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A secondary semantics for Second Order Intuitionistic Propositional Logic

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In this paper we propose a Kripke-style semantics for second order intuitionistic propositional logic and we provide a semantical proof of the disjunction and the explicit definability property. Moreover, we provide a tableau calculus which is sound and complete with respect to such a semantics.

1 Introduction

We propose a Kripke-style semantics for second order intuitionistic propositional logic. Our semantics can be viewed as a *secondary semantics with nested domains* in the sense of Skvortsov [6]. Namely, let $\mathcal{F}$ be a *Kripke frame*, that is a partially ordered set, and let $D(\mathcal{F})$ be the Heyting algebra of the upward-closed subsets of $\mathcal{F}$. In principal semantics, quantifiers ranges over all the elements of $D(\mathcal{F})$ and, as proved in [6], the set of formulas valid in such a semantics is non-recursively axiomatizable (according to [4] such a set is even *non-arithmetical*). On the other hand, in secondary semantics propositional quantifiers range over proper subsets of $D(\mathcal{F})$, and in [6] some examples of axiomatizable logics with a secondary semantics are given.

The logic $\mathbf{Ipl}^2$ generated from our semantics corresponds to H_2 of [6] and can be seen as a variant of the ones of Gabbay [2, 3] and Sobolev [7]. Such a semantics has an impredicative character connected with the distinction between *pseudomodels* and *models*, the latter being pseudomodels where every closed formula is simulated by an appropriate propositional constant. The domain of every element of a model is a set of propositional constants and propositional quantifiers range over these sets.

In this paper we prove that $\mathbf{Ipl}^2$ meets the disjunction property ($A \vee B \in \mathbf{Ipl}^2$ implies $A \in \mathbf{Ipl}^2$ or $B \in \mathbf{Ipl}^2$) and the explicit definability property ($\exists p\, A(p) \in \mathbf{Ipl}^2$ implies $A(H/p) \in \mathbf{Ipl}^2$ for some formula H). Our proof is semantical and, as far as we know, no semantical proof of constructivity for a secondary semantics has been given in the literature. In the paper we also provide a tableau calculus $\mathcal{T}$-$\mathbf{Ipl}^2$ for $\mathbf{Ipl}^2$ obtained by adding to a standard tableau calculus for intuitionistic propositional logic (see, e. g., [1]) the rules for quantifiers and a special rule. In the last section of the paper we show that $\mathcal{T}$-$\mathbf{Ipl}^2$ is sound and complete.

2 Preliminaries

In this paper we consider the propositional second order language $\mathcal{L}$ generated by a (possibly empty) countable set $\mathcal{C}$ of constant symbols, the set of logical constants $\wedge$, $\vee$, $\rightarrow$, $\perp$, $\exists$, $\forall$ and a denumerable set of propositional

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variables $\mathcal{V}$; $a, b, \dots$ (possibly with indexes) denote propositional constants and $p, q, \dots$ denote propositional variables. The set of second order propositional formulas (*wff*'s for short) is defined as follows: $\bot$, any propositional variable and any propositional constant is a wff; if A and B are wff's and p is a propositional variable, then $A \wedge B$, $A \vee B$, $A \to B$, $\exists p\, A$ and $\forall p\, A$ are wff's. Moreover, we use $\neg A$ as an abbreviation for $A \to \bot$ and $A \leftrightarrow B$ as a shorthand for $(A \to B) \wedge (B \to A)$.

The notions of *free* and *bounded* variable and the notions of *open* and *closed* wff are the usual ones. If A is a wff, $A(p_1, \dots, p_n)$ means that A contains at most the free variables $p_1, \dots, p_n$ and $A(H_1/p_1, \dots, H_n/p_n)$ denotes the simultaneous substitution of all the free occurrences of $p_1, \dots, p_n$ in A with the wff's $H_1, \dots, H_n$, respectively. Given a wff A, we denote with $\forall A$ the universal closure of A, that is the wff $\forall p_1 \dots \forall p_n A(p_1, \dots, p_n)$, where $p_1, \dots, p_n$ are all the free variables occurring in A.

Let $\mathcal{L}$ be a language for second order propositional logic generated by a (possibly empty) countable set of propositional constants $\mathcal{C}$; a *pseudomodel* for $\mathcal{L}$ is a quadruple $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$, where

1. $\underline{P} = \langle P, \leq\rangle$ is a poset;
2. $\mathcal{D} : P \longrightarrow 2^{\overline{\mathcal{C}}}$, where $\overline{\mathcal{C}}$ is a set of constants including $\mathcal{C}$, is the *domain function* such that, for every $\alpha \in P$, $\mathcal{C} \subseteq \mathcal{D}(\alpha)$ and, if $\alpha \leq \beta$, then $\mathcal{D}(\alpha) \subseteq \mathcal{D}(\beta)$;
3. $\mathcal{F} \subseteq P \times \overline{\mathcal{D}}$, where $\overline{\mathcal{D}} = \bigcup_{\alpha \in P} \mathcal{D}(\alpha)$, is the *valuation relation* such that, if $\langle\alpha, a\rangle \in \mathcal{F}$ then $a \in \mathcal{D}(\alpha)$, and if $\alpha \leq \beta$, then $\langle\beta, a\rangle \in \mathcal{F}$.

We associate with each element α of $\underline{K}$ the language $\mathcal{L}_{\underline{K}}(\alpha)$ built over the set of propositional constants $\mathcal{D}(\alpha)$. Since the model has nested domains, $\alpha \leq \beta$ implies $\mathcal{L} \subseteq \mathcal{L}_{\underline{K}}(\alpha) \subseteq \mathcal{L}_{\underline{K}}(\beta)$. We denote with $\mathcal{L}_{\underline{K}}$ the language $\bigcup_{\alpha \in P} \mathcal{L}_{\underline{K}}(\alpha)$.

The forcing relation $\Vdash$ between an element $\alpha \in P$ and a closed wff $A \in \mathcal{L}_{\underline{K}}(\alpha)$ is inductively defined as follows:

1. $\alpha \Vdash c$ iff $\langle\alpha, c\rangle \in \mathcal{F}$;
2. $\alpha \nVdash \bot$;
3. $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$;
4. $\alpha \Vdash A \vee B$ iff $\alpha \Vdash A$ or $\alpha \Vdash B$;
5. $\alpha \Vdash A \to B$ iff, for all $\beta \in P$ such that $\alpha \leq \beta$, if $\beta \Vdash A$, then $\beta \Vdash B$;
6. $\alpha \Vdash \exists p\, A(p)$ iff there exists $c \in \mathcal{D}(\alpha)$ such that $\alpha \Vdash A(c/p)$;
7. $\alpha \Vdash \forall p\, A(p)$ iff, for all $\beta \in P$ such that $\alpha \leq \beta$ and for all $c \in \mathcal{D}(\beta)$, $\beta \Vdash A(c/p)$.

Given an open wff $A \in \mathcal{L}_{\underline{K}}(\alpha)$, $\alpha \Vdash A$ iff α forces its universal closure, i. e., $\alpha \Vdash \forall A$.

It is easy to check that the forcing relation meets the monotonicity condition:

Proposition 2.1 (Monotonicity condition) *Let $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ be a pseudomodel and let $\alpha \in P$. For each wff $A \in \mathcal{L}_{\underline{K}}(\alpha)$, if $\alpha \Vdash A$, then $\beta \Vdash A$ for every $\beta \in P$ such that $\alpha \leq \beta$.*

The semantics based on the above notion of Kripke pseudomodel is a principal semantics, according to the classification of [6]. Indeed, pseudomodels correspond to n*-structures* (where upward-closed subsets are identified with propositional constants) and the set of formulas valid in every pseudomodel coincides with the logic H_2^+ of [6], which is non-recursively axiomatizable. In [4] it is also proved that H_2^+ is non-arithmetical.

To get a secondary semantics we introduce the following notion which corresponds to *Sobolev completeness* (see [6]).

Definition 2.2 Let $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ be a pseudomodel for $\mathcal{L}$. $\underline{K}$ is a *model* for $\mathcal{L}$ iff, for every $\alpha \in P$ and for every closed wff $A \in \mathcal{L}_{\underline{K}}(\alpha)$, there exists $c \in \mathcal{D}(\alpha)$ such that $\alpha \Vdash A \leftrightarrow c$.

Given $A \in \mathcal{L}$, A is *valid*, and we write $\vDash A$, iff, for every model $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ for $\mathcal{L}$ and for every $\alpha \in P$, $\alpha \Vdash A$. Fixed a language $\mathcal{L}$, we denote with $\mathbf{Ipl}^2$ the set of all the valid wff's of $\mathcal{L}$.

The logic $\mathbf{Ipl}^2$ coincides with the logic H_2 of [6] axiomatized by adding to a Hilbert-style calculus for intuitionistic propositional logic the following axioms and rules:

Bernays' schemata: $\forall p\, A(p) \to A(c/p)$ and $A(c/p) \to \exists p\, A(p)$, where c is any term.

Comprehension schema: $\exists q\, (q \leftrightarrow A)$, where q is not free in A.

$\dfrac{A \to B}{\exists p\, A \to B}$, $\quad \dfrac{B \to A}{B \to \forall p\, A}$, where p is not free in B.

3 A tableau calculus for Ipl^2

We introduce a tableau calculus $\mathcal{T}$-$\mathbf{Ipl}^2$ for second order intuitionistic propositional logic using the signs $\mathbf{T}$ and $\mathbf{F}$. A *signed formula* (*swff* for short) is a string of the form $\mathcal{S}A$, where $\mathcal{S} \in \{\mathbf{T}, \mathbf{F}\}$ and A is a closed wff.

The meaning of the signs $\mathbf{T}$ and $\mathbf{F}$ is explained in terms of *realizability*. Given a model $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ and $\alpha \in P$. Then α *realizes* $\mathbf{T}A$ iff $A \in \mathcal{L}_{\underline{K}}(\alpha)$ and $\alpha \Vdash A$, and α *realizes* $\mathbf{F}A$ iff $A \in \mathcal{L}_{\underline{K}}(\alpha)$ and $\alpha \nVdash A$. Given a swff H we write $\alpha \rhd H$ to mean that α realizes H; α realizes a set S of swff's (and we write $\alpha \rhd S$) iff $\alpha \rhd H$ for every $H \in S$. A set S of swff's is *realizable* iff there is an element α of a model $\underline{K}$ such that $\alpha \rhd S$. A *configuration* is any finite sequence $S_1|\dots|S_j|\dots|S_n$ (with $n \geq 1$), where every S_j is a set of swff's; a *configuration is realizable* iff at least an S_j is realizable. S is *contradictory* if either $\mathbf{T}\bot \in S$ or there exists a wff A such that $\{\mathbf{T}A, \mathbf{F}A\} \subseteq S$. The following fact is immediate:

Proposition 3.1 *If a set of swff's is contradictory, then it is not realizable.*

The rules of the calculus $\mathcal{T}$-$\mathbf{Ipl}^2$ are the following:

$$\frac{S, \mathbf{T}(A \wedge B)}{S, \mathbf{T}A, \mathbf{T}B}\,\mathbf{T}\wedge, \quad \frac{S, \mathbf{F}(A \wedge B)}{S, \mathbf{F}A \mid S, \mathbf{F}B}\,\mathbf{F}\wedge, \quad \frac{S, \mathbf{T}(A \vee B)}{S, \mathbf{T}A \mid S, \mathbf{T}B}\,\mathbf{T}\vee, \quad \frac{S, \mathbf{F}(A \vee B)}{S, \mathbf{F}A, \mathbf{F}B}\,\mathbf{F}\vee,$$

$$\frac{S, \mathbf{T}(A \to B)}{S, \mathbf{F}A, \mathbf{T}(A \to B) \mid S, \mathbf{T}B}\,\mathbf{T}\!\to, \quad \frac{S, \mathbf{F}(A \to B)}{S_T, \mathbf{T}A, \mathbf{F}B}\,\mathbf{F}\!\to,$$

$$\frac{S, \mathbf{T}\forall p\, A(p)}{S, \mathbf{T}A(c/p), \mathbf{T}\forall p\, A(p)}\,\mathbf{T}\forall, \quad \frac{S, \mathbf{F}\forall p\, A(p)}{S_T, \mathbf{F}A(a/p)}\,\mathbf{F}\forall \quad \text{with } a \text{ new},$$

$$\frac{S, \mathbf{T}\exists p\, A(p)}{S, \mathbf{T}A(a/p)}\,\mathbf{T}\exists \quad \text{with } a \text{ new}, \quad \frac{S, \mathbf{F}\exists p\, A(p)}{S, \mathbf{F}A(c/p), \mathbf{F}\exists pA(p)}\,\mathbf{F}\exists,$$

$$\frac{S}{S, \mathbf{T}(H \leftrightarrow a)}\,\textit{special} \quad \text{with } H \text{ any closed wff, } a \text{ new and not occurring in } H.$$

In the above rules we use the notation $S,\ H$, where S is a set of swff's and H is a swff, to denote the set $S \cup \{H\}$. Every rule of the calculus but the *special*-rule applies to a *main swff*, which is the swff that is in evidence in the premise of the rule. As an example, $\mathbf{T}(A \wedge B)$ is the main swff of the rule $\mathbf{T}\wedge$ while $\mathbf{F}\exists pA(p)$ is the main swff of the rule $\mathbf{F}\exists$. The rules $\mathbf{F} \to$ and $\mathbf{F}\forall$ narrow the set S of swff's to the *certain part* of S, that is the set $S_T = \{\mathbf{T}X : \mathbf{T}X \in S\}$.

Given a set S of swff's, a rule R of $\mathcal{T}$-$\mathbf{Ipl}^2$ is *applicable* to S if S contains a swff that can be used as main swff of an application of R. A *proof table* is a finite sequence of applications of the rules of the calculus $\mathcal{T}$-$\mathbf{Ipl}^2$, starting from some configuration. The rules $\mathbf{F}\forall$, $\mathbf{T}\exists$ and *special* introduce as a parameter a new propositional constant symbol a, that is a constant symbol not occurring in the previous configurations of the proof.

A proof table is *closed* iff all the sets S_j of its final configuration are contradictory. A *proof of a wff B* in $\mathcal{T}$-$\mathbf{Ipl}^2$ is a closed proof table in $\mathcal{T}$-$\mathbf{Ipl}^2$ starting from the configuration $\{\mathbf{F}\forall B\}$.

As an exercise, the reader could build the proofs of the wff's

$$\forall p\,((A \to p) \to ((B \to p) \to p)) \to A \vee B \quad \text{and} \quad \forall p\,(\forall q\,(A(q) \to p) \to p) \to \exists q\, A(q).$$

Such wff's are the non trivial sides of the equivalences showing that in $\mathbf{Ipl}^2$ the logical constants $\vee$ and $\exists$ can be expressed in terms of $\to$ and $\forall$. We remark that the proofs of such formulas use in an essential way the *special*-rule.

Definition 3.2 (Consistent set) A set of swff's S is *consistent* iff no proof table starting from any finite subset of S is closed.

As we show in Section 5, the calculus $\mathcal{T}$-$\mathbf{Ipl}^2$ is valid and complete with respect to our semantics. In particular we prove the following version of the Completeness Theorem:

Theorem 3.3 (Completeness Theorem) *Let S be a countable set of swff's. S is consistent iff S is realizable.*

We point out that, as a consequence of Completeness Theorem and the definition of consistent set of swff's, the Compactness Theorem holds:

Theorem 3.4 (Compactness Theorem) *Let S be a countable set of swff's. S is realizable iff every finite subset of S is realizable.*

4 Disjunction and explicit definability properties

In this section we prove that $\mathbf{Ipl}^2$ satisfies the *disjunction property* (if $A \vee B \in \mathbf{Ipl}^2$, then $A \in \mathbf{Ipl}^2$ or $B \in \mathbf{Ipl}^2$) and the *explicit definability property* (if $\exists p\, A(p) \in \mathbf{Ipl}^2$, then $A(H/p) \in \mathbf{Ipl}^2$ for some wff H). We remark that the syntactical constructivity proof for first order tableau calculus for intuitionistic logic (see, e. g., [5] cannot be extended to $\mathcal{T}$-$\mathbf{Ipl}^2$ for the presence of the *special*-rule. Indeed, a closed tableau proof for $\mathbf{F}\exists p\, A(p)$ might not start with an application of the $\mathbf{F}\exists$-rule. Our proof is semantical and, as far as we know, no semantical constructivity proof for a secondary semantics has been given in the literature.

For the sake of simplicity we prove the disjunction property and the explicit definability property for closed wff's; the results can be easily generalized to the open case.

Theorem 4.1 *If $A \vee B \in \mathbf{Ipl}^2$, with $A \vee B$ a closed wff, then either $A \in \mathbf{Ipl}^2$ or $B \in \mathbf{Ipl}^2$.*

Proof. Let $\mathcal{L}$ be the language over the set of constant symbols $\mathcal{C}$. Let $A \vee B$ be a closed formula of $\mathcal{L}$ such that $A \notin \mathbf{Ipl}^2$ and $B \notin \mathbf{Ipl}^2$; we prove that $A \vee B \notin \mathbf{Ipl}^2$. Let $\underline{K}_1 = \langle P_1, \leq_1, \mathcal{D}_1, \mathcal{F}_1\rangle$ be a model for $\mathcal{L}$ with root ϱ_1 such that $\varrho_1 \nVdash A$, and let $\underline{K}_2 = \langle P_2, \leq_2, \mathcal{D}_2, \mathcal{F}_2\rangle$ be a model for $\mathcal{L}$ with root ϱ_2 such that $\varrho_2 \nVdash B$. We can assume, without loss of generality, that $P_1 \cap P_2 = \emptyset$. Let ϱ be an element not belonging to $P_1 \cup P_2$ and let t^* be a constant symbol not belonging to $\mathcal{L}_{\underline{K}_1} \cup \mathcal{L}_{\underline{K}_2}$. We define the structure $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ as follows:

$$P = P_1 \cup P_2 \cup \{\varrho\} \qquad \leq \, = \, \leq_1 \cup \leq_2 \cup \{(\varrho, \alpha) : \alpha \in P\}$$

$$\mathcal{D}(\alpha) = \begin{cases} (\mathcal{D}_1(\varrho_1) \times \mathcal{D}_2(\varrho_2)) \cup \{t^*\} \cup \mathcal{C} & \text{if } \alpha \equiv \varrho, \\ \mathcal{D}_1(\alpha) \cup (\mathcal{D}_1(\varrho_1) \times \mathcal{D}_2(\varrho_2)) \cup \{t^*\} & \text{if } \alpha \in P_1, \\ \mathcal{D}_2(\alpha) \cup (\mathcal{D}_1(\varrho_1) \times \mathcal{D}_2(\varrho_2)) \cup \{t^*\} & \text{if } \alpha \in P_2, \end{cases}$$

$$\begin{aligned} \mathcal{F} = \; & \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{\langle \alpha, (c,d)\rangle : \alpha \in P_1 \text{ and } \langle \alpha, c\rangle \in \mathcal{F}_1\} \\ & \cup \{\langle \alpha, (c,d)\rangle : \alpha \in P_2 \text{ and } \langle \alpha, d\rangle \in \mathcal{F}_2\} \cup \{\langle \alpha, t^*\rangle \mid \alpha \in P\}. \end{aligned}$$

It is easy to check that $\underline{K}$ is a pseudomodel for $\mathcal{L}$. Now, let H be an intuitionistically valid wff. Since $\varrho_1 \Vdash H$ in $\underline{K}_1$ and $\underline{K}_1$ is a model, there exists $t_1 \in \mathcal{D}_1(\varrho_1)$ such that $\varrho_1 \Vdash H \leftrightarrow t_1$ in $\underline{K}_1$ and hence $\langle \varrho_1, t_1\rangle \in \mathcal{F}_1$; similarly, there exists $t_2 \in \mathcal{D}_2(\varrho_2)$ such that $\langle \varrho_2, t_2\rangle \in \mathcal{F}_2$. Let $\alpha \in P$ such that $\varrho_1 \leq \alpha$ and $H \in \mathcal{L}_{\underline{K}}(\alpha)$, we denote with $\tau_1 H$ the wff of $\mathcal{L}_{\underline{K}_1}(\alpha)$ obtained by replacing every occurrence of t^* in H with t_1 and every occurrence of (c,d) in H with c. Analogously, given $\alpha \in P$ such that $\varrho_2 \leq \alpha$ and $H \in \mathcal{L}_{\underline{K}}(\alpha)$, we denote with $\tau_2 H$ the wff of $\mathcal{L}_{\underline{K}_2}(\alpha)$ obtained by replacing every occurrence of t^* in H with t_2 and every occurrence of (c,d) in H with d. It is easy to prove, by induction on the structure of H, the following facts:

(i) For every $\alpha \in P$ such that $\varrho_1 \leq \alpha$ and every $H \in \mathcal{L}_{\underline{K}}(\alpha)$, $\alpha \Vdash H$ in $\underline{K}$ iff $\alpha \Vdash \tau_1 H$ in $\underline{K}_1$.

(ii) For every $\alpha \in P$ such that $\varrho_2 \leq \alpha$ and every $H \in \mathcal{L}_{\underline{K}}(\alpha)$, $\alpha \Vdash H$ in $\underline{K}$ iff $\alpha \Vdash \tau_2 H$ in $\underline{K}_2$.

To prove that $\underline{K}$ is a model for $\mathcal{L}$ we have to show that, for every $\alpha \in P$ and every closed $H \in \mathcal{L}_{\underline{K}}(\alpha)$, there exists a constant $c_H \in \mathcal{D}(\alpha)$ such that $\alpha \Vdash H \leftrightarrow c_H$ in $\underline{K}$. Let us suppose that $\varrho_1 \leq \alpha$. Since $\underline{K}_1$ is a model and $\tau_1 H \in \mathcal{L}_{\underline{K}_1}(\alpha)$, there exists $c \in \mathcal{D}_1(\alpha)$ such that $\alpha \Vdash \tau_1 H \leftrightarrow c$ in $\underline{K}_1$. Since $\tau_1(H \leftrightarrow c) \equiv \tau_1 H \leftrightarrow c$, by (i) it follows that $\alpha \Vdash H \leftrightarrow c$ in $\underline{K}$. The proof is similar if $\varrho_2 \leq \alpha$. Now, let us suppose that $\alpha \equiv \varrho$. If $\varrho \Vdash H$ in $\underline{K}$, then $c_H \equiv t^*$ is the required constant. Let us assume that $\varrho \nVdash H$. Since $H \in \mathcal{L}_{\underline{K}}(\varrho_1) \cap \mathcal{L}_{\underline{K}}(\varrho_2)$, by the above discussion there exist $c \in \mathcal{D}(\varrho_1)$ and $d \in \mathcal{D}(\varrho_2)$ such that $\varrho_1 \Vdash H \leftrightarrow c$ in $\underline{K}$ and $\varrho_2 \Vdash H \leftrightarrow d$ in $\underline{K}$. Since $\tau_1(H \leftrightarrow c) \equiv \tau_1(H \leftrightarrow (c,d))$, by (i) we get $\varrho_1 \Vdash H \leftrightarrow (c,d)$ in $\underline{K}$; similarly $\varrho_2 \Vdash H \leftrightarrow (c,d)$ in $\underline{K}$. Let us prove that $\varrho \Vdash H \leftrightarrow (c,d)$ in $\underline{K}$. Let $\alpha \geq \varrho$ such that $\alpha \Vdash H$; then $\alpha \geq \varrho_1$ or $\alpha \geq \varrho_2$, hence $\alpha \Vdash (c,d)$. Let $\alpha \geq \varrho$ such that $\alpha \Vdash (c,d)$; then (by definition of $\mathcal{F}$) $\alpha \geq \varrho_1$ or $\alpha \geq \varrho_2$, hence $\alpha \Vdash H$ in $\underline{K}$. This proves that $\underline{K}$ is a model. To conclude the proof we must show that $\varrho \nVdash A \vee B$ in $\underline{K}$. But, $\varrho_1 \nVdash A$ by (i) (being $\tau_1 A \equiv A$) and $\varrho_2 \nVdash B$ by (ii) (being $\tau_2 A \equiv A$); therefore $\varrho \nVdash A \vee B$ in $\underline{K}$. □

We remark that, differently from the case of first order intuitionistic logic, the above proof cannot be directly extended to the case of explicit definability. We need some auxiliary definitions and results. Given a wff A, we denote with $A[H/c]$ the wff obtained by replacing every occurrence of the constant symbol c in A with the wff H. This notation extends in the obvious way to swff's and sets of swff's. It is easy to prove the following result:

Lemma 4.2 *Let $\underline{K}$ be a model, let S be a set of swff's and let H_1, H_2 be any two wff's. For every element α of $\underline{K}$, $\alpha \rhd S[H_1/c], \mathbf{T}(H_1 \leftrightarrow H_2)$ implies $\alpha \rhd S[H_2/c]$.*

In the proof of explicit definability we use the following fact:

Lemma 4.3 *Let S be a set of swff's, let H be a closed wff and let c be a constant symbol not occurring in H. $S[H/c]$ is realizable iff $S, \mathbf{T}(H \leftrightarrow c)$ is realizable.*

P r o o f. Let $\underline{K}$ be a model and α an element of $\underline{K}$ such that $\alpha \rhd S, \mathbf{T}(H \leftrightarrow c)$. From Lemma 4.2 it immediately follows that $S[H/c]$ is realizable in $\underline{K}$. Conversely, let $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ and $\alpha \in P$ such that $\alpha \rhd S[H/c]$. Since $\underline{K}$ is a model, there exists $d \in \mathcal{D}(\alpha)$ such that $\alpha \Vdash H \leftrightarrow d$. If $d \equiv c$, by Lemma 4.2 we get $\alpha \rhd S$, hence $S, \mathbf{T}(H \leftrightarrow c)$ is realizable in $\underline{K}$. If $d \not\equiv c$, let c' be a constant symbol not occurring in $\mathcal{L}_{\underline{K}}$ and let $\underline{K}' = \langle P, \leq, \mathcal{D}', \mathcal{F}'\rangle$ be the model where $\mathcal{D}'$ and $\mathcal{F}'$ are defined as follows:

$$\mathcal{D}'(\gamma) = \begin{cases} \mathcal{D}(\gamma) & \text{if } c \notin \mathcal{D}(\gamma), \\ \mathcal{D}(\gamma) \cup \{c'\} & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \langle \gamma, b\rangle \in \mathcal{F}' \text{ iff one of following cases holds: } & b \not\equiv c \text{ and } b \not\equiv c' \text{ and } \langle \gamma, b\rangle \in \mathcal{F}, \\ & b \equiv c \text{ and } \langle \gamma, d\rangle \in \mathcal{F}, \\ & b \equiv c' \text{ and } \langle \gamma, c\rangle \in \mathcal{F}. \end{aligned}$$

Given a wff A of $\mathcal{L}_{\underline{K}'}$, let us denote with τA the wff of $\mathcal{L}_{\underline{K}}$ obtained by replacing every occurrence of c with d and every occurrence of c' with c. It is easy to check that, for every $\beta \in P$ and for every closed $A \in \mathcal{L}_{\underline{K}'}(\beta)$, $\beta \Vdash A$ in $\underline{K}'$ iff $\beta \Vdash \tau A$ in $\underline{K}$. Since $\alpha \Vdash H \leftrightarrow d$ in $\underline{K}$ and $\tau(H \leftrightarrow c) \equiv H \leftrightarrow d$ (indeed, c and c' do not occur in H), we get $\alpha \Vdash H \leftrightarrow c$ in $\underline{K}'$. Since $\alpha \rhd S[H/c]$ in $\underline{K}$ and $\tau(S[H/c]) \equiv S[H/c]$ (indeed, c and c' do not occur in $S[H/c]$), we get $\alpha \rhd S[H/c]$ in $\underline{K}'$. From Lemma 4.2 we deduce that $\alpha \rhd S$ in $\underline{K}'$ and hence $\alpha \rhd S, \mathbf{T}(H \leftrightarrow c)$ in $\underline{K}'$. □

To prove the explicit definability property, we show how to construct a countermodel for a closed wff $\exists p\, A(p)$ of $\mathcal{L}$ over the set $\mathcal{C}$ of constant symbols, assuming that

(i) For every closed wff H of $\mathcal{L}$, $A(H/p) \notin \mathbf{Ipl}^2$.

We assume, without loss of generality, that $\mathcal{C}$ is denumerable.

Our proof shows how to build a countermodel for $\exists p\, A(p)$ starting from a family $\mathcal{K}$ of countermodels for the formulas $A(H/p)$. The essential point of the proof is to select the countermodels of $\mathcal{K}$ in such a way that, for every closed formula H of $\mathcal{L}$ there exists a constant c_H such that $H \leftrightarrow c_H$ is valid in every model of $\mathcal{K}$. This means that every closed wff H is simulated by the *same* constant symbol c_H in every model of $\mathcal{K}$. To build up the countermodel $\underline{K}$ for $\exists p\, A(p)$, we glue together the models of $\mathcal{K}$ adding a root ϱ whose domain contains the constant symbols c_H.

Let $H_1, H_2, \ldots$ be an enumeration of the closed wff's of $\mathcal{L}$. Since $\mathcal{C}$ is denumerable, we can define the set of swff's $S = \{\mathbf{T}(H_1 \leftrightarrow c_1),\ \mathbf{T}(H_2 \leftrightarrow c_2), \ldots\}$, where $c_1, c_2, \ldots$ are constant from $\mathcal{C}$ and, for every $n \geq 1$, c_n does not occur in $H_1 \leftrightarrow c_1, \ldots, H_{n-1} \leftrightarrow c_{n-1}, H_n$ and in $\exists p\, A(p)$. We remark that S establishes a one-to-one correspondence between the closed formulas of $\mathcal{L}$ and the constant symbols of $\mathcal{C}$. Let $d_1, d_2, \ldots$ be an enumeration of the constant symbols of $\mathcal{C}$. For every $n \geq 1$, let $S_n = \{\mathbf{F}A(d_n/p)\} \cup S$. We prove the following non-trivial fact:

(ii) For every $n \geq 1$, S_n is realizable.

Indeed, let us suppose that S_n is not realizable. Then, by the Compactness Theorem, there exists $k \geq 1$ such that $\Phi = \{\mathbf{F}A(d_n/p), \mathbf{T}(H_1 \leftrightarrow c_1), \ldots, \mathbf{T}(H_k \leftrightarrow c_k)\}$ is not realizable. We have two cases.

C a s e 1 . $d_n \not\equiv c_1, \ldots, d_n \not\equiv c_k$. By Lemma 4.3, $\{\mathbf{F}A(d_n/p)\}$ is not realizable, hence $A(d_n/p) \in \mathbf{Ipl}^2$ and this contradicts (i).

C a s e 2 . $d_n \equiv c_j$ for some $1 \leq j \leq k$. In this case Φ can be rewritten as

$$\Phi = \{\mathbf{F}A(c_j/p), \mathbf{T}(H_1 \leftrightarrow c_1), \ldots, \mathbf{T}(H_j \leftrightarrow c_j), \ldots, \mathbf{T}(H_k \leftrightarrow c_k)\}.$$

If $j > 1$, we can apply Lemma 4.3 to get the non realizable set

$$\{\mathbf{F}A(c_j/p), \mathbf{T}(H_2^1 \leftrightarrow c_2), \ldots, \mathbf{T}(H_j^1 \leftrightarrow c_j), \ldots, \mathbf{T}(H_k^1 \leftrightarrow c_k)\},$$

where, for all $2 \leq i \leq k$, $H_i^1 \equiv H_i[H_1/c_1]$. We can iterate the above procedure $(j-2)$ times (for instance, in the next step c_2 does not occur in H_2^1) until we get the non realizable set

$$\{\mathbf{F}A(c_j/p), \mathbf{T}(H_j^{j-1} \leftrightarrow c_j), \ldots, \mathbf{T}(H_k^{j-1} \leftrightarrow c_k)\}$$

(note that, if $j = 1$ this set coincides with Φ). Applying Lemma 4.3 one more time (noticing that c_j does not occur in $\exists p\, A(p)$), we get the non realizable set $\{\mathbf{F}A(H_j^{j-1}/p), \mathbf{T}(H_{j+1}^j \leftrightarrow c_{j+1}), \ldots, \mathbf{T}(H_k^j \leftrightarrow c_k)\}$. Since $c_{j+1}, \ldots, c_k$ do not occur in $A(H_j^{j-1}/p)$, further applications of Lemma 4.3 allow us to deduce that the set $\{\mathbf{F}A(H_j^{j-1}/p)\}$ is not realizable. Thus $A(H_j^{j-1}/p) \in \mathbf{Ipl}^2$ against (i). This concludes the proof of (ii).

We use the sets S_n to define the family of countermodels $\mathcal{K}$. By (ii), for every $n \geq 1$ there exists a model $\underline{K}_n = \langle P_n, \leq_n, \mathcal{D}_n, \mathcal{F}_n\rangle$ with root ϱ_n such that $\varrho_n \rhd S_n$ in $\underline{K}_n$. Let $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ be the structure defined as follows:

$$P = \bigcup_{n\geq 1} P_n \cup \{\varrho\} \quad \text{where } \varrho \text{ is a new element,} \qquad \leq = \bigcup_{n\geq 1} \leq_n \cup \{(\varrho, \alpha) : \alpha \in P\},$$

$$\mathcal{D}(\alpha) = \begin{cases} \mathcal{D}_n(\alpha) & \text{if } \alpha \in P_n, \\ \mathcal{C} & \text{if } \alpha \equiv \varrho, \end{cases} \qquad \mathcal{F} = \bigcup_{n\geq 1} \mathcal{F}_n \cup \{\langle \varrho, c^*\rangle\},$$

where c^* is the constant symbol of $\mathcal{C}$ such that $\mathbf{T}(\forall p\,(p \to p) \leftrightarrow c^*)$ belongs to S (note that c^* is the only constant forced in ϱ).

It is immediate to check that $\underline{K}$ is a pseudomodel for $\mathcal{L}$. Moreover, for every $\alpha \in P_n$ and $B \in \mathcal{L}_{\underline{K}_n}(\alpha)$, $\alpha \Vdash B$ in $\underline{K}_n$ iff $\alpha \Vdash B$ in $\underline{K}$. Let us prove that $\underline{K}$ is a model for $\mathcal{L}$, that is, for every $\alpha \in P$ and every closed wff B of $\mathcal{L}_{\underline{K}}(\alpha)$, there exists a constant $b \in \mathcal{D}(\alpha)$ such that $\alpha \Vdash B \leftrightarrow b$ in $\underline{K}$. If $\alpha \in P_n$, the assertion holds since $\underline{K}_n$ is a model. Let us assume that $\alpha \equiv \varrho$. If $\varrho \Vdash B$ in $\underline{K}$, then $b \equiv c^*$ is the required constant. Let us assume that $\varrho \nVdash B$. Since B is a closed wff of $\mathcal{L}_{\underline{K}}(\varrho)$ and $\mathcal{L}_{\underline{K}}(\varrho) = \mathcal{L}$, there exists $k \geq 1$ such that $B \equiv H_k$ and $\mathbf{T}(H_k \leftrightarrow c_k) \in S$. It follows that B is simulated by c_k in all models $\underline{K}_n$; indeed, $\varrho_n \rhd S$ in $\underline{K}_n$ for every $n \geq 1$, hence $\varrho_n \Vdash H_k \leftrightarrow c_k$ for every $n \geq 1$. Let $\alpha \geq \varrho$ such that $\alpha \Vdash H_k$. Since $\alpha \neq \varrho$, there exists $j \geq 1$ such that $\varrho_j \leq \alpha$, hence $\alpha \Vdash c_k$. Let $\alpha \geq \varrho$ such that $\alpha \Vdash c_k$. Clearly $c_k \neq c^*$ (since $H_k \not\equiv \forall p\,(p \to p)$), hence $\varrho_j \leq \alpha$ for some $j \geq 1$; thus $\alpha \Vdash H_k$. This implies that $\varrho \Vdash H_k \leftrightarrow c_k$, hence $\underline{K}$ is a model for $\mathcal{L}$.

Now, let us assume that $\varrho \Vdash \exists p\, A(p)$. Then there exists $b \in \mathcal{D}(\varrho)$ such that $\varrho \Vdash A(b/p)$. On the other hand, there exists $j \geq 1$ such that $b \equiv d_j$, hence $\varrho_j \Vdash A(d_j/p)$. But this yields a contradiction since $\mathbf{F}A(d_j/p) \in S_j$ and $\varrho_j \rhd S_j$. We can conclude:

Theorem 4.4 *If $\exists p\, A(p) \in \mathbf{Ipl}^2$, with $\exists p\, A(p)$ a closed wff, then there exists a closed wff H such that $A(H/p) \in \mathbf{Ipl}^2$.*

5 Soundness and completeness of $\mathcal{T}$-Ipl^2

To conclude the paper we prove that the tableau calculus $\mathcal{T}$-$\mathbf{Ipl}^2$ is sound and complete with respect to $\mathbf{Ipl}^2$.

As usual, the main step of the Soundness Theorem consists in proving that the rules of the calculus preserve realizability.

Lemma 5.1 *Let S be a set of swff's, let $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ be a model, and let $\alpha \in P$ such that $\alpha \rhd S$. If R is a rule of $\mathcal{T}$-$\mathbf{Ipl}^2$ applicable to S, then there exist a set S' in the configuration obtained by applying R to S, a model $\underline{K}' = \langle P', \leq', \mathcal{D}', \mathcal{F}'\rangle$, and $\beta \in P'$ such that $\beta \rhd S'$.*

Proof. The proof requires a careful analysis of the various rules; we only analyze some relevant cases.

Special-rule: Let S, $\mathbf{T}(H \leftrightarrow a)$ be the configuration obtained by applying the *special*-rule to S. Here we assume, without loss of generality, that a is not a constant symbol of $\mathcal{L}_{\underline{K}}$. Since H is a closed wff of $\mathcal{L}_{\underline{K}}(\alpha)$ and $\underline{K}$ is a model, there exists a constant $c \in \mathcal{D}(\alpha)$ such that $\alpha \Vdash H \leftrightarrow c$. Now, let us consider the pseudomodel $\underline{K}' = \langle P, \leq, \mathcal{D}', \mathcal{F}' \rangle$ where

$$\mathcal{D}'(\gamma) = \begin{cases} \mathcal{D}(\gamma) & \text{if } c \notin \mathcal{D}(\gamma), \\ \mathcal{D}(\gamma) \cup \{a\} & \text{otherwise,} \end{cases} \qquad \mathcal{F}' = \mathcal{F} \cup \{\langle \gamma, a \rangle : \langle \gamma, c \rangle \in \mathcal{F}\}.$$

It is easy to check that, for every $\gamma \in P$ and every $B(p) \in \mathcal{L}_{\underline{K}'}(\gamma)$ such that a does not occur in $B(p)$, $\gamma \Vdash B(a/p)$ in $\underline{K}'$ iff $\gamma \Vdash B(c/p)$ in $\underline{K}$. This implies that, for all $B \in \mathcal{L}_{\underline{K}}(\gamma)$, $\gamma \Vdash B$ in $\underline{K}'$ iff $\gamma \Vdash B$ in $\underline{K}$. It follows that $\underline{K}'$ is a model. Moreover $\alpha \rhd S$ in $\underline{K}'$ and $\alpha \Vdash H \leftrightarrow a$ in $\underline{K}'$ and this concludes the proof.

F∀-rule: We have to prove that $S_T, \mathbf{F}A(a/p)$ is realizable. Let us assume, without loss of generality, that a is not a constant symbol of $\mathcal{L}_{\underline{K}}$. If $\alpha \rhd \mathbf{F}\forall p\, A(p)$, then $\alpha \nVdash \forall p\, A(p)$. Hence there exist $\beta \geq \alpha$ and $c \in \mathcal{D}(\beta)$ such that $\beta \nVdash A(c/p)$ (clearly $\beta \rhd S_T$). Now, let us consider the pseudomodel $\underline{K}' = \langle P, \leq, \mathcal{D}', \mathcal{F}' \rangle$ where $\mathcal{D}'$ and $\mathcal{F}'$ are defined as above. Reasoning as in the previous case, we can prove that $\underline{K}'$ is a model and $\beta \rhd S_T, \mathbf{F}A(a/p)$ in $\underline{K}'$. □

From the previous lemma it immediately follows that, if a configuration is realizable, then the configuration obtained by applying to the former configuration one of the rules of $\mathcal{T}$-**Ipl**^{2} is realizable. This leads to the Soundness Theorem. Indeed, let us assume that S is not consistent. Then there exist a finite subset S' of S and a closed proof-table for S'. If S is realizable, then there exist a model $\underline{K} = \langle P, \leq, \mathcal{D}, \mathcal{F} \rangle$ and $\alpha \in P$ such that $\alpha \rhd S$. Hence $\alpha \rhd S'$ and, by Lemma 5.1, a contradictory set of swff's is realizable against Proposition 3.1.

Theorem 5.2 (Soundness) *Let S be a set of swff's. If S is realizable, then S is consistent.*

The Completeness Theorem has the following form: *If a countable set S of closed swff's is consistent, then there is a model $\underline{K}$ together with an element α of $\underline{K}$ such that $\alpha \rhd S$.* Our proof is based on a general method allowing us to build up, for every consistent set S of swff's, a model $\underline{K}(S)$ whose root realizes S. The construction of $\underline{K}(S)$ consists of two main steps. In the first step, starting from a consistent set of swff's S, we construct two sets S^* and $\overline{S}$, called the *saturated set of* S and the *node set of* S, respectively. The set $\overline{S}$ will be the root of the model $\underline{K}(S)$, and the swff's in $\overline{S}$ will determine the forcing relation in $\overline{S}$. In the second step we construct the successor sets of $\overline{S}$. The model $\underline{K}(S)$ will be constructed by iterating the two steps on the new elements, and so on.

Given a swff H, we call *extension(s) of* H the set(s) $\mathcal{R}^1_H, \dots, \mathcal{R}^n_H$ (where $n \in \{1, 2\}$) coinciding with the sets in the configuration obtained by applying the rule related to H in $\mathcal{T}$-**Ipl**^{2} to the configuration $\{H\}$. Moreover, given a set S of swff's we denote with $\Pi(S)$ the set of the constant symbols occurring in the swff's of S.

Given a countable and consistent set S of closed swff's and a set Π of constant symbols including $\Pi(S)$, let $\mathcal{C}'$ be a denumerable set of constants such that $\Pi \cap \mathcal{C}' = \emptyset$. We take from $\mathcal{C}'$ the constant symbols needed to build up the saturated set of S. It is easy to check that under the above assumptions $\mathcal{C}'$ contains enough symbols to construct such a set.

Let $\overline{\mathcal{L}}$ be the language over the set of constants $\Pi \cup \mathcal{C}'$ and let us consider an enumeration ε_S: $A_1, \dots, A_n, \dots$ of the swff's of S and an enumeration $\varepsilon_{\overline{\mathcal{L}}}$: $\Phi_1, \dots, \Phi_n, \dots$ of the closed wff's of $\overline{\mathcal{L}}$. We inductively define a sequence $\{S_i\}_{i \in \omega}$ whose elements are sets of closed swff's as follows: $S_0 = \emptyset$. Given $S_i = \{H_1, H_2, \dots\}$ and given $c \in \mathcal{C}'$ such that c does not occur in $\Pi \cup \Pi(\bigcup_{h \leq i} S_h)$ and in Φ_{i+1}, then

$$S_{i+1} = \textstyle\bigcup_{H_j \in S_i} \mathcal{U}(H_j, i) \cup \{A_{i+1}, \mathbf{T}(\Phi_{i+1} \leftrightarrow c)\},$$

where, setting

$$\begin{aligned} S'_j &= \mathcal{U}(H_1, i) \cup \dots \cup \mathcal{U}(H_{j-1}, i) \cup \{H_j, H_{j+1} \dots, A_{i+1}, A_{i+2}, \dots\}, \\ \Pi'_j &= \Pi \cup \Pi(\textstyle\bigcup_{h \leq i} S_h) \cup \Pi\,(\mathcal{U}(H_1, i) \cup \dots \cup \mathcal{U}(H_{j-1}, i)) \cup \{c\} \cup \Pi(\{\Phi_{i+1}\}), \end{aligned}$$

the set $\mathcal{U}(H_j, i)$ is defined as follows:

1. If H_j is a swff of the kind $\mathbf{T}(A \wedge B)$, $\mathbf{F}(A \wedge B)$, $\mathbf{T}(A \vee B)$, $\mathbf{F}(A \vee B)$ or $\mathbf{T}(A \to B)$, then $\mathcal{U}(H_j, i) = \mathcal{R}_{H_j}$, where $\mathcal{R}_{H_j}$ is any extension of H_j such that $(S'_j \setminus \{H_j\}) \cup \mathcal{R}_{H_j}$ is consistent.
2. If $H_j \equiv \mathbf{T}\forall p\, A(p)$, then $\mathcal{U}(H_j, i)$ is $\{\mathbf{T}\forall p\, A(p), \mathbf{T}A(c_1/p), \mathbf{T}A(c_2/p), \dots\}$, where $\{c_1, c_2, \dots\} = \Pi'_j$.
3. If $H_j \equiv \mathbf{F}\exists p\, A(p)$, then $\mathcal{U}(H_j, i)$ is $\{\mathbf{F}\exists\, pA(p), \mathbf{F}A(c_1/p), \mathbf{F}A(c_2/p), \dots\}$, where $\{c_1, c_2, \dots\} = \Pi'_j$.
4. If $H_j \equiv \mathbf{T}\exists p\, A(p)$, then $\mathcal{U}(H_j, i)$ is $\{\mathbf{T}A(d/p)\}$, where $d \in \mathcal{C}'$ and $d \notin \Pi'_j$.
5. In all the other cases $\mathcal{U}(H_j, i) = \{H_j\}$.

It is easy to prove, by induction on $i \geq 0$, that if S is consistent, then every S_i is consistent. Hence, it is always possible to find a set $\mathcal{U}(H_j, i)$ satisfying the above conditions.

The *saturated set of* S *w. r. t.* Π is the set $S^* = \bigcup_{i \geq 0} S_i$. We remark that such a set is not unique since different choices of the enumerations ε_S and $\varepsilon_{\overline{\mathcal{L}}}$ give rise to different S^*.

Given a set of swff's V, we say that a swff $H \in V$ is *final in* V iff one of the following cases holds:

- H is either of the form $\mathbf{T}c$ or $\mathbf{F}c$, with c a constant symbol or $c \equiv \bot$;
- H is either of the form $\mathbf{F}(A \to B)$, $\mathbf{T}\forall pA(p)$ or $\mathbf{F}\forall p\, A(p)$;
- H is of the form $\mathbf{T}(A \to B)$ and $\mathbf{T}B \notin V$.

The *node set of* S *w. r. t.* Π is the set $\overline{S} = \{H : H \text{ is final in } S^*\}$ We also call $\overline{S}$ the *node set related to* S^*. Since any finite subset of $\overline{S}$ is contained in some S_i, it follows that $\overline{S}$ is consistent.

Given a node set $\overline{S}$ of a consistent set of swff's S, the *successor sets of* $\overline{S}$ are defined as follows:

- If $\mathbf{F}(A \to B) \in \overline{S}$, then $\overline{S}_T \cup \{\mathbf{T}A, \mathbf{F}B\}$ is a successor set of $\overline{S}$.
- If $\mathbf{F}\forall p\, A(p) \in \overline{S}$ and c is a constant symbol such that $c \notin \Pi(S^*)$, then $\overline{S}_T \cup \{\mathbf{F}A(c/p)\}$ is a successor set of $\overline{S}$.

Given a consistent set S of closed swff's of $\mathcal{L}$, the structure $\underline{K}(S) = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ is defined as follows:

- Let $\overline{S}$ be a node set of S w. r. t. $\Pi(S)$, then $\overline{S} \in P$ and $\mathcal{D}(\overline{S}) = \Pi(S^*)$.
- For every $\overline{\Gamma} \in P$ and every successor set U of $\overline{\Gamma}$, let $\overline{U}$ be a node set of U w. r. t. $\mathcal{D}(\overline{\Gamma}) \cup \Pi(U)$. Then $\overline{U}$ belongs to P, $\overline{U}$ is an immediate successor of $\overline{\Gamma}$ and $\mathcal{D}(\overline{U}) = \Pi(U^*)$.
- $\leq$ is the transitive and reflexive closure of the immediate successor relation.
- For every $\overline{\Gamma} \in P$ and for every $c \in \mathcal{D}(\overline{\Gamma})$, $\langle\overline{\Gamma}, c\rangle \in \mathcal{F}$ iff $\mathbf{T}c \in \overline{\Gamma}$.

It is easy to check that $\underline{K}(S)$ is a pseudomodel for $\overline{\mathcal{L}}$. In particular, $\overline{\Gamma} \leq \overline{\Delta}$ implies $\mathcal{D}(\overline{\Gamma}) \subseteq \mathcal{D}(\overline{\Delta})$. Indeed, if $\overline{\Delta}$ is an immediate successor of $\overline{\Gamma}$, then $\overline{\Delta}$ is the node set w. r. t. $\mathcal{D}(\overline{\Gamma}) \cup \Pi(\Delta)$ of a successor set Δ of $\overline{\Gamma}$. Hence, the set Δ^* contains a swff of the kind $\mathbf{T}(c \leftrightarrow d)$ for every constant symbol $c \in \mathcal{D}(\overline{\Gamma})$, thus $\mathcal{D}(\overline{\Gamma}) \subseteq \Pi(\Delta^*) = \mathcal{D}(\overline{\Delta})$. As for the valuation relation, if $\langle\overline{\Gamma}, c\rangle \in \mathcal{F}$, then $\mathbf{T}c \in \overline{\Gamma}$ and, being $\mathbf{T}c \in \overline{\Gamma}_T$, $\mathbf{T}c$ belongs to every successor set U of $\overline{\Gamma}$. Moreover, since $\mathbf{T}c \in U$ implies that $\mathbf{T}c$ is final in U^*, $\mathbf{T}c \in \overline{U}$. Thus, $\langle\overline{\Gamma}, c\rangle \in \mathcal{F}$ implies $\langle\overline{\Delta}, c\rangle \in \mathcal{F}$ for every $\overline{\Delta} \geq \overline{\Gamma}$.

Now, we prove the main lemma:

Lemma 5.3 *Let S be a countable and consistent set of closed swff's and let $\underline{K}(S) = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ be the pseudomodel defined above. For every $\overline{\Gamma} \in P$ and for every swff $H \in \Gamma^*$, $\overline{\Gamma} \rhd H$.*

P r o o f. The proof is by induction on the structure of H.

Basis. If $H \equiv \mathcal{S}c \in S^*$ with c a constant symbol and $\mathcal{S} \in \{\mathbf{T}, \mathbf{F}\}$, then, since $\mathcal{S}c$ is a final swff in Γ^*, $\mathcal{S}c \in \overline{\Gamma}$. If $\mathcal{S}$ is $\mathbf{T}$, then $\mathbf{T}c$ is realized in $\overline{\Gamma}$ by the definition of forcing; if $\mathcal{S}$ is $\mathbf{F}$, then, since $\overline{\Gamma}$ is consistent, $\mathbf{T}c$ cannot belong to $\overline{\Gamma}$, hence $\overline{\Gamma} \not\Vdash c$, which implies that $\mathbf{F}c$ is realized in $\overline{\Gamma}$.

Step. The proof goes on by cases according to the structure of H. Here we only give some illustrative examples.

C a s e $H \equiv \mathbf{T}(A \to B)$. We have to prove that, for every $\overline{\Delta}$ such that $\overline{\Gamma} \leq \overline{\Delta}$, either $\overline{\Delta} \not\Vdash A$ or $\overline{\Delta} \Vdash B$. Let us consider $\overline{\Delta} \geq \overline{\Gamma}$, we have two cases:

1. If $\mathbf{F}A \in \Delta^*$ then, by induction hypothesis, $\overline{\Delta} \not\Vdash A$.
2. If $\mathbf{F}A \notin \Delta^*$, then there exists $\overline{\Theta} \in P$ such that $\overline{\Gamma} \leq \overline{\Theta} \leq \overline{\Delta}$, $\{\mathbf{T}(A \to B), \mathbf{T}B\} \subseteq \Theta^*$ and $\mathbf{T}(A \to B) \notin \overline{\Theta}$. By induction hypothesis $\overline{\Theta} \rhd \mathbf{T}B$, hence $\overline{\Delta} \Vdash B$.

Case $H \equiv \mathbf{F}\forall p\, A(p)$. Since H is final in Γ^*, H belongs to $\overline{\Gamma}$. By construction there exists an immediate successor $\overline{\Delta}$ of $\overline{\Gamma}$ which is the node set of a successor set Δ of Γ containing a swff $\mathbf{F}A(c/p)$. Since $\mathbf{F}A(c/p) \in \Delta^*$, by induction hypothesis $\overline{\Delta} \rhd \mathbf{F}A(c/p)$ and hence $\overline{\Gamma} \nVdash \forall p\, A(p)$. □

Now, given a consistent set S of closed swff's, let $\overline{\Gamma}$ be any element of $\underline{K}(S)$ and let A be any closed wff of $\mathcal{L}_{\underline{K}(S)}(\overline{\Gamma})$. By construction of Γ^* there exists a constant c of $\mathcal{L}_{\underline{K}(S)}(\overline{\Gamma})$ such that $\mathbf{T}(A \leftrightarrow c) \in \Gamma^*$. By the previous lemma $\overline{\Gamma} \rhd \mathbf{T}(A \leftrightarrow c)$, hence $\overline{\Gamma} \Vdash A \leftrightarrow c$. Thus we have:

Lemma 5.4 *Let S be a countable and consistent set of closed swff's. The structure $\underline{K}(S) = \langle P, \leq, \mathcal{D}, \mathcal{F}\rangle$ defined above is a model.*

By Lemmas 5.3 and 5.4 it immediately follows:

Theorem 5.5 (Completeness) *Let S be a countable set of swff's. If S is consistent, then S is realizable.*

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