HOW MUCH SWEETNESS IS THERE IN THE UNIVERSE?

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Abstract. We continue investigations of forcing notions with strong ccc properties introducing new methods of building sweet forcing notions. We also show that quotients of topologically sweet forcing notions over Cohen reals are topologically sweet while the quotients over random reals do not have to be such.

0. Introduction

One of the main ingredients of the construction of the model for *all projective sets of reals have the Baire property* presented in Shelah [7, §7] was a strong ccc property of forcing notions called *sweetness*. This property is preserved in amalgamations and also in compositions with the Hechler forcing notion $\mathbb{D}$ and the Universal Meager forcing $\mathbb{UM}$ (see [7, §7]; a full explanation of how this is applied can be found in [3]). Stern [10] considered a slightly weaker property, *topological sweetness*, which is also preserved in amalgamations and compositions with $\mathbb{D}$ and $\mathbb{UM}$. We further investigated the sweet properties of forcing notions in [6, §4], where we introduced a new property called *iterable sweetness* and we showed how one can build sweet forcing notions. New examples of iterably sweet forcing notions can be used in constructions like [7, §7], [9], but it could be that there is no need for this — the old forcing notions could be adding generic objects for all of them. In [4] we proved that this is exactly what happens with the natural examples of sweet forcing notions determined by the universality parameters as in [6, §2.3]: a sequence Cohen real — dominating real — Cohen real produces generic filters for many of them.

In the present paper we show that sweetness is not so rare after all and we give more constructions of sweet forcing notions. In the first section we present a new method of building sweet forcing notions and we give our first example: a forcing notion $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$ associated with scattered subtrees of $2^{<\omega}$. We do not know if the iterations of "old" forcing notions add generic objects for $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$, but in Proposition 1.8 we present an indication that this does not happen. In the second section we use our method to introduce two large families of sweet forcing notions, in some sense generalizing the known examples from [6]. This time we manage to show that *some* of our forcing notions are really new by showing that we have too many different examples (in Theorems 2.9, 2.14).

In the last section of the paper we investigate the preservation of topological sweetness under some operations. We note that a complete subforcing of a topologically sweet separable partial order is equivalent to a topologically sweet forcing (in

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Proposition 3.5). We also show that the quotient of a topologically sweet forcing notion by a Cohen subforcing is topologically sweet (Theorem 3.7), but quotients by random real do not have to be topologically sweet (Corollary 3.10).

0.1. **Notation.** Our notation is rather standard and compatible with that of classical textbooks (like Jech [2] or Bartoszyński and Judah [1]). In forcing we keep the older convention that *a stronger condition is the larger one*. Our main conventions are listed below.

(1) For a forcing notion $\mathbb{P}$, $\Gamma_{\mathbb{P}}$ stands for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$. With this one exception, all $\mathbb{P}$–names for objects in the extension via $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$).
The weakest element of $\mathbb{P}$ will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one and that there is no other condition equivalent to it).
(2) The complete Boolean algebra determined by a forcing notion $\mathbb{P}$ is denoted by $\mathbf{BA}(\mathbb{P})$. For a complete Boolean algebra $\mathbb{B}$, $\mathbb{B}^+$ is $\mathbb{B}\setminus\{\mathbf{0}_{\mathbb{B}}\}$ treated as a forcing notion (so the order is the reverse Boolean order). Also, for a formula φ, the Boolean value (with respect to $\mathbb{B}$) of φ will be denoted by $[\![\varphi]\!]_{\mathbb{B}}$.
(3) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet ($\alpha,\beta,\gamma,\delta\ldots$) and also by i,j (with possible sub- and superscripts). Cardinal numbers will be called κ,λ,μ.
(4) For two sequences η,ν we write $\nu\vartriangleleft\eta$ whenever ν is a proper initial segment of η, and $\nu\trianglelefteq\eta$ when either $\nu\vartriangleleft\eta$ or $\nu=\eta$. The length of a sequence η is denoted by $\mathrm{lh}(\eta)$.
(5) The quantifier $(\exists^\infty n)$ is an abbreviation for $(\forall m\in\omega)(\exists n>m)$.
(6) The Cantor space 2^ω and the Baire space ω^ω are the spaces of all functions from ω to 2, ω, respectively, equipped with the natural (Polish) topology.

0.2. **Background on sweetness.** Let us recall basic definitions related to sweet forcing notions.

Definition 0.1 (Shelah [7, Def. 7.2])**.** A pair $(\mathbb{P},\bar{E})$ is *model of sweetness* whenever:

(i) $\mathbb{P}$ is a forcing notion,
(ii) $\bar{E}=\langle E_n:n<\omega\rangle$, each E_n is an equivalence relation on $\mathbb{P}$ such that $\mathbb{P}/E_n$ is countable,
(iii) equivalence classes of each E_n are $\leq_{\mathbb{P}}$–directed, $E_{n+1}\subseteq E_n$,
(iv) if $\{p_i:i\leq\omega\}\subseteq\mathbb{P}$, $p_i\ E_i\ p_\omega$ (for $i\in\omega$), then
$$(\forall n\in\omega)(\exists q\geq p_\omega)(q\ E_n\ p_\omega\ \&\ (\forall i\geq n)(p_i\leq q)),$$
(v) if $p,q\in\mathbb{P}$, $p\leq q$ and $n\in\omega$, then there is $k\in\omega$ such that
$$(\forall p'\in[p]_{E_k})(\exists q'\in[q]_{E_n})(p'\leq q').$$

If there is a model of sweetness based on $\mathbb{P}$, then we say that $\mathbb{P}$ is *sweet*.

Definition 0.2 (Stern [10, Def. 1.2])**.** *A model of topological sweetness* is a pair $M=(\mathbb{P},\mathcal{B})$ such that $\mathbb{P}=(\mathbb{P},\leq)$ is a forcing notion, $\mathcal{B}$ is a countable basis of a topology τ on $\mathbb{P}$ and

(i) $\emptyset_{\mathbb{P}}$ is an isolated point in τ,
(ii) if a sequence $\langle p_n:n<\omega\rangle\subseteq\mathbb{P}$ is τ–converging to $p\in\mathbb{P}$, $q\geq p$ and W is a τ–neighbourhood of q, then there is a condition $r\in\mathbb{P}$ such that
  (a) $r\in W$, $r\geq q$,

(b) the set $\{n\in\omega : p_n\leq r\}$ is infinite.

If there is a model of topological sweetness $(\mathbb{P},\mathcal{B})$, then the forcing notion $\mathbb{P}$ is *topologically sweet.*

Lemma 0.3 (See [6, Lemma 4.2.3])**.** *Assume that $(\mathbb{P},\mathcal{B})$ is a model of topological sweetness.*

(1) *If $p,q\in\mathbb{P}$, $p\leq q$ and $q\in U\in\mathcal{B}$, then there is an open neighbourhood V of p such that*
$$(\forall r\in V)(\exists r'\in U)(r\leq r').$$
(2) *If $m\in\omega$, $p\in U\in\mathcal{B}$, then there is an open neighbourhood V of p such that any $p_0,\ldots,p_m\in V$ have a common upper bound in U.*

Definition 0.4 (See [6, Def. 4.2.1])**.** Let $\mathcal{B}$ be a countable basis of a topology on a forcing notion $\mathbb{Q}$. We say that $(\mathbb{Q},\mathcal{B})$ is *a model of iterable sweetness* if

(i) $\mathcal{B}$ is closed under finite intersections,
(ii) each $U\in\mathcal{B}$ is directed and $p\leq q\in U\ \Rightarrow\ p\in U$,
(iii) if $\langle p_n : n\leq\omega\rangle\subseteq U$ and the sequence $\langle p_n : n<\omega\rangle$ converges to p_ω (in the topology generated by $\mathcal{B}$), then there is a condition $p\in U$ such that $(\forall n\leq\omega)(p_n\leq p)$.

Proposition 0.5 (See [6, Proposition 4.2.2])**.** *If $\mathbb{P}$ is a sweet forcing notion in which any two compatible conditions have a least upper bound, then $\mathbb{P}$ is iterably sweet.*

1. SW–CLOSED FAMILIES AND SCATTERED TREES

In this section we present a new method of building sweet forcing notions. This method is, essentially, a generalization of that determined by the universality parameters of [6, §2.3].

Definition 1.1. (1) A *tree* is a family T of finite sequences such that for some $\mathrm{root}(T)\in T$ we have
$$(\forall\nu\in T)(\mathrm{root}(T)\trianglelefteq\nu)\quad\text{ and }\quad\mathrm{root}(T)\trianglelefteq\nu\trianglelefteq\eta\in T\ \Rightarrow\ \nu\in T.$$
(2) If η is a node in the tree T then
$$\begin{array}{lcl}\mathrm{succ}_T(\eta) & = & \{\nu\in T:\eta\vartriangleleft\nu\ \&\ \mathrm{lh}(\nu)=\mathrm{lh}(\eta)+1\}\ \text{ and}\\ T^{[\eta]} & = & \{\nu\in T:\eta\trianglelefteq\nu\}.\end{array}$$
(3) For a tree T, the family of all ω–branches through T is denoted by $[T]$, and we let
$$\max(T)\stackrel{\rm def}{=}\{\nu\in T:\text{ there is no }\rho\in T\text{ such that }\nu\vartriangleleft\rho\}$$
and
$$\mathrm{split}(T)\stackrel{\rm def}{=}\{\nu\in T:|\mathrm{succ}_T(\nu)|\geq 2\}.$$
(4) A tree T is *normal* if $\max(T)=\emptyset$ and $\mathrm{root}(T)=\langle\rangle$.

Definition 1.2. Suppose that $\mathcal{T}$ is a family of normal subtrees of $\omega^{<\omega}$. We say that $\mathcal{T}$ is *sw–closed* whenever

(1) if $T_1\in\mathcal{T}$, $T_2\subseteq T_1$ and T_2 is a normal tree, then $T_2\in\mathcal{T}$,
(2) if $T_1,T_2\in\mathcal{T}$, then $T_1\cup T_2\in\mathcal{T}$, and

(3) if $\langle T_n : n \leq \omega\rangle \subseteq \mathcal{T}$ is such that $(\forall n < \omega)\big(T_\omega \cap \omega^{\leq n} = T_n \cap \omega^{\leq n}\big)$, then $\bigcup\limits_{n\leq\omega} T_n \in \mathcal{T}$.

Definition 1.3. For a family $\mathcal{T}$ of normal subtrees of $\omega^{<\omega}$ we define a forcing notion $\mathbb{Q}^{\mathcal{T}}$ as follows.
A condition in $\mathbb{Q}^{\mathcal{T}}$ is a pair $p = (N^p, T^p)$ such that $N^p < \omega$ and $T^p \in \mathcal{T}$.
The order $\leq_{\mathbb{Q}^{\mathcal{T}}}$ **of** $\mathbb{Q}^{\mathcal{T}}$ is given by
$p \leq_{\mathbb{Q}^{\mathcal{T}}} q$ if and only if
$N^p \leq N^q$, $T^p \subseteq T^q$ and $T^q \cap \omega^{N^p} = T^p \cap \omega^{N^p}$.

The relation between the forcing $\mathbb{Q}^{\mathcal{T}}$ and the family $\mathcal{T}$ is similar to that in the case of the Universal Meager forcing notion $\mathbb{UM}$ and nowhere dense subtrees of $2^{<\omega}$. Note that $\mathbb{Q}^{\mathcal{T}}$ does not have to be ccc in general, however in many natural cases it is.

Proposition 1.4. *Assume that $\mathcal{T}$ is an sw–closed family of normal subtrees of $\omega^{<\omega}$ such that every $T \in \mathcal{T}$ is finitely branching. Then $\mathbb{Q}^{\mathcal{T}}$ is a sweet forcing notion in which any two compatible conditions have a least upper bound (and consequently $\mathbb{Q}^{\mathcal{T}}$ is iterably sweet).*

Proof. One easily verifies that $\mathbb{Q}^{\mathcal{T}}$ is indeed a forcing notion and that any two compatible conditions in $\mathbb{Q}^{\mathcal{T}}$ have a least upper bound.
For an integer $n < \omega$ let E_n be a binary relation on $\mathbb{Q}^{\mathcal{T}}$ defined by
$q\ E_n\ p$ if and only if
$N^q = N^p$ and $T^q \cap \omega^{\leq N^q + n} = T^p \cap \omega^{\leq N^q + n}$,
and let $\bar{E} = \langle E_n : n < \omega\rangle$. We claim that $(\mathbb{Q}^{\mathcal{T}}, \bar{E})$ is a model of sweetness. Conditions 0.1(i–iii) should be clear. To verify 0.1(iv) suppose that $p_i \in \mathbb{Q}^{\mathcal{T}}$ for $n \leq i \leq \omega$ are such that $p_i\ E_i\ p_\omega$ (for $i < \omega$). Thus, for $n \leq i < \omega$, $N^{p_i} = N^{p_\omega}$ and

$$T^{p_i} \cap \omega^{\leq N^{p_i} + i} = T^{p_\omega} \cap \omega^{\leq N^{p_\omega} + i}.$$

Put $N = N^{p_\omega}$ and $T = \bigcup\{T^{p_i} : n \leq i \leq \omega\}$. It follows from 1.2(3) that $T \in \mathcal{T}$, and plainly $q = (N, T) \in \mathbb{Q}^{\mathcal{T}}$, $q\ E_n\ p_\omega$ and $(\forall i \geq n)(p_i \leq q)$, finishing justification of 0.1(iv).

Finally, to check 0.1(v) suppose that $p, q \in \mathbb{Q}^{\mathcal{T}}$, $p \leq q$ and $n < \omega$. Let $k = N^q + n$. It should be clear that $(\forall p' \in [p]_{E_k})(\exists q' \in [q]_{E_n})(p' \leq q')$. $\square$

Now we are going to present our first example of an sw–closed family: the family of scattered subtrees of $2^{<\omega}$.

Definition 1.5. (1) For a closed set $A \subseteq 2^\omega$, let $\mathrm{rk}(A)$ be the Cantor–Bendixson rank of A, that is

$$\mathrm{rk}(A) = \min\{\alpha < \omega_1 : A^\alpha = A^{\alpha+1}\},$$

where A^α denotes the α^{th} Cantor–Bendixson derivative of A.

(2) We say that a tree $T \subseteq 2^{<\omega}$ is *scattered* if it is normal and $[T]$ is countable. The family of all scattered subtrees of $2^{<\omega}$ will be denoted by $\mathcal{T}^{\mathrm{sc}}$.

(3) For a scattered tree $T \subseteq 2^{<\omega}$, let $g^T : [T] \longrightarrow \mathrm{rk}([T])$ and $h^T : [T] \longrightarrow \omega$ be such that for each $\eta \in [T]$ we have

$$g^T(\eta) = \min\{\alpha < \mathrm{rk}(T) : \eta \notin [T]^{\alpha+1}\}$$

and

$$h^T(\eta)=\min\{m<\omega:\big(\forall\nu\in[T]\big)\big(\nu{\restriction}m=\eta{\restriction}m\ \Rightarrow\ (\eta=\nu\ \vee\ g^T(\nu)<g^T(\eta))\big)\}.$$

Proposition 1.6. *Let $T\subseteq 2^{<\omega}$ be a normal tree. Then T is scattered if and only if there is a mapping $\varphi:T\longrightarrow\omega_1$ such that*

$(\circledast)^0_{\varphi,T}$ $\big(\forall\eta,\nu\in T\big)\big(\nu\vartriangleleft\eta\ \Rightarrow\ \varphi(\nu)\geq\varphi(\eta)\big)$, *and*

$(\circledast)^1_{\varphi,T}$ $\big(\forall\eta\in\mathrm{split}(T)\big)\big(\varphi(\eta^\frown\langle 0\rangle)<\varphi(\eta)\ \vee\ \varphi(\eta^\frown\langle 1\rangle)<\varphi(\eta)\big)$.

Proof. It should be clear that if there is a function $\varphi:T\longrightarrow\omega_1$ such that $(\circledast)^0_{\varphi,T}+(\circledast)^1_{\varphi,T}$ holds true, then the tree T contains no perfect subtree and hence T is scattered.

We will show the converse implication by induction on $\mathrm{rk}(T)$.

Suppose that T is a scattered tree. Choose $\{\eta_\ell:\ell<n\}\subseteq[T]$, $n<\omega$, such that $F\stackrel{\rm def}{=}\{\eta_\ell{\restriction}h^T(\eta_\ell):\ell<n\}$ is a front of T and let

$$A\stackrel{\rm def}{=}\big\{\rho\in T:\big(\exists\ell<n\big)\big(h^T(\eta_\ell)<\mathrm{lh}(\rho)\ \&\ \rho{\restriction}(\mathrm{lh}(\rho)-1)\vartriangleleft\eta_\ell\ \&\ \rho\ntriangleleft\eta_\ell\big)\big\}.$$

Note that if $\ell<n$, $\nu\in[T]\setminus\{\eta_\ell\}$ and $\nu{\restriction}h^T(\eta_\ell)=\eta_\ell{\restriction}h^T(\eta_\ell)$, then $g^T(\nu)<g^T(\eta_\ell)$. Hence $\big(\forall\rho\in A\big)\big(\mathrm{rk}(T^{[\rho]})<\mathrm{rk}(T)\big)$, so by the inductive hypothesis for each $\nu\in A$ we may choose $\varphi_\nu:T^{[\nu]}\longrightarrow\omega_1$ such that $(\circledast)^0_{\varphi_\nu,T^{[\nu]}}+(\circledast)^1_{\varphi_\nu,T^{[\nu]}}$ holds true. Put $\alpha^*=\sup\{\varphi_\nu(\nu):\nu\in A\}<\omega_1$, $k^*=\max\{h^T(\eta_\ell):\ell<n\}+1$ and let $\varphi:T\longrightarrow\omega_1$ be defined by

$$\varphi(\eta)=\begin{cases}\alpha^*+k^*-\mathrm{lh}(\eta) & \text{if no initial segment of } \eta \text{ belongs to } F\text{, and}\\ \alpha^*+1 & \text{if an initial segment of } \eta \text{ belongs to } F\\ & \text{but no initial segment of } \eta \text{ belongs to } A\text{, and}\\ \varphi_\nu(\eta) & \text{if } \nu\in A \text{ and } \nu\trianglelefteq\eta.\end{cases}$$

One easily verifies that the function φ (is well defined and) satisfies $(\circledast)^0_{\varphi,T}+(\circledast)^1_{\varphi,T}$. □

Proposition 1.7. *$\mathcal{T}^{\rm sc}$ is an sw–closed family and consequently $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$ is iterably sweet.*

Proof. Plainly $\mathcal{T}^{\rm sc}$ satisfies the conditions (1) and (2) of 1.2.

To verify 1.2(3) suppose that $\langle T_n:n\leq\omega\rangle\subseteq\mathcal{T}^{\rm sc}$ is a sequence of scattered trees such that $\big(\forall n<\omega\big)\big(T_\omega\cap 2^{\leq n}=T_n\cap 2^{\leq n}\big)$. Let $T=\bigcup\limits_{n\leq\omega}T_n$. We are going to show that T is a scattered tree, and for this we have to show that $[T]$ is countable.

Note that if $n<\omega$, $\nu\in 2^{<\omega}\setminus T_\omega$ and $\mathrm{lh}(\nu)\leq n$, then $\nu\notin T_n$. Therefore, if $\nu\in 2^{<\omega}\setminus T_\omega$ then $[\nu]\cap[T]\subseteq\bigcup\{[T_n]:n<\mathrm{lh}(\nu)\}$, so $[\nu]\cap[T]$ is countable. Hence $[T]\setminus[T_\omega]$ is countable and thus (since $[T_\omega]$ is countable) so is $[T]$.

The "consequently" part follows from 1.4 (remember that members of $\mathcal{T}^{\rm sc}$ are subtrees of $2^{<\omega}$ so finitely branching). □

Recall that a forcing notion $\mathbb{P}$ has $\aleph_1$–caliber if for every uncountable family $\mathcal{F}\subseteq\mathbb{P}$ there is a condition $p\in\mathbb{P}$ such that $\big|\{q\in\mathcal{F}:q\leq p\}\big|=\aleph_1$ (see Truss [11]).

Proposition 1.8. (1) *If a forcing notion $\mathbb{P}$ has $\aleph_1$–caliber, then in $\mathbf{V}^{\mathbb{P}}$ there is no tree $T\subseteq 2^{<\omega}$ such that*

(a) *for every $\alpha<\omega_1$ there is a countable closed set $A\subseteq 2^\omega$ coded in $\mathbf{V}$ such that $\mathrm{rk}(A)=\alpha$ and $A\subseteq[T]$, and*

(b) *T includes no perfect subtree from $\mathbf{V}$.*
Consequently, $\mathbb{P}$ does not add generic object for $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$.

(2) *If $\mathfrak{b} > \aleph_1$, then neither the Hechler forcing notion $\mathbb{D}$ nor its composition $\mathbb{D} * \mathbb{C}$ with the Cohen real forcing add generic objects for $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$.*

Proof. (1) Suppose toward contradiction that $\mathbb{P}$ has an $\aleph_1$–caliber, $p \in \mathbb{P}$ and $\underaccent{\tilde}{T}$ is a $\mathbb{P}$–name for a subtree of $2^{<\omega}$ such that the condition p forces that both (a) and (b) of 1.8(1) hold true for $\underaccent{\tilde}{T}$. Then for each $\alpha < \omega_1$ we may choose a scattered tree $T_\alpha \subseteq 2^{<\omega}$ and a condition $p_\alpha \in \mathbb{P}$ such that ($T_\alpha \in \mathbf{V}$ and)

$$\mathrm{rk}([T_\alpha]) = \alpha \quad \text{ and } \quad p \leq p_\alpha \quad \text{ and } \quad p_\alpha \Vdash_{\mathbb{P}} \text{“ } T_\alpha \subseteq \underaccent{\tilde}{T} \text{ ”}.$$

Since $\mathbb{P}$ has an $\aleph_1$–caliber we find a condition $p^* \in \mathbb{P}$ such that the set

$$Y \stackrel{\rm def}{=} \{\alpha < \omega_1 : p_\alpha \leq p^*\}$$

is uncountable. Put $T^* = \bigcup\limits_{\alpha \in Y} T_\alpha$. Clearly T^* is a non-scattered tree and ($T^* \in \mathbf{V}$ and) $p^* \Vdash T^* \subseteq \underaccent{\tilde}{T}$, contradicting (b).

Concerning the “consequently” part it is enough to note that if $\underaccent{\tilde}{T}^{\rm sc}$ is the canonical $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$–name for a subset of $2^{<\omega}$ such that

$$\Vdash_{\mathbb{Q}^{\mathcal{T}^{\rm sc}}} \text{“ } \underaccent{\tilde}{T}^{\rm sc} = \bigcup \big\{ T^p : p \in \Gamma_{\mathbb{Q}^{\mathcal{T}^{\rm sc}}} \big\} \text{ ”},$$

then $\Vdash$“ $\underaccent{\tilde}{T}^{\rm sc}$ is a tree satisfying 1.8(1)(a,b) ”.

(2) If the unbounded number $\mathfrak{b}$ is greater than $\aleph_1$, then both $\mathbb{D}$ and $\mathbb{D} * \mathbb{C}$ have the $\aleph_1$–caliber, so part (1) applies. $\square$

Remark 1.9. The forcing notion $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$ is somewhat similar to the universal forcing notions discussed in [6, §2.3] and [4]. However it follows from 1.8(2) that if MA holds true, then the composition $\mathbb{C}*\mathbb{D}*\mathbb{C}$ does not add generic real for $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$. This is somewhat opposite to the result presented in [4, Theorem 2.1] and it may indicate that the answer to the following question is negative.

Problem 1.10. Can a finite composition (or, in general, an FS iteration) of the Hechler forcing notions add a generic object for $\mathbb{Q}^{\mathcal{T}^{\rm sc}}$?

2. More sweet examples

In this section we will present two classes of sw–closed families of trees, producing many new examples of sweet forcing notions. Let us start with extending the framework of universality parameters to that of sw–closed families.

The sweet forcing notions determined by the universality parameters were introduced in [6, §2.3]. In [4] we showed that, unfortunately, the use of them may be somewhat limited because the composition of, say, the Universal Meager forcing notions adds generic reals for many examples of the forcing notions determined by universality parameters. However, as we will show here, *families* of universality parameters may determine forcing notions which cannot be embedded into the known examples of sweet forcing notions.

Let us start with recalling definitions concerning universality parameters and the related forcing notions. We will cut down the generality of [6, §2.3] and we will quote here the somewhat simpler setting of [4]. Let $\mathbf{H}$ be a function from ω to $\omega \setminus 2$.

Definition 2.1. (1) *A finite* $\mathbf{H}$*–tree* is a tree $S \subseteq \bigcup_{n\le N}\prod_{i<n}\mathbf{H}(i)$ with $N<\omega$, $\mathrm{root}(S)=\langle\rangle$ and $\max(S)\subseteq\prod_{i<N}\mathbf{H}(i)$. The integer N may be called *the level of the tree* S and it will be denoted by $\mathrm{lev}(S)$.

(2) *An infinite* $\mathbf{H}$*–tree* is a normal tree $T\subseteq\bigcup_{n<\omega}\prod_{i<n}\mathbf{H}(i)$.

Definition 2.2. *A simplified universality parameter* $\mathfrak{p}$ *for* $\mathbf{H}$ is a pair $(\mathcal{G}^{\mathfrak{p}},F^{\mathfrak{p}})=(\mathcal{G},F)$ such that

(α) elements of $\mathcal{G}$ are triples $(S,n_{\rm dn},n_{\rm up})$ such that S is a finite $\mathbf{H}$–tree and $n_{\rm dn}\le n_{\rm up}\le\mathrm{lev}(S)$, $(\{\langle\rangle\},0,0)\in\mathcal{G}$;

(β) **if:** $(S^0,n^0_{\rm dn},n^0_{\rm up})\in\mathcal{G}$, S^1 is a finite $\mathbf{H}$–tree, $\mathrm{lev}(S^0)\le\mathrm{lev}(S^1)$, and $S^1\cap\prod_{i<\mathrm{lev}(S^0)}\mathbf{H}(i)\subseteq S^0$, and $n^1_{\rm dn}\le n^0_{\rm dn}$, $n^0_{\rm up}\le n^1_{\rm up}\le\mathrm{lev}(S^1)$,

then: $(S^1,n^1_{\rm dn},n^1_{\rm up})\in\mathcal{G}$,

(γ) $F\in\omega^\omega$ is increasing,

(δ) **if:**

- $(S^\ell,n^\ell_{\rm dn},n^\ell_{\rm up})\in\mathcal{G}$ (for $\ell<2$), $\mathrm{lev}(S^0)=\mathrm{lev}(S^1)$,
- S is a finite $\mathbf{H}$–tree, $\mathrm{lev}(S)<\mathrm{lev}(S^\ell)$, and $S^\ell\cap\prod_{i<\mathrm{lev}(S)}\mathbf{H}(i)\subseteq S$ (for $\ell<2$),
- $\mathrm{lev}(S)<n^0_{\rm dn}$, $n^0_{\rm up}<n^1_{\rm dn}$, $F(n^1_{\rm up})<\mathrm{lev}(S^1)$,

then: there is $(S^*,n^*_{\rm dn},n^*_{\rm up})\in\mathcal{G}$ such that

- $n^*_{\rm dn}=n^0_{\rm dn}$, $n^*_{\rm up}=F(n^1_{\rm up})$, $\mathrm{lev}(S^*)=\mathrm{lev}(S^0)=\mathrm{lev}(S^1)$, and
- $S^0\cup S^1\subseteq S^*$ and $S^*\cap\prod_{i<\mathrm{lev}(S)}\mathbf{H}(i)=S$.

Definition 2.3. Let $\mathfrak{p}=(\mathcal{G},F)$ be a simplified universality parameter for $\mathbf{H}$. We say that an infinite $\mathbf{H}$–tree T is $\mathfrak{p}$*–narrow* if for infinitely many $n<\omega$, for some $n=n_{\rm dn}<n_{\rm up}$ we have

$$(T\cap\bigcup_{n\le n_{\rm up}+1}\prod_{i<n}\mathbf{H}(i),n_{\rm dn},n_{\rm up})\in\mathcal{G}.$$

The family of all $\mathfrak{p}$–narrow infinite $\mathbf{H}$–trees will denoted by $\mathcal{T}^*(\mathfrak{p},\mathbf{H})$.

Proposition 2.4. *If* $\mathfrak{p}$ *is a simplified universality parameter, then* $\mathcal{T}^*(\mathfrak{p},\mathbf{H})$ *is an sw–closed family (of finitely branching normal trees). Consequently,* $\mathbb{Q}^{\mathcal{T}(\mathfrak{p},\mathbf{H})}$ *is an iterably sweet forcing notion.*

Proof. It is should be clear that $\mathcal{T}^*(\mathfrak{p},\mathbf{H})$ satisfies 1.2(1,2). The proof of 1.2(3) is, basically, included in the proof of [6, Proposition 4.2.5(3)]. $\square$

The examples of simplified universality parameters include the following.

Definition 2.5 (Compare [4, Definition 1.7, Example 1.9(2)])**.** Suppose that the function $\mathbf{H}$ is increasing and $g\in\omega^\omega$ is such that $(\forall i\in\omega)(0<g(i)<\mathbf{H}(i))$. Let $A\in[\omega]^\omega$. We define $\mathcal{G}^{g,A}_{\mathbf{H}}$ as the family consisting of $(\{\langle\rangle\},0,0)$ and of all triples $(S,n_{\rm dn},n_{\rm up})$ such that

(α) S is a finite $\mathbf{H}$–tree, $n_{\rm dn}\le n_{\rm up}\le\mathrm{lev}(S)$, $A\cap[n_{\rm dn},n_{\rm up}]\neq\emptyset$, and

(β) for some sequence $\langle w_i:i\in A\cap[n_{\rm dn},n_{\rm up}]\rangle$ such that $w_i\in[\mathbf{H}(i)]^{\le g(i)}$ (for $i\in A\cap[n_{\rm dn},n_{\rm up}]$) we have

$$\big(\forall\eta\in\max(S)\big)\big(\exists i\in A\cap[n_{\rm dn},n_{\rm up}]\big)\big(\eta(i)\in w_i\big).$$

Proposition 2.6. *Assume that $\mathbf{H}, g, A$ are as in 2.5, and $F(n) = \prod\limits_{i\leq n} \mathbf{H}(i)^2$ (for $n \in \omega$). Then $\mathfrak{p}^{g,A}_{\mathbf{H}} \stackrel{\rm def}{=} (\mathcal{G}^{g,A}_{\mathbf{H}}, F)$ is s simplified universality parameter (and even it is a regular universality parameter in the sense of* [4, Definition 1.14]*).*

The universality parameters $\mathfrak{p}^{g,A}_{\mathbf{H}}$ from 2.6 are related to the strong PP–property (see [8, Ch VI, 2.12*], compare also with [5, §7.2]). Note that an infinite $\mathbf{H}$–tree T is $\mathfrak{p}^{g,A}_{\mathbf{H}}$–narrow if and only if there exist sequences $\bar{w} = \langle w_i : i \in A\rangle$ and $\bar{n} = \langle n_k : k < \omega\rangle$ such that

- $\big(\forall i \in A\big)\big(w_i \subseteq \mathbf{H}(i)\ \&\ |w_i| \leq g(i)\big)$, and
- $n_k < n_{k+1} < \omega$ for each $k < \omega$, and
- $\big(\forall \eta \in [T]\big)\big(\forall k < \omega\big)\big(\exists i \in A \cap [n_k, n_{k+1})\big)\big(\eta(i) \in w_i\big)$.

It should be clear that the intersection of a family of sw–closed sets of normal trees is sw–closed. So now we are going to look at the intersections of the families of $\mathfrak{p}^{g,A}_{\mathbf{H}}$–narrow trees.

Definition 2.7. Let $\mathbf{H}, g$ be as in 2.5 and let $\emptyset \neq \mathcal{B} \subseteq [\omega]^{\omega}$.

(1) Put $\mathcal{T}(\mathcal{B}) = \mathcal{T}^g_{\mathbf{H}}(\mathcal{B}) \stackrel{\rm def}{=} \bigcap \big\{\mathcal{T}^*(\mathfrak{p}^{g,B}_{\mathbf{H}}, \mathbf{H}) : B \in \mathcal{B}\big\}$ and $\mathbb{P}_{\mathcal{B}} = \mathbb{Q}^{\mathcal{T}(\mathcal{B})}$.

(2) Let $\underset{\sim}{T}_{\mathcal{B}}$ be a $\mathbb{P}_{\mathcal{B}}$–name such that
$$\Vdash_{\mathbb{P}_{\mathcal{B}}} \text{“ } \underset{\sim}{T}_{\mathcal{B}} = \bigcup \big\{T^p : p \in \Gamma_{\mathbb{P}_{\mathcal{B}}}\big\} \text{ ”}.$$

(3) For a set $A \in [\omega]^{\omega}$ put
$$S_A = \big\{\eta \in \bigcup_{n<\omega} \prod_{i\leq n} \mathbf{H}(i) : \big(\forall i \in \mathrm{lh}(\eta) \cap A\big)\big(\eta(i) = 0\big)\big\}.$$

Lemma 2.8. *Suppose that $\mathbf{H}, g$ are as in 2.5.*

(1) *Let $A, C \in [\omega]^{\omega}$. Then the tree S_A is $\mathfrak{p}^{g,C}_{\mathbf{H}}$–narrow if and only if $A \cap C$ is infinite.*

(2) *Let $\emptyset \neq \mathcal{B} \subseteq [\omega]^{\omega}$. Then, in $\mathbf{V}^{\mathbb{P}_{\mathcal{B}}}$, $\underset{\sim}{T}_{\mathcal{B}}$ is an infinite $\mathbf{H}$–tree such that*

(a) *if $T \in \mathbf{V}$ is an infinite $\mathbf{H}$–tree which is $\mathfrak{p}^{g,B}_{\mathbf{H}}$–narrow for all $B \in \mathcal{B}$, then there is an $n < \omega$ such that*
$$\big(\forall \nu \in \underset{\sim}{T}_{\mathcal{B}}\big)\big(\forall \eta \in T\big)\big(n = \mathrm{lh}(\nu) < \mathrm{lh}(\eta) \ \Rightarrow\ \nu^\frown\eta{\restriction}[n, \mathrm{lh}(\eta)) \in \underset{\sim}{T}_{\mathcal{B}}\big),$$

(b) *if an infinite $\mathbf{H}$–tree $T \in \mathbf{V}$ is not $\mathfrak{p}^{g,B}_{\mathbf{H}}$–narrow for some $B \in \mathcal{B}$, then*
$$\big(\forall n < \omega\big)\big(\exists \eta \in T\big)\big(\mathrm{lh}(\eta) > n\ \&\ (\forall \nu \in \prod_{i<n} \mathbf{H}(i))(\nu^\frown\eta{\restriction}[n, \mathrm{lh}(\eta)) \notin \underset{\sim}{T}_{\mathcal{B}})\big).$$

Theorem 2.9. *Suppose that $\mathbb{P}$ is a ccc forcing notion, $\Vdash_{\mathbb{P}}$ “ $2^{\aleph_0} = \kappa$ ”, $\kappa < 2^{2^{\aleph_0}}$. Then there is a family $\mathcal{B} \subseteq [\omega]^{\omega}$ such that $\mathbb{P}$ does not add the generic object for the (iterably sweet) forcing notion $\mathbb{P}_{\mathcal{B}}$.*

Proof. Note that if $\mathcal{U}$ is a uniform ultrafilter on ω, then $(\forall A, B \in \mathcal{U})(|A \cap B| = \omega)$ and hence, by 2.8(1), for every $A \in \mathcal{U}$ and every $B \in \mathcal{U}$, the tree S_A is $\mathfrak{p}^{g,B}_{\mathbf{H}}$–narrow. Also by 2.8(1), for every $A \in [\omega]^{\omega}$ the tree S_A is not $\mathfrak{p}^{g,\omega\setminus A}_{\mathbf{H}}$–narrow.

Now, if $\mathcal{U}', \mathcal{U}'' \subseteq [\omega]^{\omega}$ are two distinct uniform ultrafilters on ω, then we may pick $A \in [\omega]^{\omega}$ such that $A \in \mathcal{U}'$ and $\omega \setminus A \in \mathcal{U}''$. Then the tree S_A

- is $\mathfrak{p}^{g,B}_{\mathbf{H}}$–narrow for every $B \in \mathcal{U}'$, but
- is not $\mathfrak{p}^{g,\omega\setminus A}_{\mathbf{H}}$–narrow, $\omega \setminus A \in \mathcal{U}''$.

Therefore, by 2.8(2), the interpretations of the names $\underset{\sim}{T}_{\mathcal{U}'}$, $\underset{\sim}{T}_{\mathcal{U}''}$ by the corresponding generic filters must be different. Since there are $2^{2^{\aleph_0}}$ ultrafilters on ω we easily get the conclusion. $\square$

Corollary 2.10. *There exists an iterably sweet forcing notion $\mathbb{Q}$ which cannot be embedded into the forcing notion constructed in* [7, §7].

Let us present now a different class of sw–closed families of normal trees and corresponding forcing forcing notions.

Definition 2.11. The *sw–closure* $\mathrm{cl}^{\mathrm{sw}}(\mathcal{T})$ of the family $\mathcal{T}$ is the smallest family $\mathcal{T}^*$ of subtrees of $\omega^{<\omega}$ which includes $\mathcal{T}$ and is sw–closed.

Clearly, $\mathrm{cl}^{\mathrm{sw}}(\mathcal{T})$ is well defined for any family $\mathcal{T}$ of normal subtrees of $\omega^{<\omega}$.

Lemma 2.12. (1) *Suppose that T^* is a normal subtree of $\omega^{<\omega}$ and let $\mathcal{T}^*$ be the family of all normal subtrees of T^*. Then $\mathcal{T}^*$ is sw–closed. Consequently, if $\mathcal{T}\subseteq\mathcal{T}^*$, then $\mathrm{cl}^{\mathrm{sw}}(\mathcal{T})\subseteq\mathcal{T}^*$.*

(2) *Assume that $\mathcal{T}$ is an sw–closed family of normal subtrees of $\omega^{<\omega}$ and $A\subseteq\omega^{\omega}$ is a closed set. Let*
$$\mathcal{T}^-(A)=\big\{T\in\mathcal{T}:[T]\cap A \textit{ is nowhere dense in } A\big\}.$$
Then $\mathcal{T}^-(A)$ is sw–closed.

(3) *If $\mathcal{T}$ is a family of normal subtrees of $\omega^{<\omega}$, $T\subseteq\omega^{<\omega}$ is a normal tree and*
$$\big(\forall T'\in\mathcal{T}\big)\big([T]\cap[T'] \textit{ is nowhere dense in } [T]\big),$$
then $T\notin\mathrm{cl}^{\mathrm{sw}}(\mathcal{T})$.

Proof. (1) Should be clear.

(2) Clearly $\mathcal{T}^-(A)$ is closed under finite unions. Assume now that $T_n,T_\omega\in\mathcal{T}^-(A)$ are such that $\big(\forall n<\omega\big)\big(T_\omega\cap\omega^{\leq n}=T_n\cap\omega^{\leq n}\big)$ and let $T=\bigcup\limits_{n\leq\omega}T_n$. We want to show that $T\in\mathcal{T}^-(A)$. Since $\mathcal{T}$ is sw–closed we see that $T\in\mathcal{T}$, so we need to show that $[T]\cap A$ is nowhere dense in A. To this end let $S\subseteq\omega^{<\omega}$ be a normal tree such that $A=[S]$ and suppose that $\nu\in S$. Since $T_\omega\in\mathcal{T}^-(A)$, we may find $\eta_0\in S$ such that $\nu\vartriangleleft\eta_0$ and $\eta_0\notin T_\omega$. Then, by our assumptions on $\langle T_n:n\leq\omega\rangle$, also for each $k\geq\mathrm{lh}(\eta_0)$ we have $\eta_0\notin T_k$. Since $T_n\in\mathcal{T}^-(A)$ (for $n<\mathrm{lh}(\eta_0)$), the set $\bigcup\limits_{n<\mathrm{lh}(\eta_0)}[T_n]\cap A$ is nowhere dense in A and hence we may find $\eta\in S$ such that $\eta_0\vartriangleleft\eta$ and $\eta\notin\bigcup\limits_{n<\mathrm{lh}(\eta_0)}T_n$. Then we also have $\nu\vartriangleleft\eta\in S$ and $\eta\notin T$.

(3) Follows from (2). $\square$

Definition 2.13. (1) For a set $A\in[\omega]^{\omega}$ let $\mathcal{T}^A$ be the collection of all normal subtrees T of $2^{<\omega}$ such that $\big(\forall\nu\in\mathrm{split}(T)\big)\big(\mathrm{lh}(\nu)\in A\big)$.

(2) For a family $\mathcal{A}\subseteq[\omega]^{\omega}$ let $\mathcal{T}_{\mathcal{A}}=\mathrm{cl}^{\mathrm{sw}}\big(\bigcup\{\mathcal{T}^A:A\in\mathcal{A}\}\big)$.

Theorem 2.14. *Suppose that $\mathbb{P}$ is a ccc forcing notion, $\Vdash_{\mathbb{P}}$ " $2^{\aleph_0}=\kappa$ ", $\kappa<2^{2^{\aleph_0}}$. Then there is a family $\mathcal{A}\subseteq[\omega]^{\omega}$ such that $\mathbb{P}$ does not add the generic object for the (iterably sweet) forcing notion $\mathbb{Q}^{\mathcal{T}_{\mathcal{A}}}$.*

Proof. Let us start with some observations of a more general character.

Claim 2.14.1. *Assume that $\mathcal{T}$ is an sw–closed family of subtrees of $2^{<\omega}$ such that*

($\circledast$) *for every* $\eta\in 2^\omega$ *and* $T\in\mathcal{T}$ *we have*

$$T -_2 \eta \stackrel{\rm def}{=} \{\nu\in 2^{<\omega}:\nu +_2 (\eta{\restriction}{\rm lh}(\nu))\in T\}\in\mathcal{T}.$$

Let $\underset{\sim}{T}^{\mathcal{T}}$ *be a* $\mathbb{Q}^{\mathcal{T}}$*–name such that*

$$\Vdash_{\mathbb{Q}^{\mathcal{T}}} \text{“ } \underset{\sim}{T}^{\mathcal{T}} = \bigcup\big\{T^p : p\in\Gamma_{\mathbb{Q}^{\mathcal{T}}}\big\} \text{ ”}.$$

Then, in $\mathbf{V}^{\mathbb{Q}^{\mathcal{T}}}$, $\underset{\sim}{T}^{\mathcal{T}}$ *is a subtree of* $2^{<\omega}$ *such that*

(1) *for every* $T\in\mathcal{T}$ *there is an* $n<\omega$ *such that*
if $\nu_0\in T\cap 2^n$, $\nu_1\in\underset{\sim}{T}^{\mathcal{T}}\cap 2^n$, *and* $\nu_0\vartriangleleft\eta\in T$, then $\nu_1{}^\frown\eta{\restriction}[n,{\rm lh}(\eta))\in\underset{\sim}{T}^{\mathcal{T}}$,

(2) *for every normal tree* $T\subseteq 2^{<\omega}$ *such that* $T\notin\mathcal{T}$, $T\in\mathbf{V}$, *we have*

$$\big(\forall n<\omega\big)\big(\exists\eta\in T\big)\big({\rm lh}(\eta)>n\ \&\ (\forall\nu\in 2^n)(\nu^\frown\eta{\restriction}[n,{\rm lh}(\eta))\notin\underset{\sim}{T}^{\mathcal{T}})\big).$$

Proof of the Claim. (1) Suppose that $p\in\mathbb{Q}^{\mathcal{T}}$ and $T\in\mathcal{T}$. Let $\langle\eta_\ell:\ell<2^{N^p}\rangle$ list all elements of 2^ω which are constantly zero on $[N^p,\omega)$. It follows from our assumption ($\circledast$) that

$$\big(\forall\ell<2^{N^p}\big)\big(T -_2 \eta_\ell\in\mathcal{T}\ \&\ T^p -_2 \eta_\ell\in\mathcal{T}\big).$$

Since $\mathcal{T}$ is sw–closed we may now conclude that (by 1.2(2))

$$T_0 \stackrel{\rm def}{=} \bigcup_{\ell<2^{N^p}}(T -_2 \eta_\ell)\cup\bigcup_{\ell<2^{N^p}}(T^p -_2 \eta_\ell)\in\mathcal{T},$$

and hence also (by 1.2(1))

$$T_1 \stackrel{\rm def}{=} \{\eta\in T_0 : ({\rm lh}(\eta)\le N^p\ \&\ \eta\in T^p)\ \vee\ ({\rm lh}(\eta)>N^p\ \&\ \eta{\restriction}N^p\in T^p)\}\in\mathcal{T}.$$

Now, letting $N^q=N^p$ and $T^q=T_1$ we get a condition $q\in\mathbb{Q}^{\mathcal{T}}$ stronger than p and such that

$$q\Vdash(\forall\nu_0\in T\cap 2^{N^q})(\forall\nu_1\in\underset{\sim}{T}^{\mathcal{T}}\cap 2^{N^q})(\forall\eta\in T^{[\nu_0]})(\nu_1{}^\frown\eta{\restriction}[N^q,{\rm lh}(\eta))\in\underset{\sim}{T}^{\mathcal{T}}).$$

(2) Now suppose that $p\in\mathbb{Q}^{\mathcal{T}}$, $n<\omega$ and $T\subseteq 2^{<\omega}$ is a normal tree which does not belong to $\mathcal{T}$. Let $N=N^p+n$ and let $\langle\eta_\ell:\ell<2^N\rangle$ list all elements of 2^ω which are constantly zero on $[N,\omega)$. It follows from ($\circledast$) that $T_0\stackrel{\rm def}{=}\bigcup\limits_{\ell<2^N}(T^p -_2 \eta_\ell)\in\mathcal{T}$ and since $T\notin\mathcal{T}$ we may conclude by 1.2(1) that $T\setminus T_0\neq\emptyset$. Pick $\eta\in T\setminus T_0\neq\emptyset$ and note that necessarily ${\rm lh}(\eta)>N\geq n$. Letting $N^q={\rm lh}(\eta)$ and $T^q=T_0$ we get a condition $q\in\mathbb{Q}^{\mathcal{T}}$ stronger than p and such that

$$q\Vdash(\forall\nu\in 2^n)(\nu^\frown\eta{\restriction}[n,{\rm lh}(\eta))\notin\underset{\sim}{T}^{\mathcal{T}})\big).$$

$\square$

Claim 2.14.2. *If* $\mathcal{T}$ *is a collection of normal subtrees of* $2^{<\omega}$ *such that the demand in 2.14.1($\circledast$) holds for* $\mathcal{T}$, *then also* ${\rm cl}^{\rm sw}(\mathcal{T})$ *satisfies this condition. Consequently, for each* $\mathcal{A}\subseteq[\omega]^\omega$, ($\circledast$) *of 2.14.1 holds true for* $\mathcal{T}_{\mathcal{A}}$.

Proof of the Claim. Should be clear. $\square$

Claim 2.14.3. *Suppose that* $A\in[\omega]^\omega$ *and* $\mathcal{A}\subseteq[\omega]^\omega$ *are such that*

$$\big(\forall B\in\mathcal{A}\big)\big(|A\setminus B|=\omega\big).$$

Then $\mathcal{T}^A\nsubseteq\mathcal{T}_{\mathcal{A}}$.

Proof of the Claim. Let $T = \{\nu \in 2^{<\omega} : (\forall n < \mathrm{lh}(\nu))(\nu(n) = 1 \;\Rightarrow\; n \in A)\}$. Plainly $T \in \mathcal{T}^A$. Also, for every $B \in \mathcal{A}$ and $T' \in \mathcal{T}^B$ the set $[T] \cap [T']$ is nowhere dense in $[T]$, so by 2.12(3) $T \notin \mathrm{cl}^{\mathrm{sw}}\big(\bigcup\{\mathcal{T}^B : B \in \mathcal{A}\}\big) = \mathcal{T}_{\mathcal{A}}$. $\square$

Now choose a family $\mathcal{I} \subseteq [\omega]^\omega$ of almost disjoint sets, $|\mathcal{I}| = 2^{\aleph_0}$.

Suppose that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$, $\mathcal{A} \neq \mathcal{B}$, say $A \in \mathcal{A} \setminus \mathcal{B}$. Then $(\forall B \in \mathcal{B})(|A \setminus B| = \omega)$ and hence (by Claim 2.14.3) we get $\mathcal{T}^A \nsubseteq \mathcal{T}_{\mathcal{B}}$, so we have a normal tree $T \in \mathcal{T}_{\mathcal{A}} \setminus \mathcal{T}_{\mathcal{B}}$. Now look at Claim 2.14.1 — by 2.14.2 it is applicable to $\mathbb{Q}^{\mathcal{T}_{\mathcal{A}}}$, $\mathbb{Q}^{\mathcal{T}_{\mathcal{B}}}$ and we get from it that if $T_{\mathcal{A}}, T_{\mathcal{B}} \subseteq 2^{<\omega}$ are trees generic over $\mathbf{V}$ for $\mathbb{Q}^{\mathcal{T}_{\mathcal{A}}}$, $\mathbb{Q}^{\mathcal{T}_{\mathcal{B}}}$, respectively, then

- $\big(\exists n < \omega\big)\big(\forall \nu \in T_{\mathcal{A}} \cap 2^n\big)\big(\forall \eta \in T\big)\big(\mathrm{lh}(\eta) > n \;\Rightarrow\; \nu^\frown\eta{\restriction}[n, \mathrm{lh}(\eta)) \in T_{\mathcal{A}}\big)$,
- $\big(\forall n < \omega\big)\big(\exists \eta \in T\big)\big(\mathrm{lh}(\eta) > n \;\&\; (\forall \nu \in 2^n)(\nu^\frown\eta{\restriction}[n, \mathrm{lh}(\eta)) \notin T_{\mathcal{B}}\big)$.

Hence $T_{\mathcal{A}} \neq T_{\mathcal{B}}$. Since $\mathbb{P}$ satisfies the ccc and $\Vdash_{\mathbb{P}}$" $2^{\aleph_0} = \kappa$ " and $\kappa < 2^{2^{\aleph_0}}$, we may find a family $\mathcal{F}$ of subsets of $\mathcal{I}$ such that $|\mathcal{F}| = \kappa$ and

$$\Vdash_{\mathbb{P}} \text{" for no } \mathcal{A} \subseteq \mathcal{I} \text{ with } \mathcal{A} \notin \mathcal{F}, \text{ there is a } \mathbb{Q}^{\mathcal{T}_{\mathcal{A}}}\text{–generic filter over } \mathbf{V} \text{ ".}$$

$\square$

One should note that the examples of sweet forcing notions which cannot be embedded into the one constructed in [7, §7] which we gave in this section are not very nice — it may well be that the parameters $\mathcal{A}, \mathcal{B}$ needed to define them are not definable from a real. Even the candidate for a somewhat definable example from the previous section, the forcing notion $\mathbb{Q}^{\mathcal{T}^{\mathrm{sc}}}$, is not Souslin. Thus the following variant of [6, Problem 5.5] may be of interest.

Problem 2.15. Is there a Souslin ccc iterably sweet forcing notion $\mathbb{Q}$ such that no finite composition of the Universal Meager forcing notion adds a $\mathbb{Q}$–generic real? Such that the forcing of [7, §7] does not add $\mathbb{Q}$–generic real?

3. Subforcings, Quotients and likes

Topological sweetness, as defined in 0.2, is a property of particular representation of a forcing notion. It is only natural to ask if a forcing notion having a topologically sweet dense subforcing is topologically sweet, or, in general, if a forcing notion equivalent to a topologically sweet one is topologically sweet. We start this section with some results in these directions.

Definition 3.1. We say that a forcing notion $\mathbb{P}$ has a GLB–property provided that for every $p_0, \ldots, p_k \in \mathbb{P}$, $k < \omega$, there is $q \in \mathbb{P}$ such that

(α) $q \le p_i$ for $i \le k$, and
(β) if $q^* \in \mathbb{P}$ satisfies $(\forall i \le k)(q^* \le p_i)$, then $q^* \le q$.

Remark 3.2. If $\mathbb{B}$ is a Boolean algebra, then $\mathbb{B}^+$ is a forcing notion with the GLB–property. Also the forcing notions $\mathbb{R}$ and $\mathbb{A}$ defined in 3.8 later have this property.

Proposition 3.3. *Suppose that a forcing notion $\mathbb{P}$ has the GLB–property and $\mathbb{Q} \subseteq \mathbb{P}$ is its dense subforcing. If $\mathbb{Q}$ is topologically sweet, then so is $\mathbb{P}$.*

Proof. Let $(\mathbb{Q}, \mathcal{B})$ be a model of topological sweetness and let τ be the topology on $\mathbb{Q}$ generated by $\mathcal{B}$. For sets $U_0, \ldots, U_k \in \mathcal{B}$, $k < \omega$, define

$$W(U_0, \ldots, U_k) = \{p \in \mathbb{P} : (\forall i \le k)(\exists q \in U_i)(p \le q)\},$$

and let

$$\mathcal{B}^* = \big\{W(U_0, \ldots, U_k) : k < \omega \;\&\; U_0, \ldots, U_k \in \mathcal{B}\big\} \cup \big\{\{\emptyset_{\mathbb{P}}\}\big\}.$$

It should be clear that

- $\mathcal{B}^*$ is closed under finite intersections, and
- it is a countable basis of a topology τ^* on $\mathbb{P}$, and
- $\emptyset_{\mathbb{P}}$ is an isolated point in τ^*.

We are going to show that the topology τ^* satisfies the demand of 0.2(ii). So suppose that a sequence $\bar{p} = \langle p_n : n < \omega\rangle \subseteq \mathbb{P}$ is τ^*–converging to $p \in \mathbb{P}$ and $q \geq p$ and W is a τ^*–neighbourhood of q. Pick $U_0, \ldots, U_k \in \mathcal{B}$ such that $q \in W(U_0, \ldots, U_k) \subseteq W$ and let $q_i \in U_i$ (for $i \leq k$) be such that $q \leq q_i$. Furthermore, for $i \leq k$, let $\{V^i_n : n < \omega\}$ be a basis of τ–neighbourhoods of $q_i \in \mathbb{Q}$ such that $(\forall n_0 < n_1 < \omega)(q_i \in V^i_{n_1} \subseteq V^i_{n_0} \subseteq U_i)$.

Since $p \in W(V^0_n, V^1_n, \ldots, V^k_n) \in \mathcal{B}^*$ (for each $n < \omega$) and the sequence $\bar{p}$ τ^*–converges to p, we may choose an increasing sequence $\langle m_n : n < \omega\rangle \subseteq \omega$ such that $\big(\forall n < \omega\big)\big(p_{m_n} \in W(V^0_n, V^1_n, \ldots, V^k_n)\big)$. Then we may also pick $p^*_{n,i}$ (for $n < \omega$ and $i \leq k$) such that $p_{m_n} \leq p^*_{n,i} \in V^i_n$. Fix $i \leq k$ and look at the sequence $\bar{p}^*_i = \langle p^*_{n,i} : n < \omega\rangle$: clearly it τ–converges to q_i. Consequently, we may easily choose (be repeated application of 0.2(ii) for τ) conditions $q^*_i \in \mathbb{Q}$ such that

- $q_i \leq q^*_i \in U_i$ for $i \leq k$, and
- $(\exists^\infty n < \omega)(\forall i \leq k)(p^*_{n,i} \leq q^*_i)$.

Since $\mathbb{P}$ has the GLB–property we may pick $q^* \in \mathbb{P}$ such that

(α) $q^* \leq q_i$ for $i \leq k$, and
(β) if $r \in \mathbb{P}$ is weaker than $q^*_0, \ldots, q^*_k$, then $r \leq q^*$.

Then, plainly, $q^* \in W(U_0, \ldots, U_k)$ and $q \leq q^*$ and $(\exists^\infty n < \omega)(p_{m_n} \leq q)$. $\square$

Proposition 3.4. *Assume that $\mathbb{P}$ is a topologically sweet forcing notion. Then there is a model $(\mathbb{P}, \mathcal{B}^*)$ of topological sweetness such that all members of $\mathcal{B}^*$ are downward closed.*

Proof. Let $(\mathbb{P}, \mathcal{B})$ be a model of topological sweetness. For $U \in \mathcal{B}$ put $W(U) = \{p \in \mathbb{P} : (\exists q \in U)(p \leq q)\}$, and let $\mathcal{B}^* = \{W(U) : U \in \mathcal{B}\}$. Note that if $p \in W(U_0) \cap W(U_1)$ and $p \leq p_0 \in U_0$, $p \leq p_1 \in U_1$, then there is $V \in \mathcal{B}$ such that $p \in V$ and $V \subseteq W(U_0) \cap W(U_1)$ (remember 0.3(1)). Hence we easily conclude that $\mathcal{B}^*$ is a base of a topology τ^* on $\mathbb{P}$. Similarly as in 3.3 one shows that $(\mathbb{P}, \mathcal{B}^*)$ is a model of topological sweetness. $\square$

Proposition 3.5. *Assume that $\mathbb{P}$ is a topologically sweet and separative partial order, $\mathbb{Q}$ is a forcing notion. Suppose also that*

$$(\forall q \in \mathbb{Q})(\exists p \in \mathbb{P})(p \Vdash_{\mathbb{P}} \text{“ there is a } \mathbb{Q}\text{–generic } \underset{\sim}{G} \subseteq \mathbb{Q} \text{ over } \mathbf{V} \text{ such that } q \in \underset{\sim}{G} \text{ ”}).$$

Then $\mathbb{Q}$ is equivalent to a topologically sweet forcing notion.

Proof. It follows from our assumptions on $\mathbb{P}$ that it is (isomorphic to) a dense subset of $\mathbf{BA}(\mathbb{P})^+$ and hence, by 3.3+3.4, there is a model $(\mathbf{BA}(\mathbb{P})^+, \mathcal{B})$ of topological sweetness such that all members of $\mathcal{B}$ are downward closed. By the assumptions on $\mathbb{Q}, \mathbb{P}$ we also know that $\mathbf{BA}(\mathbb{Q})$ is a complete subalgebra of $\mathbf{BA}(\mathbb{P})$; let $\pi : \mathbf{BA}(\mathbb{P}) \longrightarrow \mathbf{BA}(\mathbb{Q})$ be the projection. Put

$$\mathcal{B}' = \{U \cap \mathbf{BA}(\mathbb{Q})^+ : U \in \mathcal{B}\}.$$

We claim that $(\mathbf{BA}(\mathbb{Q})^+, \mathcal{B}')$ is a model of topological sweetness. It is easy to verify 0.2(i), so let us only argue that 0.2(ii) holds true. To this end suppose that a sequence $\bar{p} = \langle p_n : n < \omega\rangle \subseteq \mathbf{BA}(\mathbb{Q})^+$ converges to $p \in \mathbf{BA}(\mathbb{Q})^+$ (in the topology generated by $\mathcal{B}'$) and let $p \leq q \in U \cap \mathbf{BA}(\mathbb{Q})^+$, $U \in \mathcal{B}$. Then also $\bar{p}$ converges to p in the topology generated by $\mathcal{B}$ on $\mathbf{BA}(\mathbb{P})^+$, so we may find $r \in \mathbf{BA}(\mathbb{P})^+$ such that $q \leq r \in U$ and $(\exists^\infty n < \omega)(p_n \leq r)$. Let $r^* = \pi(r) \in \mathbf{BA}(\mathbb{Q})$. Then we have

- $q \leq r^*$ (as π is the projection and $q \in \mathbf{BA}(\mathbb{Q})^+$, $q \leq r$),
- $(\exists^\infty n < \omega)(p_n \leq r^*)$ (as π is the projection and $p_n \in \mathbf{BA}(\mathbb{Q})^+$),
- $r^* \in U$ (as U is downward closed, $r^* \leq r \in U$).

□

The sweetness and topological sweetness are important properties because they are preserved in amalgamations of forcing notions. Since the amalgamation can be represented as the composition with the product of two quotients (see, e.g., [3] on that), one may ask if sweetness is also preserved in quotients.

Definition 3.6. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions and suppose that $\mathbb{Q} \lessdot \mathbf{BA}(\mathbb{P})$. The quotient $(\mathbb{P} : \mathbb{Q})$ is the $\mathbb{Q}$–*name* for the subforcing of $\mathbb{P}$ consisting of all $p \in \mathbb{P}$ such that p is compatible (in $\mathbf{BA}(\mathbb{P})$) with all members of $\Gamma_{\mathbb{Q}}$. Thus for $p \in \mathbb{P}$ and $q \in \mathbb{Q}$,

$$\begin{array}{l} q \Vdash_{\mathbb{Q}} \text{`` } p \in (\mathbb{P} : \mathbb{Q}) \text{ ''} \qquad \text{if and only if} \\ (\forall r \in \mathbb{Q})(q \leq r \ \Rightarrow\ r, p \text{ are compatible in } \mathbf{BA}(\mathbb{P})). \end{array}$$

Theorem 3.7. *Let $\mathbb{C}$ be the standard Cohen forcing notion (so it is a countable atomless partial order). Suppose that $(\mathbb{P}, \mathcal{B})$ is a model of topological sweetness and $\mathbb{C} \lessdot \mathbf{BA}(\mathbb{P})$. Let $\underset{\sim}{\mathcal{B}}^{\mathbb{C}}$ be the $\mathbb{C}$-name for the family $\{U \cap (\mathbb{P} : \mathbb{C}) : U \in \mathcal{B}\}$. Then*

$$\Vdash_{\mathbb{C}} \text{`` } \big((\mathbb{P} : \mathbb{C}), \underset{\sim}{\mathcal{B}}^{\mathbb{C}}\big) \textit{ is a model of topological sweetness ''}.$$

Proof. First note that, in $\mathbf{V}^{\mathbb{C}}$, $\underset{\sim}{\mathcal{B}}^{\mathbb{C}}$ is a countable basis of a topology on $(\mathbb{P} : \mathbb{C})$, and $\emptyset_{(\mathbb{P}:\mathbb{C})} = \emptyset_{\mathbb{P}}$ is an isolated point in this topology. Thus the only thing that we should verify is the demand in 0.2(1)(ii).

Suppose that $\eta \in \mathbb{C}$ and $\mathbb{C}$–names $\langle \underset{\sim}{p}_i : i < \omega \rangle$, $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{W}$ are such that

$$\begin{array}{ll} \eta \Vdash_{\mathbb{C}} & \text{`` } \underset{\sim}{p}_i, \underset{\sim}{p}, \underset{\sim}{q} \in (\mathbb{P} : \mathbb{C}),\ \underset{\sim}{W} \in \underset{\sim}{\mathcal{B}}^{\mathbb{C}},\ \underset{\sim}{p} \leq \underset{\sim}{q} \in \underset{\sim}{W} \text{ and} \\ & \text{the sequence } \langle \underset{\sim}{p}_i : i < \omega \rangle \text{ converges to } \underset{\sim}{p} \text{ in the topology generated by } \underset{\sim}{\mathcal{B}}^{\mathbb{C}} \text{ ''} \end{array}$$

Passing to a stronger than η condition in $\mathbb{C}$ (if necessary), we may assume that for some $p, q \in \mathbb{P}$ and $W \in \mathcal{B}$ we have

$$\eta \Vdash_{\mathbb{C}} \text{`` } \underset{\sim}{p} = p \ \&\ \underset{\sim}{q} = q \ \&\ \underset{\sim}{W} = W \cap (\mathbb{P} : \mathbb{C}) \text{ ''}.$$

Then also $\eta \Vdash_{\mathbb{C}}$ " $p, q \in (\mathbb{P} : \mathbb{C})$ " and $p \leq q \in W$. Let us choose a condition $q^+ \in \mathbb{P}$ which is (in $\mathbf{BA}(\mathbb{P})$) stronger than both q and η, and let $U \in \mathcal{B}$ be a neighborhood of q^+ such that any two members of U are compatible in $\mathbb{P}$ (remember 0.3(2)). Next, choose $W^+ \in \mathcal{B}$ such that $q \in W^+ \subseteq W$ and every member of W^+ has an upper bound in U (possible by 0.3(1)).

Pick $V_i \in \mathcal{B}$ (for $i < \omega$) such that $\{V_i : i < \omega\}$ forms a neighbourhood basis at p (for the topology generated by $\mathcal{B}$) such that for each $i < \omega$:

(α) $p \in V_{i+1} \subseteq V_i$,
(β) any $i + 1$ elements of V_{i+1} have a common upper bound in V_i.

[The choice is clearly possible; remember 0.3.]

Clearly $\eta \Vdash_{\mathbb{C}}$ " $\{V_i \cap (\mathbb{P} : \mathbb{C}) : i < \omega\}$ forms a neighbourhood basis at p (for the topology generated by $\underset{\sim}{\mathcal{B}}^{\mathbb{C}}$) ". Hence, without loss of generality, we may assume that $\eta \Vdash_{\mathbb{C}}$" $\underset{\sim}{p}_i \in V_i$ " (as we may change the names $\underset{\sim}{p}_i$ reflecting a passage to a subsequence). Let us fix a list $\{\nu_\ell : \ell < \omega\}$ of all conditions in $\mathbb{C}$ stronger than η, and for every $i,\ell<\omega$ let us pick $p_{i,\ell}\in\mathbb{P}$ such that $\nu_\ell \nVdash_{\mathbb{C}}$ " $\underset{\sim}{p}_i \neq p_{i,\ell}$ ". Note that then $p_{i,\ell}\in V_i$, so by clause (β) above we may choose $p^*_i\in V_i$ such that for each $i>0$ we have

$$(\forall \ell\leq i)(p_{i+1,\ell}\leq p^*_i).$$

The sequence $\langle p^*_i : i<\omega\rangle$ converges to p so (by 0.2(1)(ii) for $(\mathbb{P},\mathcal{B})$) there are a condition $r\in\mathbb{P}$ and an infinite set $A\subseteq\omega$ such that

$$r\in W^+ \ \text{ and } \ q\leq r \ \text{ and } \ (\forall i\in A)(p^*_i\leq r).$$

By the choice of W^+, the condition r has an upper bound in U and hence (by the choice of U) r,q^+ are compatible in $\mathbb{P}$. Therefore, as q^+ is stronger than η (in $\mathbf{BA}(\mathbb{P})$), there is $\nu\in\mathbb{C}$ stronger than η such that $\nu\Vdash_{\mathbb{C}}$" $r\in(\mathbb{P}:\mathbb{C})$ ". Now the proof follows from the following Claim.

Claim 3.7.1. $\nu\Vdash_{\mathbb{C}}$" $(\exists^\infty i<\omega)(\underset{\sim}{p}_i\leq r)$ ".

Proof of the Claim. If not, then we may find $\nu'\in\mathbb{C}$ stronger than ν and $i'<\omega$ such that $\nu'\Vdash_{\mathbb{C}}$" $(\forall i\geq i')(\underset{\sim}{p}_i\nleq r)$ ". Let $\ell<\omega$ be such that $\nu'=\nu_\ell$ and let $i\in A$ be larger than $\ell+i'+1$. Look at our choices before - we know that:

(i) $p^*_i\leq r$,
(ii) $p_{i+1,\ell}\leq p^*_i$,
(iii) $\nu_\ell\nVdash_{\mathbb{C}}$ " $\underset{\sim}{p}_{i+1}\neq p_{i+1,\ell}$ ".

Therefore some condition $\nu^*\in\mathbb{C}$ stronger than ν_ℓ forces that $\underset{\sim}{p}_{i+1}\leq r$, contradicting the choice of $\nu'=\nu_\ell$ (as $i+1>i'$). □

□

In the rest of this section we are going to show that the result of 3.7 cannot be very much improved: when taking a quotient over a random real forcing we may loose topological sweetness. Let us start with recalling some notation and definitions, which we will need later.

Definition 3.8. (1) The Lebesgue (product) measure on 2^ω is denoted by $\mu^{\rm Leb}$, $\mathbf{Borel}(2^\omega)$ is the σ–field of Borel subsets of 2^ω and $\mathbb{L}$ is the σ–ideal of Lebesgue null subsets of 2^ω. The quotient complete Boolean algebra $\mathbb{B}=\mathbf{Borel}(2^\omega)/\mathbb{L}$ is called *the random algebra.*

(2) The random forcing notion $\mathbb{R}$ is defined as follows:
a condition in $\mathbb{R}$ is a closed subset of 2^ω of positive Lebesgue measure,
the order of $\mathbb{R}$ is the reverse inclusion.

(3) The amoeba for measure forcing notion $\mathbb{A}$ is defined as follows:
a condition in $\mathbb{A}$ is a closed subset F of 2^ω such that $\mu^{\rm Leb}(F)>\frac{1}{2}$,
the order of $\mathbb{A}$ is the reverse inclusion.

Of course, $\mathbb{B}=\mathbf{BA}(\mathbb{R})$. Let us also recall that both $\mathbb{R}$ and $\mathbb{A}$ are topologically sweet (see [10, 1.3.3]).

Proposition 3.9. (1) $\Vdash_{\mathbb{B}}$" $\mathbb{A}^{\mathbf{V}}$ *is not topologically sweet* ".

(2) $\Vdash_{\mathbb{B}}$" $\mathbb{R}^{\mathbf{V}}$ *is not topologically sweet* ".

Proof. (1) Suppose toward contradiction that

$$[\![\text{ there is a model of topological sweetness based on } \mathbb{A}^{\mathbf{V}}]\!]_{\mathbb{B}}\neq\mathbf{0}_{\mathbb{B}}.$$

Since the random algebra is homogeneous, we may assume that we have $\mathbb{B}$–names $\underset{\sim}{U}_n$ for subsets of $\mathbb{A}^{\mathbf{V}}$ such that

$(*)_0$ $\Vdash_{\mathbb{B}}$ " $(\mathbb{A}^{\mathbf{V}}, \{\underset{\sim}{U}_n : n < \omega\})$ is a model of topological sweetness ".

For $i < \omega$ let $m_i = \lfloor -\frac{i \cdot 2^i}{\log_2(1-2^{-2^{i+1}})} \rfloor + 2$, so $\frac{m_i}{2^i} > \frac{-i}{\log_2(1-2^{-2^{i+1}})}$ and thus

$(*)_1$ $(1-2^{-2^{i+1}})^{m_i/2^i} < 2^{-i}$.

Let μ be the product Lebesgue measure on the space $\prod\limits_{i<\omega} m_i$, and let μ^* be the corresponding outer measure.

Define $\langle n_i : i < \omega\rangle$ by $n_0 = 0$, $n_{i+1} = n_i + m_i \cdot 2^{i+1}$, and for $i < \omega$, $j < m_i$ put

$$t^i_j \stackrel{\rm def}{=} \big\{\sigma \in 2^{[n_i, n_{i+1})} : \big(\exists \ell < 2^{i+1}\big)\big(\sigma(n_i + j \cdot 2^{i+1} + \ell) = 1\big)\big\}.$$

Note that

$(*)^i_2$ if $j_0 < j_1 < \ldots < j_k < m_i$, then

$$\big|t^i_{j_0} \cap t^i_{j_1} \cap \ldots \cap t^i_{j_k}\big| = \big(1 - 2^{-2^{i+1}}\big)^{k+1} \cdot 2^{m_i \cdot 2^{i+1}}.$$

For $x \in \prod\limits_{i<\omega} m_i$ let

$$Z_x \stackrel{\rm def}{=} \big\{\eta \in 2^\omega : \big(\forall i < \omega\big)\big(\eta{\restriction}[n_i, n_{i+1}) \in t^i_{x(i)}\big)\big\}$$

and note that Z_x is a closed set and $\mu^{\rm Leb}(Z_x) > \frac{1}{2}$, so $Z_x \in \mathbb{A}$. For each $x \in \prod\limits_{i<\omega} m_i$ and $n < \omega$ we may pick a Borel set $B(x,n) \subseteq 2^\omega$ such that $[\![Z_x \in \underset{\sim}{U}_n]\!]_{\mathbb{B}} = [B(x,n)]_{\mathbb{L}}$. Next, for each $k < \omega$ (and $x \in \prod\limits_{i<\omega} m_i$ and $n < \omega$) choose a clopen set $C(x,n,k) \subseteq 2^\omega$ such that $\mu^{\rm Leb}\big(B(x,n) \triangle C(x,n,k)\big) < 2^{-k}$. Now, for $n < \omega$, consider a binary relation $\sim_n$ on $\prod\limits_{i<\omega} m_i$ given by

$$x \sim_n y \quad \text{ if and only if } \quad \big(\forall k, \ell \leq n\big)\big(C(x,\ell,k) = C(y,\ell,k)\big).$$

It should be clear that (for each $n < \omega$) $\sim_n$ is an equivalence relation on $\prod\limits_{i<\omega} m_i$ such that

$(*)^n_3$ $x \sim_{n+1} y \ \Rightarrow\ x \sim_n y$ (for each $x, y \in \prod\limits_{i<\omega} m_i$), and

$(*)^n_4$ $\prod\limits_{i<\omega} m_i / \sim_n$ is countable.

Consequently we may pick $x^* \in \prod\limits_{i<\omega} m_i$ such that for each $n < \omega$ we have

$$\lim_{\ell\to\infty} \frac{\mu^*\big(\big\{x \in \prod\limits_{i<\omega} m_i : x{\restriction}\ell = x^*{\restriction}\ell \ \ \& \ \ x \sim_n x^*\big\}\big)}{\mu\big(\big\{x \in \prod\limits_{i<\omega} m_i : x{\restriction}\ell = x^*{\restriction}\ell\big\}\big)} = 1.$$

So now we may choose an increasing sequence $\langle \ell_i : i < \omega\rangle \subseteq \omega$ such that for $i < \omega$ we have

$$\mu^*\big(\big\{x \in \prod_{j<\omega} m_j : x{\restriction}\ell_i = x^*{\restriction}\ell_i \ \ \& \ \ x \sim_i x^*\big\}\big) > \frac{1}{2}\mu\big(\big\{x \in \prod_{j<\omega} m_j : x{\restriction}\ell_i = x^*{\restriction}\ell_i\big\}\big),$$

and then for each $i < \omega$ we may choose $v_i \subseteq m_{\ell_i}$ and $\langle y^i_k : k \in v_i\rangle \subseteq \prod\limits_{j<\omega} m_j$ such that

$(*)^i_5$ $|v_i| > \frac{1}{2} m_{\ell_i}$,

$(*)^i_6$ $y^i_k{\restriction}\ell_i = x^*{\restriction}\ell_i$, $y^i_k(\ell_i) = k$ and $y^i_k \sim_i x^*$ for $k \in v_i$.

It follows from the definition of the relations $\sim_n$ and from $(*)_6^i$ that for each $k\in v_i$ and all $\ell\leq i$ we have

$$\mu^{\rm Leb}\big(B(x^*,\ell)\bigtriangleup B(y^i_k,\ell)\big)<2^{1-i}.$$

Thus, for each $i<\omega$, we may pick a partition $\langle B^i_k: k\in v_i\rangle$ of 2^ω into disjoint Borel sets such that for all $k\in v_i$ we have

$(*)_7^{i,k}$ $\mu^{\rm Leb}(B^i_k)=\frac{1}{|v_i|}$, and

$(*)_8^{i,k}$ $\mu^{\rm Leb}\big(B^i_k\cap (B(xj,\ell)\bigtriangleup B(y^i_k,\ell))\big)<2^{1-i}/|v_i|$ for all $\ell\leq i$.

Let $\underaccent{\tilde}{x}_i$ be a $\mathbb{B}$–name for a member of $\mathbf{V}\cap\prod\limits_{j<\omega} m_j$ such that

$$\big(\forall k\in v_i\big)\big([\![\underaccent{\tilde}{x}_i=y^i_k]\!]_{\mathbb{B}}=[B^i_k]_{\mathbb{L}}\big).$$

Claim 3.9.1.

$$\Vdash_{\mathbb{B}} \text{“}\ \big(\forall n<\omega\big)\big(\forall^\infty i<\omega\big)\big(Z_{x^*}\in\underaccent{\tilde}{U}_n\ \Rightarrow\ Z_{\underaccent{\tilde}{x}_i}\in \underaccent{\tilde}{U}_n\big).$$

Proof of the Claim. Note that for $n,i<\omega$ we have

$$[\![Z_{\underaccent{\tilde}{x}_i}\notin\underaccent{\tilde}{U}_n]\!]_{\mathbb{B}}=\Big[\bigcup_{k\in v_i}B^i_k\setminus B(y^i_k,n)\Big]_{\mathbb{L}},$$

and thus $[\![Z_{x^*}\in\underaccent{\tilde}{U}_n\ \&\ Z_{\underaccent{\tilde}{x}_i}\notin\underaccent{\tilde}{U}_n]\!]_{\mathbb{B}}=\big[\bigcup\limits_{k\in v_i}\big(B(x^*,n)\setminus B(y^i_k,n)\big)\cap B^i_k\big]_{\mathbb{L}}$. It follows from $(*)_8^{i,k}$ that (for $n\leq i<\omega$) we have

$$\mu^{\rm Leb}\big(\bigcup_{k\in v_i}\big(B(x^*,n)\setminus B(y^i_k,n)\big)\cap B^i_k\big)<2^{1-i}.$$

Hence for, each $n<\omega$,

$$\mu^{\rm Leb}\Big(\bigcap_{m<\omega}\bigcup_{i>m}\big(\bigcup_{k\in v_i}\big(B(x^*,n)\setminus B(y^i_k,n)\big)\cap B^i_k\big)\Big)=0,$$

so

$$[\![(\exists^\infty i<\omega)(Z_{x^*}\in\underaccent{\tilde}{U}_n\ \&\ Z_{\underaccent{\tilde}{x}_i}\notin\underaccent{\tilde}{U}_n]\!]_{\mathbb{B}}=\mathbf{0}_{\mathbb{B}},$$

and the Claim follows. $\square$

It follows from $(*)_0$ and 3.9.1 that

$$[\![\big(\exists F\in\mathbb{A}^{\mathbf{V}}\big)\big(F\subseteq Z_{x^*}\ \&\ (\exists^\infty i<\omega)(F\subseteq Z_{\underaccent{\tilde}{x}_i})\big)]\!]_{\mathbb{B}}=\mathbf{1}_{\mathbb{B}},$$

and therefore we may find $F\in\mathbb{A}\cap\mathbf{V}$ such that $F\subseteq Z_{x^*}$ and $a\stackrel{\rm def}{=}[\![(\exists^\infty i<\omega)(F\subseteq Z_{\underaccent{\tilde}{x}_i})]\!]_{\mathbb{B}}\neq\mathbf{0}_{\mathbb{B}}$. For $i<\omega$ put

$$w_i=\{k\in v_i: F\subseteq Z_{y^i_k}\}\quad\text{ and }\quad C_i=\bigcup_{k\in w_i}B^i_k.$$

Plainly, $a=\big[\bigcap\limits_{m<\omega}\bigcup\limits_{i>m}C_i\big]_{\mathbb{L}}$ so (as $a\neq\mathbf{0}_{\mathbb{B}}$) $\sum\limits_{i=1}^\infty\mu^{\rm Leb}(C_i)=\infty$, and hence the set

$$I\stackrel{\rm def}{=}\{i<\omega:\mu^{\rm Leb}(C_i)>2^{1-i}\}$$

is infinite.

Fix $i\in I$ for a moment. Then

$$2^{1-i}<\mu^{\rm Leb}(C_i)=\sum_{k\in w_i}\mu^{\rm Leb}(B^i_k)=\frac{|w_i|}{|v_i|},$$

and thus (by $(*)^i_5$)

$$|w_i| > |v_i| \cdot 2^{1-i} > \frac{1}{2} \cdot m_{\ell_i} \cdot 2^{1-i} \geq m_{\ell_i}/2^{\ell_i}.$$

Hence, by $(*)^{\ell_i}_1$, we get $\big(1-2^{-2^{\ell_i+1}}\big)^{|w_i|} < 2^{-\ell_i} \leq 2^{-i}$. Now (for our $i \in I$) consider the closed set $Y_i \stackrel{\rm def}{=} \bigcap\limits_{k\in w_i} Z_{y^i_k}$ and note that

$$Y_i \subseteq \big\{\eta \in 2^\omega : (\forall k \in w_i)(\eta{\restriction}[n_{\ell_i}, n_{\ell_i+1}) \in t^{\ell_i}_k)\big\}.$$

Thus, by $(*)^{\ell_i}_2$, we may conclude that (for our $i \in I$)

$$\mu^{\rm Leb}(Y_i) \leq \frac{|\bigcap\limits_{k\in w_i} t^{\ell_i}_k|}{2^{m_{\ell_i}\cdot 2^{\ell_i+1}}} = \big(1-2^{-2^{\ell_i+1}}\big)^{|w_i|} < 2^{-i}.$$

Since I is infinite and for every $i \in I$ we have $F \subseteq \bigcap\limits_{k\in w_i} Z_{y^i_k} = Y_i$ we may now conclude that $\mu^{\rm Leb}(F) = 0$, contradicting $F \in \mathbb{A}$.

(2) The same proof as for (1) works here too. $\square$

Putting together 3.3 and 3.9 we may easily conclude the following.

Corollary 3.10. *Both* $\mathbb{R}\times\mathbb{R}$ *and* $\mathbb{A}\times\mathbb{A}$ *are topologically sweet, but*

$\Vdash_{\mathbb{R}}$ *" no dense subforcing of* $(\mathbb{R}\times\mathbb{R} : \mathbb{R})$ *(*$(\mathbb{R}\times\mathbb{A} : \mathbb{R})$*, respectively) is topologically sweet ".*

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