COMPLEXITY OF COUNTABLE CATEGORICITY IN FINITE LANGUAGES

ALEKSANDER IVANOV

Abstract. We study complexity of the index set of countably categorical theories and Ehrenfeucht theories in finite languages.

S.Lempp and T.Slaman proved in [7] that indexes of decidable ω-categorical theories form a Π^0_3-subset of the set of indexes of all computably enumerable theories. Moreover there is an infinite language so that the property of ω-categoricity distinguishes a Π^0_3-complete subset of the set of indexes of computably enumerable theories of this language. Steffen Lempp asked the author if this could be done in a finite language. In this paper we give a positive answer (see Section 4). The crucial element of our proof is a theorem of Hrushovski on coding of ω-categorical theories in finite languages (see [3], Section 7.4, pp. 353 - 355). Since we apply the method which was used in the the proof of this theorem, we present all the details in Section 1. Sections 2 - 3 contain several other applications of this theorem. In particular in the very short Section 2 we give an example of a non-G-compact ω-categorical theory in a finite language. In Section 3 we show that there is a finite language such that the indexes of Ehrenfeucht theories with exactly three countable models form a Π^1_1-hard set. Here we also use the idea of Section 4 of [7] where a similar statement is proved in the case of infinite languages.

The main results of the paper are available both for computability theorists and model theorists. The only place where a slightly advanced model-theoretical material appears is Section 2. On the other hand the argument applied in this section is very easy and all necessary preliminaries are presented.

1. Hrushovski on ω-categorical structures and finite languages

The material of this section is based on Section 7.4 of [3], pp. 353 - 355 (and preliminary notes of W.Hodges). We also give some additional modifications and remarks.

Let N be a structure in the language L with a unary predicate P. For any family of relations $\mathcal{R}$ on P definable in N over $\emptyset$ one may consider the structure $M = (P, \mathcal{R})$. We say that M is a *dense relativised reduct* if the image of the homomorphism $Aut(N) \rightarrow Aut(M)$ (defined by restriction) is dense in $Aut(M)$.

Let L be the language consisting of four unary symbols P, Q, λ, ρ, a two-ary symbol H and a four-ary one S. We will consider only L-structures where P and Q define a partition of the basic sort and λ, ρ and H are defined on Q. Moreover when $S(a, b, c, d)$ holds we have that $a, c \in P$ and $b, d \in Q$.

Theorem 1. *If M_0 is any countable ω-categorical structure then there is a countable ω-categorical L-structure N such that M_0 is a dense relativised reduct of N. In particular M_0 is interpretable in N over $\emptyset$.*

For every set of sentences Φ axiomatising $Th(M_0)$ the theory $Th(N)$ is axiomatised by a set of axioms which is computable with respect to Φ and the Ryll-Nardzewski function of $Th(M_0)$.

Proof (E.Hrushovski). Let M_0 be any countable ω-categorical structure in a language L_0. We remind the reader that the *Ryll-Nardzewski function* of an ω-categorical theory T assigns to any natural n the number of n-types of T. So by the set Φ as in the formulation and by the Ryll-Nardzewski function of $Th(M_0)$ one can find an effective list of all pairwise non-equivalent formulas. Thus w.l.o.g. we may assume that L_0 is 1-sorted, relational and M_0 has quantifier elimination. In fact we can suppose that $L_0 = \{R_1, R_2, ..., R_n, ...\}$ where each R_n describes a complete type in M_0 of arity not greater than n. We may also assume that for $m < n$ the arity of R_m is not greater than the arity of R_n. We admit that tuples realising R_n may have repeated coordinates.

We now use standard material about Fraïssé limits, see [2]. Note that the class of all finite substructures of M_0 (say $\mathcal{K}_0$) has the joint embedding and the amalgamation properties. Moreover for every n the number of finite substructures of size n is finite (this is the place where we use the assumption that each R_n describes a complete type).

Let us consider structures of the language $L \cup L_0$ which satisfy the property that all the relations R_n are defined on P. For such a structure M we call a tuple $(a_0, ..., a_{m-1}, c_0, ..., c_{n-1})$ of elements of M, an *n-pair of arity m* if :

(1) $m \leq n$ and $M \models \bigwedge\{P(a_i) : i < m\} \wedge \bigwedge\{Q(c_j) : j < n\}$;

(2) the elements c_i are paiwise distinct and $M \models H(c_i, c_j)$ iff $(j = i+1) mod(n)$;

(3) $M \models \lambda(c_i)$ iff $i = 0$ and $M \models \rho(c_i)$ iff $i = m-1$;

(4) $M \models S(a_i, c_j, a_k, c_l)$ iff $a_i = a_j$.

In this case we say that the n-pair $\bar{a}\bar{c}$ labels the tuple $\bar{a}$.

We now define a class $\mathcal{K}$ of finite $(L \cup L_0)$-structures as follows.

(i) In each structure of $\mathcal{K}$ all the relations R_n are defined on P;

(ii) The P-part of any structure from $\mathcal{K}$ is isomorphic to a finite substructure of M_0;

(iii) For any $D \in \mathcal{K}$, any n and any n-pair from D labelling a tuple $\bar{a}$ we have $R_n(\bar{a})$.

It is obvious that $\mathcal{K}$ is closed under substructures and there is a function $f : \omega \to \omega$ so that for every n the number of non-isomorphic sructures of $\mathcal{K}$ of size n is bounded by $f(n)$. The function f is computable with respect to Φ and the Ryll-Nardzewski function.

Lemma 2. *The class $\mathcal{K}$ has the amalgamation (and the joint embedding) property.*

Proof. Let D_1 and D_2 be structures in $\mathcal{K}$ with intersection C. By induction it is enough to deal with the case where $|D_1 \setminus C| = |D_2 \setminus C|$ =1. Let $D_i \setminus C = \{d_i\}$ and $d_1 \neq d_2$. There are three cases.

Case 1. d_1 and d_2 both satisfy P. Using that M_0 has quantifier elimination we amalgamate the P-parts of D_1 and D_2 remaining the Q-part and S the same as before. By (4) there are no new n-pairs in the amalgam, for any n.

Case 2. d_1 and d_2 both satisfy Q. In this case we just take the free amalgamation (without any new tuples in relations). By (4) there are no new n-pairs in the amalgam, for any n.

Case 3. d_1 satisfies P and d_2 satisfies Q. In this case we again take the free amalgamation and by (4) we again have that there are no new n-pairs in the amalgam, for any n. $\square$

We now see that by Fraïssé's theorem, the class $\mathcal{K}$ has a universal homogeneous (and ω-categorical) structure U. In particular $\mathcal{K}/\cong$ coincides with $Age(U)$ (= collection of all types of finite substructures of U).

Since M_0 is the Fraïssé limit of the class of all P-parts of structures from $\mathcal{K}$, we see that the P-part of U is isomorphic to M_0. Let N be the reduct of U to the language L. Note that U (thus M_0) is definable in N. Indeed each R_n is definable by the rule: $U \models R_n(\bar{a})$ if and only if there is an n-pair in N which labels $\bar{a}$ (this follows from the fact that $\mathcal{K}$ contains an n-pair for such $\bar{a}$).

If two tuples $\bar{a}$ and $\bar{b}$ in M_0 realise the same type in M_0 they realise the same quantifier free type in U. So by quantifier elimination there is an automorphism of U (and of N) which takes $\bar{a}$ to $\bar{b}$. This shows that M_0 is a dense relativised reduct of N.

To see the last statement of the theorem consider a set Φ axiomatising $Th(M_0)$. Thus the P-part of U must satisfy Φ with respect to the relations R_n defined in N as above. The remaining axioms of $Th(N)$ (and of $Th(U)$) are just the axioms of the universal homogeneous structures of the corresponding class satisfying (i) - (iii) as above. $\square$

Remark 3. The structure U produced in the proof is axiomatised as follows.

Axiomatisation of $Th(U)$.

(a) all universal axioms forbidding finite substructures which cannot occur in M_0;

(b) all universal axioms stating property (iii) from the proof ;

(c) all $\exists$-axioms for finite substructures of M_0;

(d) all $\forall\exists$-axioms which realise the property of universal homogeneous structures that for any $\mathcal{K}$-structures $A < B$ with $A < U$ there is an A-embedding of B into U.

Note that for every pair of natural numbers n and l the axioms of (a), (b) and (c) with at most n quantifiers in the sublanguage of $L \cup L_0$ of arity $\leq l$ determine all n-element structures from $\mathcal{K}$ in this sublanguage. On the other hand by the Ryll-Nardzewski function of $Th(M_0)$ we can find the arity l_n so that all $\mathcal{K}$-embeddings between structures of size $\leq n$ are determined by their relations of arity $\leq l_n$. Thus the axioms of (d) with at most n quantifiers can be effectively found by the corresponding axioms (a - c) and the Ryll-Nardzewski function. Moreover there is an effective procedure which for every natural numbers n produces all $\forall\exists$-sentences of $Th(U)$ with at most n quantifiers, when one takes as the input the axioms of (a) and (c) of U with at most n quantifiers.

2. Finite language and non-G-compact theories

The following definitions and facts are partially taken from [1]. Let $\mathbf{C}$ be a monster model of the teory $Th(\mathbf{C})$. For $\delta \in \{1, 2, ..., \omega\}$ let E_L^δ be the finest bounded $Aut(\mathbf{C})$-invariant equivalence relation on δ-tuples (i.e. the cardinality of the set of equivalence classes is bounded). The classes of E_L^δ are called Lascar strong types. The relation E_L^δ can be characterized as follows: $(\bar{a}, \bar{b}) \in E_L^\delta$ if there are δ-tuples $\bar{a}_0(=\bar{a}), \bar{a}_1, ..., \bar{a}_n(=\bar{b})$ such that each pair $\bar{a}_i, \bar{a}_{i+1}$, $0 \leq i < n$, extends to an infinite indiscernible sequence. In this case denote by $d(\bar{a}, \bar{b})$ the minimal n such that some $\bar{a}_0(=\bar{a}), \bar{a}_1, ..., \bar{a}_n(=\bar{b})$ are as above.

Let E_{KP}^δ be the finest bounded type-definable equivalence relation on δ-tuples. Classes of this equivalence relation are called KP-strong types. The theory $Th(\mathbf{C})$ is called *G-compact* if $E_L^\delta = E_{KP}^\delta$ for all δ. The first example of a non-G-compact theory was found in [1]. The first example of an ω-categorical non-G-compact theory was found by the author in [4]. The following proposition is a straightforward application of Theorem 1.

Proposition 4. *There is a countably categorical structure N in a finite language such that $Th(N)$ is not G-compact.*

Proof. Let L be defined as in the proof of Theorem 1. Corollary 1.9(2) of [8] states that G-compactness is equivalent to existence of finite bound on the diameters of Lascar strong types. Let M_0 be an ω-categorical structure which is not G-compact, see [4]. In [4] for every n a pair $\bar{a}_n$, $\bar{b}_n$ of finite tuples of the same Lascar strong type is explicitely found so that $d(\bar{a}_n, \bar{b}_n) > n$.

Let N be an L-structure, so that M_0 is a dense relativised reduct in N defined by P. Then $Th(N)$ is not G-compact. Indeed for every n, the pair $\bar{a}_n, \bar{b}_n$ is of the same Lascar strong type and $d(\bar{a}_n, \bar{b}_n) > n$ with respect to the theory of N. To see this notice that if in $\bar{c}_0(=\bar{a}_n), \bar{c}_1, ..., \bar{c}_m(=\bar{b}_n)$ each $\bar{c}_i, \bar{c}_{i+1}$ extends to an indiscernible sequence in $Th(M_0)$, then this still holds in $Th(N)$ by density of the image of $Aut(N)$ in $Aut(M_0)$. On the other hand since $Aut(N) \leq Aut(M_0)$ on $P(M)$, we cannot find in N such a sequence with $m \leq n$. $\square$

3. Finite language and Ehrenfeucht theories

In this section we consider the situation where M_0 is obtained by an ω-sequence of ω-categorical expansions. We will see that under some natural assumptions the construction of Section 1 still works in this situation. Using this we will prove that there is a finite language such that the indexes of Ehrenfeucht theories with exactly three countable models form a Π^1_1-hard set.

Let M_0 be a countable structure of a 1-sorted, relational language $L_0 = \{R_1, R_2, ..., R_n, ...\}$. Suppose $L_0 = \bigcup_{i>0} L_i$, where for each $i > 0$, $L_i = \{R_1, ..., R_{l_i}\}$ and the L_i-reduct of M_0 admits quantifier elimination (and thus ω-categorical). We may assume that the arity of R_n is not greater than n. Admitting R_n with repeated coordinats, we may also assume that for all $m < n$ the arity of R_m is not greater than the arity of R_n and the arity of R_{l_i} is less than the arity of R_{l_i+1}.

We now admit that M_0 is not ω-categorical. On the other hand the theory of M_0 can be axiomatised as follows. For each i consider the L_i-reduct of M_0 and its age $Age(M_0|L_i)$. Then this reduct is axiomatised by the standard axioms of a universal homogeneous structure (i.e. the versions of (a),(c),(d) from Remark 3 with respect to $Age(M_0|L_i)$). The collection of all systems of axioms of this kind gives an axiomatisation of $Th(M_0)$.

Applying the proof of Theorem 1 we associate to each L_i-reduct of M_0, a class $\mathcal{K}_i$ of $(L \cup L_i)$-structures obtained by conditions (i)-(iii) from this proof. Since the L_i-reduct of M_0 has quantifier elimination, repeating the argument of Theorem 1 we obtain an ω-categorical $(L \cup L_i)$-structure U_i and the corresponding L-reduct N_i (since the language is finite, we do not need the assumption that each R_i describes a type). Notice that the construction forbids n-pairs for R_n of arity greater than the arity of L_i.

Lemma 5. *(1) For any $i < j$ the structures U_i and U_j satisfy the same axioms of the form (a) - (d) of Remark 3 where the language of the P-part is restricted to L_i and the number of variables of the Q-part is bounded by the arity of L_i.*

(2) The corresponding structures N_i and N_j satisfy the same sentences which are obtained by rewriting of the axioms of statement (1) as L-sentences (using the interpretation of U_i in N_i).

Proof. Let m be the arity of L_i. To see statement (1) let us prove that the classes $\mathcal{K}_i$ and $\mathcal{K}_j$ consist of the same $(L \cup L_i)$-structures among those with the Q-part of size $\leq m$. The direction $j \to i$ is clear: the $(L \cup L_i)$-reduct of an $(L \cup L_j)$-structure of this form obviously satisfies the requirements (i) - (iii) corresponding to $\mathcal{K}_j$ (and to $\mathcal{K}_i$ too). To see the direction $i \to j$ note that the assumption that the size of the Q-part is not geater than m implies that such a structure from $\mathcal{K}_i$ has an expansion to an $(L \cup L_j)$-structure from $\mathcal{K}_j$.

Now the case of axioms of the form (a),(b),(c) is easy. Consider case (d). Since the L_i-reduct of M_0 admits elimination of quantifiers, for any finite L_j-substructure $A < M_0$ and any embedding of the L_i-reduct of A into any $B \in Age(M_0|L_i)$ there is an L_j-substructure of M_0 containing A with the L_i-reduct isomorphic to B. This obviously implies that for any substructure $A' < U_j$ without n-pairs for arities greater than $arity(L_i)$, any embedding of the $(L \cup L_i)$-reduct of A' into any $B' \in \mathcal{K}_i$ can be realised as a substructure of U_j containing A' with the $(L \cup L_i)$-reduct isomorphic to B'. This proves (1).

Statement (2) follows from statement (1). $\square$

We now additionally assume that M_0 is a generic structure with respect to the class $\mathcal{K}_0$ of all finite L_0-substructures of M_0. This means that $\mathcal{K}_0$ has the joint embedding and amalgamation properties (JEP and AP), $(\mathcal{K}_0/\cong) = Age(M_0)$ and M_0 is a countable union of an increasing chain of structures from $\mathcal{K}_0$ so that any isomorphism between finite substructures extends to an automorphism of M_0.

Let $\mathcal{K}$ be the class of all finite $(L_0 \cup L)$-structures satisfying the conditions (i)-(iii) with respect to $\mathcal{K}_0$. In particular it obviously contains only countably many isomorphism types and the class $\mathcal{K}_0$ appears as the class of all P-parts of $\mathcal{K}$. Applying the proof of Theorem 1 we see that $\mathcal{K}$ is closed under substructures and has the joint embedding and amalgamation properties. By Theorem 1.5 of [6] the class $\mathcal{K}$ has a unique (up to isomorphism) generic structure (i.e. a structure which is a countable union of an increasing chain of structures from $\mathcal{K}$ and satisfies axioms (a) - (d) of Remark 3). Note that this structure can be non-ω-categorical.

Lemma 6. *Under the circumstances of this section let U be a generic $(L \cup L_0)$-structure for $\mathcal{K}$ as above.*

Then the P-part of U is isomorphic to M_0. The structure M_0 is a dense relativised reduct of U.

Proof. The firts statement is obvious. The second statement is an application of back-and-forth. □

It is worth noting here that for every m the amalgamation of Theorem 1 preserves the subclass of $\mathcal{K}$ consisting of structures without n-pairs for arities greater than m (for example structures with the size of the Q-part less than $m+1$). If m is the arity of the language L_i then $(L_i \cup L)$-reducts of these structures form the Fraïssé class corresponding to the universal homogeneous structure U_i.

Proposition 7. *(1) All axioms of U_i of the form (a),(c),(d) of Remark 3 also hold in U.*
(2) The theory $Th(U)$ is model complete and is axiomatised by axioms of the form (b) of Remark 3 together with the union of all axioms of the form (a),(c),(d) for all $Th(U_i)$.
(3) For any axiom ϕ of $Th(U)$ of the form (a)-(d) as in (2) there is a number i so that ϕ holds in all U_j for $j > i$.

Proof. (1) The case of axioms of the form (a),(c) is easy. Consider case (d). Since the L_i-reduct of M_0 admits elimination of quantifiers, for any substructure $A < M_0$ and any embedding of the L_i-reduct of A into any $B \in Age(M_0|L_i)$ there is a substructure of M_0 containing A with the L_i-reduct isomorphic to B. This obviously implies that for any substructure $A' < U$ without n-pairs of arity greater than $arity(L_i)$, any embedding of the $(L \cup L_i)$-reduct of A' into any $B' \in \mathcal{K}_i$ can be realised as a substructure of U containing A' with the $(L \cup L_i)$-reduct isomorphic to B'. This proves (1).

(2) Let U' and U'' satisfy axioms as in the formulation of (2). Then obviously the $(L_i \cup L)$-reducts of U' and U'' satisfy the axioms of $Th(U_i)$ as in statement (1). In particular $P(U') \cong P(U'')$ in each L_i. Moreover if $U' < U''$, then by axioms (d) one can easily verify that this embedding is $\forall$-elementary. Thus U' is an elementary substructure of U'' by a theorem od Robinson. It is also clear that U is embeddable into any structure satisfying axioms as in (2).

(3) By Lemma 5 we see that for every sentence $\theta \in Th(U)$ of the form (a) - (d) of (2) there is a number i such that for all $j > i$, θ holds in U_j. $\square$

Some typical examples of Ehrenfeucht theories (i.e. with finitely many countable models) are build by the method of this section: the theory of all expansions of $(\mathbb{Q}, <)$ by infinite discrete sequences $c_1 < c_2 < ... < c_n < ...$, is Ehrenfeucht and can be easily presented in an appropriate L_0 as above.

Proposition 8. *Under the circumstances of this section assume that M_0 is a generic structure with respect to the class $\mathcal{K}_0$ of all finite substructures of M_0. Assume that $Th(M_0)$ is an Ehrenfeucht theory. Let U be a generic $(L \cup L_0)$-structure for $\mathcal{K}$ as above.*

Then $Th(U)$ is also Ehrenfeucht.

Proof. Let U', U'' be countable models of $Th(U)$. Assume that the P-parts of U' and U'' (say M' and M'') are isomorphic. Identifying them let us show that U' is isomorphic to U''. For this we fix a sequence of finite substructures $A_1 < A_2 < ... < A_i < ...$ so that $M' = \bigcup A_i$. Having enumerations of the Q-parts of U' and U'' we build by back-and-forth, sequences $B'_1 < B'_2 < ... < B'_i < ...$ and $B''_1 < B''_2 < ... < B''_i < ...$ with $B'_i > A_i < B''_i$, $U' = \bigcup B'_i$ and $U'' = \bigcup B''_i$ so that B'_i is isomorphic to B''_i over A_i. By Proposition 7(2) using the fact that $U', U'' \models Th(U)$ we see that such sequences exist. $\square$

We now prove that there is a finite language L such that the set of Ehrenfeucht L-theories with exactly three models is Π^1_1-hard.

Theorem 9. *There is a finite language L such that for every $B \in \Pi^1_1$ there is a Turing reduction of B to the set $3Mod_L$ of all indexes of decidable Ehrenfeucht L-theories with exactly three countable models.*

Proof. Let L be the language defined in Section 1. We use the idea of Section 4 of [7]. In particular we can reduce the theorem to the case when B coincides with the index set $NoPath$ of the property of being a computable tree $\subseteq \omega^{\omega}$ having no infinite path. The Turing reduction of this set to $3Mod_L$ which will be built below, is a composition of the procedure described in [10] and [7], and the construction of this section. The former one is as follows. Having an index e of a computable tree $Tr_e \subset \omega^{\omega}$, R.Reed defines a complete decidable theory T_e of the language

$$\langle \wedge, <_L, \leq_H, E^{\eta}_{\xi}, L^{\eta}_{\xi}, H_{\eta}, A_{\eta}, B_{\eta}, c_{\eta} \ (\ \eta, \xi \in Tr_e \ ) \ \rangle,$$

where $\wedge$ is the function of the greatest lower bound of a tree, $<_L$ is a Kleene-Brouwer ordering of this tree and $\leq_H$ is a binary relation measuring 'heights' of nodes. Constants c_{η}, $\eta \in Tr_e$, define embeddings of Tr_e into models of T_e. The remaining relations are binary.

For each natural n define $T_e|_n$ to be the restriction of T_e to the sublanguage corresponding to the indexes from the finite subtree $Tr_e \cap n^{<n}$. The proof of Lemma 9 from [10] shows that $T_e|_n$ admits effective quantifier elimination. Lemma 6 of [10] asserts that every quantifier-free formula of $T_e|_n$ is equivalent to a Boolean combination of atomic formulas of the following form:

$$u \wedge v = w \wedge z \ , \ u <_L w \ , \ u \wedge v \leq_H w \wedge z \ , \ E^{\eta}_{\xi}(u, w) \ ,$$

$$L^{\eta}_{\xi}(u, w) \ , \ H_{\eta}(u \wedge v, w) \ , \ A_{\eta}(u \wedge v, w) \ ,$$

where u, v, w, z is either a variable or a constant in $T_e|_n$. By Lemma 8 of [10] the corresponding Boolean combination can be found effectively. This implies that replacing the function $\wedge$ by the first, third, sixth and seventh relations of the list above we transform the language of each T_e into an equivalent relational language. In particular we have that each $T_e|_n$ is ω-categorical.

Note that extending the set of relations we can eliminate constants c_{η} from our language. Admitting empty relations we may assume that all T_e have the same language (where $\omega^{<\omega}$ is the set of indexes). Admitting repeated coordinates we may assume that this language $L_0 = \{R_1, ..., R_i, ...\}$ satisfies the assumptions of the beginning of the section and each sublanguage L_n of the presentation $L_0 = \bigcup_{i>0} L_i$ corresponds to $T_e|_n$.

We now apply Lemma 5 to all $T_e|_n$. Since each $T_e|_n$ is computably axiomatisable uniformly in e and n, we obtain an effective enumeration of computable axiomatisations of L-expansions of all $T_e|_n$ (with $T_e|_n$ on the P-part). For each e taking the axioms which hold in almost all L-expansions of $T_e|_n$ we obtain by Lemma 5(1) a computable axiomatisation of a theory of L-expansions of T_e.

When T_e is an Ehrenfeucht theory with exactly three models (i.e. $e \in NoPath$), the prime model of T_e is generic with respect to its age. Applying Proposition 7 to T_e and all $T_e|_n$ we obtain a generic $(L_0 \cup L)$-structure such that its theory is computably axiomatised as above. This theory has exactly three countable models by Proposition 8.

When we take L-reducts of all T_e and the corresponding computable axiomatisations we obtain a computable enumeration of L-theories which gives the reduction as in the formulation of the theorem. □

Remark 10. In the proof above we used Proposition 7 in order to obtain a complete L-expansions of Ehrenfeuch T_e's. We cannot apply it in the case when T_e does not have an appropriate generic model, for example when the corresponding Tr_e has continuum many paths. Nevertheless the author hopes that the proof can be modified so that the reduction as above also shows that the set of all L-theories with continuum many models is Σ^1_1-hard. In the case of infinite languages this is shown in Section 4 of [7].

4. Coding ω-categorical theories

The main theorem of this section improves the corresponding result of [7] (where the authors do not demand that the language is finite). It is worth noting that the author together with Barbara Majcher-Iwanow have found some other improvements in [5].

Theorem 11. *There is a finite language L such that the property of ω-categoricity distinguishes a Π^0_3-complete subset of the set of all decidable complete L-theories.*

Proof. In the formulation of the theorem L is the language defined in Section 1. It is shown in [7] that the property of ω-categoricity is Π^0_3. The proof of Π^0_3-completeness in the case of L is based on Theorem 1, Section 3 and the idea of Section 2 of [7]. The latter one will be presented in some special form, the result of a fusion with some ideas from [9].

Let us fix the standard enumeration p_n of prime numbers and a Gödel 1-1-enumeration of the set of pairs $\langle i, j\rangle$. Let $a(x)$ be a computable increasing function from ω to $\omega \setminus \{0,1,2\}$ so that if natural numbers $x_1 < x_2$ enumerate pairs $\langle i_1, j_1\rangle$ and $\langle i_2, j_2\rangle$ then $p_{i_1} a(x_1) < p_{i_2} a(x_2)$.

Let L_E consist of $2p_n$-ary relational symbols E_n, $n \in \omega$, and T_E be the $\forall\exists$-theory of the universal homogeneous structure of the universal theory saying that each E_n is an equivalence relation on the set of p_n-tuples which does not depend on the order of tuples and such that all p_n-tuples with at least one repeated coordinate lie in one isolated E_n-class (Remark 4.2.1 in [9]). It is worth mentioning here that the joint embedding property and the amalgamation property are easily verified by an appropriate version of free amalgamation (modulo transitivity of E_n-s). Note also that T_E is ω-categorical and decidable.

We now define an auxiliary language L_{ESP}. We firstly extend L_E by countably many sorts S_n, $n \in \omega$. Start with a countable model $M_E \models T_E$ and take the expansion of M_E to the language $L_E \cup \{S_1, ..., S_n, ...\} \cup \{\pi_1, ..., \pi_n, ...\}$, where each S_n is interpreted by the non-diagonal elements of M^{p_n}/E_n and π_n by the corresponding projection. To define L_{ESP} we extend $L_E \cup \{S_1, ..., S_n, ...\} \cup \{\pi_1, ..., \pi_n, ...\}$ by an ω-sequence of relations P_m, $m \in \omega$, with the following properties. If m is the Gödel number of the pair $\langle n, i\rangle$ then we interpret P_m as a subset of the diagonal of $S_n^{a(m)}$. Let T_{ESP} be the L_{ESP}-theory axiomatized by T_E together with the natural axioms for all π_n and P_m as above.

Having a structure $M \models T_{ESP}$ (which is an expansion of M_E) we now build another expansion M^* of M_E (in the 1-sorted language). For each relational symbol P_m of the sort $S_n^{a(m)}$ we add a new relational symbol P^*_m on $M_E^{a(m)p_n}$ interpreted in the following way:

$$M^* \models P^*_m(\bar{a}_1, ..., \bar{a}_{a(m)}) \Leftrightarrow M \models P_m(\pi_n(\bar{a}_1), ..., \pi_n(\bar{a}_{a(m)})).$$

It is clear that M^* and M are bi-interpretable.

By T^*_{ESP} we denote the theory of all M^* with $M \models T_{ESP}$. Let L_0 be the corresponding language. Then M_E is the L_E-reduct of any countable $M^* \models T^*_{ESP}$. It is clear that T^*_{ESP} is axiomatized by the $\forall\exists$-axioms of T_E, $\forall$-axioms of E_n-invariantness of all P^*_m and $\forall$-axioms that every P_m is a subset of an appropriate diagonal. Moreover for every natural l we have ≤ 1 relations of arity l in L_0 and the function of arities of P^*_m is increasing. Admitting empty relations (say R_j) we may think that for every natural number $l > 0$ the language L_0 contains exactly one relation of arity l. In particular L_0 satisfies basic requirments on L_0 from Section 3. We present L_0 as the union of a sequence of finite languages $L_1 \subset L_2 \subset ... \subset L_m \subset ...$ of arities $l_1 < l_2 < ... < l_m < ...$ where L_m consists of all relations of arity $\leq p_n a(m)$ $(= l_m)$ with n to be the first coordinate of the pair enumerated by m. Note that when m codes a pair $\langle n, j\rangle$ the relation E_n is also in L_m.

For every $m \in \{1, ..., i, ..., \omega\}$ and a finite set D of indexes of relations P^*_i of arity $\leq l_m$ we consider the class $\mathcal{K}_D$ of all finite substructures of models of T^*_{ESP} satisfying the property that all P^*_i with $i \notin D$, are empty. It is clear that for any natural number k the number of structures of $\mathcal{K}_D$ of size k is finite. We will also denote $\mathcal{K}_{\omega,D} := \mathcal{K}_D$. When $m < \omega$ we define $\mathcal{K}_{m,D}$ as the class of all reducts of $\mathcal{K}_D$ to the sublanguage L_m.

By an appropriate version of free amalgamation we see that $\mathcal{K}_{m,D}$ has the joint embedding property and the amalgamation property. Let $M_{m,D}$ be the corresponding universal homogeneous structure and let $T^*_{m,D}$ be the theory of $M_{m,D}$. It follows from $T^*_{\omega,D}$ that $T^*_{ESP} \subset T^*_{\omega,D}$ and for every n the family of all P_i [1] , with $i \in D$ coding some $\langle n, j\rangle$, freely generates a Boolean algebra of infinite subsets of the sort S_n (we may interpret such P_i as a unary predicate on S_n).

By the definition of the class $\mathcal{K}_D$ we see that for any $t < m$ and any two finite sets D' and D'' satisfying

$$D' \cap \{0, ..., l_t\} = D'' \cap \{0, ..., l_t\}$$

the reducts of $M_{m,D'}$ and $M_{m,D''}$ to L_t are isomorphic.

Let us apply the construction of Theorem 1 to $M_{m,D}$. Then we obtain the $(L_m \cup L)$-structure $U_{m,D}$ and the corresponding L-reduct $N_{m,D}$, where L is the language as in Theorem 1. It follows from the proof of that theorem that in the situation of the previous paragraph the structures $U_{m,D'}$ and $U_{m,D''}$ satisfy the same axioms of the form (a) - (d) of Remark 3, where the language of the P-part is restricted to L_t and the number of variables of the Q-part is bounded by l_t. When we rewrite these axioms as L-sentences (using the corresponding definition of the relations of L_m) we obtain that $N_{m,D'}$ and $N_{m,D''}$ satisfy the same axioms of this kind.

[1] L_{ESP}-predicates corresponding to P^*_i

Let $\varphi(x, y)$ be a universal computable function, i.e. $\varphi(e, x) = \varphi_e(x)$. Find a computable function ρ (with $Dom(\rho) = \omega$) enumerating $Dom(\varphi(\varphi(y, z), x))$, i.e. the set of all triples $\langle e, n, x\rangle$ with $x \in W_{\varphi_e(n)}$.

For any natural e, s we define a finite set D^s_e of codes $m \leq l_s$ of all pairs $\langle n, k\rangle$ such that

$$(\exists x)(\rho(k) = \langle e, n, x\rangle \wedge (\forall k' < k)(\rho(k') \neq \langle e, n, x\rangle)).$$

Let T_e and T^*_e be the L_{ESP}-theory and the corresponding 1-sorted version (containing T^*_{ESP}) such that for all natural s the reduct of T^*_e to L_s coincides with the corresponding reduct of T^*_{s,D^s_e}. Since for any $s < t$ we have $D^t_e \cap \{0, ..., l_s\} = D^s_e$, the definition of T_e and T^*_e is correct. It is clear that both T_e and T^*_e are axiomatisable by computable sets of axioms uniformly in e. Since for each s the reduct of T^*_e as above is ω-categorical, the theories T_e and T^*_e are complete. Thus T_e and the corresponding theory T^*_e are decidable uniformly in e. It is worth noting that for each m the L_m-reduct of T^*_e admits elimination of quantifiers (it is of the form $T^*_{m,D}$ as above). Moreover, the class $\bigcup_l \mathcal{K}_{\omega,D^l_e}$ considered as a class of L_0-structures where almost all P^*_m are empty, is a countable class with JEP and AP. It is clear that T^*_e is the theory of the corresponding universal homogeneous structure M^*_e.

Applying Proposition 7 to M^*_e and all M_{l,D^l_e} we obtain the $(L_0 \cup L)$-structures U_e and their approximations U_{l,D^l_e} (and N_{l,D^l_e}), which for $l \to \infty$ give a computable axiomatisation of the complete L-theory T^L_e of the corresponding L-reducts N_e. By Remark 3 applied to all U_{l,D^l_e} (with decidable theories), this axiomatisation (the corresponding decidability of T^L_e) can be found by an effective uniform in e procedure.

We now fix a Gödel coding of the language L, and identify decidable complete L-theories with computable functions from $\{sgn(\varphi_e(x)) : e \in \omega\}$ realising the corresponding characteristic functions (by $sgn(x)$ we denote the function which is equal to 1 for all non-zero numbers and $sgn(0) = 0$). We want to prove that the set of all natural numbers e satisfying the relation

"$sgn(\varphi_e(x))$ codes a decidable ω-categorical theory"

is Π^0_3-complete.

Fix a Turing machine $\kappa(x, y)$ which decides when for a pair d, e the number d codes a sentence which belongs to T^L_e (in this case $\kappa(d, e) = 1$). The following procedure defines a computable function $\xi(z)$ and a computably enumerable set Z. At step e we take the Turing machine for $sgn(\varphi_e(x))$ and check if any replacement of some parameter e' in that program by a variable y makes it the Turing machine $\kappa(x,y)$. If this happens we put e into Z and define $e' := \xi(e)$. As a result we obtain a computably enumerable set Z and a computable function ξ with $Dom(\xi) \supset Z$ and $Rng(\xi) = \omega$ such that for every $e \in Z$ the function $sgn(\varphi_e(x))$ is computed by the machine $\kappa(x, \xi(e))$ (for $T^L_{\xi(e)}$).

By Ryll-Nardzewski's theorem the L_{ESP}-theory $T_{\xi(e)}$ is ω-categorical if and only if all $W_{\varphi_{\xi(e)}(n)}$ are finite (i.e. the set of 1-types (pairwise non-equivalent Boolean combinations of P_m) of each S_n is finite). If we consider the corresponding $T^L_{f(e)}$, then this property remains true.

Since for any Turing machine computing $\varphi_{e'}(x)$ we can effectively find a Turing machine deciding $T^L_{e'}$ (i.e. in fact we can find $sgn(\varphi_e(x))$ with $\xi(e) = e'$), we see that the Π^0_3-set $\{e' : \forall n(W_{\varphi_{e'}(n)}$ is finite$)\}$ is reducible to $\{e : sgn(\varphi_e(x))$ codes an ω-categorical L-theory$\}$. Since the former one is Π^0_3-complete (see [7]) we have the theorem. $\square$

References

[1] E.CASANOVAS, D.LASCAR, A.PILLAY and M.ZIEGLER, *Galois groups of first order theories*, **J. Math. Logic**, 1(2001), 305 - 319.

[2] D.EVANS, *Examples of $\aleph_0$-categorical structures.* In: R.Kaye, D.Macpherson (eds.) **Automorphisms of First-Order Structures**. pp. 33 - 72. Oxford University Press, 1994.

[3] W.HODGES, **Model Theory**, Cambridge University Press, 1993.

[4] A.IVANOV, *A countably categorical theory which is not G-compact*, **Siberian Advances in Math.**, 20(2010), no.2, 75 - 82.

[5] A.IVANOV and B.MAJCHER-IWANOW, *Polish group actions and computability*, arXiv:0903.1070 math.LO

[6] D.W.KUEKER and C.LASKOWSKI, *On generic structures*, **Notre Dame J. Formal Logic**, 33(1992), 175 - 183.

[7] S.LEMPP and Th.SLAMAN, *The complexity of the index set of $\aleph_0$-categorical theories and Ehrenfeucht theories*, **Contemporary Mathematics**, vol. 425 (2007), pp. 43 - 47.

[8] L.NEWELSKI, *The diameter of the Lascar strong type*, **Fund. Math.**, 176(2003), 157 - 170.

[9] R. PELAÉZ, *About the Lascar group*, Tesis Doctoral, (57 pgs), Departmento de Lógika, Historia y Filosofia de la Ciencia, Universidad de Barcelona, 2008.

[10] R.REED, *A decidable Ehrenfeucht theory with exactly two hyperarithmetic models*, **Ann. Pure Appl. Logic**, 53(1991), 135 - 168.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW,

pl.GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND

E-mail: ivanov@math.uni.wroc.pl