EFFECTIVELY AND NONEFFECTIVELY NOWHERE SIMPLE SETS

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Abstract. R. Shore proved that every recursively enumerable (r.e.) set can be split into two (disjoint) nowhere simple sets. Splitting theorems play an important role in recursion theory since they provide information about the lattice $\mathcal{E}$ of all r.e. sets. Nowhere simple sets were further studied by D. Miller and J. Remmel, and we generalize some of their results. We characterize r.e. sets which can be split into two (non)effectively nowhere simple sets, and r.e. sets which can be split into two r.e. non-nowhere simple sets. We show that every r.e. set is either the disjoint union of two effectively nowhere simple sets or two noneffectively nowhere simple sets. We characterize r.e. sets whose every nontrivial splitting is into nowhere simple sets, and r.e. sets whose every nontrivial splitting is into effectively nowhere simple sets.

R. Shore proved that for every effectively nowhere simple set A, the lattice $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$, and that there is a nowhere simple set A such that $\mathcal{L}^*(A)$ is not effectively isomorphic to $\mathcal{E}^*$. We prove that every nonzero r.e. Turing degree contains a noneffectively nowhere simple set A with the lattice $\mathcal{L}^*(A)$ effectively isomorphic to $\mathcal{E}^*$.

1. Introduction and notation

Shore [5] introduced the concepts of nowhere simple and effectively nowhere simple sets. Let $A \subseteq \omega$. A is *nowhere simple* if A is r.e. and for every r.e. set B with $B - A$ infinite, there is an infinite r.e. set W such that $W \subseteq B - A$. Let $W_0, W_1, \dots$ be a standard recursive enumeration of all r.e. sets such that $(\forall x)(\forall t)[x \in W_{e,t} \Rightarrow x \leq t]$. A is *effectively nowhere simple* if A is r.e. and there is a unary recursive function f such that for every $e \in \omega$, $W_{f(e)} \subseteq W_e - A$ and ($W_e - A$ is infinite $\Rightarrow W_{f(e)}$ is infinite). Since simple sets are not nowhere simple, every nonrecursive r.e. degree contains a set which is not nowhere simple. Shore [5] and Miller and Remmel [3] have proven that every nonrecursive r.e. Turing degree contains a nowhere simple set which is not effectively nowhere simple.

We denote the set of all nowhere simple sets by NS and the set of all effectively nowhere simple sets by ENS. It is easy to show that NS and ENS are closed under finite intersections. The set (NS - ENS) of all *noneffectively nowhere simple* sets is denoted by NENS. Let $\mathcal{E}$ be the lattice of all r.e. sets, and for an r.e. set A, let $\mathcal{L}(A)$ be the lattice of all r.e. supersets of A. Let $\mathcal{E}^*$ ($\mathcal{L}^*(A)$) be $\mathcal{E}(\mathcal{L}(A))$ modulo the ideal of finite sets. The properties of nowhere simplicity, effective nowhere simplicity, and noneffective nowhere simplicity are definable both in $\mathcal{E}$ and $\mathcal{E}^*$ [3].

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For fixed recursive enumerations of r.e. sets W and B, $W \searrow B$ is the set of all elements first enumerated in W and later in B, and $W \backslash B =_{def} (W \searrow B) \cup (W - B)$. We fix $\langle \cdot, \cdot \rangle$ to be a recursive bijection from ω^2 onto ω, which is strictly increasing with respect to both coordinates. Let $X, Y \subseteq \omega$. By $X \leq_T Y$ ($X \leq_{1-1} Y$) we denote that X is Turing (1-1) reducible to Y. As usual, $\emptyset'$ ($\emptyset''$) is the first (second) jump of $\emptyset$. For $n \in \omega$, $\omega^{[n]} =_{def} \{\langle n, i \rangle : i \in \omega\}$.

2. Splitting r.e. sets into nowhere simple sets

Shore Splitting Theorem [5]. *Every r.e. set B can be split into two disjoint nowhere simple sets A_0 and A_1.*
Now we recall
Friedberg Splitting Theorem. *Every r.e. set B can be split into two disjoint r.e. sets A_0 and A_1 such that for every r.e. set W:*
(1) ($W - A_0$ *is r.e.* $\vee$ $W - A_1$ *is r.e.*) $\Rightarrow W - B$ *is r.e..*
Friedberg's strategy of splitting B into A_0 and A_1 is based on satisfying the following requirement for every r.e. set W:
(F) $W \searrow B$ is infinite $\Rightarrow (W \cap A_0 \neq \emptyset \wedge W \cap A_1 \neq \emptyset)$.
Shore's strategy of splitting B into A_0 and A_1 is based on satisfying the following requirement for every r.e. set W:
(S) $W \searrow B$ is infinite $\Rightarrow$ ($W \cap A_0$ is infinite $\wedge$ $W \cap A_1$ is infinite).
Requirement (S) implies the following condition, which guarantees nowhere simplicity of A_0 and A_1. For every r.e. set W, we have
(2) ($W \cap A_0$ is finite $\vee$ $W \cap A_1$ is finite) $\Rightarrow W - B$ is r.e..
For $W = \omega - A_i, i \in \{0, 1\}$, it follows from (2) that
(3) (A_0 is recursive $\vee$ A_1 is recursive) $\Rightarrow B$ is recursive.
Since $W = (W - A_i) \cup (W \cap A_i)$ for $i \in \{0, 1\}$, (2) and hence (3) follow from (1). Thus, we have the following

Proposition 2.1. [1] *The sets A_0 and A_1 obtained in Friedberg Splitting Theorem are nowhere simple.*

Miller and Remmel [3] have established the following equivalence:
(4) $A \in \mathrm{ENS} \Leftrightarrow (\exists \text{ r.e. } T)[T \cap A = \emptyset \wedge \forall i(W_i - A \text{ is infinite} \Rightarrow W_i \cap T \text{ is infinite})]$.
An r.e. set T satisfying the matrix of the right-hand side of formula (4) is called a *witness set for A.*

Proposition 2.2. *In formula* (4), ($W_i \cap T$ *is infinite*) *can be replaced by* ($W_i \cap T \neq \emptyset$).

Proof. Let $A \in \mathrm{ENS}$ and let T be such that $\forall i(W_i - A$ is infinite $\Rightarrow W_i \cap T \neq \emptyset)$. For $i \in \omega$, let $h(\cdot, \cdot)$ be a (recursive) function such that $W_{h(i,0)} = W_i$ and $W_{h(i,x)} = W_i - \{0, \dots, x-1\}$ for $x \in \omega - \{0\}$. Assume that $W_i - A$ is infinite. Then $\forall x(W_{h(i,x)} - A$ is infinite). Hence $\forall x(W_{h(i,x)} \cap T \neq \emptyset)$. Thus, $W_i \cap T$ is infinite. ∎

Rodgers [4, Section 8.7] has introduced the following sets of r.e. sets:
$\mathcal{C}_0 =$ the set of all recursive sets,
$\mathcal{C}_1 =$ the set of all simple sets,
$\mathcal{C}_2 = \{A \in \mathcal{E} - \mathcal{C}_0 : (\exists \text{ r.e. } W)(W \text{ is infinite} \wedge W \cap A = \emptyset \wedge W \cup A \text{ is simple})\}$,
$\mathcal{C}_3 = \{A \in \mathcal{E} : A$ is not creative $\wedge$ $(\forall$ r.e. $B)[B \subseteq \overline{A} \Rightarrow (\exists$ r.e. $W)(W$ is infinite $\wedge$ $W \subseteq \overline{A} \wedge W \cap B = \emptyset)]\}$,

$\mathcal{C}_4 =$ the set of all creative sets.

Clearly, $\mathcal{C}_0 \subseteq$ ENS and $\mathcal{C}_1 \cap NS = \emptyset$. Shore [5] has shown that $\mathcal{C}_2 \cap NS \neq \emptyset$, $\mathcal{C}_3 \cap NS \neq \emptyset$, and $\mathcal{C}_4 \cap NS = \emptyset$.

Theorem 2.3. *The following are equivalent for an r.e. set A:*
(i) $A \in NENS$;
(ii) $A \in NS \wedge A \notin \mathcal{C}_0 \wedge (\forall \text{ r.e. } W)[A \cap W = \emptyset \Rightarrow A \cup W \text{ is not simple}]$;
(iii) $A \in NS \wedge A \notin \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$.

Proof. (ii) $\Leftrightarrow$ (iii) follows immediately.

(i) $\Rightarrow$ (ii): Assume that $A \in$ NENS. Clearly, $A \in$ NS and $A \notin \mathcal{C}_0$. Let an r.e. set W be such that $A \cap W = \emptyset$. Then, since W is not a witness set for A, there is an r.e. set U such that $U - A$ is infinite and $U \cap W = \emptyset$. Since $A \in$ NS, there is an infinite r.e. set R such that $R \subseteq U - A$. Since $R \cap (A \cup W) = \emptyset$, we conclude that $A \cup W$ is not simple.

(ii) $\Rightarrow$ (i): Let A be as in (ii). Let an r.e. set W be such that $A \cap W = \emptyset$. It follows that $A \cup W$ is coinfinite, since otherwise $A =^* W$, which contradicts the assumption that A is nonrecursive. Since $A \cup W$ is not simple, there is an infinite r.e. set R such that $R \cap (A \cup W) = \emptyset$. Since $R \cap W = \emptyset$, W is not a witness set for A. Thus, $A \in$ NENS. ∎

Theorem 2.4. *The following are equivalent for an r.e. set A:*
(i) A is the disjoint union of two noneffectively nowhere simple sets;
(ii) $A \notin \mathcal{C}_0 \wedge (\forall \text{ r.e. } W)[A \cap W = \emptyset \Rightarrow A \cup W \text{ is not simple}]$;
(iii) $A \notin \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$.

Proof. (ii) $\Leftrightarrow$ (iii) follows immediately.

(i) $\Rightarrow$ (ii): Assume that $A = A_0 \cup A_1$, where $A_0 \cap A_1 = \emptyset$ and $A_0, A_1 \in$ NENS. It follows that A is nonrecursive, since A_0 is nonrecursive and $\overline{A_0} = A_1 \cup \overline{A}$. Let an r.e. set W be such that $A \cap W = \emptyset$. Since $A \cup W = A_0 \cup (A_1 \cup W)$, $A_0 \cap (A_1 \cup W) = \emptyset$ and $A_0 \in$ NENS, it follows, by Theorem 2.3, that $A \cup W$ is not simple.

(ii) $\Rightarrow$ (i): Let A be as in (ii). Let A be split into nowhere simple sets A_0 and A_1 using Shore's splitting strategy. Then (2) is satisfied and, by (3), A_0 and A_1 are nonrecursive. We will prove that $A_0 \in$ NENS by showing that it satisfies (ii) of Theorem 2.3. Let an r.e. set W be such that $A_0 \cap W = \emptyset$. By (2), we conclude that $W - A$ is r.e.. It follows that $A \cup W$ is coinfinite, since otherwise $\overline{A} =^* W - A$, hence $\overline{A}$ is r.e., contradicting the fact that A is nonrecursive. Since $A \cup W$ is not simple, there is an infinite r.e. set R such that $R \cap (A \cup W) = \emptyset$ and, thus, $R \cap A_0 = \emptyset$. Therefore, $A_0 \cup W$ is not simple, so $A_0 \in$ NENS. Similarly, $A_1 \in$ NENS. ∎

It follows from Theorems 2.3 and 2.4 that every noneffectively nowhere simple set is the disjoint union of two noneffectively nowhere simple sets.

Theorem 2.5. *The following are equivalent for an r.e. set A:*
(i) A is the disjoint union of two effectively nowhere simple sets;
(ii) $A \in \mathcal{C}_0 \vee (\exists \text{ r.e. } W)[A \cap W = \emptyset \wedge A \cup W \text{ is simple}]$;
(iii) $A \in \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$;
(iv) Every splitting of A into two nowhere simple sets is into effectively nowhere simple sets.

Proof. (ii) $\Longleftrightarrow$ (iii) and (iv) $\Rightarrow$ (i) follow immediately.

(i) $\Rightarrow$ (ii): Assume that $A = A_0 \cup A_1$, where $A_0 \cap A_1 = \emptyset$ and $A_0, A_1 \in$ ENS. Let T_i be a witness set for $A_i, i \in \{0, 1\}$. Let $W =_{def} T_0 \cap T_1$. Clearly, $A \cap W = \emptyset$ since

$A_i \cap T_i = \emptyset$ for $i \in \{0,1\}$. Let $S =_{def} A \cup W$. If S is cofinite, then $\overline{A} =^* W$, hence A is recursive. Now assume that S is coinfinite. We will show that S is simple. Assume otherwise. Then there is an infinite r.e. set U such that $U \cap S = \emptyset$. Since $U \cap A_0 = \emptyset$, it follows that $U \cap T_0$ is infinite. Similarly, since $(U \cap T_0) \cap A_1 = \emptyset$, it follows that $(U \cap T_0) \cap T_1 = U \cap S$ is infinite, which is a contradiction.

(ii) $\Rightarrow$ (iv): Let (ii) be satisfied. Assume that A is split into nowhere simple sets B_0 and B_1. We will show that $B_0 \in$ ENS. If B_0 is recursive, then $B_0 \in$ ENS. Assume that B_0 is not recursive. Then A is not recursive. Let W be a corresponding r.e. set for A. Since $A \cup W = B_0 \cup (B_1 \cup W)$ and $B_0 \cap (B_1 \cup W) = \emptyset$, it follows that $B_0 \cup (B_1 \cup W)$ is simple. Thus, by Theorem 2.3, $B_0 \notin$ NENS. Hence $B_0 \in$ ENS. Similarly, $B_1 \in$ ENS. ∎

Corollary 2.6. *Every r.e. set is either the disjoint union of two effectively nowhere simple sets or of two noneffectively nowhere simple sets.*

Miller and Remmel [3, Thm. 12] have proven that in every nonzero r.e. degree, there is an r.e. set A which is not nowhere simple, such that A is not the disjoint union of two effectively nowhere simple sets (hence A is not simple). As a consequence of this result and Theorem 2.5, we have the following

Corollary 2.7. *In every nonzero r.e. degree, there is an r.e. set A which is not nowhere simple and which is not half of a splitting of a simple set.*

Definition 2.1. (*i*) *A splitting of an r.e. set into two disjoint nonempty sets is called nontrivial if both subsets are nonrecursive.*
(*ii*) *A set A is maximal in a set B if A is r.e., $A \subseteq B$, $B - A$ infinite, and there is no r.e. set W such that $A \subseteq W \subseteq B$ and both $B - W$ and $W - A$ are infinite.*
(*iii*) *A set A is simple in a set B if A is r.e., $A \subseteq B$, $B - A$ infinite, and there is no infinite r.e. set W such that $W \subseteq B - A$.*

Thus, A is nowhere simple if and only if it is not simple in any r.e. set B. Shore [5, Thm. 11] has constructed an r.e. set A such that A is not nowhere simple and A is not simple in any nowhere simple set.

Downey and Stob [1, Thm. 1.7] have proven that there is a nonrecursive effectively nowhere simple set A_0 which is not half of a nontrivial splitting of a simple set B such that condition (F) is satisfied.

Theorem 2.8. *Let A be an r.e. set.*
Every nontrivial splitting of A into two r.e. sets is into nowhere simple sets
$\Leftrightarrow$ ($A \in$ *NS* $\vee$ A *is maximal in some recursive set*).

Proof. $\Leftarrow$-part: Assume that $A \in$ NS and that $A = A_0 \cup A_1$, where $A_0 \cap A_1 = \emptyset$. In order to prove that A_0 is nowhere simple, assume that W is an r.e. set such that $A_0 \subseteq W$ and $W - A_0$ is infinite. If $W \cap A_1$ is infinite, $W \cap A_1$ is a required r.e. set since $W \cap A_1 \subseteq W - A_0$. Otherwise, $W - A$ must be infinite. Since $A \in$ NS, there is an infinite r.e. set R such that $R \subseteq W - A$, and hence $R \subseteq W - A_0$. Thus $A_0 \in$ NS.

Now, assume that A is maximal in a recursive set B. Then $A \subseteq B$ and $B - A$ is infinite. Let $A = A_0 \cup A_1$, where $A_0 \cap A_1 = \emptyset$ and A_0, A_1 are nonrecursive r.e. sets. We will show that $A_0 \in$ ENS by showing that $A_1 \cup \overline{B}$ is a corresponding witness set. Assume that $A_1 \cup \overline{B}$ is not a witness set. Then there is an r.e. set W such that $A_0 \subseteq W$, $W - A_0$ is infinite and $W \cap (A_1 \cup \overline{B}) = \emptyset$. Since A is maximal in B, we conclude that $B - W$ is finite, hence $\overline{A_1} =^* W \cup \overline{B}$. Thus $\overline{A_1}$ is r.e., contradicting the fact that $\overline{A_1}$ is nonrecursive. Similarly, we prove that $A_1 \in$ ENS.

$\Rightarrow$-part: Assume that A is not nowhere simple and that A is not maximal in any recursive set. We will prove that there is a nontrivial splitting of A into two r.e. sets, at least one of which is not nowhere simple. Since A is not nowhere simple, there is an r.e. set W such that $A \subseteq W$, $W - A$ is infinite and $W - A$ does not contain an infinite r.e. subset. If W is nonrecursive, let $U =_{def} W$. If W is recursive, then A is not maximal in W, so there exists an r.e. set U such that $A \subseteq U \subseteq W$ and both $W - U$ and $U - A$ are infinite. U is nonrecursive, since otherwise $W - U = W \cap \overline{U}$ would be an infinite r.e. set contained in $W - A$.

In any case, there is a nonrecursive set U such that $A \subseteq U$, $U - A$ is infinite and $U - A$ does not contain an infinite r.e. subset. Now we split U into two nonrecursive r.e. sets, U_0 and U_1. Let $A_i =_{def} U_i \cap A$ for $i \in \{0, 1\}$. Assume that both $U_0 - A_0$ and $U_1 - A_1$ are infinite. Then $A_0 \notin$ NS and $A_1 \notin$ NS. Hence both A_0 and A_1 are nonrecursive, as required in a nontrivial splitting. Now, for example, assume that $U_0 - A_0$ is infinite and $U_1 - A_1$ is finite. Clearly, $A_0 \notin$NS, so A_0 is not recursive. In addition, A_1 is nonrecursive since $A_1 =^* U_1$. ∎

Theorem 2.9. *Let A be an r.e. set.*
Every nontrivial splitting of A into two r.e. sets is into effectively nowhere simple sets
$\Leftrightarrow$ ($A \in$ *ENS* $\vee$ A *is maximal in some recursive set*).

Proof. $\Leftarrow$-part: Assume that $A \in$ ENS. It follows, by Theorem 2.5 (also by Proposition 6 in [3]), that every splitting of A into two r.e. sets is into effectively nowhere simple sets.

If A is maximal in a recursive set then, by the $\Leftarrow$-part of the proof of Theorem 2.8, every nontrivial splitting of A into two r.e. sets is into effectively nowhere simple sets.

$\Rightarrow$-part: Assume that A is not effectively nowhere simple and that A is not maximal in any recursive set. We will prove that there is a nontrivial splitting of A into two r.e. sets, at least one of which is not effectively nowhere simple. If A is not nowhere simple, then the conclusion follows by the $\Rightarrow$-part of the proof of Theorem 2.8. Now assume that $A \in$ NENS. Hence A is nonrecursive. Moreover, by Theorem 2.3, $A \notin \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$. Now we split A into two nonrecursive r.e. sets A_0 and A_1. By Theorem 2.5, A is not the disjoint union of two effectively nowhere simple sets, so not both A_0 and A_1 are effectively nowhere simple. ∎

Corollary 2.10. (i)[3, *Thm.* 8(a)] *Every nontrivial splitting of a maximal set M into two r.e. sets is into effectively nowhere simple sets.*
(ii) [3, *Prop.* 11] *Let M be a maximal set and let R be an infinite recursive subset of M. Then every nontrivial splitting of $M - R$ is into effectively nowhere simple sets.*

Proof. (i) M is maximal in ω.
(ii) $M - R$ is maximal in $\overline{R}$. ∎

Theorem 2.11. *Let A be an r.e. set.*
A is the disjoint union of two r.e. sets, neither of which is nowhere simple
$\Leftrightarrow$ ($\exists$ *r.e.* X, Y)[$X \cap Y = A \wedge A$ *is simple both in* X *and* Y].

Proof. $\Rightarrow$-part: Let $A = A_0 \cup A_1$, where $A_0 \cap A_1 = \emptyset$ and $A_0, A_1 \notin$NS. Since A_0 is not nowhere simple, there is an r.e. set W_0 such that $W_0 - A_0$ is an infinite set which does not contain any infinite r.e. subset. Hence $W_0 \cap A_1$ is finite. Let $U_0 =_{def} W_0 - A_1$. Since A_1 is not nowhere simple, there is an r.e. set W_1 such that $W_1 - A_1$ is an infinite set which does not contain any infinite r.e. subset. Hence $W_1 \cap U_0$ is finite. Let $U_1 =_{def} W_1 - U_0$. Now we define $X = U_0 \cup A$ and $Y = U_1 \cup A$. Clearly, A is simple in both X and Y, and $X \cap Y = A$.

$\Leftarrow$-part: Assume that A is simple in both X and Y, where X and Y are r.e. sets such that $X \cap Y = A$. By the reduction principle for r.e. sets, there are r.e. sets U and V such that $U \subseteq X$, $V \subseteq Y$, $U \cup V = X \cup Y$ and $U \cap V = \emptyset$. Now we define $A_0 = A \cap U$ and $A_1 = A \cap V$. Clearly, A_0 and A_1 are not nowhere simple. ∎

Recall that a set $A \subseteq \omega$ is *r-maximal* if A is r.e., coinfinite and there is no recursive set R such that both $R \cap \overline{A}$ and $\overline{R} \cap \overline{A}$ infinite. Clearly, every maximal set is r-maximal. The converse does not hold. Also, there are simple sets which are not r-maximal (see [6, p. 211]).

Corollary 2.12. (i) *Every nonmaximal hh-simple set A is the disjoint union of two r.e. sets, neither of which is nowhere simple.*
(ii) *Every simple set A which is not r-maximal is the disjoint union of two r.e. sets, neither of which is nowhere simple.*

Proof. (i) Since A is not maximal, there exists an r.e. set X such that $A \subseteq X$ and both $X - A$ and $\overline{X}$ are infinite. Since A is hh-simple, there exists an r.e. set Y such that $A \subseteq Y$, $X \cap Y = A$ and $X \cup Y = \omega$. Hence $Y - A$ is infinite. Since A is simple, it is simple both in X and Y.

(ii) Let R be a recursive set such that both $R \cap \overline{A}$ and $\overline{R} \cap \overline{A}$ are infinite. Define $X = R \cup A$ and $Y = \overline{R} \cup A$. ∎

3. Semilowness properties of nowhere simple sets

We recall that a set
X is *semilow* iff $\{e : W_e \cap X \neq \emptyset\} \leq_T \emptyset'$,
X is *semilow*$_{1.5}$ iff $\{e : W_e \cap X \text{ is infinite}\} \leq_{1-1} \{e : W_e \text{ is infinite}\}$, and
X is *semilow*$_2$ iff $\{e : W_e \cap X \text{ is infinite}\} \leq_T \emptyset''$.

Maass [2, Lemma 2.1] has shown that if A is r.e., then $\overline{A}$ is semilow$_{1.5}$ if and only if there is an enumeration of A, simultaneously with an enumeration of r.e. sets $(U_e)_{e\in\omega}$, such that for every $e \in \omega$, $W_e =^* U_e$ and ($U_e \setminus A$ is infinite $\Rightarrow U_e \cap \overline{A}$ is infinite). Maass [2] has further proven that for an r.e. set A, $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$ ($\mathcal{L}^*(A) \cong_{eff} \mathcal{E}^*$) if and only if $\overline{A}$ is infinite and semilow$_{1.5}$.

Shore [5] has proven that the complement of every nowhere simple set is semilow$_2$. The converse is not true since maximal sets are not nowhere simple and their complements are semilow$_2$. Shore [5] has proven that for every effectively nowhere simple set A, $\overline{A}$ is semilow$_{1.5}$, and hence, if $\overline{A}$ is infinite, $\mathcal{L}^*(A)$ is effectively isomorphic to $\mathcal{E}^*$. Shore [5] has also constructed a (noneffectively) nowhere simple set A such that $\mathcal{L}^*(A)$ is not effectively isomorphic to $\mathcal{E}^*$ (see also [3, Thm. 3]). The following theorem supplements these results.

Theorem 3.1. *Every nonzero r.e. Turing degree contains a noneffectively nowhere simple set A such that $\overline{A}$ is semilow$_{1.5}$ (and hence $\mathcal{L}^*(A) \cong_{eff} \mathcal{E}^*$).*

Proof. Let δ be a nonzero r.e. Turing degree and let C be an r.e. set of degree δ. Let $B =_{def} \{\langle n,i\rangle : n \in C \ \wedge \ i \in \omega\}$. We consider the following enumeration of B.

Let $B_{-1} =_{def} \emptyset$.

Stage s: For every $n \in C_s$, let k be the greatest number such that $\langle n,k\rangle \leq s+1$. Enumerate all elements among $\langle n,0\rangle, \ldots, \langle n,k\rangle$ which are not in B_{s-1} into B_s.

Now A is constructed as in Theorem 4 of [5], to be a thick subset of B. For every $e \in \omega$, we have a unique marker Δ_e.

Construction

Stage s. For every $e \leq s$, check whether there is an element $x \in \omega^{[n]}$ for some $n > e$, such that $x \in W_{e,s} - A_s$ and no element of $\omega^{[n]}$ is marked by Δ_e. Place Δ_e on the least such element x, if it exists. Enumerate all unmarked elements of $B_s - B_{s-1}$ into A_s.

First, we show that the enumeration of B has the property:

(P) if $n \in C$ and $e \in \omega$, then for all but finitely many $i \in \omega$, $\langle n,i\rangle \in B \setminus W_e$.

Assume that $n \in C_t$. Let $x \in \omega^{[n]}$ be such that $x > t$. Then $x \notin W_{e,t}$ (since $x > t$). Thus, if $x \in W_e$, then there is $s > t$ such that $x \in W_{e,s+1} - W_{e,s}$. Hence $x \leq s+1$, and $x \in B_s$.

In order to prove that A is semilow$_{1.5}$, it is enough to show that for every $e \in \omega$, ($W_e \setminus A$ is infinite $\Rightarrow$ $W_e \cap \overline{A}$ is infinite). Assume otherwise. That is, for some $e, W_e \setminus A$ is infinite and $W_e \cap \overline{A}$ is finite. Then $W_e \searrow A$ is infinite. Since $A_s \subseteq B_s$, by (P) there is no single $n \in C$ such that $(W_e \searrow A) \cap \omega^{[n]}$ is infinite. Hence there are infinitely many $n \in C$ such that $(W_e \searrow A) \cap \omega^{[n]} \neq \emptyset$. Let $x \in \omega^{[n]}$ be such that $n > e$ and $x \in W_e \searrow A$. Then for some s, $x \in W_{e,s} - A_s$. Therefore, there is a marked element $y \in \omega^{[n]}$ such that $y \in W_e$. Hence $y \in \overline{A}$. Thus, $W_e \cap \overline{A}$ is infinite, which is a contradiction.

Now, as in Theorem 4 of [5], we show that A is a nowhere simple set of degree δ and, as in Theorem 3 of [3], we show that A is not effectively nowhere simple. ∎

Harrington and Soare (unpublished) have recently proven that there is a nowhere simple set A such that $\mathcal{L}^*(A)$ is not isomorphic to $\mathcal{E}^*$.

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