A Localic Theory of Lower and Upper Integrals

Vickers, Steven

DOI:
10.1002/malq.200710028

Citation for published version (Harvard):
Vickers, S 2008, 'A Localic Theory of Lower and Upper Integrals', *Mathematical Logic Quarterly*, vol. 54, no. 1, pp. 109-123. https://doi.org/10.1002/malq.200710028

A localic theory of lower and upper integrals

Steven Vickers
School of Computer Science, University of Birmingham,
Birmingham, B15 2TT, UK
s.j.vickers@cs.bham.ac.uk

March 30, 2009

Abstract

An account of lower and upper integration is given. It is constructive in the sense of geometric logic. If the integrand takes its values in the non-negative lower reals, then its lower integral with respect to a valuation is a lower real. If the integrand takes its values in the non-negative upper reals, then its upper integral with respect to a covaluation and with domain of integration bounded by a compact subspace is an upper real. Spaces of valuations and of covaluations are defined.

Riemann and Choquet integrals can be calculated in terms of these lower and upper integrals.

This is a preprint version of the article published as –
Mathematical Logic Quarterly **54** (1) (2008), pp. 109 - 123.
doi:10.1002/malq.200710028

1 Introduction

Despite being one of the most fundamental constructions of analysis, integration is remarkably protean. Even in classical mathematics one finds a range of definitions of integration of an integrand, over a domain of integration and with respect to a measure, with a variety of more or less complicated calculations to cope with integrands that may be signed or even non-functional, and domains or measures that may be infinite. One also finds lower and upper integrals.

Our aim is to give a constructive localic account, and here the complexity is worsened by the need to choose carefully the kind of real numbers to be used. Even if (as is localically natural) one takes Dedekind sections as the usual reals, there is often still a need to consider upper or lower reals, i.e. like Dedekind sections but with only the right or left parts.

Here we provide minimal primitive forms of integration, in terms of which the others can be defined.

For lower integration, which is approximated from below, we use lower reals throughout. The integrand takes its values as lower reals. The measure is taken in the form of a valuation, which values the opens continuously in the lower reals – "continuously" means with respect to the Scott topology, so that a valuation transforms directed joins to sups.

For upper integration, which is approximated from above, we use upper reals. The integrand takes its values as upper reals. The measure is taken in the form of a *co*valuation, which in effect values the closeds cocontinuously (transforming directed meets to infs) in the upper reals. At the same time, we shall need a compact bound for the domain of integration.

For both of these we find ourselves constrained to integrating non-negative integrands, in order to avoid trying to negate lower or upper reals. However, when we strengthen the context to one involving Dedekind reals we find we can simplify the discussion with a form of Riemann integration.

2 Geometricity

The paper is "constructive" in a loose sense of conforming with the choice-free, predicative reasoning of *geometric* logic. More precisely, it is preserved under change of base in topos theory, along the inverse image functors of geometric morphisms. This goes somewhat beyond pure logic, and the techniques are explained in more detail in [Vic07] and [Vic04]; some sample applications appear in [Vic99], [Vic05a]. This makes it intuitionistic (topos-valid), but also allows it to conform with predicative type theory – specifically, the powerset construction is inapplicable, since it is not geometric.

An important aspect is that although it is presented in terms of "spaces", and can indeed be read as referring to ordinary topological spaces in classical mathematics, it also has a covert reading in terms of point-free topology. In each description of a space, the points are described in effect as the models of some geometric theory, and the geometric propositions then supply a subbase of opens for the topology. The theory can be used to present a frame by generators and relations, and so describes a locale.

As part of the spatial presentation, we shall commonly use symbols of set theory ($\cup, \cap, \subseteq \cdots$) with opens, rather than those of lattice theory ($\vee, \wedge, \leq \cdots$).

A map $f : X \to Y$ between spaces is described by how it transforms points. It should generally be clear how the inverse image function can be derived so as to give a locale map. However, at a deeper level the description of the transformation is itself geometric and can be applied to the generic point of X in the topos of sheaves over X. Topos theory then tells us that we have a geometric morphism between the toposes of sheaves, and hence a map between the locales.

The work is also intended to be amenable to formulation in predicative formal topology [Sam87], though the details have not been worked through. A geometric theory gives an inductively generated formal topology [CSSV03] in an obvious way, though with extra complexity due to the standard practice of presenting with a base of opens rather than a subbase. Regarding maps, we cannot apply in any obvious way the topos techniques using generic points.

Nonetheless, the process by which an inverse image function is extracted from a point transformation described geometrically should also yield continuous maps between the formal topologies.

3 Spaces of reals

Various kinds of real numbers will be used. We write $\mathbb{Q}$ for the set of rationals and Q_+ for the set of positive rationals.

Definition 1 *1. A* lower real *is a rounded, down-closed subset of* $\mathbb{Q}$*. Note that this includes* $\emptyset$ *and* $\mathbb{Q}$ *(*$-\infty$ *and* ∞*). If* $\underline{a}$ *is a lower real and* $q \in \mathbb{Q}$*, then we write* $q < \underline{a}$ *if* q *is an element of* $\underline{a}$*.*

2. An upper real *is a rounded, up-closed subset of* $\mathbb{Q}$*. This time,* $\emptyset$ *and* $\mathbb{Q}$ *are* ∞ *and -*∞*. We write* $\overline{a} < q$ *if* q *is an element of the upper real* $\overline{a}$*.*

3. A Dedekind real x *is a pair* $\langle \underline{x}, \overline{x} \rangle$ *where* $\underline{x}$ *is a non-empty lower real,* $\overline{x}$ *is a non-empty upper real, and* $\underline{x}$ *and* $\overline{x}$ *are disjoint but come arbitrarily close (or, if* $q < r$ *are rationals, then either* $q < \underline{x}$ *or* $\overline{x} < r$*).*

(Here and elsewhere, we shall use symbols such as $\overline{a}$ and $\underline{b}$ for variables of type upper and lower real respectively.)

Though classically these are all equivalent, they are still distinguished by their implicit topologies: the Scott topology for the lower reals, the opposite Scott topology for the upper reals (these are, respectively, the topologies of lower and upper semicontinuity), and the usual Hausdorff topology for the Dedekind reals. The specialization orders are, respectively, numerical order, its opposite, and equality.

For Dedekind reals we take the common notation for intervals, with a round bracket for an open end and a square bracket for a closed end. We can also extend this by allowing $\underline{x}$ and/or $\overline{x}$ to be empty, thus adjoining infinities. Hence the Dedekind real line (as a locale) is $(-\infty, \infty)$ and we may extend it with an infinity at either end, or both, writing $(-\infty, \infty]$ etc.

We adapt the same notation for lower or upper reals by writing an arrow to show the direction of the specialization order. Thus, for instance, $\overleftarrow{[0, \infty)}$ denotes the space of non-negative upper reals, excluding ∞ – concretely, the inhabited rounded upsets of positive rationals. Similarly, $\overrightarrow{[0, \infty]}$ comprises those rounded downsets in $\mathbb{Q}$ that contain all negative rationals. Note that the interval *must* be closed at the arrowhead end. This is because locales have directed joins of points with respect to the specialization order.

Lemma 2 *The map* $\overleftarrow{[0, \infty)} \times \overrightarrow{[0, \infty]} \to \mathbb{S}$*, defined by* $\langle \overline{a}, \underline{b} \rangle \mapsto \overline{a} < \underline{b}$ *(i.e. there is some rational* p *in* $\overline{a} \cap \underline{b}$*), gives a homeomorphism* $\overrightarrow{[0, \infty]} \cong \mathbb{S}^{\overleftarrow{[0, \infty)}}$*. Similarly, we get* $\overleftarrow{[0, \infty]} \cong \mathbb{S}^{\overrightarrow{(0, \infty]}}$*.*

Note how in each case the exponential includes an endpoint not in the exponent. For example, ∞ in $\overrightarrow{[0,\infty]}$ corresponds to the entire space $\overleftarrow{[0,\infty)}$ as open in itself.

Although we have addition maps for $\overrightarrow{[-\infty,\infty]}$ and $\overleftarrow{[-\infty,\infty]}$ (and also addition and multiplication in $\overrightarrow{[0,\infty]}$ and $\overleftarrow{[0,\infty]}$), we do not have subtraction. This is obvious because $a-b$ would be antitone in b with respect to the specialization order. However, we do have mixed subtraction.

Definition 3 *The subtraction map* $- : \overrightarrow{[-\infty,\infty]} \times \overleftarrow{[-\infty,\infty]} \to \overrightarrow{[-\infty,\infty]}$ *is defined as follows. Let* $\underline{a}$ *and* $\overline{b}$ *be lower and upper reals respectively. Then* $\underline{a} - \overline{b} = \{q - r \mid q < \underline{a}, r > \overline{b}\}$.

It is easy to see that, for any rational s, we have $s < \underline{a} - \overline{b}$ iff $s + \overline{b} < \underline{a}$. Note that, since ∞ in $\overleftarrow{[-\infty,\infty]}$ is empty, $\underline{a} - \infty = -\infty$ for all $\underline{a}$ including ∞. similarly, $(-\infty) - \overline{b} = -\infty$ for all $\overline{b}$ including $-\infty$.

Similarly, we have $- : \overleftarrow{[-\infty,\infty]} \times \overrightarrow{[-\infty,\infty]} \to \overleftarrow{[-\infty,\infty]}$.

Lemma 4

1. *Suppose* $\underline{a}_i$ *is in* $\overrightarrow{[-\infty,\infty]}$ *and* $\overline{b}_i$ *is in* $\overleftarrow{[-\infty,\infty]}$ $(i = 1, 2)$. *Then*
$$(\underline{a}_1 + \underline{a}_2) - (\overline{b}_1 + \overline{b}_2) = (\underline{a}_1 - \overline{b}_1) + (\underline{a}_2 - \overline{b}_2).$$

2. *Let* $\underline{a}$ *and* $\underline{c}$ *be lower reals, and* $\overline{b}$ *an upper real. Then* $\underline{a} - (\overline{b} - \underline{c}) = (\underline{a} - \overline{b}) + \underline{c}$.

3. *Let* $\langle \underline{x}, \overline{x} \rangle$ *be a Dedekind section. Then* $\underline{x} - \overline{x} = 0$.

The duals (interchanging $\overrightarrow{[-\infty,\infty]}$ *and* $\overleftarrow{[-\infty,\infty]}$*) also hold.*

Proof. Easy. ∎

4 A space of valuations

Broadly speaking, a *valuation* (or *evaluation*) is like a measure, but defined just on the opens. It has origins in Birkhoff's lattice theoretic definition using the modular law [Bir84] but can be seen more directly in the definition of the *probabilistic powerdomain* [Jon89], [JP89]. For us, a valuation will always be assumed to be Scott continuous. In detail, a valuation on a space X is a Scott continuous function $\mu : \Omega X \to [0,\infty]$ such that $\mu(\emptyset) = 0$, and for which the *modular* law
$$\mu U + \mu V = \mu(U \cup V) + \mu(U \cap V)$$
holds.

The implied use here of the Scott topology on $[0,\infty]$ shows that we are in fact dealing with $\overrightarrow{[0,\infty]}$, and it follows that a valuation μ is determined by the pairs $\langle p, U \rangle \in \mathbb{Q} \times \Omega X$ for which $p < \mu U$. To be precise, a valuation is equivalent to a subset I of $\mathbb{Q} \times \Omega X$ such that (i) $\langle p, U \rangle \in I$ if $p < 0$, (ii) $\langle p, U \rangle \in I$ iff $\langle p', U \rangle \in I$ for some $p' > p$, (iii) $\langle p, \bigvee^{\uparrow}_i U_i \rangle \in I$ iff $\langle p, U_i \rangle \in I$ for some i (here as elsewhere the use of "$\uparrow$" in $\bigvee^{\uparrow}$ denotes that the family of which the join is taken is directed), (iv) $\langle p, \emptyset\rangle$ is not in I for any $p \geq 0$, (v) if $\langle q, U\rangle, \langle r, V\rangle \in I$ then there are q', r' with $q' + r' = q + r$ and $\langle q', U \cap V\rangle, \langle r', U \cup V\rangle \in I$, and (vi) if $\langle q', U \cap V\rangle, \langle r', U \cup V\rangle \in I$ then there are q, r with $q + r = q' + r'$ and $\langle q, U\rangle, \langle r, V\rangle \in I$. Here (i) and (ii) determine a function from ΩX to the lower reals, (iii) says that it is Scott continuous, (iv) says it is 0 on $\emptyset$ and (v) and (vi) are the modular law.

We write $\mathfrak{V}X$ for the space whose points are the valuations on X. The above criteria on I are geometric in nature, and so we can consider $\mathfrak{V}X$ to be a locale. Explicitly, the frame $\Omega\mathfrak{V}X$ can be presented as

$$
\begin{aligned}
\mathrm{Fr}\langle [p : U]\ (p \in \mathbb{Q}, U \in \Omega X) \mid & \\
\top &\leq [p : U]\ (p < 0) \\
[p : U] &= \bigvee_{p' > p} [p' : U] \\
[p : \textstyle\bigvee^{\uparrow}_i U_i] &= \textstyle\bigvee^{\uparrow}_i [p : U_i] \\
[p : \emptyset] &\leq \emptyset\ (0 \leq p) \\
\bigvee_{q+r=p} ([q : U] \wedge [r : V]) &= \bigvee_{q+r=p} ([q : U \cup V] \wedge [r : U \cap V]) \\
& (0 \leq p \in \mathbb{Q},\ U, V \in \Omega X)\rangle.
\end{aligned}
$$

This idea for presenting a valuation locale has already been noted in [Hec94] for the probabilistic case ($\mu X = 1$). Note however that Heckmann assumes $p \geq 0$ in the symbol $[p : U]$, with the convention that $0 < 0$ so that $\top \leq [0 : U]$ for all U. This is introduced to ensure that the modular law is correctly dealt with in cases where $\mu(U \cap V) = 0$. The underlying problem is as follows. We should like to define addition on lower reals as $I + J = \{p + q \mid p \in I, q \in J\}$. If we are dealing with non-negative lower reals, then we can ignore the negative rationals since they are automatically in the rounded downsets. But then the lower real 0 is determined by the empty set $I = \emptyset$, and then the previous definition of addition gives the wrong answer $0 + J = 0$. It can be corrected as $I + J = I \cup J \cup \{p + q \mid p \in I, q \in J\}$, or it can be saved by Heckmann's convention of $0 < 0$ so that 0 is always included. Instead we shall here just include all the negative rationals.

The frame ΩX is used as one of the indexing sets for the generators in $\mathfrak{V}X$, and this is ungeometric since frame structure is not preserved by inverse image functors. The situation is similar to that of the powerlocale constructions, which in [Vic04] are shown to be geometric by reducing to geometric constructions on frame presentations by generators and relations. We shall use a similar technique here. The proof is somewhat technical, and will not add much to the broad story of integration. Nonetheless, it does have the consequence that the valuation spaces can be dealt with as formal topologies: if a formal topology is given for X, then one can be constructed, predicatively, for $\mathfrak{V}X$. (The corresponding results for powerlocales are in [Vic06] and [Vic05b].)

Proposition 5 *The valuation space construction $\mathfrak{V}$ is geometric.*

Proof. By [VT04], it suffices to consider ΩX presented in a form in which the generators form a distributive lattice L, whose lattice structure is preserved in ΩX, and the relations are all of the form $a \leq \bigvee_i^{\uparrow} a_i$. We write $\eta : L \to \Omega X$ for the injection of generators. Now consider the frame

$$\begin{aligned}
B = \mathrm{Fr}\langle [p:a]' \ (p \in \mathbb{Q}, a \in L) \mid \\
\top &\leq [p:a]' \ (p < 0) \\
[p:a]' &= \bigvee_{p'>p} [p':a]' \\
[p:a]' &\leq [p:b]' \ (\text{if } a \leq b) \\
[p:a]' &\leq \bigvee\nolimits_i^{\uparrow} [p:a_i]' \ (\text{if } a \leq \bigvee\nolimits_i^{\uparrow} a_i \text{ a relation}) \\
[p:0]' &\leq \emptyset \ (0 \leq p) \\
\bigvee_{q+r=p} ([q:a]' \wedge [r:b]') &= \bigvee_{q+r=p} ([q:a \vee b]' \wedge [r:a \wedge b]') \\
& (0 \leq p \in \mathbb{Q}, \ a, b \in L)\rangle.
\end{aligned}$$

Clearly there is a frame homomorphism $\alpha : B \to \Omega X$ taking $[p:a]' \mapsto [p:\eta(a)]$.

If $p \in \mathbb{Q}$, the function $a \mapsto [p:a]'$ is monotone and respects the relations, and so by [VT04] extends uniquely to a Scott continuous function $\beta_p : \Omega X \to B$ with $\beta_p \circ \eta(a) = [p:a]'$. We show that the assignment $[p,U] \mapsto \beta_p(U)$ respects the relations for ΩX and so defines a frame homomorphism $\beta : \Omega X \to B$. Most of it is obvious from the definition (and Scott continuity) of β_p, and the fact that each $U \in \Omega X$ is a directed join of elements $\eta(a)$. However, we need to take a little more care over the modular law.

Suppose $U, V \in \Omega X$, and $q, r \in \mathbb{Q}$, $p = q + r$. If $\eta(a) \subseteq U$ and $\eta(b) \subseteq V$ then

$$\begin{aligned}
[q:a]' \wedge [r:b]' &\leq \bigvee_{q'+r'=p} ([q':a \vee b]' \wedge [r':a \wedge b]') \\
&\leq \bigvee_{q'+r'=p} (\beta_{q'}(U \cup V) \wedge \beta_{r'}(U \cap V)).
\end{aligned}$$

The reverse direction is somewhat more intricate. Suppose we have $\eta(c) \subseteq U \cup V$ and $\eta(d) \subseteq U \cap V$. Replacing c by $c \vee d$, we may assume without loss of generality that $d \leq c$. Then

$$\begin{aligned}
\eta(c) &= (\eta(c) \cap U) \cup (\eta(c) \cap V) \\
&= \bigcup\nolimits^{\uparrow} \{\eta(a \vee b) \mid \eta(a) \subseteq \eta(c) \cap U, \eta(b) \subseteq \eta(c) \cap V\}.
\end{aligned}$$

Replacing a and b by $a \vee d$ and $b \vee d$, we may assume without loss of generality that $d \leq a \wedge b$. Then

$$\begin{aligned} [q : a \vee b]' \wedge [r : d]' &\leq [q : a \vee b]' \wedge [r : a \wedge b]' \\ &\leq \bigvee_{q'+r'=p} ([q' : a]' \wedge [r' : b]') \\ &\leq \bigvee_{q'+r'=p} (\beta_{q'}(U) \wedge \beta_{r'}(V)). \end{aligned}$$

Now that both α and β are defined as frame homomorphisms, it is straightforward to show that they are mutually inverse. ■

Remark 6 *If X is locally compact, then the Scott continuous maps from ΩX to $\overrightarrow{[0,\infty]}$ are the points of $\overrightarrow{[0,\infty]}^{\mathbb{S}^X}$. The sublocale on which the modular law holds can be described as the equalizer of two maps $\overrightarrow{[0,\infty]}^{\mathbb{S}^X} \to \overrightarrow{[0,\infty]}^{\mathbb{S}^X \times \mathbb{S}^X}$, described by $\mu \mapsto \lambda U.\lambda V.(\mu U + \mu V)$ and $\mu \mapsto \lambda U.\lambda V.(\mu(U \cup V) + \mu(U \cap V))$. The zero law $\mu\emptyset = 0$ is similar. Hence we can describe $\mathfrak{V}X$ by equalizers as a sublocale of $\overrightarrow{[0,\infty]}^{\mathbb{S}^X}$. Since exponentiation of locales is trivially geometric [Vic04], this is enough to give geometricity of $\mathfrak{V}$ in the locally compact case. Now*

$$\overrightarrow{[0,\infty]}^{\mathbb{S}^X} \cong (\mathbb{S}^{\overleftarrow{[0,\infty)}})^{\mathbb{S}^X} \cong (\mathbb{S}^{\mathbb{S}^X})^{\overleftarrow{[0,\infty)}} \cong (\mathbb{P}X)^{\overleftarrow{[0,\infty)}}$$

where $\mathbb{P}X$ is the double powerlocale [Vic04]. Similarly, $\overrightarrow{[0,\infty]}^{\mathbb{S}^X \times \mathbb{S}^X} \cong (\mathbb{P}(X + X))^{\overleftarrow{[0,\infty)}}$. Since $\mathbb{P}X$ is defined for arbitrary X, and since [VT04] has described a general sense in which $\mathbb{P}X \cong \mathbb{S}^{\mathbb{S}^X}$, it is plausible that those simple geometricity arguments can also be made valid beyond the locally compact case. However, the necessary metatheory does not appear to exist yet.

If $f : X \to Y$ is a map, then we have $\mathfrak{V}f : \mathfrak{V}X \to \mathfrak{V}Y$ defined by

$$\mathfrak{V}f(\mu)(V) = \mu(f^*V).$$

In terms of subbasics, its inverse image takes $[q : V]$ to $[q : f^*V]$, and it is clear that this respects the relations for $\mathfrak{V}X$ and $\mathfrak{V}Y$. Hence the definition by points also leads to a localic definition. We also see that $\mathfrak{V}$ is functorial.

I am indebted to Martín Escardó for pointing out the importance of the following concept.

Definition 7 *A valuation μ on X is* finite *if μX is a Dedekind real, i.e. an upper real $\overline{a}$ is provided such that $\langle \mu X, \overline{a} \rangle$ is a Dedekind section.*

Note that $\overline{a}$ must be finite – as a rounded upset of postive rationals it must be inhabited. Also, if $\overline{a}$ exists at all then it is unique.

Definition 8 *The space $\mathfrak{V}_f X$ of finite valuations on X is defined as a subspace of $\mathfrak{V}X \times \overleftarrow{[0,\infty)}$. It is presented by relations*

$$[q : X] \times \overleftarrow{[0,q)} \leq \emptyset$$
$$\mathfrak{V}X \times \overleftarrow{[0,\infty)} \leq [q : X] \odot \overleftarrow{[0,r)} \ \textit{(if } q < r\textit{)}.$$

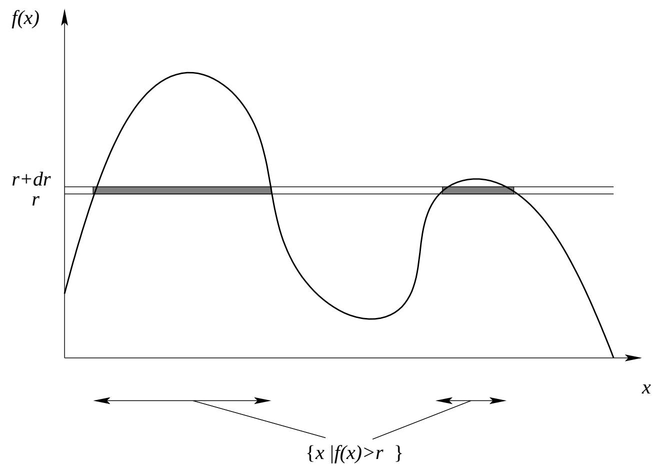

Figure 1: Illustrating Choquet integration.

(Note – in a product space $X \times Y$, the notation $U \odot V$ denotes $U \times Y \cup X \times V$.)

For $\langle \mu, \overline{a} \rangle$ to be in the subspace, the first relation says that we cannot have $\overline{a} < q < \mu X$, and the second that if $q < r$ then either $q < \mu X$ or $\overline{a} < r$.

The composite map $\mathfrak{V}_f X \hookrightarrow \mathfrak{V}X \times \overleftarrow{[0,\infty)} \to \mathfrak{V}X$ is monic, but not an inclusion. To see this, consider the specialization orders. In $\mathfrak{V}X$ we have $\mu \sqsubseteq \mu'$ iff $\mu U \leq \mu' U$ for all U. In $\mathfrak{V}_f X$ we must have, in addition, that $\mu X = \mu' X$.

5 Lower integrals

We now describe how to obtain lower integrals for maps $f : X \to \overrightarrow{[0,\infty]}$ with respect to valuations μ on X. In Section 9 we shall look more closely at the *Choquet integral*, defined classically (see, e.g., [AMJK04], which deals with valuations in a classical setting) by

$$\int f \; d\mu = \int_0^{+\infty} \mu([f > r]) \; dr$$

where $[f > r]$ denotes the open set $f^{-1}(r, \infty]$. The essential idea behind this is illustrated in fig. 1. From the definition it is clear that one would then expect the following results: that if $g : Y \to X$ and μ' is a valuation on Y, then

$$\begin{aligned}\int f \circ g \; d\mu' &= \int_0^{+\infty} \mu'([f \circ g > r]) \; dr = \int_0^{+\infty} \mu'(g^{-1}[f > r]) \; dr \\ &= \int_0^{+\infty} \mathfrak{V}g(\mu')([f > r]) \; dr = \int f \; d\mathfrak{V}g(\mu')\end{aligned}$$

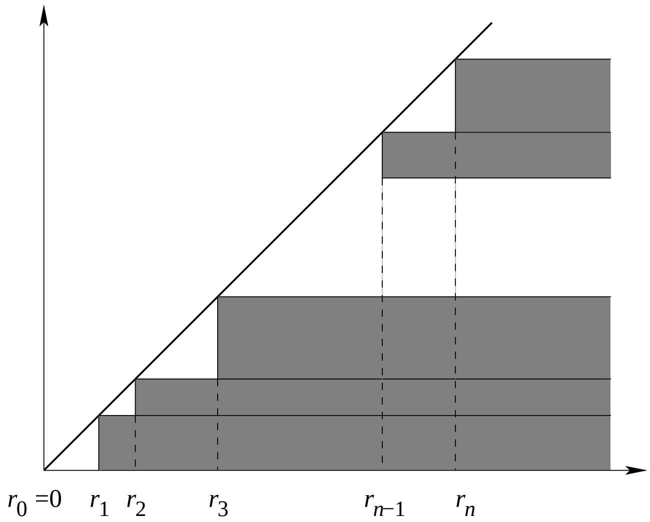

Figure 2: Illustrating lower integration.

and, as a consequence,

$$\int f \; d\mu = \int \mathrm{Id} \;\; d\mathfrak{V}f(\mu).$$

Definition 9 *Let* μ *be a valuation on* $\overrightarrow{[0,\infty]}$. *We define the* lower integral $\underline{\int} \mathrm{Id}\, d\mu$ *to be*

$$\sup\{\sum_{i=1}^{n}(r_i - r_{i-1})\mu\overrightarrow{(r_i,\infty]} \mid$$
$$0 = r_0 < \cdots < r_n \text{ are rationals } (n \geq 1)\}.$$

If $f : X \to \overrightarrow{[0,\infty]}$ *and* μ *is a valuation on* X, *then we define the lower integral* $\underline{\int} f d\mu$ *to be* $\underline{\int} \mathrm{Id}\, d\mathfrak{V}f(\mu)$.

This is illustrated in fig. 2, which shows how horizontal slabs are used to approximate the area from below.

Note that because the rational coefficients $(r_i - r_{i-1})$ are all positive, the terms in the sum are all still in $\overrightarrow{[0,\infty]}$. Also, $\overrightarrow{[0,\infty]}$ has all sups of set-indexed families. Hence the definition is valid.

One might have expected an approximation by vertical slabs along the line of Riemann integration, giving

$$\sum_{i=1}^{n-1} r_i(\mu\overrightarrow{(r_i,\infty]} - \mu\overrightarrow{(r_{i+1},\infty]}) + r_n\mu\overrightarrow{(r_n,\infty]}.$$

However, this would be illegitimate because it subtracts lower reals.

As anticipated, we see that if $f : X \to \overrightarrow{[0,\infty]}$ and $g : Y \to X$ are two maps, and μ is a valuation on Y then

$$\underline{\int} f \circ g \; d\mu = \underline{\int} \mathrm{Id} \, d\mathfrak{V}f(\mathfrak{V}g(\mu)) = \underline{\int} f \; d\mathfrak{V}g(\mu).$$

6 Covaluations

By a *covaluation* on a topological space X we mean in effect a valuation on the closed subspaces, so for each open it describes the mass of its complement. We therefore have a contravariant function $\nu : \Omega X \to [0,\infty]$ such that $\nu(X) = 0$, and again we expect the *modular* law

$$\nu U + \nu V = \nu(U \cup V) + \nu(U \cap V).$$

We do not expect the mass function on closed sets to preserve directed unions, for the same reasons as measure theory only requires countable unions to be preserved in general – every closed set is a directed union of finite sets. However, it is more reasonable to suppose that the mass of closed sets preserves down-directed intersections, so we suppose ν transforms directed joins to infs. Thus we take ν to be a Scott continuous map from ΩX to $\overleftarrow{[0,\infty]}$.

A covaluation ν is now determined by the pairs $\langle p, U\rangle \in Q_+ \times \Omega X$ for which $p > \mu U$ (we write Q_+ for the set of positive rationals). Then a covaluation is equivalent to a subset I of $Q_+ \times \Omega X$ such that (i) $\langle p, U\rangle \in I$ iff $\langle p', U\rangle \in I$ for some $p' < p$, (ii) $\langle p, \bigvee_i^{\uparrow} U_i\rangle \in I$ iff $\langle p, U_i\rangle \in I$ for some i, (iii) $\langle p, X\rangle$ is in I for all p, (iv) if $\langle q, U\rangle, \langle r, V\rangle \in I$ then there are q', r' with $q' + r' = q + r$ and $\langle q', U \cap V\rangle, \langle r', U \cup V\rangle \in I$, and (v) if $\langle q', U \cap V\rangle, \langle r', U \cup V\rangle \in I$ then there are q, r with $q + r = q' + r'$ and $\langle q, U\rangle, \langle r, V\rangle \in I$. Here (i) determines a function from ΩX to the upper reals, (ii) says that it is Scott continuous, (iii) says it is 0 on X and (iv) and (v) are the modular law.

We write $\mathfrak{C}X$ for the space of covaluations on X. Again, an explicit frame presentation for $\Omega\mathfrak{C}X$ can be given as

$$\begin{aligned}
&\mathrm{Fr}\langle [p : U] \; (p \in Q_+, U \in \Omega X) \mid \\
&\qquad [p : U] = \bigvee_{p' > p} [p' : U] \\
&\qquad [p : \textstyle\bigvee_i^{\uparrow} U_i] = \textstyle\bigvee_i^{\uparrow} [p : U_i] \\
&\qquad \top \leq [p : X] \\
&\qquad \bigvee_{q+r=p} ([q : U] \wedge [r : V]) = \bigvee_{q+r=p} ([q : U \cup V] \wedge [r : U \cap V]) \\
&\qquad (0 \leq p \in Q_+, \; U, V \in \Omega X)\rangle.
\end{aligned}$$

Its geometricity argument is completely analogous to Proposition 5.

If $f : X \to Y$ is a map, then we define $\mathfrak{C}f : \mathfrak{C}X \to \mathfrak{C}Y$ by $\mathfrak{C}f(\nu)(V) = \nu(f^*V)$. Localically we have $(\mathfrak{C}f)^*([q : V]) = [q : f^*V]$, and this clearly respects the relations for $\mathfrak{C}X$ and $\mathfrak{C}Y$. $\mathfrak{C}$ is functorial.

Definition 10 *A covaluation ν on X is* finite *if $\nu\emptyset$ is a Dedekind real, i.e. a lower real $\underline{a}$ is provided such that $\langle \underline{a}, \nu\emptyset\rangle$ is a Dedekind section.*

Again, if $\underline{a}$ exists at all then it is unique. Also, $\nu\emptyset$ must be finite.

Definition 11 *The space $\mathfrak{C}_f X$ of finite covaluations on X is defined as a subspace of $\mathfrak{C}X \times \overrightarrow{[0,\infty]}$. It is presented by relations*

$$\mathfrak{C}X \times \overrightarrow{[0,\infty]} \leq \bigvee_{q \in Q_+} [q : \emptyset] \times \overrightarrow{[0,\infty]}$$

$$[q : \emptyset] \times \overrightarrow{(q,\infty]} \leq \emptyset \ (q \in Q_+)$$

$$\mathfrak{C}X \times \overrightarrow{[0,\infty]} \leq [q : \emptyset] \odot \overrightarrow{(r,\infty]} \ (\text{if } 0 \leq r < q \text{ in } \mathbb{Q}).$$

For $\langle \nu, \underline{a}\rangle$ to be in the subspace, the first relation says that $\nu\emptyset$ is finite, i.e. $\nu\emptyset < q$ for some q. The second says that we cannot have $\nu X < q < \underline{a}$, and the third that if $r < q$ then either $r < \underline{a}$ or $\nu X < q$.

The composite map $\mathfrak{C}_f X \hookrightarrow \mathfrak{C}X \times \overrightarrow{[0,\infty]} \to \mathfrak{C}X$ is monic, but not an inclusion. To see this, again consider the specialization orders. In $\mathfrak{C}X$ we have $\nu \sqsubseteq \nu'$ iff $\nu U \geq \nu' U$ for all U. (Note the order reversal. But the specialization order is in the same direction as numerical order on the covaluations considered as valuations on *closed* subspaces.) In $\mathfrak{C}_f X$ we must have, in addition, that $\nu\emptyset = \nu'\emptyset$.

Proposition 12 *Let X be a space. Then $\mathfrak{V}_f X \cong \mathfrak{C}_f X$.*

Proof. First we define $\alpha : \mathfrak{V}_f X \to \mathfrak{C}_f X$. For a point $\langle \mu, \overline{a}\rangle$ of $\mathfrak{V}_f X$, we define ν by $\nu U = \overline{a} - \mu U$. The modular law follows from Lemma 4 (1), and $\nu X = \overline{a} - \mu X = 0$ because $\langle \mu X, \overline{a}\rangle$ is a Dedekind section. To see that $\langle \nu, \mu X\rangle$ is in $\mathfrak{C}_f X$, we have $\nu\emptyset = \overline{a} - \mu\emptyset = \overline{a} - 0 = \overline{a}$, so $\langle \mu X, \nu\emptyset\rangle$ is the Dedekind section $\langle \mu X, \overline{a}\rangle$.

Similarly, $\langle \nu, \underline{a}\rangle \longmapsto \langle \mu, \nu\emptyset\rangle$, with $\mu U = \underline{a} - \nu U$, defines a map $\beta : \mathfrak{C}_f X \to \mathfrak{V}_f X$.

It remains to show that these are mutually inverse. For $\beta \circ \alpha = \mathrm{Id}$ this amounts to showing that, for each $\langle \mu, \overline{a}\rangle$ in $\mathfrak{V}_f X$, $\mu U = \mu X - (\overline{a} - \mu U)$, and this follows from Lemma 4 (2) and (3). $\alpha \circ \beta = \mathrm{Id}$ is similar. ■

If μ is a finite valuation on X then we shall write $\overline{\mu}$ for the corresponding covaluation (i.e. $\alpha(\mu)$ in the above proof). Similarly, if ν is a finite covaluation then we shall write $\underline{\nu}$ for the corresponding valuation.

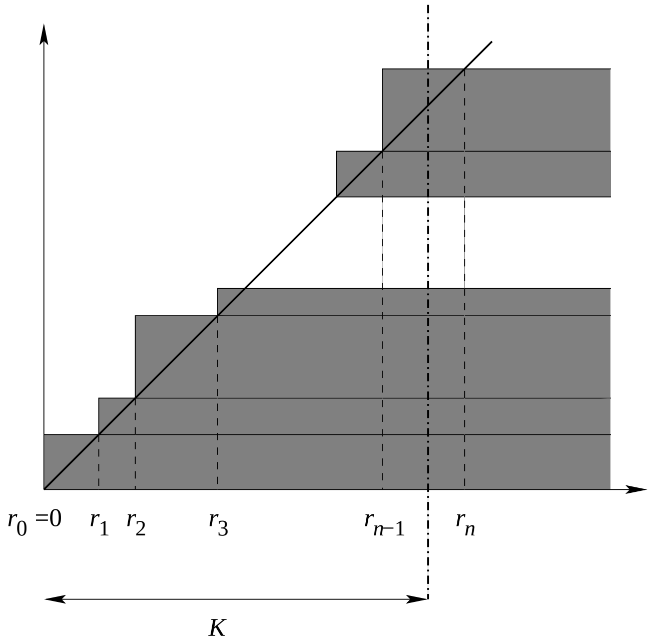

Figure 3: Illustrating upper integration.

7 Upper integrals

This time we describe how to obtain upper integrals for maps $f : X \to \overleftarrow{[0,\infty]}$ with respect to covaluations ν on X. An upper integral is always going to be an inf, and hence requires upper reals. However, to get upper bounds on the integral from finite decompositions we find we need extra conditions. We shall assume that X is compact.

For the same reasons as before, we in effect define $\overline{\int} f d\nu = \overline{\int} \mathrm{Id}\, d\mathfrak{C}f(\nu)$, reducing to the problem of integrating the identity map on $\overleftarrow{[0,\infty]}$. However, to obtain finite approximations from *above* we find we must restrict to a compact saturated domain of integration, for which we use the saturation (the intersection of the open neighbourhoods) of the image $f(X)$ of f.

Definition 13 *Let ν be a covaluation on $\overleftarrow{[0,\infty]}$, and let K be a compact subspace of $\overleftarrow{[0,\infty]}$. We define the* upper integral $\overline{\int}_K \mathrm{Id}\, d\nu$ *to be*

$$\inf\{\sum_{i=1}^{n}(r_i - r_{i-1})\nu\overleftarrow{[0,r_{i-1})} \mid$$
$$0 = r_0 < \cdots < r_n \text{ are rationals } (n \geq 1), \text{ and } K \subseteq \overleftarrow{[0,r_n)}\}.$$

(Even if K is not saturated, the definition will give the same answer as for its saturation.)

If $f : X \to \overleftarrow{[0,\infty]}$, with X compact, and ν is a covaluation on X, then we define the upper integral $\overline{\int} f d\nu$ to be $\overline{\int}_{f(X)} \mathrm{Id}\, d\mathfrak{C}f(\nu)$.

Fig. 3 shows how this is approximating from above the area under that part of the graph that lies above a compact saturated subspace K. Note, however, that it is not a good approximation unless $\nu\overleftarrow{[0,r_n)} = 0$, since $r_n\nu\overleftarrow{[0,r_n)}$ is the area of the infinite rectangle to the right and it lies beyond K. Fortunately, in the case of interest for $\overline{\int} f d\nu$, where $K = f(X) \subseteq \overleftarrow{[0,r_n)}$, we do have

$$\mathfrak{C}f(\nu)\overleftarrow{[0,r_n)} = \nu f^*(\overleftarrow{[0,r_n)}) = \nu X = 0$$

Since $\overleftarrow{[0,\infty]}$ is itself compact, we could in principle take $K = \overleftarrow{[0,\infty]}$. In that case there are no rationals r_n with $K \subseteq \overleftarrow{[0,r_n)}$, and we see that $\overline{\int}_K \mathrm{Id}\, d\nu = \infty$ regardless of ν. Hence the upper integral $\overline{\int} f d\nu$ is only of interest when $f : X \to \overleftarrow{[0,\infty)}$.

A Riemann decomposition into vertical slabs would give

$$\sum_{i=1}^{n} r_i(\nu\overleftarrow{[0,r_{i-1})} - \nu\overleftarrow{[0,r_i)}),$$

which does not include the surplus $r_n\nu\overleftarrow{[0,r_n)}$. But again one sees that this is illegitimate because it subtracts upper reals.

In general the role of K is to license us to ignore an infinite part under the graph, at upper right. Without K, one might try to add on a term $\infty \cdot \nu\overleftarrow{[0,r_n)}$ with the hope then that in the case of interest – where $\nu\overleftarrow{[0,r_n)} = 0$ for sufficiently large r_n – this would give 0 as $\infty \cdot 0$. Unfortunately, however, for the upper reals we have

$$\infty \cdot 0 = \infty \cdot \inf_{\varepsilon>0} \varepsilon = \inf_{\varepsilon>0}(\infty \cdot \varepsilon) = \infty.$$

8 Riemann integration

Suppose $f : [0,1] \to \mathbb{R} = (-\infty,\infty)$. We can define $f_+ = \max(f,0)$, $f_- = -\min(f,0)$, so that $f = f_+ - f_-$ with f_+ and f_- both non-negative. We shall denote by $x \mapsto \underline{x}$ and $x \mapsto \overline{x}$ the maps $\mathbb{R} \to \overrightarrow{(-\infty,\infty]}$ and $\mathbb{R} \to \overleftarrow{[-\infty,\infty)}$ giving the lower and upper parts (i.e. the left and right parts of a Dedekind section), so we have $\underline{f}_\pm : [0,1] \to \overrightarrow{[0,\infty]}$ and $\overline{f}_\pm : [0,1] \to \overleftarrow{[0,\infty)}$. Let λ be the Lebesgue valuation on $[0,1]$, defined by

$$\lambda U = \sup\{\sum_{i=1}^{n}(r_i - q_i) \mid \text{the } (q_i,r_i)\text{s are disjoint rational intervals included in } U\},$$

and let $\overline{\lambda}$ be the corresponding covaluation.

In terms of our previous work, we should be able to define

$$\underline{\int}_0^1 f(x)dx = \underline{\int} \underline{f}_+ d\lambda - \overline{\int} \overline{f}_- d\overline{\lambda} = \underline{\int} \mathrm{Id}\, d\mathfrak{V}(\underline{f}_+)(\lambda) - \overline{\int}_{\overline{f}_-[0,1]} \mathrm{Id}\, d\mathfrak{C}(\overline{f}_-)(\overline{\lambda}).$$

Our aim now is to prove that this gives the same result as Riemann integration. Of course, once the work is done for the interval $[0,1]$, it can be done for any other compact interval by scaling.

The lower Riemann integral $\underline{\int}_0^1 f(x)dx$ can be defined as

$$\sup\{\sum_{i=0}^{n-1}(q_{i+1}-q_i)\inf\{f(x) \mid q_i \leq x \leq q_{i+1}\} \mid \\ 0 = q_0 \leq q_1 \leq \cdots \leq q_n = 1, q_i \in \mathbb{Q}\}.$$

In fact, this can be defined even if f takes its values as lower reals, for we can define $\inf\{f(x) \mid q_i \leq x \leq q_{i+1}\}$ as a lower real by (for $p \in \mathbb{Q}$)

$$p < \inf\{f(x) \mid q_i \leq x \leq q_{i+1}\} \text{ if } [q_i, q_{i+1}] \subseteq f^{-1}\overrightarrow{(p,\infty]}.$$

To see that this set of rationals is rounded, note that the localic Heine-Borel theorem holds constructively. Hence if $[q_i, q_{i+1}]$ is covered by $f^{-1}\overrightarrow{(p,\infty]} = \bigcup_{p<p'} f^{-1}\overrightarrow{(p',\infty]}$, then it is covered by some $f^{-1}\overrightarrow{(p',\infty]}$. Of course, for geometricity the inclusion $[q_i, q_{i+1}] \subseteq f^{-1}\overrightarrow{(p,\infty]}$ has to be interpreted localically. [Vic03] shows explicitly how bounded closed intervals such as $[q_i, q_{i+1}]$ can be understood as point of the Vietoris powerlocale $V\mathbb{R}$, and then inclusion in an open U is equivalent to being in an open $\Box U$ of $V\mathbb{R}$.

Similarly, the upper Riemann integral $\overline{\int}_0^1 f(x)dx$ is

$$\inf\{\sum_{i=0}^{n-1}(q_{i+1}-q_i)\sup\{f(x) \mid q_i \leq x \leq q_{i+1}\} \mid \\ 0 = q_0 \leq q_1 \leq \cdots \leq q_n = 1, q_i \in \mathbb{Q}\},$$

and this can be defined even if f takes its values in the upper reals. Note that $\overline{\int}_0^1 f(x)dx = -\underline{\int}_0^1 -f(x)dx$

It is a non-trivial combinatorial calculation to prove the equivalence of these with expressions in terms of our previous definitions. The reason is that the Riemann integral divides the area into vertical slabs, whereas the previous definitions in effect divide it into horizontal slabs.

Lemma 14 *Let* $f : [0,1] \to \overrightarrow{[0,\infty]}$. *Then*

$$\underline{\int}_0^1 f(x)dx = \underline{\int} f \; d\lambda$$

where the left-hand integral is Riemann, and the right-hand is as in Definition 9.

Proof. To show $\geq$, suppose $0 < s < \underline{\int} f\ d\lambda < \underline{\int} \mathrm{Id}\, d\mathfrak{V}(f)(\lambda)$ with s rational. Then there are rationals $0 = r_0 < \cdots < r_n$ such that

$$s < \sum_{i=1}^{n} (r_i - r_{i-1}) \lambda(f^{-1} \overrightarrow{(r_i, \infty]}).$$

For each $i \geq 1$ we can find a finite set of disjoint rational intervals in $f^{-1}\overrightarrow{(r_i, \infty]}$ such that we can replace each $\lambda(f^{-1}\overrightarrow{(r_i, \infty]})$ in the above expression by the sum of the sizes of those rational intervals, and still have the expression bigger than s. On the face of it these intervals are open, but by shrinking slightly we can assume they are closed. Let us refer to these intervals as "at level i". We shall also refer to $(0, 1)$ as being at level 0. Since $\overrightarrow{(r_i, \infty]} \subseteq \overrightarrow{(r_{i-1}, \infty]}$, we can assume without loss of generality that each of the intervals at level i is enclosed (i.e. included) in one at level $i-1$. Now consider the pairs $\langle q, i \rangle$ where q is an endpoint of an interval at level i. Thinking of these as brackets for their intervals, we can order them so that the bracketing is properly nested. More precisely, we can order them as $\langle q_j, i'_j \rangle$ $(0 \leq j \leq m)$ so that the q_js are in ascending order, and if $\langle q_j, i'_j \rangle$ and $\langle q_k, i'_k \rangle$ are the endpoints for some interval at level $i'_j = i'_k$ then all the enclosed intervals at higher levels have indexes strictly between j and k. Now let us write

$$i_j = \begin{cases} i'_j & \text{if } q_j \text{ is the start of its interval} \\ i'_j - 1 & \text{if } q_j \text{ is the finish of its interval} \end{cases}$$

so that $[q_j, q_{j+1}] \subseteq f^{-1}\overrightarrow{(r_{i_j}, \infty]}$.

Then

$$\begin{aligned} s &< \sum_{i=1}^{n} (r_i - r_{i-1}) \sum_{i_j \geq i} (q_{j+1} - q_j) \\ &= \sum_{j=0}^{m-1} (q_{j+1} - q_j) \sum_{i=1}^{i_j} (r_i - r_{i-1}) = \sum_{j=0}^{m-1} (q_{j+1} - q_j) r_{i_j} \\ &\leq \sum_{j=0}^{m-1} (q_{j+1} - q_j) \inf\{f(x) \mid q_j \leq x \leq q_{j+1}\} \\ &\leq \underline{\int_0^1} f(x) dx. \end{aligned}$$

To show $\leq$, suppose $s < \underline{\int_0^1} f(x) dx$. Then we can find $0 = q_0 \leq q_1 \leq \cdots \leq q_m = 1$, with each $q_j \in \mathbb{Q}$, such that

$$s < \sum_{j=0}^{m-1} (q_{j+1} - q_j) \inf\{f(x) \mid q_j \leq x \leq q_{j+1}\}.$$

We can then find $r'_j \in \mathbb{Q}$ such that $s < \sum_{j=0}^{m-1}(q_{j+1} - q_j)r'_j$ and $r'_j < \inf\{f(x) \mid q_j \leq x \leq q_{j+1}\}$, i.e. $[q_j, q_{j+1}] \subseteq f^{-1}\overrightarrow{(r'_j, \infty]}$. Let $0 < r_1 < \cdots < r_n$ be the positive elements amongst the r'_js, with $r'_j = r_{i_j}$ if $r'_j > 0$. If $r'_j \leq 0$ then we put $i_j = 0$ and write $r_0 = 0$. Then $s < \sum_{j=0}^{m-1}(q_{j+1} - q_j)r_{i_j}$ and a reversal of the argument above shows that $s < \underline{\int} \mathrm{Id}\, d\mathfrak{V}(f)(\lambda)$. ■

Lemma 15 *Let* $f : [0,1] \to \overleftarrow{[0, \infty)}$. *Then*

$$\overline{\int_0^1} f(x)dx = \overline{\int} f \; d\overline{\lambda}$$

where the left-hand integral is Riemann, and the right-hand is as in Definition 13.

Proof. For $\leq$, suppose $\overline{\int} f d\overline{\lambda} = \overline{\int}_{f[0,1]} \mathrm{Id}\, d\mathfrak{C}(f)(\overline{\lambda}) < t$. Then there is some sequence of rationals $0 = r_0 < \cdots < r_n$ $(n \geq 1)$, with $f[0,1] \subseteq \overleftarrow{[0, r_n)}$, such that

$$\sum_{i=1}^{n}(r_i - r_{i-1})\mathfrak{C}(f)(\overline{\lambda})\overleftarrow{[0, r_{i-1})} < t.$$

It follows that we can find rationals s_i $(1 \leq i \leq n)$ with $\mathfrak{C}(f)(\overline{\lambda})\overleftarrow{[0, r_{i-1})} < s_i$ and $\sum_{i=1}^{n}(r_i - r_{i-1})s_i < t$. Now

$$\mathfrak{C}(f)(\overline{\lambda})\overleftarrow{[0, r_{i-1})} = \overline{\lambda}(f^{-1}\overleftarrow{[0, r_{i-1})}) = 1 - \lambda(f^{-1}\overleftarrow{[0, r_{i-1})})$$

and so $1 - s_i < \lambda(f^{-1}\overleftarrow{[0, r_{i-1})})$.

For each $i \geq 1$ we can find a finite set of disjoint closed rational intervals ("at level i") in $f^{-1}\overleftarrow{[0, r_{i-1})}$ such that the sum of their sizes is $> 1 - s_i$. Since $\overleftarrow{[0, r_{i-1})} \subseteq \overleftarrow{[0, r_i)}$, we can assume without loss of generality that each of the intervals at level $i - 1$ is enclosed in one at level i. The process is like that in Lemma 14, except that now the enclosing intervals are at higher levels instead of lower. Again we get a sequence of rationals $0 = q_0 \leq q_1 \leq \cdots \leq q_{m-1} \leq q_m = 1$ with $[q_j, q_{j+1}] \subseteq f^{-1}\overleftarrow{[0, r_{i_j})}$.

Then $1 - s_i < \sum_{i_j \leq i-1} (q_{j+1} - q_j)$, so

$$\begin{aligned}
t &> \sum_{i=1}^{n} (r_i - r_{i-1}) s_i \\
&\geq \sum_{i=1}^{n} (r_i - r_{i-1}) \left(1 - \sum_{i_j \leq i-1} (q_{j+1} - q_j) \right) \\
&= r_n - \sum_{j=0}^{m-1} (q_{j+1} - q_j) \sum_{i=i_j+1}^{n} (r_i - r_{i-1}) \\
&= r_n - \sum_{j=0}^{m-1} (q_{j+1} - q_j)(r_n - r_{i_j}) \\
&= r_n(1 - q_m + q_0) + \sum_{j=0}^{m-1} (q_{j+1} - q_j) r_{i_j} \\
&\geq \sum_{j=0}^{m-1} (q_{j+1} - q_j) \sup\{f(x) \mid q_j \leq x \leq q_{j+1}\} \\
&\geq \overline{\int}_0^1 f(x) dx.
\end{aligned}$$

For $\geq$, suppose $\overline{\int}_0^1 f(x) dx < t$. Then we can find $0 = q_0 \leq q_1 \leq \cdots \leq q_m = 1$, with each $q_j \in \mathbb{Q}$, such that

$$t > \sum_{j=0}^{m-1} (q_{j+1} - q_j) \sup\{f(x) \mid q_j \leq x \leq q_{j+1}\}.$$

We can then find $r'_j \in \mathbb{Q}$ such that $t > \sum_{j=0}^{m-1} (q_{j+1} - q_j) r'_j$ and $r'_j > \sup\{f(x) \mid q_j \leq x \leq q_{j+1}\}$, i.e. $[q_j, q_{j+1}] \subseteq f^{-1}\overleftarrow{[0, r'_j)}$. Let r be an upper bound of the r'_js with $[0,1] \subseteq f^{-1}\overleftarrow{[0, r)}$, let $r_0 = 0$ and let r_i $(1 \leq i \leq n)$ be the distinct values of the r'_js and r, in ascending order, with $r'_j = r_{i_j}$. Then by the above calculation, $t > \sum_{i=1}^{n} (r_i - r_{i-1}) s_i$ where

$$s_i = 1 - \sum_{i_j \leq i-1} (q_{j+1} - q_j).$$

We find

$$\sum_{i_j \leq i-1} (q_{j+1} - q_j) \leq \lambda(f^{-1}\overleftarrow{[0, r_{i-1})}),$$

and then reversing the argument above gives $t > \overline{\int}_{f[0,1]} \mathrm{Id}\, d\mathfrak{C}(f)(\overline{\lambda})$. ■

Theorem 16 *Let $f : [0,1] \to (-\infty, \infty)$. Then*

$$\underline{\int}_0^1 f(x)dx = \underline{\int} \underline{f}_+ \; d\lambda - \overline{\int} \overline{f}_- \; d\overline{\lambda}$$

and

$$\overline{\int}_0^1 f(x)dx = \overline{\int} \overline{f}_+ \; d\overline{\lambda} - \underline{\int} \underline{f}_- \; d\lambda.$$

Proof. It is standard that $\underline{\int}_0^1$ and $\overline{\int}_0^1$ preserve sums in the integrand, and so

$$\underline{\int}_0^1 f(x)dx = \underline{\int}_0^1 f_+(x)dx + \underline{\int}_0^1 -f_-(x)dx.$$

By Lemma 14, $\underline{\int}_0^1 f_+(x)dx = \underline{\int}_0^1 \underline{f}_+(x)dx = \underline{\int} \underline{f}_+ \; d\lambda$. Also,

$$\begin{aligned} \underline{\int}_0^1 -f_-(x)dx &= 0 - \overline{\int}_0^1 \overline{f}_-(x)dx \\ &= 0 - \overline{\int} \overline{f}_- \; d\overline{\lambda} \end{aligned}$$

using Lemma 15.

The other equation follows by considering $\underline{\int}_0^1 -f(x)dx$. ∎

9 Choquet integration

In Section 5 we motivated our lower integral through a Choquet definition $\int g \; d\mu = \int_0^{+\infty} \mu([g > r]) \; dr$. In [AMJK04] this is used as definition of lower integral where $g : X \to [0, \infty)$ is bounded, lower semicontinuous and μ is a continuous valuation. The integral on the right is intended to be Riemann integration, but a generalization of Lemma 14 would suggest replacing this by a lower integral $\underline{\int} \mu([g > -]) \; d\lambda$. We now show that the equation of the Choquet definition then does indeed hold.

If $g : X \to \overrightarrow{[0,\infty]}$ and $\overline{r}$ is an upper real, then $\mu([g > \overline{r}])$ is a lower real and so we can interpret $\mu([g > -])$ as a map $\overleftarrow{[0,\infty)} \to \Omega X \to \overrightarrow{[0,\infty]}$. The space $\overleftarrow{[0,\infty)}$ has a Lebesgue valuation λ – in fact this is nothing other than the homeomorphism $\mathbb{S}^{\overleftarrow{[0,\infty)}} \to \overrightarrow{[0,\infty]}$. Hence $\underline{\int} \mu([g > -]) \; d\lambda$ can be defined as a lower integral.

Proposition 17 *Let $g : X \to \overrightarrow{[0,\infty]}$ and let μ be a valuation on X. Then $\underline{\int} g \; d\mu = \underline{\int} \mu([g > -]) \; d\lambda$.*

Proof. We must show that

$$\underline{\int} \mathrm{Id} \; d\mathfrak{V}g(\mu) = \underline{\int} \mathrm{Id} \; d\mathfrak{V}(\mu([g > -]))(\lambda).$$

Note that

$$\mathfrak{V}(\mu([g > -]))(\lambda)(q, \infty] = \lambda((\mu([g > -]))^{-1}(q, \infty]).$$

Treating this as a rounded downset of positive rationals, it follows that

$$r < \mathfrak{V}(\mu([g > -]))(\lambda)(q, \infty] \iff q < \mu([g > r]) = \mathfrak{V}g(\mu)(r, \infty].$$

Now suppose $t < \underline{\int} \mathrm{Id} \;\; d\mathfrak{V}g(\mu)$. We can then find rationals $0 = r_0 < r_1 < \cdots < r_n$ and $q_i < \mathfrak{V}g(\mu)(r_i, \infty]$ $(1 \leq i \leq n)$ such that $t < \sum_{i=1}^{n}(r_i - r_{i-1})q_i$. Since $\mathfrak{V}g(\mu)(r_i, \infty]$ increases as i decreases, we may assume that $q_0 \geq q_1 \geq \cdots \geq q_n$. We may also omit any terms for which $q_i \leq 0$, and hence assume that $q_n > 0$. Now

$$\sum_{i=1}^{n}(r_i - r_{i-1})q_i = \sum_{i=1}^{n} r_i q_i - \sum_{i=0}^{n-1} r_i q_{i+1} = \sum_{i=1}^{n-1} r_i(q_i - q_{i+1}) + r_n q_n.$$

Reversing the sequence of q_is, and omitting duplicate elements, we find this is less than or equal to $\underline{\int} \mathrm{Id} \;\; d\mathfrak{V}(\mu([g > -]))(\lambda)$.

Now suppose we have $t < \underline{\int} \mathrm{Id} \;\; d\mathfrak{V}(\mu([g > -]))(\lambda)$. We can then find rationals $0 = q_0 < q_1 < \cdots < q_n$ and $r_i < \mathfrak{V}(\mu([g > -]))(\lambda)(q_i, \infty]$ $(1 \leq i \leq n)$ such that $t < \sum_{i=1}^{n}(q_i - q_{i-1})r_i$. More or less by reversing the argument above, we find that $t < \underline{\int} \mathrm{Id} \;\; d\mathfrak{V}g(\mu)$. ■

10 Conclusions

In any constructive localic treatment of analysis, the question arises of what versions of the reals to use. The constructive issue is that of analysing whether a real is approximated from below (in the lower reals $\overrightarrow{[-\infty, \infty]}$), from above (in the upper reals $\overleftarrow{[-\infty, \infty]}$) or both (in the Dedekind reals $\mathbb{R} = (-\infty, \infty)$). In integration this affects both the measure and the integrand. We have defined lower and upper integrals for which the integrand (in both cases required to be non-negative) takes its values in the lower and upper reals respectively. For the lower, the measure is a *valuation* (on opens), while for the upper the measure is a *covaluation* (essentially a valuation on closed subspaces), with a compact bound needed for the range of integration. Using these we can then recover lower and upper Riemann integrals of maps from $[0, 1]$ to $\mathbb{R}$, and also a form of Choquet integration.

A significant issue in a localic treatment is that of continuity: Is integration continuous in its various ingredients? For the valuation (or covaluation) we have defined spaces $\mathfrak{V}(X)$ and $\mathfrak{C}(X)$ whose points are the valuations and covaluations on X. (The locale $\mathfrak{V}(X)$ has already appeared in a slightly different form in [Hec94].) Then continuity with respect to these is captured by expressing integration as a map from $\mathfrak{V}(X)$ or $\mathfrak{C}(X)$. However, continuity with respect to the integrand has not been discussed. On the face of it we need a function space

R^X (where R is an appropriate variant of the reals), which can exist only if X is locally compact. Escardó (private communication) has given an argument to show how, in the case where X is locally compact, continuity with respect to the integrand can be derived from Proposition 17.

We conjecture that there is a more general account of this continuity. If X is not locally compact then $\mathbb{S}^X$ does not exist, but $\mathbb{S}^{\mathbb{S}^X}$ does – it is the double powerlocale $\mathbb{P}X$ [VT04]. We may then hope to capture the space R^{R^X} in the case $R = \overrightarrow{[0,\infty]} \cong \mathbb{S}^{\overleftarrow{[0,\infty)}}$ by saying $R^X = \mathbb{S}^{\overleftarrow{[0,\infty)} \times X}$ (or would do, if it existed), so

$$R^{R^X} = \mathbb{S}^{\overleftarrow{[0,\infty)} \times \mathbb{S}^{\overleftarrow{[0,\infty)} \times X}} \cong (\mathbb{S}^{\mathbb{S}^{\overleftarrow{[0,\infty)} \times X}})^{\overleftarrow{[0,\infty)}} \cong (\mathbb{P}(\overleftarrow{[0,\infty)} \times X)^{\overleftarrow{[0,\infty)}}.$$

In this context we expect to show integration as a map from valuations to functionals in such a way as to give, in the case locally compact case, a map from $\mathfrak{V}X \times R^X$ to R.

10.1 Acknowledgements

This paper has benefited greatly from stimulating and wide-ranging discussions with Martín Escardó. Thanks are also due to the anonymous referee for his or her helpful suggestions regarding the presentation of the material.

References

[AMJK04] M. Alvarez-Manilla, A. Jung, and K. Keimel, *The probabilistic powerdomain for stably compact spaces*, Theoretical Computer Science **328** (2004), 221–244.

[Bir84] G. Birkhoff, *Lattice theory*, AMS Colloquium Publications, no. 25, AMS, Providence, Rhode Island, 1984.

[CSSV03] T. Coquand, G. Sambin, J. Smith, and S. Valentini, *Inductively generated formal topologies*, Annals of Pure and Applied Logic **124** (2003), 71–106.

[Hec94] Reinhold Heckmann, *Probabilistic power domains, information systems, and locales*, Mathematical Foundations in Programming Semantics – 9th International Conference, 1993 (Berlin / Heidelberg) (Stephen Brookes, Michael Main, Austin Melton, and David Schmidt, eds.), Lecture Notes in Computer Science, no. 802, Springer, 1994, pp. 410–437.

[Jon89] C. Jones, *Probabilistic non-determinism*, Ph.D. thesis, University of Edinburgh, 1989.

[JP89] C. Jones and G. Plotkin, *A probabilistic powerdomain of evaluations*, LICS '89, IEEE Computer Society Press, 1989, pp. 186–195.

[Sam87] G. Sambin, *Intuitionistic formal spaces – a first communication*, Mathematical Logic and its Applications (Dimiter G. Skordev, ed.), Plenum, 1987, pp. 187–204.

[Vic99] Steven Vickers, *Topical categories of domains*, Mathematical Structures in Computer Science **9** (1999), 569–616.

[Vic03] ———, *Localic completion of generalized metric spaces II: Powerlocales*, Submitted for publication. Preprint available on web at `http://www.cs.bham.ac.uk/~sjv`, 2003.

[Vic04] ———, *The double powerlocale and exponentiation: A case study in geometric reasoning*, Theory and Applications of Categories **12** (2004), 372–422.

[Vic05a] ———, *Localic completion of generalized metric spaces I*, Theory and Applications of Categories **14** (2005), 328–356.

[Vic05b] ———, *Some constructive roads to Tychonoff*, From Sets and Types to Topology and Analysis: Towards Practicable Foundations for Constructive Mathematics (Laura Crosilla and Peter Schuster, eds.), Oxford Logic Guides, no. 48, Oxford University Press, 2005, pp. 223–238.

[Vic06] ———, *Compactness in locales and in formal topology*, Annals of Pure and Applied Logic **137** (2006), 413–438.

[Vic07] ———, *Locales and toposes as spaces*, Handbook of Spatial Logics (Marco Aiello, Ian E. Pratt-Hartmann, and Johan F.A.K. van Benthem, eds.), Springer, 2007, pp. 429–496.

[VT04] S.J. Vickers and C.F. Townsend, *A universal characterization of the double powerlocale*, Theoretical Computer Science **316** (2004), 297–321.