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An invariant of spatial graphs

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Introduction. Some useful invariants for links have appeared in the last few years, e. g. Jones polynomial, 2-variable Jones polynomial, Kawffman polynomial, etc. But these invariants are not defined for spatial graphs. The Alexander ideals and the Alexander polynomials of spatial graphs [3,4,5,7] are determined by the fundamental groups of the complements of spatial graphs. Therefore they are neighborhood equivalence class invariant of spatial graphs. So, the two spatial graphs shown in Figure 7 can not be distinguished by them.

In this paper, we will introduce a 1-variable Laulent polynomial invariant for non-directed spatial graphs. It is a simple and useful invariant. We will define two types of spatial graphs, one is of spatial graphs with flat vertices, and the other is of spatial graphs with pliable vertices. Our polynomial is an invariant for flat vertex graphs. Also, it is an invariant for pliable vertex graphs whose maximum degrees are less than 4. In the case of θ_n-curves, our argument will be made more precisely by the twisting number of diagrams of θ_n-curves.

The restriction of our invariant to 2-regular graphs is an invariant of links. We will show that it is a specialization of

Kauffman's Dubrovnik polynomial, moreover, it is the Jones polynomial of the (2,0)-cabling for a knot.

1. Spatial graphs, diagrams and Reidemeister moves.

Throughout this paper we work in the piecewise-linear category. Let $G=(V,E)$ be a graph embedded in $\mathbb{R}^3$, we say G is a *spatial graph*. If for each vertex v of G, there exist a neighborhood B_v of v and a small flat plane P_v such that $G\cap B_v\subset P_v$, then we say that G is a *flat vertex graph*. For two spatial graphs G, G', if there exists an isotopy $h_t:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ $t\in[0,1]$ such that $h_0=id$ and $h_1(G)=G'$ then we say that G and G' are *ambient isotopic as pliable vertex graphs (pliably isotopic)*. For two flat vertex graphs G, G', if there exists an isotopy $h_t:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ $t\in[0,1]$ such that $h_0=id$, $h_1(G)=G'$ and $h_t(G)$ is a flat vertex graph for each $t\in[0,1]$ then we say that G and G' are *ambient isotopic as flat vertex graphs (flatly isotopic)*.

Let $G\subset\mathbb{R}^3$ be a spatial graph. We say that a projection $p:\mathbb{R}^3\longrightarrow\mathbb{R}^2$ is a regular projection corresponding to G if each multi point of $p(G)$ is a double point of transversal two edges. Then we say the image $p(G)$ with the informations about the over crossings at all crossings of $p(G)$ a *diagram* of G.

We shall define fundamental moves of diagrams, called Reidemeister moves, as in Figure 1.

Figure 1

It is easy to see that (0) is generated by (I), (I) is included in (VI), (II) is included in (IV) and that (V) is generated by (II), (III) and (VI). We say the deformation generated by (I)~(VI) *pliable deformation*, one generated by (I)~(V) *flat deformation*, and we say one generated by (0), (II)~(IV) *regular deformation*.

The next is a primitive lemma in this paper.

Lemma 1. *Let* G_0 *and* G_1 *be two spatial graphs.* G_0 *is pliably isotopic (resp. flatly isotopoc) to* G_1 *if and only if a diagram of* G_0 *is deformable to a diagram of* G_1 *by pliable deformation (resp. flat deformation).*

(Proof) Let $h_t:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ be an ambient isotopy which transforms G to G'. We can assume that for any time t_0, there exist a positive real number ε and a 0-simplex v_0 of the PL-structure of G such that for any $t\in(t_0-\varepsilon,t_0+\varepsilon)$ and for any 0-simplex v of the PL-structure of G except for v_0, $h_{t_0}(v)=h_t(v)$. Then we can take a suitable projection $p:\mathbb{R}^3\longrightarrow\mathbb{R}^2$ such that the state of the deformation of $p\circ h_t(G)$ around v is one of the Redemeister moves (I)~(VI) through $t\in(t_0-\varepsilon,t_0+\varepsilon)$. Therefore we can trace the isotopic deformation of $h_t(G)$ by a pliable deformation on diagrams.

Let $h_t:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ be an ambient isotopy which transforms G to G' as flat vertex graphs. Let v be a vertex of G and P_v be the small plane which contains $G \cap B_v$ where B_v is a neighborhood of v. Let $p:\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be a projection. Assume that for a real number $t_0 \in [0,1]$, $p \circ h_{t_0}(P_v)$ degenerates and it is turned over at the time t_0. Then the diagram $p \circ h_t(G)$ is deformed as shown in Figure 2 through $t \in [t_0-\varepsilon, t_0+\varepsilon]$ for some positive real number ε. Figure 3 shows that this deformation is generated by Reidemeister moves (I)~(V).

The sufficiency is trivial. □

Figure 2 Figure 3

2. An invariant of spatial graphs.

Let $G=(V,E)$ be a graph , where V is the vertex set and E is the edge set of G. Let $\mu(G)$ and $\beta(G)$ be the number of connected components of G and the first Betti number of G, respectively. Put $f(G) = x^{\mu(G)}y^{\beta(G)}$ and define a 2-variable Laulent polynomial by

$$h(G) = h(G)(x,y) = \sum_{F \subset E} (-x)^{-|F|} f(G-F),$$

where F ranges over the family of subsets of E, $|F|$ is the number of elements of F, $G-F = (V,E-F)$ and x and y are indeterminates. In particular, define $h(\phi)=1$.

Of course, $h(G)$ is a invariant of graph G. And it is a specialization of Negami's polynomial [6] of G. This has the following properties.

Proposition 1. *Let* e *be a not loop edge of a graph* G. *Then* $h(G) = h(G/e) - 1/x\ h(G-e)$. *Where* G/e *is the graph obtained from* G *by contracting* e *to a point and* $G-e = G-\{e\}$.

(Proof)
$$h(G) = \sum_{e\in F\subset E} (-x)^{-|F|} f(G-F) + \sum_{e\notin F\subset E} (-x)^{-|F|} f(G-F)$$
$$= \sum_{F\subset E-e} (-x)^{-|F|} f(G/e-F) - 1/x \sum_{F'\subset E-e} (-x)^{-|F'|} f(G-e-F)$$
$$= h(G/e) - 1/x\ h(G-e). \quad \square$$

For two graphs G_1 and G_2, $G_1 \amalg G_2$ denotes the disjoint union of G_1 and G_2 and $G_1 \vee G_2$ denotes a wedge at a vertex of G_1 and G_2, i. e. $G_1 \vee G_2 = G_1 \cup G_2$ and $G_1 \cap G_2 = \{\text{a vertex}\}$. The symbol $\vee$ is quoted from [7]. Then the following proposition holds.

Proposition 2.

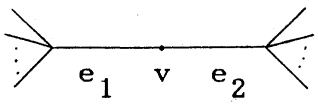
(1) $h(G_1 \amalg G_2) = h(G_1)h(G_2)$,

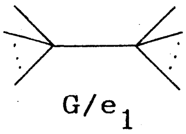
(2) $h(G_1 \vee G_2) = 1/x\ h(G_1)h(G_2)$,

(3) If G has a cut edge then $h(G)=0$.

(Proof) (1) and (2) are trivial.
(3): Let e is the cut edge of G. Then $G-e = G_1 \amalg G_2$ and $G/e = G_1 \vee G_2$ for some graphs G_1 and G_2. By proposition 1, $h(G) = h(G_1 \vee G_2) - 1/x\ h(G_1 \amalg G_2) = 0$. $\square$

Theorem 1. *Let* v *be a vertex of a graph* $G=(V,E)$ *which is a terminal point of just two not loop edges* e_1 *and* e_2. *Then* $h(G) = h(G/e_1)$,

i. e. h(

) = h(

).

(Proof) The graph $G-e_1$ has a cut edge e_2, so $h(G-e_1)=0$. Therefore from the previous propositions,

$h(G/e_1) = h(G) - 1/x\ h(G-e_1) = h(G)$. □

Corollary 1. $h(G)$ *is a topological invariant of a graph* G. *i. e. if* G *is homeomorphic to a graph* G' *then* $h(G)=h(G')$.

Figure 4

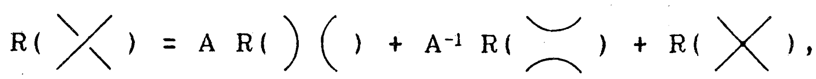
Now, we will define an invariant of spatial graphs. Let g be a diagram of a graph. For a crossing c of g, we define s_+, s_- and s_0 called the spin of +1, -1 and 0, as shown in Figure 4. Let S be the plane graph obtained from g by replacing each crossing with a spin. S is called a state on g and $\mathcal{S}(g)$ denotes the set of states on g. . Put $\{g|S\} = A^{p-q}$, where p and q are the numbers of crossings with spin of +1 and spin of -1 in S respectively and A is an indeterminate. We define a 1-variable Laulent polynomial $R(g)(A)$ as follows.

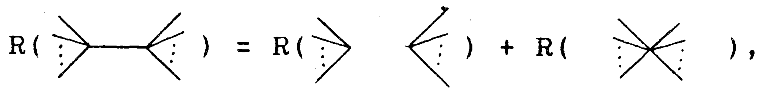
$$R(g) = R(g)(A) = \sum_{S \in \mathcal{S}(g)} \{g|S\}\ H(S),$$

where $H(S) = h(S)(-1, -A-2-A^{-1})$. In particular, define that $R(\phi)=1$.

This polynomial has the following properties. The next proposition is derived from the definition of $R(g)$ and previous propositions.

Proposition 3.

(1) 

(2) 

(3) $R(g_1 \amalg g_2) = R(g_1)R(g_2)$,

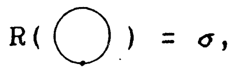
(4) $R(g_1 \vee g_2) = -R(g_1)R(g_2)$,

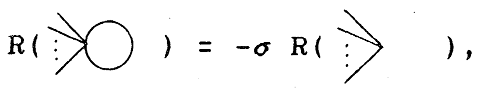
(5) If g has a cut edge then $R(g) = 0$.

(Remark) *Those figures in a equation represent diagrams that differ only as indicated in the figures.*

The next proposition is very useful for the proof of invariance of $R(g)$ and the calculation of it.

Proposition 4.

(1)  where $\sigma = A+1+A^{-1}$,

(2) 

(3) R( ) = $-A$ R( ) $-$ (A^2+A) R( ),

(4) R( ) = $-A^{-1}$ R( ) $-$ $(A^{-2}+A^{-1})$ R( ),

(5) R( ) = $-A$ R( ), R( ) = $-A^{-1}$ R( ),

(6) R( ) = A^2 R( ), R( ) = A^{-2} R( ).

(Proof) (1): h( )(x,y) = $xy-1$, so R( ) = H( ) h( )$(-1, -A-2-A^{-1})$ = $A+1+A^{-1}$. Others are easy calculation using Proposition 3. □

Theorem 2. R(g) *is a regular deformation invariant of a diagram* g.

(Proof) We shall show that R(g) does not change under the Reidemeister moves (0),(II)~(IV).

(0): It is derived from Proposition 4-(6).

(II): R( ) = R( ) + $(A^2+A^{-2}+\sigma)$ R( ) + $(A+A^{-1})$ R( )

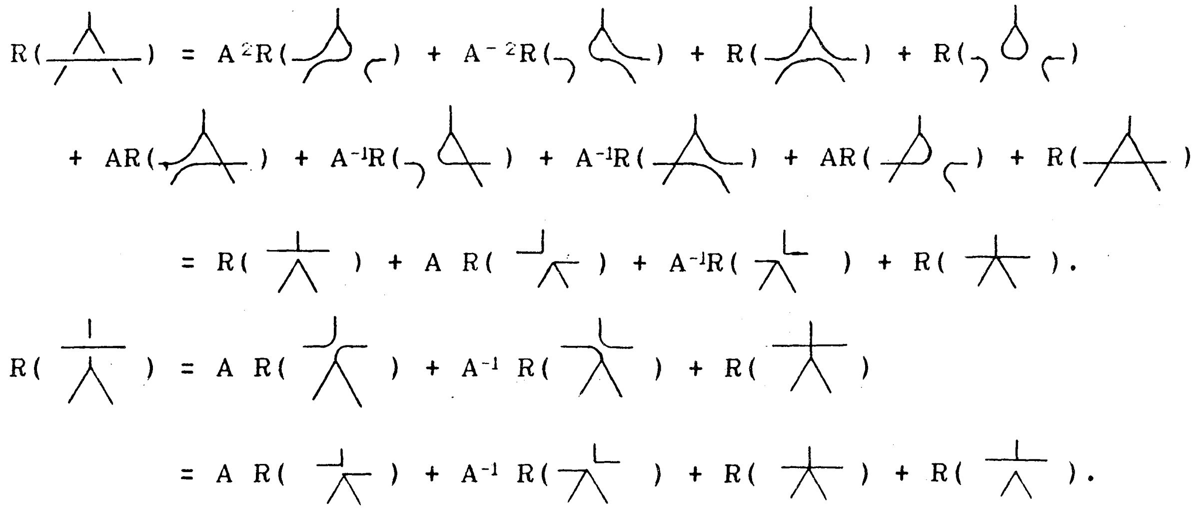
+ A R( ) + A^{-1} R( ) + R( )

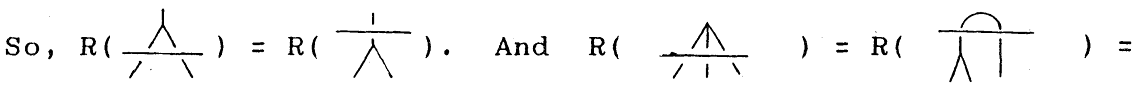
= R( ) + $(A^2+A^{-2}+\sigma-A\sigma-A^{-1}\sigma+1)$ R( )

= R( ).

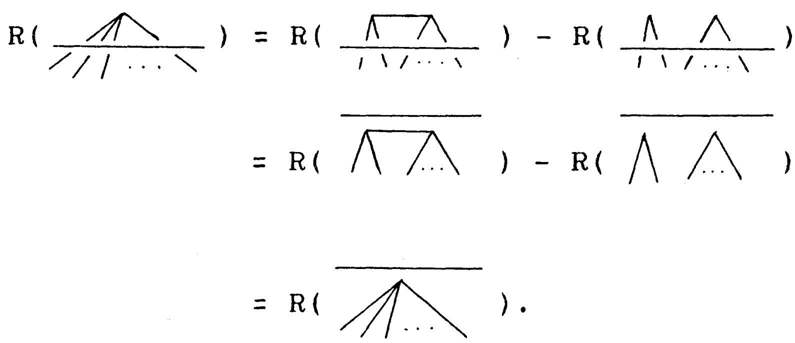
(IV): Let v be the moving vertex in the figure of Reidemeister move (IV). Our proof is an induction on the degree of

v. If degree(v)=1 then such diagrams have cut edges, so both of their polynomials are zero. If degree(v)=2 then it is shown in the case of (II) of this proof. If degree(v)=3,



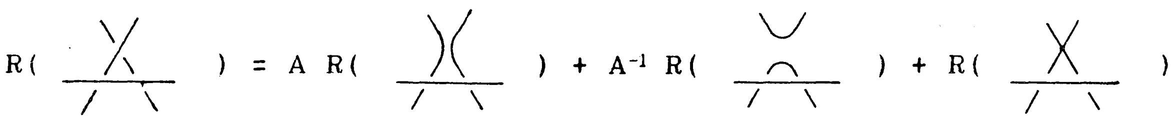


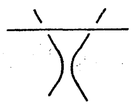
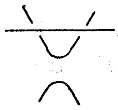
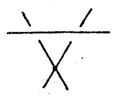
 If degree(v)>3, from Proposition 3-(2) and the hypothesis of the induction,

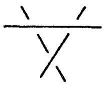


The other equation is shown similarly.

(III): From the definition of R(g) and its invariance under the Reidemeister moves (II) and (IV),

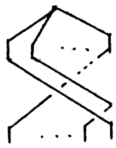
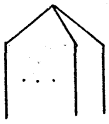


$=$ A R() + A^{-1} R() + R()

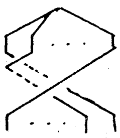
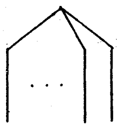
$=$ R().

This completes the proof. □

Proposition 5.

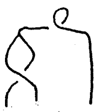
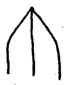
R() = $(-A)^n$ R(),

n

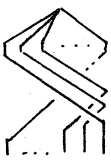
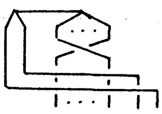
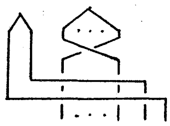
R() = $(-A)^{-n}$ R().

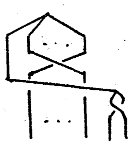
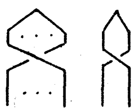
n

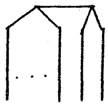
(Proof) Our proof is an induction on n. If $n=1$ then such diagrams have a cut edge, so both of their polynomials are zero. If $n=2$ then it is shown in Proposition 4-(6). If $n=3$,

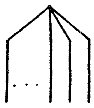
R() = R() = A^2 R() = $-A^3$ R().

If $n>3$ then from the hypothesis of the induction,

R() = R() − R()

= R() − R()

$= (-A)^n R($  $) - (-A)^n R($  $)$

$= (-A)^n R($  $)$.

The other equation is shown similarly. This completes the proof.

The above proposition implies the next theorem.

Theorem 3. $R(g)$ *is a flat deformation invariant of a diagram* g *up to multiplying* $(-A)^n$ *for some integer* n.

Let g_1 and g_2 be diagrams of a graph whose maximum degree is less than 4, where the *maximum degree* of a graph $G=(V,E)$ is $\max\{\mathrm{degree}(v) \mid v \in V\}$. Then, g_1 is pliabley deformable to g_2 if and only if g_1 is flatly deformable to g_2. Because the Reidemeister move (VI) is generated by (I)~(V) for such diagrams. So, we get the next theorem.

Theorem 4. *If* g *is a diagram of a graph whose maximum degree is less than* 4 *then* $R(g)$ *is a pliable deformation invariant of* g *up to multiplying* $(-A)^n$ *for some integer* n.

For a spatial graph G, define $\bar{R}(G) = (-A)^{-m}R(g)$ where g is a diagram of G and m is the lowest degree of $R(g)$. By Theorem 3, $\bar{R}(G)$ is a flat isotopy invariant, moreover by Theorem 4, if G is a graph whose maximum degree is less than 4 then

$\bar{R}(G)$ is a pliable isotopy invariant of G.

3. Connected sum of graphs.

For a positive integer n and graphs G and G', let v (resp. v') be vertex of G (resp. G') of degree n and e_1, ... , e_n (resp. e_1', ... , e_n') be the edges which are adjacent to v (resp. v'). Then, we construct a graph $G\#_n G' = (V\cup V'\cup\{v_1,\ldots,v_n\}\setminus\{v,v'\}, E\cup E')$ by removing v and v' from $G\amalg G'$ and adding n vertices $v_1, \ldots, v_n$ and changing the end point v and v' of e_i and e_i' to v_i for $i=1\ldots n$. We say $G\#_n G'$ a *connected sum* of G and G' of order n. See Figure 5.

/Figure 5/

For a positive integer n and spatial graphs G, G', v (resp. v') be vertex of G (resp. G') of degree n. Assume that G is in the upper half-space $\mathbb{R}^3_+=\{(x,y.z)\in\mathbb{R}^3 \mid z\geq 0\}$ except for some small neighborhood of v, and G' is in the lower half-space $\mathbb{R}^3_-=\{(x,y,z)\in\mathbb{R}^3 \mid z\leq 0\}$ except for some small neighborhood of v', and $G\cap\mathbb{R}^2_0 = G'\cap\mathbb{R}^2_0 = \{n \text{ points}\}$, where $\mathbb{R}^2_0$ is the boundary plane of those half-space. Then the spatial graph $(G\cap\mathbb{R}^3_+)\cup(G'\cap\mathbb{R}^3_-)\in\mathbb{R}^3$ is called a *connected sum* of G and G' of order n and it is denote by $G\#_n G'$.

For a positive integer n and diagrams g, $g'\subset\mathbb{R}^2$, diagrams of v (resp. v') be vertex of g (resp. g') of degree n. Assume that g is in upper half-plane $\mathbb{R}^2_+$ except for some small neighbourhood of v, and g' is in under half-plane $\mathbb{R}^2_-$ except for some small neighbourhood of v', and $g\cap\mathbb{R}_0 = g'\cap\mathbb{R}_0 = \{n$ points$\}$, where $\mathbb{R}_0$ is the boundary line of those half-planes. Then the diagram $(g\cap\mathbb{R}^2_+)\cup(g'\cap\mathbb{R}^2_-)$ is called a *connected sum* of g and g' and it is denoted by $g\#_n g'$.

Let Θ_n be the graph which consists of two vertices and n edges, which are not loops. We say Θ_n the θ_n-*graph*.

Proposition 6.

(1) $h(G\#_2G') = h(G)h(G')/h(\Theta_2)$,

(2) $h(G\#_3G') = h(G)h(G')/h(\Theta_3)$.

(Proof) In this proof let $n=2$ or 3. Let v, v' be the vertices of G, G' which are removed when construct the connected sum $G\#_nG'$ and $e_1, \ldots, e_n$, $e'_1, \ldots, e'_n$ be the edges of G, G' which are adjacent to v, v', respectively. It suffices to assume that G and G' are connected and has no cut edge. Our proof is an induction on the number $k = |E\cup E'|-2n$, where E and E' are the edge sets of G and G', respectively.

If $k=0$, G, G' and $G\#_nG'$ are homeomorphic to the θ_n-graph Θ_n, hence the equalities hold. If $k>0$, let e be an edge which is neither of $e_1, \ldots, e_n, e'_1, \ldots, e'_n$. It suffices to assume that e is an edge of G. We shall prove in the next two cases.

If e is a loop, from Proposition 2 and the hypothesis of the induction,

$$h(G\#_n G') = (y-1/x)h((G-e)\#_n G')$$

$$= (y-1/x)h(G-e)h(G')/h(\Theta_n)$$

$$= h(G)R(G')/h(\Theta_n).$$

If e is not a loop, by Proposition 1 and the hypothesis of the induction,

$$h(G\#_n G') = h((G/e)\#_n G') - 1/x\ h((G-e)\#_n G')$$

$$= h(G/e)h(G')/h(\Theta_n) - 1/x\ h(G-e)h(G')/h(\Theta_n)$$

$$= (h(G/e) - 1/x\ h(G-e))h(G')/h(\Theta_n)$$

$$= h(G)h(G')/h(\Theta_n).$$

This completes the proof. □

This proposition implies the next theorem.

Theorem 5.

(1) $R(g\#_2 g') = R(g)R(g')/\sigma$,

(2) $R(g\#_3 g') = R(g)R(g')/(\sigma-\sigma^2)$, where $\sigma = A+1+A^{-1}$.

(Proof) In this proof let $n = 2$ or 3. By Proposition 6,

$$R(g\#_n g') = \sum_{\substack{S\in\mathcal{S}(g)\\ S'\in\mathcal{S}(g')}} \{g\#_n g' | S\#_n S'\} H(S\#_n S')$$

$$= \sum_{\substack{S \in \mathcal{S}(g) \\ S' \in \mathcal{S}(g')}} \{g|S\}H(S)\{g'|S'\}H(S')/H(\Theta_n)$$

$$= \sum_{S \in \mathcal{S}(g)} \{g|S\}H(S) \sum_{S' \in \mathcal{S}(g')} \{g'|S'\}H(S')/H(\Theta_n)$$

$$= R(g)R(g')/H(\Theta_n),$$

And $H(\Theta_2) = \sigma$, $H(\Theta_3) = \sigma - \sigma^2$. Hence we complete the proof. $\square$

4. Twisting number and the invariant of θ_n-curves.

Let k be a knot diagram i. e. a diagram of a 2-regular 1-component graph. We define the *twisting number* $t(k)$ as follows. We fix an orientation on k, and put $t(k) = \sum_c \mathrm{sign}(c)$, where c ranges over the all crossings of k and sign( )=+1, sign( )=-1. $t(k)$ is not depend on the choice of the orientation of k.

Let $\Theta_n = (\{u,v\}, \{e_1, \ldots, e_n\})$ be a spatial θ_n-graph. We say Θ_n a θ_n-*curve*. In paticular we say Θ_3 a θ-*curve*. Let C_{ij} be the cycle $u \cdot e_i \cdot v \cdot e_j \cdot u$ of Θ_n $(i \neq j)$. Let θ_n be a diagram of Θ_n and c_{ij} be the subdiagram of θ_n corresponding to C_{ij}. Then, we define the *twisting number* of θ_n by

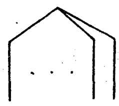
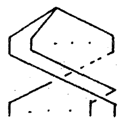
$$t(\theta_n) = \sum_{i<j} t(c_{ij}) \,/(n-1).$$

More generally, let $\Xi = \Theta_{n_1} \cup \cdots \cup \Theta_{n_s}$ be a link of some θ_n-curves and ξ be a diagram of Ξ and θ_{n_i} be the subdiagram of ξ corresponding to Θ_{n_i}. We define the *twisting number* $t(\xi)$

of ξ by $t(\xi) = \sum_{i=1}^{s} t(\theta_{n_i})$. It is easy to see that $t(\xi')=-t(\xi)$ where ξ' is the mirror image of ξ.

Now we define that $S(\xi) = (-A)^{-2t(\xi)}R(\xi)$.

Theorem 6. $S(\xi)$ *is a flat deformation invariant of* ξ.

(Proof) It is easy to see that the twisting number is a regular deformation invariant. So, $S(\xi)$ does not change under the regular deformation. We shall show the invariance of $R(\xi)$ under the Reidemeister move (V). Put $\xi =$  , $\xi' =$  .

n

Then $t(\xi') = t(\xi)+n/2$. By proposition 5, $R(\xi') = A^n R(\xi)$. So,

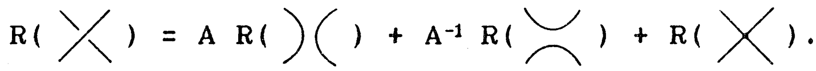
$R(\xi') = (-A)^{-2(t(\xi)+n/2)}(-A)^n R(\xi) = (-A)^{-2t(\xi)}R(\xi) = S(\xi)$. The other equation is shown similarly. □

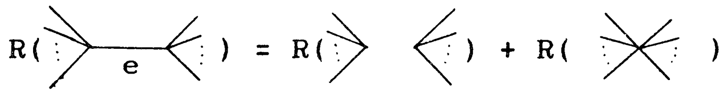
So, we define $S(\Xi)=S(\xi)$. Then $S(\Xi)$ is a flat isotopy invariant of Ξ. Theorem 4 and the above theorem imply the next.

Theorem 7. *Let* Ξ *be a link of some* θ*-curves and knots. Then* $S(\Xi)$ *is a pliable isotopy invariant of* Ξ.

5. Recursive formula of R(g) and invariants of links.

From the definition of $R(g)$ and the previous propositions we get the following formulas.

(1) 

(2)  where e is a not loop edge.

(3) $R(g_1 \amalg g_2) = R(g_1)R(g_2)$.

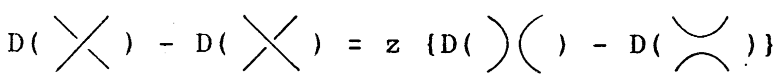
(4) $R(B_n) = -(-\sigma)^n$, where B_n is the n-leafed bouquet (See Figure 6) and $\sigma = A+1+A^{-1}$.

(5) $R(\phi) = 1$, $R(\cdot) = R(B_0) = -1$, $R(\bigcirc) = R(B_1) = \sigma$.

Figure 6

We can adopt the above formulas for the definition of $R(g)$. In fact, for any diagram g, we can resolve $R(g)$ to a summation of the invariant of some disjoint union of some bouquets with some coefficients by using (1) and (2) of the above formulas.

Kauffman discovered a regular deformation invariant $D(\ell)(a,z)$ of a link diagram ℓ (i. e. a diagram of 2-regular graph) [2]. That is colled Dubrovnik polynomial and defined by the following recursive definition.



$D(\;\;) = a\, D(\;\;)$

$D(\;\;) = a^{-1}\, D(\;\;)$

$D(\phi) = 1, \quad D(\;\;) = (a-a^{-1})/z+1.$

The above definition is different from the original one. The original one defines that $D(\;\;) = 1$.

The next equation (1)' is derived from (1) of the recursive definition of $R(g)$.

$$(1)' \quad R(\;\;) = A^{-1} R(\;\;) + A\, R(\;\;) + R(\;\;).$$

By (1)-(1)',

$$R(\;\;) - R(\;\;) = (A-A^{-1})\, \{R(\;\;) - R(\;\;)\}.$$

Moreover $R(\;\;) = A^{2}\, R(\;\;)$, $R(\;\;) = A^{-2}\, R(\;\;)$ and $R(\;\;) = A+1+A^{-1}$. So $R(\ell)(A)$ satisfies the defining formulas of $D(\ell)(A^{2}, A-A^{-1})$. Now we get the next theorem.

Theorem 8. *Let ℓ be a link diagram, then* $R(\ell)(A) = D(\ell)(A^{2}, A-A^{-1})$.

It is shown in [8] that $-(t^{\frac{1}{2}}+t^{-\frac{1}{2}})V_{K^{p(2)}}(t) = D(\hat{K})(t^{-2}, t^{-1}-t)+1$, where $V_{K^{p(2)}}(t)$ is the Jones polynomial [1] of the (2,0)-cabling of a knot K and $\hat{K}$ is a diagram of K such that $t(\hat{K})=0$. Then we get the next.

Corollary 2. $\quad -(t^{\frac{1}{2}}+t^{-\frac{1}{2}})V_{K^{p(2)}}(t) = R(\hat{K})(t^{-1})+1.$

6. Applications.

Let g_1 and g_2 be the diagrams shown in Figure 7. Then $R(g_1)=0$ and $R(g_2)=-A^5-A^4-A^3-A^2+A^{-1}+A^{-2}+A^{-3}+A^{-4}$. Therefore, the two spatial graphs G_1 and G_2 presented by g_1 and g_2 are not pliably isotopic. Note that $(\mathbb{R}^3, G_1)$ and $(\mathbb{R}^3, G_2)$ are also neighbourhood equivalent (after S. Suzuki [7]), i. e. $(\mathbb{R}^3, N(G_1)) \cong (\mathbb{R}^3, N(G_2))$.

Proposition 6. *Let* g' *be the mirror image of a diagram* g. *Then* $R(g')(A) = R(g)(A^{-1})$.

This proposition implies the following theorems.

Theorem 9. *Let* G *be a spatial graph. If* G *is amphicheiral (i. e. isotopic to the mirror image of it) as a flat vertex graph then* $\bar{R}(G)(A) = (-A)^{-d}\,\bar{R}(G)(A^{-1})$, *where* d *is the degree of* $\bar{R}(G)(A)$.

Theorem 10. *Let* G *be a spatial graph whose maximum degree is less than* 4. *If* G *is amphicheiral as a pliable vertex graph then* $\bar{R}(G)(A) = (-A)^{-d}\,\bar{R}(G)(A^{-1})$, *where* d *is the degree of* $\bar{R}(G)(A)$.

Theorem 11. *Let* θ *be an amphicheiral* θ*-curve then* $S(\theta)(A^{-1}) =$

$S(\theta)(A)$.

Let θ_1 and θ_2 be the diagram shown in Figure 8. Let Θ_1 and Θ_2 be the spatial graphs presented by θ_1 and θ_2. Then $t(\theta_1)=0$, $R(\theta_1)=-A^2-A-2-A^{-1}-A^{-2}$, $t(\theta_2)=-3/2$,

$R(\theta_2)=A^9-A^8-2A^7+A^6-A^5+2A^3+A^2+2A+A^{-1}-A^{-3}+A^{-4}+A^{-5}-A^{-6}+A^{-7}+A^{-8}$.

Therefore Θ_2 is not pliably isotopic to trivial θ-curve Θ_1. Moreover, by Theorem 9, Θ_2 is not amphicheiral as a pliable vertex graph. Note that each of the three cycles of Θ_2 is a trivial knot.

Those θ-curves shown in Figure 7 and Figure 8 are presented in [3,4].

Figure 7　　Figure 8

Acknowledgement.

I wish to thank Prof. J. Murakami for his helpful advice.

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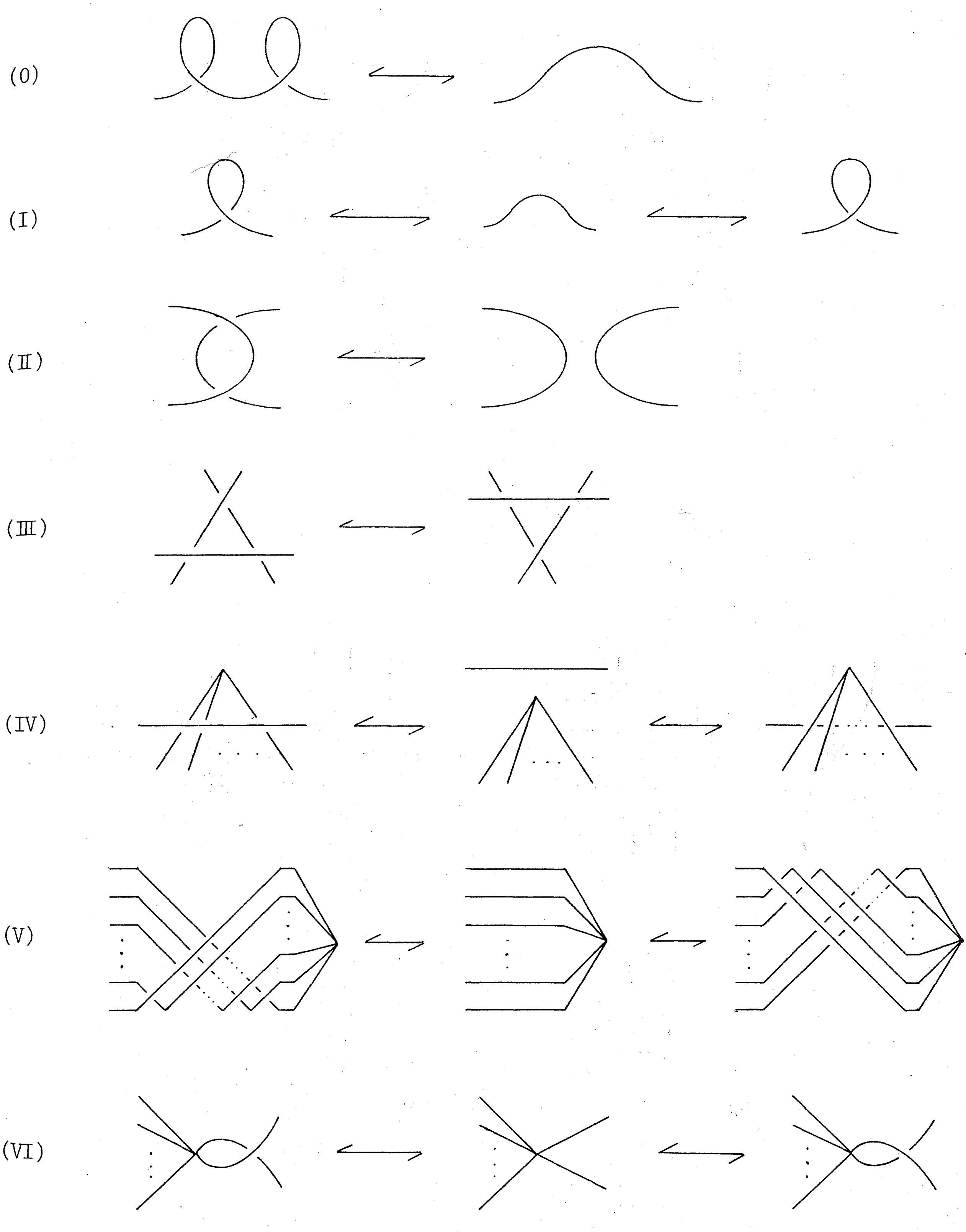


Figure 1

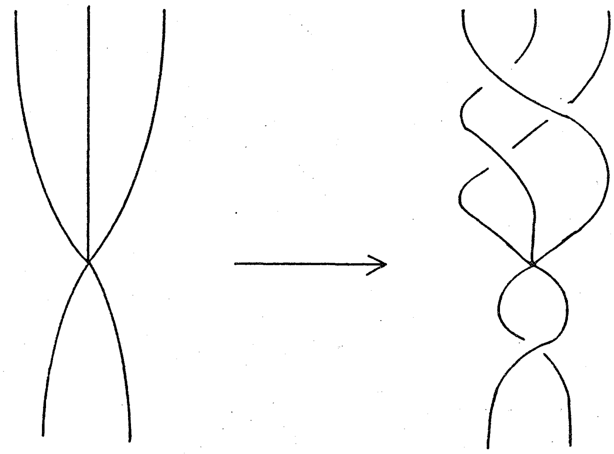


Figure 2

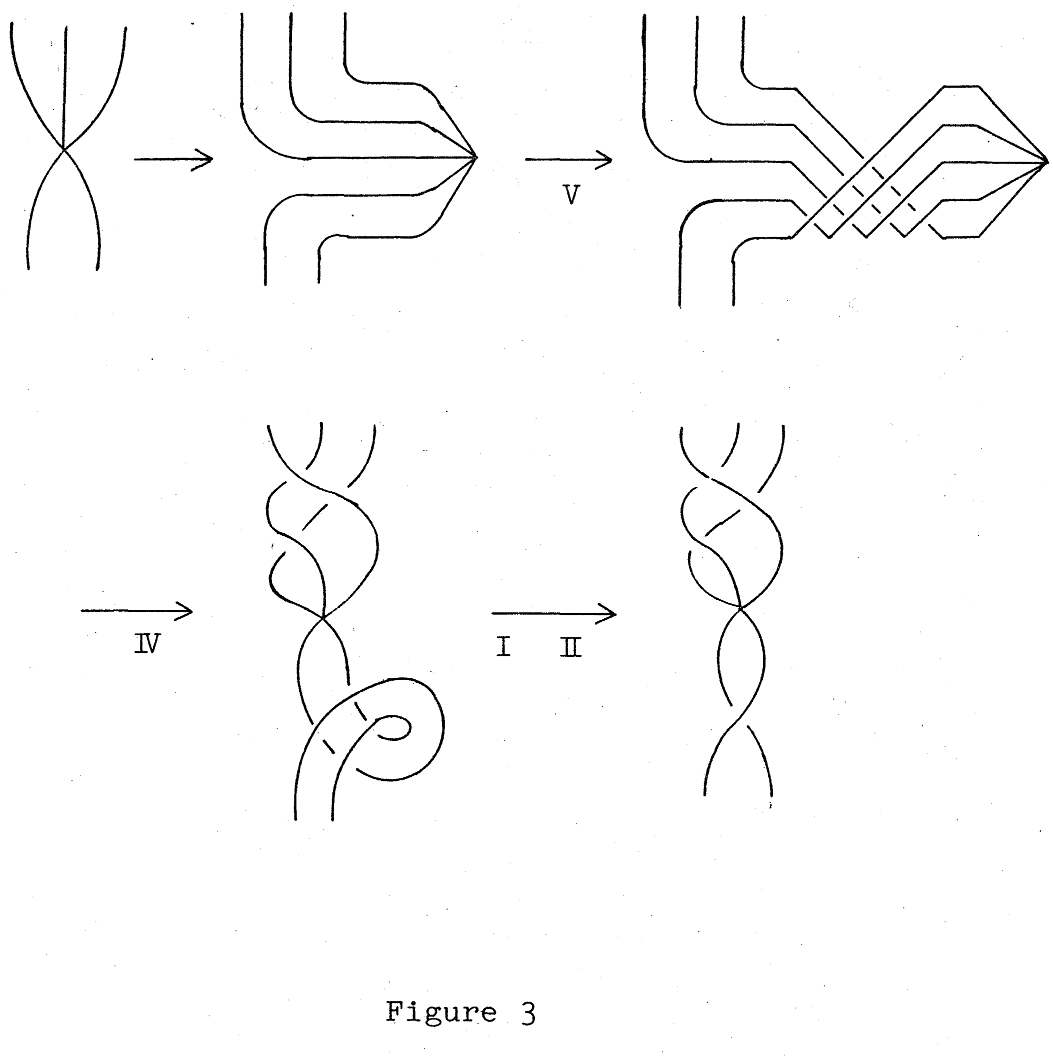
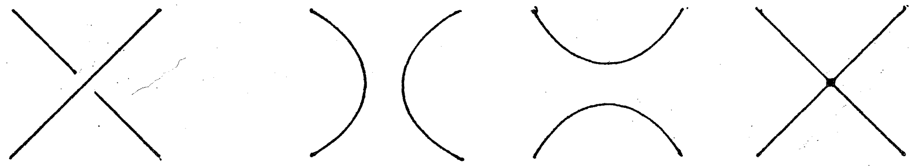
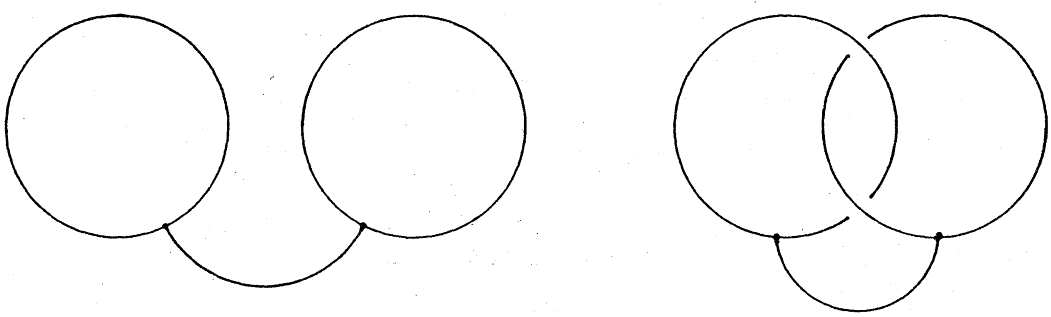


Figure 3



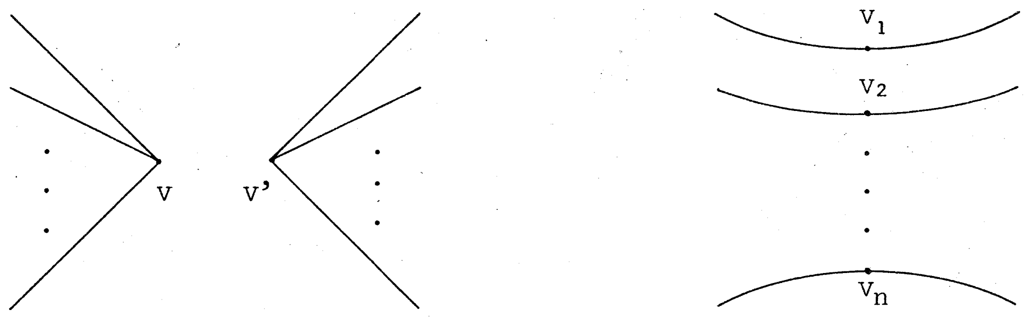
c s_+ s_- s_0

Figure 4



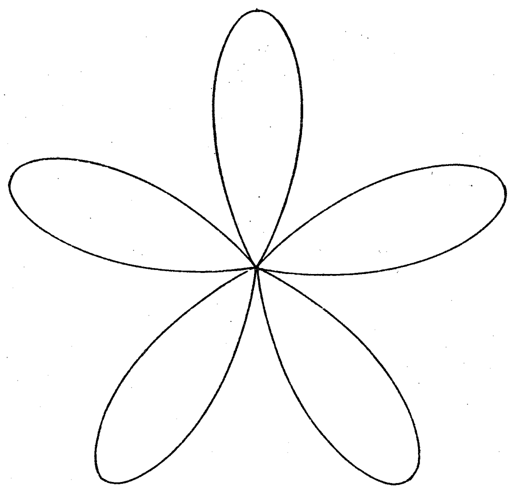
θ_1 θ_2

Figure 7



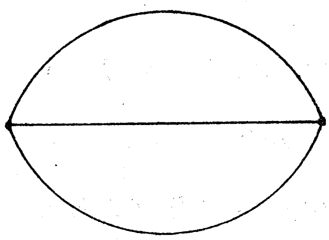
G_1 G_2 $G_1 \#_n G_2$

Figure 5

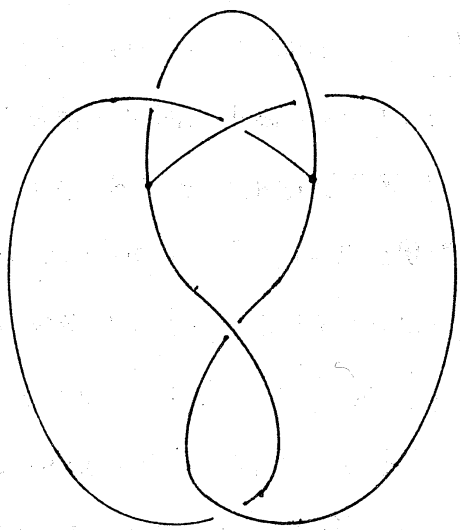


B_5

Figure 6



θ_1



θ_2

Figure 8