Girth in digraphs

Jean-Claude Bermond, A. Germa, Marie-Claude Heydemann, Dominique Sotteau

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J. C. Bermond
A. Germa
M. C. Heydemann
D. Sotteau
UNIVERSITE DE PARIS SUD[1]

ABSTRACT

For an integer $k>2$, the best function $m(n, k)$ is determined such that every strong digraph of order n with at least $m(n, k)$ arcs contains a circuit of length k or less.

INTRODUCTION

Let D be a digraph (without loops or multiple arcs) with vertex set $V(D)$ and arc set $E(D)$ where $|V(D)|=n$. The girth $g(D)$ of a digraph D which has at least one circuit (directed cycle) is the length of the smallest circuit in D. Definitions not given here can be found in [3].

A problem which has been studied is to find the minimum number $f(r, g)$ of vertices an r-regular digraph ($d^+(x)=d^-(x)=r$ for all x) with girth g may possess. In particular it has been conjectured that:

Conjecture 1. $f(r, g)=r(g-1)+1$ [1].
If $h(r, g)$ is the minimum number of vertices in a digraph of girth g in which every vertex has out degree at least r it has been conjectured that:

Conjecture 2. $h(r, g)=r(g-1)+1$ [4].

These conjectures are respectively equivalent to the following:

Conjecture 1′. If $d^+(x)=d^-(x)=r$ for every vertex x and $n\leq kr$ then $g(D)\leq k$ [1].

Conjecture 2′. If $d^+(x)\geq r$ for every vertex x and $n\leq kr$ then $g(D)\leq k$ [4].

(Note that Conjecture 2 implies Conjecture 1.)

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[1] 91405—ORSAY, France, Informatique bât, 490.

Only particular cases of these conjectures are solved: essentially the cases $r=2$, 3, any k, for the Conjecture 1, plus some couples of values (r, k) (see [2, 4]); the case $r=2$ any k for Conjecture 2 (see [5]).

Thomassen [6] asked the problem of finding the best function $m(n, k)$ such that a strong digraph of order n, with at least $m(n, k)$ arcs satisfies $g(D)\le k$. We solve this problem by showing

Theorem. Let D be a strong digraph of order n, and let $k\ge 2$. Then

$$|E(D)|\ge\frac{n^2+(3-2k)n+k^2-k}{2},$$

implies that $g(D)\le k$.

REMARK. We will also show that the theorem is the best possible.

Proof. (1) Since D is strong $g(D)$ exists and $2\le g(D)\le n$. Thus the *theorem is true for* $n=k$. The theorem is also *true for* $k=2$, because

$$|E(D)|\ge\frac{n^2-n+2}{2}=\frac{n(n-1)}{2}+1.$$

(2) Thus we will suppose $k\ge 3$ in what follows and prove the theorem *by induction* on $|V(D)|=n$. The theorem is true for $n=k$; we suppose that it is true for every n' with $k\le n'\le n-1$ and let D be a strong digraph with n vertices, where then $n\ge k+1$.

Exactly we will prove that if $g(D)\ge k+1$ then

$$|E(D)|\le\frac{n^2+(3-2k)n+k^2-k}{2}-1=\varphi(n, k).$$

Furthermore since $k+1\ge 3$, D is antisymmetric: let

$$\tilde{E}(D)=\{\{x, y\}, x, y\in V(D), (x, y)\notin E(D) \quad\text{and}\quad (y, x)\notin E(D)\}$$

then

$$|\tilde{E}(D)|=\frac{n(n-1)}{2}-|E(D)|.$$

Thus it suffices to *prove that if* $g(D)\ge k+1$, then

$$|\tilde{E}(D)|\ge(k-2)n-\frac{(k+1)(k-2)}{2}.$$

(3) If $g(D)=n$, then D is a circuit of length n and $|E(D)|=n$. The condition $n\le\varphi(n, k)$ reduces to $(n-k+2)(n-k-1)\ge 0$, which is true for every $n\ge k+1$.

Thus we will suppose that $k+1\le g(D)\le n-1$.

(4) *Let D' be a strong proper subdigraph of D of maximum order.* A subdigraph is called proper if it has strictly less than n vertices. Strong

proper subdigraphs of D exist: for instance the subdigraph generated b the vertices of a smallest circuit of D.

Let $p=|V(D')|$ then $k+1\leq p\leq n-1$. Since $g(D')\geq g(D)\geq k+1$ w have by induction hypothesis that

$$|\tilde{E}(D')|\geq(k-2)p-\frac{(k+1)(k-2)}{2}.$$

To prove the theorem it suffices thus to prove that

$$|\tilde{E}(D)|-|\tilde{E}(D')|\geq(k-2)(n-p).$$

(5) *Suppose that* $|V(D')|=p=n-1$. Let $V(D)=V(D')\cup\{a\}a\notin D'$. As D is strong and $g(D)\geq k+1$, there exists a circuit of length at least $k+1$ containing a. Let $C_l=(a, y_1, y_2, \ldots, y_l, a)$ be such a circuit of minimum length $l+1$ with $l\geq k$. As C_l is of minimum length for $2\leq i\leq l-1$ $\{a, y_i\}\in\tilde{E}(D)-\tilde{E}(D')$ and thus

$$|\tilde{E}(D)|-|\tilde{E}(D')|\geq l-2\geq k-2.$$

(6) *Thus we can assume that* $|V(D')|=p\leq n-2$. Let P be a directed path having only its end vertices in common with D' and containing a vertex of $D-D'$ (such a path exists as D is strong and antisymmetric). *Every other vertex of $D-D'$ belongs to P* otherwise $D'\cup P$ would be a larger strong proper subdigraph of D.

Thus let us write $P=(a', x_1, \ldots, x_i, \ldots, x_{n-p}, b')$, where $a'\in D'$, $b'\in D'$, $x_i\in D-D'$ (with eventually $a'=b'$).

(7) *Suppose, now that* $p\leq n-3$. (i) As D' is of maximum cardinality $\{x_i, u\}\in\tilde{E}(D)$ for $2\leq i\leq n-p-1$ and $u\in V(D')$. We have thus exhibited $p(n-p-2)$ pairs of vertices of $\tilde{E}(D)-\tilde{E}(D')$. (ii) Furthermore let $C=(x_1, \ldots, x_{n-p}, b'=x_{n-p+1}, \ldots, x_{l-1}, a'=x_l, x_1)$ be a circuit of length l containing P; thus $l\geq k+1$. For $2\leq i\leq k$ there is no arc (x_i, x_1) in D, otherwise $(x_1, \ldots, x_i, x_1)$ would be a circuit of length at most k, and $k<g(D)$. For $3\leq i\leq l$, there is no arc (x_1, x_i), otherwise D' would not be of maximum order. Thus for $3\leq i\leq k$, $\{x_i, x_1\}\in\tilde{E}(D)-\tilde{E}(D')$. Similarly $\{x_{n-p}, x_i\}\in\tilde{E}(D)-\tilde{E}(D')$ for $n-p-k+1\leq i\leq n-p-2$, where the indices are to be taken modulo l. Thus we have exhibited $2(k-2)$ pairs of vertices of $\tilde{E}(D)-\tilde{E}(D')$ which are distinct, except possibly we have counted $\{x_1, x_{n-p}\}$ twice, and which are distinct from the $p(n-p-2)$ pairs exhibited in (i). In summary

$$\begin{aligned}|\tilde{E}(D)|-|\tilde{E}(D')|&\geq p(n-p-2)+2(k-2)-1=(k-2)(n-p)\\&\quad+(p-k+2)(n-p-2)-1\\&\geq(k-2)(n-p)+3(n-p-2)-1>(k-2)(n-p),\end{aligned}$$

and thus the theorem is proved for $p\leq n-3$ (with a strict inequality).

(8) *The only remaining case is* $p=n-2$. As we have seen in (6) D consists of D' plus an arc (a, b) with at least an arc from D' to a and one arc from b to D'. Let us consider now a circuit of minimum length containing the arc $(a, b): C=(a, b, y_1, y_2, \ldots, y_{l-2}a)$ of length $l \geq k+1$.

The arc $(a, y_i) \notin E(D)$ otherwise D' will not be of minimum order and the arc $(y_i, a) \notin E(D)$ for $1 \leq i \leq l-3$ otherwise the circuit C will not be of minimum length. Thus for $1 \leq i \leq l-3$ $\{a, y_i\} \in \tilde{E}(D)-\tilde{E}(D')$. Similarly for $2 \leq i \leq l-2$ $\{b, y_i\} \in \tilde{E}(D)-\tilde{E}(D')$. Thus

$$|\tilde{E}(D)|-|\tilde{E}(D')| \geq 2(l-3)=2(k-2)+2(l-k-1).$$

Thus the theorem is also true for $p=n-2$. In order to characterize the extremal graphs we will show that in the case $p=n-2$ we have in fact $|\tilde{E}(D)|-|(\tilde{E}(D')|>2(k-2)$. That is the case if $l>k+1$. If $l=k+1$, then since $n-1 \geq k+1$ there exists a vertex z not on the circuit C. Neither the arcs (a, z) nor (z, b) belongs to $E(D)$ otherwise D' will not be of maximum order. Furthermore at least one of the arcs (z, a) and (b, z) does not belong to $E(D)$ otherwise we will have a circuit of length 3 contradicting $g(D) \geq k+1 \geq 4$. So we have one more pair of vertices in $\tilde{E}(D)-\tilde{E}(D')$ and thus the inequality is strict. ∎

REMARK. The theorem is the best possible in the sense that there exist strong digraphs of order n, which have $\varphi(n, k)$ arcs and girth $k+1$. For example let D be the digraph consisting of a Hamiltonian circuit $(x_1, x_2, \ldots, x_n, x_1)$ plus the arcs (x_i, x_j), where $k-1 \leq i<j-1 \leq n-1$ except the arc (x_{k-1}, x_n). For $n=k+1$, this digraph reduces to a circuit of length $k+1$ and is the only strong digraph of order $k+1$, having $\varphi(k+1, k)=k+1$ arcs and girth $k+1$. However for $n>k+1$, the digraph D above is not the unique digraph of order n with $\varphi(n, k)$ arcs and girth $k+1$. Such digraphs can be constructed recursively from the circuit of length $k+1$. Indeed the proof of the theorem, in particular the fact that the equality $|\tilde{E}(D)|-|\tilde{E}(D')|=(k-2)(n-p)$ occurs only for $n=p+1$ shows that a strong digraph of order n, with $\varphi(n, k)$ arcs and girth $k+1$ is obtained from a strong digraph D' of order $n-1$, with $\varphi(n-1, k)$ arcs and girth $k+1$ by adding a vertex of degree $n-k+1$ in such a way that no circuit of length less than or equal to k is created.

Note added in proof: Other partial results on Conjectures 1 and 2 have been obtained by Y. O. Hamidoune, A note on the girth of digraphs, to appear.

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