Metric complements of overt closed sets

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Abstract

We show that the set of points of an overt closed subspace of a metric completion of a Bishop-locally compact metric space is located. Consequently, if the subspace is, moreover, compact, then its collection of points is Bishop compact.

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1 Introduction

We continue our investigations into the relations between Bishop's constructive mathematics and formal topology [CS05, Pal05, Pal07, Spi07, CS09, Pal09]. Previously, we gave a formal definition of locatedness [Spi07] and showed that an overt closed subspace of a compact formal space is (formally) located. Here we consider a generalization of the pointwise side of this result. We use the real numbers as a running example. We work in informal Bishop-style mathematics, including the axiom of dependent choice.

2 Preliminaries

Bishop

We assume familarity with Bishop's constructive mathematics [BB85], but we recall some relevant notions.

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Definition 2.1. *A set is* finite *if it is in bijective correspondence with a set* $\{0, \ldots, n\}$, $n \geqslant 0$. *A set is* Kuratowski finite *(*K-finite, finitely enumerable*) if it is the image of a finite set.*

A subset of a set X if K-finite iff it can written as $\{x_1, \ldots, x_n\}$ with $x_1, \ldots, x_n$ in X. Such a set does not need to have a cardinality if the equality on X is not decidable. For example, the set $\{a, b\}$ is K-finite. However, it is finite iff we can decide whether $a = b$.

Definition 2.2. *A metric space is said to be* totally bounded *if for each* $\varepsilon > 0$ *the space can be covered by a K-finite set of balls with radius at most* ε. *A subset of a metric space is* Bishop-compact *if it is complete and totally bounded.*
A metric space is said to be locally totally bounded *if for each ball and each* $\varepsilon > 0$ *the ball can be covered by a K-finite set of balls with radius at most* ε. *A metric space is* Bishop-locally compact *if it is complete and locally totally bounded.*

The closed unit interval is compact. The real numbers are locally compact.

Definition 2.3. *A subset* A *of a metric space* (X, ρ) *is located if for each* x *in* X *the distance* $\inf\{\rho(x, a) \mid a \in A\}$ *exists as a (Dedekind) real number.*

In classical mathematics all sets are located. Constructively this is not the case, as the following Brouwerian counterexample shows.

Example 2.4. *Consider the set*

$$\{x \in \mathbb{R} \mid x > 1 \text{ or } (x > 0 \text{ and } P)\}.$$

This set will only be located if we can decide whether the proposition P *holds.*

Definition 2.5. *A subset of a metric space is* Bishop-closed *if it contains all its limit points, i.e. if it coincides with its closure.*

The closed unit interval [0,1] is Bishop-closed.

A Bishop closed located subset of a metric space coincides with the complement of its complement: a Bishop closed located set coincides with the set of all points which have zero distance to it.

Formal topology

Definition 2.6. *A* formal topology *[Sam03] consists of a pre-order* $(S, \leqslant)$ *of basic opens and a relation* $\lhd \subset S \times \mathcal{P}(S)$*, the covering relation, which satisfies:*

Ref $a \in U$ *implies* $a \lhd U$*;*

Tra $a \lhd U, U \lhd V$ *implies* $a \lhd V$*, where* $U \lhd V$ *means* $u \lhd V$ *for all* $u \in U$*;*

Loc $a \lhd U$*,* $a \lhd V$ *implies* $a \lhd U \wedge V = \{x \mid \exists u \in U \exists v \in V. x \leqslant u, x \leqslant v\}$*;*

Ext $a \leqslant b$ *implies* $a \lhd \{b\}$*.*

These axioms are known as Reflexivity, Transitivity, Localization and Extensionality. ***Ref*** *and* ***Ext*** *say that if a basic open belongs to a family, then the family covers it.* ***Tra*** *is the transitivity of the cover.* ***Loc*** *is the distributive rule for frames.*

The formal intersection $U \wedge V$ *is defined as* $U_{\leqslant} \cap V_{\leqslant}$*, where* $Z_{\leqslant}$ *is the set* $\{x \mid \exists z \in Z. x \leqslant z\}$*. Another common notation for* $Z_{\leqslant}$ *is* $Z_{\downarrow}$*. We write* $a \lhd b$ *for* $a \lhd \{b\}$*. We write* $U \equiv V$ *iff* $U \lhd V$ *and* $V \lhd U$*.*

Definition 2.7. *Let* $(S, \lhd)$ *be a formal topology. A* point *is an inhabited subset* $\alpha \subset S$ *which is filtering with respect to* $\leqslant$*, and such that* $U \cap \alpha$ *is inhabited, whenever* $a \lhd U$ *for some* $a \in \alpha$*. The collection of points is denoted by* $\mathrm{Pt}(S)$*. Let* U *be an open in* S*. Then* U_* *denotes the class of points* α *such that* $\alpha \in U$*.*

Example 2.8. *The formal reals are inductively defined by the following relation on the open rational intervals ordered by inclusion.*

1. $(p, s) \lhd \{(p, r), (q, s)\}$ *if* $p \leqslant q < r \leqslant s$*;*
2. $(p, q) \lhd \{(p', q') \mid p < p' < q' < q\}$*.*

The points of this space are precisely the (Dedekind) real numbers.

Definition 2.9. *Let* $(S, \lhd)$ *be a formal topology. A* sublocale *is a formal topology* $(S, \lhd')$ *such that* $\lhd \subset \lhd'$ *and* $a \wedge' b \lhd' a \wedge b$*.*
Let $U \subset S$*. The* closed *sublocale* $S \setminus U$ *is* $u \lhd_{-U} V$ *iff* $u \lhd V \cup U$*.*

Example 2.10. *The set* $\{(p, q) \mid q \leqslant 0 \vee p \geqslant 1\}$ *represents the closed unit interval as a subspace of the real line.*

Definition 2.11. Pos *is called a* positivity predicate *on a formal topology* S *if it satisfies:*

Pos $U \lhd U^+$*, where* $U^+ := \{u \in U \mid \mathrm{Pos}(u)\}$*.*

Mon *If* $\mathrm{Pos}(u)$ *and* $u \lhd V$*, then* $\mathrm{Pos}(V)$ *— that is,* $\mathrm{Pos}(v)$ *for some* $v \in V$*.*

A formal space is overt *if it carries a positivity predicate.*

Impredicatively, a formal topology is overt iff the locale it generates is overt, or open.

Classically, all formal topologies are overt. Constructively this is not the case, as the following formal analogue of Example 2.4 shows.

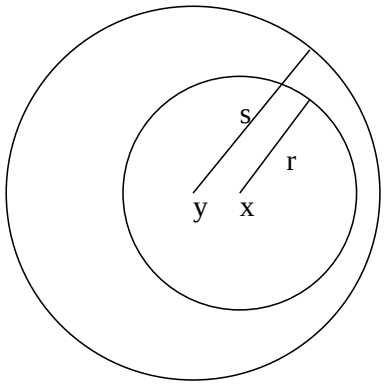
Example 2.12. *The closed sublocale defined by the open*

$$\{(p,0) \mid p < 0\} \cup \{(2,q) \mid q > 2\} \cup \{(1,q) \mid q > 1 \text{ and } P\}$$

is overt if we can decide whether the proposition P holds; see [Spi07].

Metric completion

Definition 2.13. *To any metric space X, we define, following Vickers [Vic05] and Palmgren [Pal07], a formal topology $\mathcal{M}(X)$ called the* localic completion *of X. A formal open is a pair $(x,r) \in X \times \mathbb{Q}^{>0}$, written $\mathsf{b}(x,r)$. We define the relation $\mathsf{b}(x,r) < \mathsf{b}(y,s)$ iff $d(x,y) < s - r$ as illustrated below.*



The order $\leqslant$ is defined by $\mathsf{b}(x,r) \leqslant \mathsf{b}(y,s)$ iff $d(x,y) < t$ for all $t > s - r$. The covering relation $\lhd$ is inductively generated by the axioms

M1 $u \lhd \{v \mid v < u\}$;

M2 $\mathcal{M}(X) \lhd \{\mathsf{b}(x,r) \mid x \in X\}$ *for any r.*

***M1**: Every ball is covered by all the balls strictly inside it (since the ball is open).* ***M2**: For each $r > 0$, the space is covered by all balls of size r.*
We define $U < V := \forall u \in U \exists v \in V. u < v$.

Example 2.14. *Consider the formal unit interval $[0,1]$. Then $[0,1] = \mathsf{b}(0,3) = \mathsf{b}(0,2)$, but $\mathsf{b}(0,3) > \mathsf{b}(0,2)$.*

Similarly, $\mathsf{b}(0,3) \lhd \mathsf{b}(0,2)$, but it is not the case that $\mathsf{b}(0,3) \leqslant \mathsf{b}(0,2)$. This shows that $a \lhd b$ does not imply $a \leqslant b$.

Proposition 2.15. *The localic completion of a metric space is always overt.*

The formal reals are the metric completion of the rational numbers.

Elementary description of the cover of $\mathcal{M}(X)$

The $\lessdot$ cover relation below, introduced in [Pal07], generalises the one introduced by Vermeulen [Ver86] and Coquand [CN96] for $\mathbb{R}$:

$$\begin{aligned} p \sqsubseteq_\varepsilon U &:= (\forall q \leq p)[\mathrm{radius}(q) \leq \varepsilon \Rightarrow \{q\} \leq U] \\ p \sqsubseteq U &:= p \sqsubseteq_\varepsilon U \text{ for some } \varepsilon \in \mathbb{Q}_+ \\ A(b,c) &:= \{C \in \mathcal{P}_{\text{K-fin}}(\mathcal{M}_X) : b \sqsubseteq C < c\} \\ a \lessdot U &:= (\forall b < c < a)\,(\exists U_0 \in A(b,c))\, U_0 < U. \end{aligned}$$

Theorem 2.16 ([Pal07])**.** *If X is a Bishop locally compact metric space, then*

$$a \lhd U \iff a \lessdot U$$

($\Leftarrow$ holds for any metric X space.)

Theorem 2.17 ([Pal07])**.** *Let X be a complete metric space. Then there exist a metric isomorphism j between X and* $\mathrm{Pt}(\mathcal{M}(X))$.

In particular, this holds for the real numbers.

3 Main results

We write $B(x,r)$ for the set $\{y \mid d(x,y) < r\}$. Then $\mathsf{b}(x,r)_* = B(x,r)$.

Lemma 3.1. *Let X be a metric space. Then an inhabited set $S \subseteq X$ is located if, and only if, for all $x \in X$ and all positive $\delta < \varepsilon$ we have*

$$S \cap B(x,\delta) = \emptyset \text{ or } S \between B(x,\varepsilon).$$

Where $A \between B$ means that $A \cap B$ is inhabited.

Lemma 3.2. *Let X be a metric space and let $M = \mathcal{M}(X)$ be its localic completion. If $O \subseteq M$ and the sublocale $M \setminus O$ is overt, then any positive neighbourhood contains a point of $M \setminus O$.*

Proof. Suppose that P is the positivity predicate of $M \setminus O$. Denote the cover relation of $M \setminus O$ by $\lhd'$.

Suppose $a = \mathsf{b}(x,\delta) \in P$. Let $a_1 = a$. Suppose we have constructed in P:

$$a_1 \geq a_2 \geq \cdots \geq a_n,$$

so that $\mathrm{radius}(a_{k+1}) \leq \mathrm{radius}(a_k)/2$.

By (M1) and localisation we get

$$a_n \lhd' \{a_n\} \wedge \{\mathsf{b}(y,\rho) : y \in X\}$$

where $\rho = \text{radius}(a_n)/2$. Since $a_n \in P$ we obtain some $b \in \{a_n\} \wedge \{\mathsf{b}(y,\rho) : y \in X\}$ with $b \in P$. Clearly $\text{radius}(b) \leq \text{radius}(a_n)/2$. Let $a_{n+1} = b$.

Let

$$\alpha = \{p \in M : (\exists n) a_n \leq p\}.$$

Since the radii of a_n are shrinking, this defines a point in $\text{Pt}(M)$. (Note that we used Dependent Choice.)

We claim that $\alpha \in \text{Pt}(M \setminus O) = \text{Pt}(M) \setminus O_*$. Suppose that $\alpha \in O_*$, i.e. for some $c \in O$: $c \in \alpha$. Hence there is n with $a_n \leq c$. Thus $a_n \lhd O$, that is $a_n \lhd' \emptyset$. But since a_n is positive, this is impossible! So $\alpha \in \text{Pt}(M \setminus O)$. □

The following theorem can be conveniently formulated using the following definition. However, no futher facts about this definition are needed.

Definition 3.3. *[Spi07] Let X be a metric space. A predicate* Pos *on $S = \{\mathsf{b}(x,r) \mid x \in X, r \in \mathbb{Q}^+\}$ is called* located *if*

- $\text{Pos}(u)$ *and* $u \lhd V$ *imply that* $\text{Pos}(v)$ *for some v in V;*
- $v < u$ *implies that* $\neg \text{Pos}\, v$ *or* $\text{Pos}\, u$.

Let T be a closed sublocale of $\mathcal{M}(X)$. Then T is called located *if there is a located predicate* Pos *such that T coincides with the closed sublocale defined by the open $\neg \text{Pos} \subset S$.*

Theorem 3.4. *Let X be a Bishop locally compact metric space and let $M = \mathcal{M}(X)$ be its localic completion. Let $O \subseteq M$. If a sublocale $M \setminus O$ is overt, then $M \setminus O$ is (formally) located. Consequently, the set of points $Y = j^{-1}[\text{Pt}(M \setminus O)]$ is located as a subset of X and moreover O_* is the metric complement of Y in X.*

Proof. That $M \setminus O$ is overt means that there is an inhabited subset $P \subseteq M$ (a set of positive formal neighbourhoods) so that

(P1) $a \lhd_M O \cup U$ and $a \in P$ implies $U \cap P$ inhabited,

(P2) $U \lhd_M O \cup (U \cap P)$.

Now since P is inhabited, Lemma 3.2 ensures that Y is inhabited. Let $x \in X$ be arbitrary. Consider positive rational numbers $\delta < \varepsilon$. Take ε' with $\delta < \varepsilon' < \varepsilon$ and let $\theta = \varepsilon - \varepsilon'$. Using the (P2), (M2) and the localisation we get

$$\mathsf{b}(x,\varepsilon) \lhd_M O \cup (P \cap \{c \in M : \rho(c) = \theta/2\}).$$

Thus also

$$\mathsf{b}(x,\varepsilon) \lessdot O \cup (P \cap \{c \in M : \rho(c) = \theta/2\}).$$

and by definition there is a K-finite $W \in A(\mathsf{b}(x,\delta), \mathsf{b}(x,\varepsilon'))$ with

$$W < O \cup (P \cap \{c \in M : \rho(c) = \theta/2\}).$$

Since W is K-finite we have one of the cases

(C1) $W < O$,

(C2) $(\exists d \in W) d < P \cap \{c \in M : \rho(c) = \theta/2\}$.

In case (C1) we have $\mathsf{b}(x,\delta) \lhd O$ and hence $\mathsf{b}(x,\delta)_* \subseteq O_*$. Thus $\mathsf{b}(x,\delta)_* \cap Y = \emptyset$.

In case (C2) there is $d \in W$ and $c \in P$ with $d < c$ and $\rho(c) = \theta/2$. Suppose $c = \mathsf{b}(y, \theta/2)$ and $d = \mathsf{b}(z,\tau)$. Now $W < \mathsf{b}(x,\varepsilon')$. Hence $d(z,x) < \varepsilon'$. Moreover $d < c$ implies $d(z,y) + \tau < \theta/2$. Thus

$$d(x,y) \leq d(x,z) + d(z,y) < \varepsilon' + \theta/2 - \tau = \varepsilon - \theta/2 - \tau < \varepsilon - \theta/2.$$

Thereby $c < \mathsf{b}(x,\varepsilon)$, and so $\mathsf{b}(x,\varepsilon) \in P$. By Lemma 3.1, $Y \between B(x,\varepsilon)$.

We have thus showed that $M \setminus O$, and hence Y, is located. Using Lemma 3.2, we have that

$$d(x,Y) > 0 \iff (\exists \delta > 0) B(x,\delta) \cap Y = \emptyset.$$

We claim that O_* is the metric complement of Y, i.e.

$$x \in O_* \iff d(x,Y) > 0.$$

If $x \in O_*$ then for some $\delta > 0$, $B(x,\delta) \subseteq O_*$. Thus $B(x,\delta) \cap Y$ cannot be inhabited. Conversely, suppose that $B(x,\delta) \cap Y = \emptyset$ for some $\delta > 0$. We have by (P2), that

$$\mathsf{b}(x,\delta) \lhd O \cup (\{\mathsf{b}(x,\delta)\} \cap P)$$

Thus $B(x,\delta) \subseteq O_* \cup (\{\mathsf{b}(x,\delta)\} \cap P)_*$, and hence $x \in O_*$ or $x \in (\{\mathsf{b}(x,\delta)\} \cap P)_*$. In the latter case $\mathsf{b}(x,\delta) \in P$, which contradicts $B(x,\delta) \cap Y = \emptyset$. Thus $x \in O_*$. □

Theorem 3.5. *Let X be a metric space and let $M = \mathcal{M}(X)$ be its localic completion. Let $O \subseteq M$ be such that $M \setminus O$ is compact and overt. Then* $\mathrm{Pt}(M \setminus O)$ *is Bishop-compact.*

Proof. Let P and $\lhd'$ be as in the proof of Lemma 3.2. Let $\varepsilon > 0$ be given. Then by axiom M2 and positivity

$$M \lhd' \{\mathsf{b}(x,\varepsilon/2) : x \in X\} \lhd' \{\mathsf{b}(x,\varepsilon/2) : x \in X\} \cap P.$$

By compactness there is some K-finite

$$F = \{\mathsf{b}(x_1,\varepsilon/2), \ldots, \mathsf{b}(x_n,\varepsilon/2)\} \subseteq \{\mathsf{b}(x,\varepsilon/2) : x \in X\} \cap P$$

so that

$$M \lhd' F. \tag{1}$$

Since each $\mathsf{b}(x_i, \varepsilon/2)$ is positive there is by Lemma 3.2 some $\alpha_i \in \mathsf{b}(x_i, \varepsilon/2)_*$ which is in $\mathrm{Pt}(M \setminus O)$. By (1), each point in $\mathrm{Pt}(M \setminus O)$ has distance smaller than ε to some point α_i. Thus $\{\alpha_1, \ldots, \alpha_n\}$ is the required ε-net. □

Corollary 3.6. *If in the context of Theorem 3.4, X is Bishop-compact, and then so is Y.*

Proof. If X is Bishop-compact, then $\mathcal{M}(X)$ is a compact as a formal space [Vic05]. Hence, $M \setminus O$ is compact. By Theorem 3.4, Y is Bishop-compact. □

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