G')$ leaves, where $\Delta(G')$ is the maximum degree of G'. For each vertex g_i in G', there is a unique path, denoted by P_i, in T from g_1 to g_i. Denote by $\varphi_1(i)$ the number of those edges in P_i whose corresponding directed edges in $\overrightarrow{G'}$ are along the direction of path P_i from g_1 to g_i, and denote by $\varphi_2(i)$ the number of the remaining edges in P_i. Thus, $\varphi_1(i) + \varphi_2(i) = |E(P_i)|$.

Now, let $\varphi(i) = \varphi_1(i) - \varphi_2(i)$ for each $i \in [s]$. For the digraph $\overrightarrow{G'}$ in Figure 5 (b), if T is the spanning tree of G' with its edge set $\{g_1 g_4, g_2 g_4, g_3 g_4, g_2 g_5, g_2 g_6\}$, then

$$\varphi(1) = 0, \varphi(2) = \varphi(3) = 2, \varphi(4) = \varphi(6) = 1, \varphi(5) = 3.$$

as given in Figure 6.

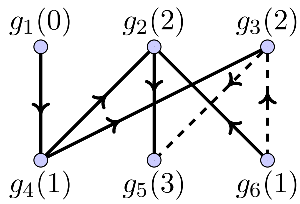


Figure 6: With a spanning tree T consisting of dense edges, the value of $\varphi(i)$ for each $i = 1, 2, \cdots, 6$ is shown beside its veterx g_i

We will complete the proof of this claim by showing the following subclaims.

Subclaim 6.1. For any edge $g_ig_j \in E(T)$, $\varphi(j) = \varphi(i) + 1$ whenever (g_i, g_j) is the corresponding directed edge of g_ig_j in $\overrightarrow{G'}$.

Assume that (g_i, g_j) is the corresponding directed edge of g_ig_j in $\overrightarrow{G'}$. As $g_ig_j \in E(T)$, either g_i is on the path P_j, or g_j is on the path P_i. If g_i is on the path P_j, then $\varphi_1(j) = \varphi_1(i) + 1$ and $\varphi_2(j) = \varphi_2(i)$. If g_j is on the path P_i, then $\varphi_1(j) = \varphi_1(i)$ and $\varphi_2(j) = \varphi_2(i) - 1$. Thus, Subclaim 6.1 follows in both cases.

For every $q \in [m]$, let S_q be the set in $\mathscr{S}|_{V_0}$ defined as follows:

$$S_q = \bigcup_{i=1}^{s} \{(v, (q + \varphi(i))(\text{mod } m)) : v \in V_i\},$$

where $(v, 0) = (v, m)$ for all $v \in V_0$. Obviously, $\{S_1, \cdots, S_m\}$ is a partition of $V(H_{G_0})$.

Subclaim 6.2. If $\varphi(j) = \varphi(i) + 1$ holds for each directed edge (g_i, g_j) in $\overrightarrow{G'}$, then $H_{G_0}[S_q] \cong G_0$ for all $q \in [m]$, and hence Claim 6 holds.

Let ϕ be the bijection from V_0 to S_q defined below: for any $v \in V_0 = \cup_{1 \le i \le s} V_i$,

$$\phi(v) = (v, (q + \varphi(i))(\text{mod } m)), \qquad \text{if } v \in V_i.$$

To show that $H_{G_0}[S_q] \cong G_0$, it suffices to prove that for each edge $uv \in E(G_0)$, $\phi(u)$ and $\phi(v)$ are adjacent in H.

For any $uv \in E_i$, where $1 \le i \le s$, we have $uv \in E_0 \setminus E^*$, implying that $(u, (q + \varphi(i))(\text{mod } m))$ and $(v, (q + \varphi(i))(\text{mod } m))$ are adjacent in H by the definition of H. Now take any edge $uv \in E_{G_0}(V_i, V_j) \subseteq E^*$, where $1 \le i, j \le s$. Without loss of generality, assume that (g_i, g_j) is an directed edge in $\overrightarrow{G'}$. Then $\varphi(j) = \varphi(i) + 1$ by the given condition in the subclaim. By the definition of H, $(u, (q + \varphi(i))(\text{mod } m))$ and $(v, (q + \varphi(j))(\text{mod } m))$ are adjacent in H.

Hence $H_{G_0}[S_q] \cong G_0$ for each $q \in [m]$, and the subclaim holds.

Subclaim 6.3. $\varphi(j) = \varphi(i) + 1$ holds for each directed edge (g_i, g_j) in $\overrightarrow{G'}$.

Suppose that $\varphi(j) \neq \varphi(i)+1$ for some directed edge (g_i, g_j) in $\overrightarrow{G'}$. By Subclaim 6.1, $g_ig_j \in E(G') \setminus E(T)$. Let C' be the fundamental cycle of edge g_ig_j in G' with respect to spanning tree T. Assume that $g_{j_1}, g_{j_2}, \cdots, g_{j_t}$ are the consecutive vertices on C', where $t \geq 3$, $j_1 = i$ and $j_t = j$.

As G_i is connected for all $i = 1, 2, \cdots, s$, we can choose a shortest cycle C in G_0 such that

$$E(C) \cap E^* \subseteq \bigcup_{q=1}^{t} E_{G_0}(V_{j_q}, V_{j_{q+1}}), \quad |E(C) \cap E_{G_0}(V_{j_q}, V_{j_{q+1}})| = 1, \quad \forall q \in [t],$$

where $V_{j_{t+1}} = V_{j_1}$. Thus, $|E(C) \cap E^*| = t$. Clearly, t is an even integer; otherwise, Claim 1 implies that G_0 is a cycle of length r_0, a contradiction.

Suppose that $|V(C)| < r_0$. By Condition (ii) in Theorem 1.5, the directed edges of $\overrightarrow{E^*}$ are balanced on C, implying that the directed edges in $\overrightarrow{G'}$ are balanced on C'. By counting the number of edges in C' which are oriented clockwise and counterclockwise along C separately, we have $\varphi_1(j_1) + \varphi_2(j_t) + 1 = \varphi_1(j_t) + \varphi_2(j_1)$, implying that $\varphi(j_1) + 1 = \varphi(j_t)$ (i.e., $\varphi(i) + 1 = \varphi(j)$), a contradiction.

Thus, $|V(C)| = r_0$, and so $V(C) = V_0$. Therefore, $t = s$ and T is a path in G' with $|V(T)| = |V(G')| = s$. Moreover, due to the choice of C, for each $q \in [s]$, $E_q \subseteq E(C)$ and $|E_{G_0}(V_{j_q}, V_{j_{q+1}})| = 1$, implying that $E_{G_0}(V_{j_q}, V_{j_{q+1}}) \subseteq E(C)$.

If G' is a cycle, then G' is C'. The above conclusion implies that $E_0 = E(C)$, and thus G_0 is a cycle of length r_0, a contradiction. Thus, G' is not a cycle, implying that G' has a spanning tree with at least three leaves, a contradiction to the choice of T.

Hence Subclaim 6.3 holds.

By Subclaims 6.2 and 6.3, $|\mathcal{G}_{\mathcal{H}}(G_0)| = m$ and Claim 6 holds. $\natural$

Claim 7 *For any $A \in \mathscr{E}$, if either $c(A) > n - r_0 + 1$ or $c(A) = n - r_0 + 1$ and $|A| \neq r_0$, then $|\mathscr{S}_A| = m^{c(A)}$ holds.*

Proof. As $A \in \mathscr{E}$, $E^* \cap \bar{\mathscr{B}}(A) \neq \emptyset$. Then, by (i) and (ii) in Section 2, it suffices to prove that for every block $G_0 = (V_0, E_0)$ of $G\langle A \rangle$ with $E_0 \cap E^* \neq \emptyset$, $|\mathcal{G}_{\mathcal{H}}(G_0)| = m$ holds.

Suppose $G_0 = (V_0, E_0)$ is a block of $G\langle A \rangle$ with $E_0 \cap E^* \neq \emptyset$ and $|\mathcal{G}_{\mathcal{H}}(G_0)| < m$. As $c(A) \geq n - r_0 + 1$, $|V_0| \leq r_0$. Then by Claims 2 and 6, $|V_0| = |E_0| = r_0$, implying that either $c(A) < n - r_0 + 1$ or $c(A) = n - r_0 + 1$ and $|A| = r_0$, a contradiction. Hence Claim 7 holds. $\natural$

Claim 8 *If $m > k$, then $\mathcal{G}_{\mathcal{H}}(C) = \emptyset$ for any cycle C in G such that $|E(C) \cap E^*|$ is odd.*

Proof. Assume that $|E(C) \cap E^*| = 2s + 1$ for some integer $s \geq 0$ and $z_1, z_2, \cdots, z_q$ are consecutive vertices in C, where $q \geq 3$. Suppose that $\mathcal{G}_{\mathcal{H}}(C) \neq \emptyset$. Then, there exists a cycle C' in H with consecutive vertices $(z_1, h_1), (z_2, h_2), \cdots, (z_q, h_q)$. By the definition of $\mathcal{H} = (L, H)$, $h_{i+1} - h_i \neq 0$ if and only if $z_iz_{i+1} \in E^*$, and $h_{i+1} - h_i \in \{0, 1, -1, m-1, 1-m\}$ for all $i \in [q]$, where $h_{q+1} = h_1$ and $z_{q+1} = z_1$. Thus, $h_{i+1} - h_i \neq 0$ holds for exactly $2s+1$ integers i's in $[q]$.

Assume that there are exactly t integers i's in $[q]$ such that $h_{i+1} - h_i \in \{m-1, 1-m\}$. Then, there are exactly $(2s+1-t)$ integers i's in $[q]$ such that $h_{i+1} - h_i \in \{1, -1\}$. It follows that

$$0 = \sum_{i=1}^{q}(h_{i+1} - h_i) = t'(m-1) + s' \times 1, \tag{4.1}$$

where t' and s' are some integers with $|t'| \leq t$ and $|s'| \leq 2s+1-t$ such that both $t - t'$ and $(2s+1-t) - s'$ are even.

Suppose that $t' \neq 0$. Without loss of generality, assume that $t' \geq 1$. Then $s' \geq -(2s+1-1) = -2s$, and (4.1) implies that

$$0 = t'(m-1) + s' \geq (m-1) - 2s \geq (m-1) - (k-1) \geq m - k > 0, \tag{4.2}$$

a contradiction. Hence $t' = 0$, implying that t is even. As $s' - (2s+1-t)$ is even, by (4.1),

$$0 = s' = (s' - (2s+1-t)) + (2s - t + 1) \equiv 1 \pmod 2, \tag{4.3}$$

a contradiction. Thus, Claim 8 holds. $\natural$

Claim 9 *The following inequality holds when $m > k$:*

$$\sum_{\substack{A \in \mathscr{E} \\ c(A) = n - r_0 + 1}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) \leq -m^{n - r_0 + 1}. \tag{4.4}$$

Proof. Let C_0 be any cycle in $\mathcal{C}'_G(E^*)$. By Claim 8, $|\mathscr{S}_{E(C_0)}| = |\mathcal{G}_{\mathcal{H}}(C_0)| = 0$ holds.

Obviously, $E(C_0)$ is a member in $\mathscr{E}$ with $|E(C_0)| = r_0$ and $c(E(C_0)) = n - r_0 + 1$. Then, due to Claim 7, the fact that r_0 is even, and (iii) in Section 2, we have

$$\begin{aligned}
\sum_{\substack{A \in \mathscr{E} \\ c(A) = n - r_0 + 1}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) &= \sum_{\substack{A \in \mathscr{E},\ |A| = r_0 \\ c(A) = n - r_0 + 1}} (-1)^{r_0}(|\mathscr{S}_A| - m^{c(A)}) \\
&= \sum_{\substack{A \in \mathscr{E},\ |A| = r_0 \\ c(A) = n - r_0 + 1}} (|\mathscr{S}_A| - m^{c(A)}) \\
&\leq |\mathscr{S}_{E(C_0)}| - m^{n - r_0 + 1} \\
&= -m^{n - r_0 + 1}.
\end{aligned} \tag{4.5}$$

$\natural$

For any $s \in \mathbb{N}$ with $s \le n - r_0$, let ϕ_s be the number of subsets $A \subseteq E(G)$ such that $c(A) = s$, $G\langle A\rangle$ is not a forest and $|A|$ is odd.

Claim 10 *For each $s \in [n - r_0]$, the following inequality holds:*

$$\sum_{\substack{A\in\mathscr{E}\\ c(A)=s}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) \le \phi_s m^s. \tag{4.6}$$

Proof. By (2.6),

$$\begin{aligned}
\sum_{\substack{A\in\mathscr{E}\\ c(A)=s}} (-1)^{|A|}(|\mathscr{S}_A| - m^{c(A)}) &\le \sum_{\substack{A\in\mathscr{E},\ c(A)=s\\ |A| \text{ is odd}}} (m^s - |\mathscr{S}_A|) \\
&\le \sum_{\substack{A\in\mathscr{E},\ c(A)=s\\ |A| \text{ is odd}}} m^s \\
&\le \phi_s m^s.
\end{aligned} \tag{4.7}$$

$\natural$

Now, by (2.4) and Claims 7, 9 and 10, we have

$$\begin{aligned}
P_{DP}(G,\mathcal{H}) - P(G,m) &= \sum_{s=1}^{n-r_0+1} \sum_{\substack{A\in\mathscr{E}\\ c(A)=s}} (-1)^{|A|}(|\mathscr{S}_A| - m^s) \\
&\le -m^{n-r_0+1} + \sum_{s=1}^{n-r_0} \phi_s m^s,
\end{aligned} \tag{4.8}$$

where the inequality holds when $m > k$. As $k, \phi_1, \cdots, \phi_{n-r_0}$ are independent of the value of m, there exists an $M \in \mathbb{N}$, such that $P_{DP}(G,\mathcal{H}) - P(G,m) < 0$ for all $m \ge M$. Hence the result is proven. □

We shall conclude this section by proving Corollary 1.6.

Proof of Corollary 1.6: Let $E^* = \{e_1, e_2, \cdots, e_k\} \subseteq E_G(V_1, V_2)$, where $e_i = u_i v_i$ with $u_i \in V_1$ and $v_i \in V_2$ for all $i \in [k]$. For each $i \in [k]$, let $\overrightarrow{e_i}$ be the directed edge (u_i, v_i), and let $\overrightarrow{E^*} = \{\overrightarrow{e_i} : i \in [k]\}$. By Theorem 1.5, it suffices to verify that $\overrightarrow{E^*}$ is balanced on every cycle C of G with $|E(C)| < r_0$.

Let C be any cycle of G such that $|E(C)| < r_0$ and $|E(C) \cap E^*|$ is positive. By the definition of r_0, $|E(C) \cap E^*| = 2r$ for some positive integer r, where $2r \le k$. Without loss of generality, assume that $E(C) \cap E^* = \{e_i : i \in [2r]\}$.

Let P be any minimal path of C which contains exactly two edges in $E(C) \cap E^*$, say e_i and e_j. Obviously, by the minimality of P, e_i and e_j must be the two edges incident with two end-vertices of P. Then, the consecutive vertices on P cannot appear in any one of the following orders:

$$u_i, v_i, \cdots, u_j, v_j \qquad \text{or} \qquad v_i, u_i, \cdots, v_j, u_j.$$

Otherwise, some component of $C - (E(C) \cap E^*)$ is either a (v_i, u_j)-path or a (u_i, v_j)-path in $G - E^*$, contradicting the given condition in the corollary. Thus, the consecutive vertices on P must appear in one of the following orders:

$$u_i, v_i, \cdots, v_j, u_j \qquad \text{or} \qquad v_i, u_i, \cdots, u_j, v_j.$$

Since $|E(C) \cap E^*| = 2r$, by the definition of directed edges in $\overrightarrow{E^*}$, the above conclusion implies that the directed edges of $\overrightarrow{E^*}$ are balanced on C.

The corollary then follows from Theorem 1.5. □

5 Study on plane graphs

By Corollary 1.4, every plane near-triangulation is DP-good and thus belongs to DP^*. In the following, we consider those plane graphs G in which at least two faces are not bounded by 3-cycles. We will first show that such a plane graph G may belong to $DP_<$ if some face of G is bounded by a 4-cycle.

Corollary 5.1 *Let G be any 2-connected plane graph in which each 3-cycle is the boundary of some face of G. If at least two faces of G are not bounded by 3-cycles and one of them is bounded by a 4-cycle, then $G \in DP_<$.*

Proof. We can choose a shortest sequence of faces $F_0, F_1, \cdots, F_t$ in G, where $t \geq 1$, F_0 is bounded by a 4-cycle and F_t is bounded by more than 3 edges, such that F_i is bounded by a 3-cycle for each $i \in [t-1]$, and faces F_{i-1} and F_i share an edge e_i on their boundaries for each $i \in [t]$. An example of the subgraph consisting of vertices and edges on boundaries of faces $F_0, F_1, \cdots, F_t$ is shown in Figure 7, where $t = 8$.

If $t = 1$, then $\ell_G(e_1) = 4$ and thus $G \in DP_<$ by Theorem 1.2. Now assume that $t \geq 2$. As $F_1, F_2, \cdots, F_{t-1}$ are all bounded by 3-cycles, e_i and e_{i+1} have a common end-vertex for each $i \in [t-1]$. Thus, e_i can be written as $e_i = u_i v_i$ for all $i \in [t]$ such that either $u_i = u_{i+1}$ (i.e., u_i and u_{i+1} are the same vertex) or $v_i = v_{i+1}$ for all $i \in [t-1]$. Let $V_1 = \{u_i : i \in [t]\}$ and $V_2 = \{v_i : i \in [t]\}$. Then $E^* := \{e_i : i \in [t]\} \subseteq E_G(V_1, V_2)$.

As F_0 is bounded by a 4-cycle, G has a 4-cycle C with $|E(C) \cap E^*| = 1$. But, as each 3-cycle in G must be the boundary of some face of G, there is no 3-cycle C in G with $|E(C) \cap E^*| = 1$. As the dual edges of the edges in E^* actually form a shortest path connecting vertices F_0^* and F_t^* in the

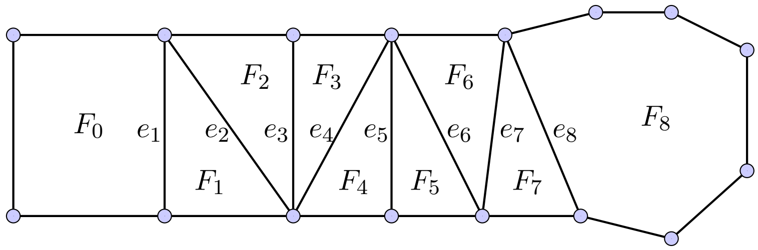


Figure 7: The graph consisting of vertices and edges on the boundaries of faces $F_0, F_1, \cdots, F_8$

dual plane graph G^* of G, there is no 3-cycle C in G with $|E(C) \cap E^*| = 3$. Therefore, $\ell_G(E^*) = 4$. Thus, by Corollary 1.7, $G \in DP_<$, and the result holds. $\square$

It is not difficult to generalize Corollary 5.1 as stated below.

Corollary 5.2 *Let G be any 2-connected plane graph. If $F_0, F_1, \cdots, F_t$ are faces in G, where $t \geq 1$, which satisfy the following conditions, then $G \in DP_<$:*

(i) *only F_0 and F_t are not bounded by 3-cycles, F_0 is bounded by an even cycle C_r and F_t is bounded by a cycle not shorter than r;*

(ii) *for each $i \in [t]$, faces F_{i-1} and F_i share an edge e_i on their boundaries; and*

(iii) *for $E^* = \{e_i : i \in [t]\}$, if C is a cycle in G with $E(C) \cap E^* \neq \emptyset$, then $|E(C)| \geq r$ holds whenever either F_0 or F_t is within cycle C.*

By Corollarys 1.6 and 5.1, it is interesting to notice that quite many graphs in $DP_<$ have their structures with a doughnut shape, as shown in Figure 8, where $E^* \subseteq E_G(V_1, V_2)$ for two disjoint vertex sets V_1 and V_2, and C is a shortest cycle in G such that $|E(C) \cap E^*|$ is odd and $|E(C)|$ is even.

However, for some other plane graphs which also look like doughnuts, we still don't know whether they belong to $DP_\approx$ or $DP_<$. For example, for a 2-connected plane graph G which is not a near-triangulation, if $\ell_G(e) = 3$ for all $e \in E(G)$, and those faces in G not bounded by 3-cycles have respectively $q_1, q_2, \cdots, q_t$ edges on their boundaries, where $4 \leq q_1 \leq q_2 \leq \cdots \leq q_t$ and q_j is even whenever $q_i < q_j$ and q_i is even, it is still unknown if G belongs to $DP_\approx$ or $DP_<$. For the particular case that q_i is odd for all $i \in [t-1]$, we guess G belongs to $DP_\approx$.

References

[1] G.D. Birkhoff, A determinant formula for the number of ways of coloring a map, *Annal. Math.* **14** (1912), 42–46.

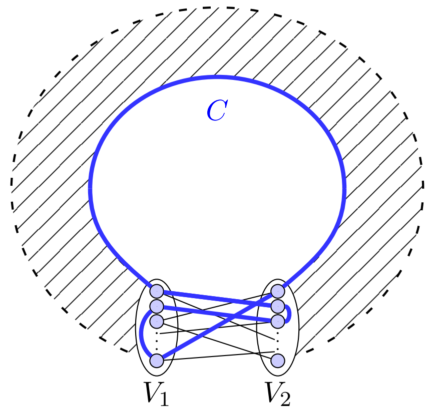


Figure 8: $E^* \subseteq E_G(V_1, V_2)$ for $V_1, V_2 \subseteq V(G)$ and $C \in \mathcal{C}'_G(E^*)$

[2] F.M. Dong and K.M. Koh, "Foundations of the chromatic polynomial," in the *Handbook on the Tutte Polynomial and Related Topics*, Jo Ellis-Monaghan and Iain Moffatt (ed.), pp 232–266, CRC press, 2022.

[3] F.M. Dong, K.M. Koh and K.L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, Singapore, 2005.

[4] F.M. Dong and Y. Yang, DP color functions versus chromatic polynomials, *Advances in Applied Mathematics* **134** (2022), article 102301.

[5] Q. Donner, On the number of list-colorings, *J. Graph Theory* **16** (1992) 239–245.

[6] Z. Dvořák and L. Postle, Correspondence coloring and its application to list-coloring planar graphs withoutcycles of lengths 4 to 8, *J. Comb. Theory, Ser. B* **129** (2018), 38–54.

[7] P. Erdős, A.L. Rubin and H. Taylor, Choosability in graphs, *Congr. Numer.* **26** (1979), 125–127.

[8] H. Kaul, M. Maxfield, J.A. Mudrock and S. Thomason, The DP Color Function of Clique-Gluings of Graphs, arXiv:2112.05316 (preprint), 2021.

[9] H. Kaul and J.A. Mudrock, On the chromatic polynomial and counting DP-colorings of graphs, *Advances in Applied Mathematics* **123** (2021), article 103121.

[10] J.A. Mudrock and S. Thomason, Answers to two questions on the DP color function, *Electron. J. Comb.* **28**(2) (2021), #P2.24.

[11] R.C. Read and W.T. Tutte, Chromatic polynomials, in *Selected Topics in Graph Theory 3*, Academic Press (1988), 15–42.

[12] G. Royle, Recent results on chromatic and flow roots of graphs and matroids, in: *Surveys in combinatorics*, London Math. Soc. Lecture Note Ser., vol. 365, Cambridge Univ. Press, Cambridge, 2009, pp. 289–327.

[13] C. Thomassen, The chromatic polynomial and list colorings, *Journal of Combinatorial Theory Series B* **99** (2009), 474–479.

[14] V.G. Vizing, Coloring the vertices of a graph in prescribed colors, *Diskret. Analiz.* no. 29, *Metody Diskret. Anal. v Teorii Kodovi Skhem* **101** (1976), 3–10.

[15] W. Wang, J. Qian and Z. Yan, When does the list-coloring function of a graph equal its chromatic polynomial, *J. Comb. Theory, Ser. B* **122** (2017) 543–549.