# Universal and complete sets in martingale theory 

Dominique LECOMTE and Miroslav ZELENÝ ${ }^{1}$

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- Université Paris 6, Institut de Mathématiques de Jussieu, Projet Analyse Fonctionnelle Couloir 16-26, 4ème étage, Case 247, 4, place Jussieu, 75252 Paris Cedex 05, France dominique.lecomte@upmc.fr
- Université de Picardie, I.U.T. de l'Oise, site de Creil, 13, allée de la faïencerie, 60107 Creil, France
- ${ }^{1}$ Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis Sokolovská 83, 18675 Prague, Czech Republic zeleny@karlin.mff.cuni.cz


#### Abstract

The Doob convergence theorem implies that the set of divergence of any martingale has measure zero. We prove that, conversely, any $G_{\delta \sigma}$ subset of the Cantor space with Lebesgue-measure zero can be represented as the set of divergence of some martingale. In fact, this is effective and uniform. A consequence of this is that the set of everywhere converging martingales is $\boldsymbol{\Pi}_{1}^{1}$-complete, in a uniform way. We derive from this some universal and complete sets for the whole projective hierarchy, via a general method. We provide some other complete sets for the classes $\boldsymbol{\Pi}_{1}^{1}$ and $\boldsymbol{\Sigma}_{2}^{1}$ in the theory of martingales.


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## 1 Introduction

The reader should see [K2] for the notation used in this paper.
Definition 1.1 We say that a map $f: 2^{<\omega} \rightarrow[0,1]$ is a martingale if $f(s)=\frac{f(s 0)+f(s 1)}{2}$ for each $s \in 2^{<\omega}$. The set of martingales is denoted by $\mathcal{M}$ and is a compact subset of $[0,1]^{2<\omega}$ (equipped with the usual product topology).

This terminology is not the standard one, but the set $\mathcal{M}$ can be interpreted as the set of all discrete martingales (in the classical sense) taking values in [0,1], as follows. If $s \in 2^{<\omega}$, then

$$
N_{s}:=\left\{\beta \in 2^{\omega} \mid s \subseteq \beta\right\}
$$

is the usual basic clopen set. Let $f \in \mathcal{M}$. If $n \in \omega$, then let $\mathcal{S}_{n}$ be the $\sigma$-algebra on $2^{\omega}$ generated by $\left\{N_{s} \mid s \in 2^{n}\right\}$, and $f_{n}: 2^{\omega} \rightarrow[0,1]$ be defined by $f_{n}(\beta):=f(\beta \mid n)$. Then the sequence $\left(f_{n}\right)_{n \in \omega}$ is a discrete martingale taking values in $[0,1]$ with respect to the sequence of $\sigma$-algebras $\left(\mathcal{S}_{n}\right)_{n \in \omega}$ and the usual Lebesgue product measure $\lambda$ on $2^{\omega}$. Conversely, if $\left(f_{n}\right)_{n \in \omega}$ is any such martingale, it can be viewed as an element of $\mathcal{M}$ by setting $f(s):=f_{|s|}(\alpha)$ if $\alpha \in N_{s}$. This definition is correct because $f_{|s|}$, as a function measurable with respect to $\mathcal{S}_{|s|}$, has a constant value on $N_{s}$.

Definition 1.2 Let $f$ be a martingale and $\beta \in 2^{\omega}$. The oscillation of $f$ at $\beta$ is the number

$$
\operatorname{osc}(f, \beta):=\inf _{N \in \omega} \sup _{p, q \geq N}|f(\beta \mid p)-f(\beta \mid q)| .
$$

The set of divergence of $f$ is $D(f):=\left\{\beta \in 2^{\omega} \mid \operatorname{osc}(f, \beta)>0\right\}$.
By definition, if $f$ is a martingale, then

$$
\beta \in D(f) \Leftrightarrow \exists r \in \omega \quad \forall N \in \omega \quad \exists p, q \geq N|f(\beta \mid p)-f(\beta \mid q)|>2^{-r} .
$$

This shows that $D(f) \in \boldsymbol{\Sigma}_{3}^{0}$. Moreover, $D(f)$ has $\lambda$-measure zero, by Doob's convergence theorem (see Chapter XI, Section 14 in [D]). So it is natural to ask whether any $\boldsymbol{\Sigma}_{3}^{0}$ subset of $2^{\omega}$ with $\lambda$ measure zero is the set of divergence of some martingale (this question was asked by Louveau). We answer positively:

Theorem 1.3 Let $B$ be a subset of $2^{\omega}$. Then the following are equivalent:
(a) $B$ is $\boldsymbol{\Sigma}_{3}^{0}$ and has $\lambda$-measure zero,
(b) there is a martingale $f$ with $B=D(f)$.

Definition 1.4 Let $\boldsymbol{\Gamma}$ be a class of subsets of Polish spaces, $X, Y$ be Polish spaces, and $\mathcal{U} \subseteq Y \times X$.
(a) We say that $\mathcal{U}$ is $Y$-universal for the $\boldsymbol{\Gamma}$ subsets of $X$ if $\mathcal{U} \in \boldsymbol{\Gamma}(Y \times X)$ and $\boldsymbol{\Gamma}(X)=\left\{\mathcal{U}_{y} \mid y \in Y\right\}$.
(b) We say that $\mathcal{U}$ is uniformly $Y$-universal for the $\boldsymbol{\Gamma}$ subsets of $X$ if $\mathcal{U}$ is $Y$-universal for the $\boldsymbol{\Gamma}$ subsets of $X$ and, for each $S \in \boldsymbol{\Gamma}\left(\omega^{\omega} \times X\right)$, there is a Borel map $b: \omega^{\omega} \rightarrow Y$ such that $S_{\alpha}=\mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$.

Corollary 1.5 Let $\mathcal{G}$ be a $G_{\delta}$ subset of $2^{\omega}$ with $\lambda(\mathcal{G})=0$. Then the set $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \beta \in D(f)\}$ is $\mathcal{M}$-universal for the $\Sigma_{3}^{0}$ subsets of $\mathcal{G}$.

In fact, we prove an effective and uniform version of the implication (a) $\Rightarrow$ (b) in Theorem 1.3 , In particular, we can associate, via a Borel map $F$, a martingale to a code $\alpha$ of an arbitrary $G_{\delta}$ subset $G$ of $\mathcal{G}$ (as in the previous corollary), in such a way that $G=D(F(\alpha))$. A consequence of this is the following:

Theorem 1.6 The set $\mathcal{P}$ of everywhere converging martingales is $\boldsymbol{\Pi}_{1}^{1}$-complete.
These statements are in the spirit of some results concerning the differentiability of functions due to Zahorski and Mazurkiewicz (see Section 4 for details). In fact, $\mathcal{P}$ is $\boldsymbol{\Pi}_{1}^{1}$-complete in a uniform way, which allows to derive some universal and complete sets for the whole projective hierarchy, in spaces of continous functions, starting from $\mathcal{P}$. More precisely, let $P_{1}:=[0,1]^{2<\omega}$ and $C_{1}:=\mathcal{P}$. We define, for each natural number $n \geq 1$,

- the space $P_{n+1}:=\mathcal{C}\left(2^{\omega}, P_{n}\right)$ of continuous functions from $2^{\omega}$ into $P_{n}$, equipped with the topology of uniform convergence (inductively),
- $C_{n+1}:=\left\{h \in P_{n+1} \mid \forall \beta \in 2^{\omega} \quad h(\beta) \notin C_{n}\right\}$ (inductively),
- $U_{n}:=\left\{(h, \beta) \in P_{n+1} \times 2^{\omega} \mid h(\beta) \in C_{n}\right\}$.

We prove the following:
Theorem 1.7 Let $n \geq 1$ be a natural number. Then
(a) the set $U_{n}$ is uniformly $P_{n+1}$-universal for the $\boldsymbol{\Pi}_{n}^{1}$ subsets of $2^{\omega}$,
(b) the set $C_{n}$ is $\boldsymbol{\Pi}_{n}^{1}$-complete.

In fact, our method is more general and works if we start with a $\Pi_{1}^{1}$ set which is complete in a uniform way.

Let $f$ be a martingale. As $D(f)$ has $\lambda$-measure zero, we can associate to $f$ the partial function $\psi(f)$ defined $\lambda$-almost everywhere by $\psi(f)(\beta):=\lim _{l \rightarrow \infty} f(\beta \mid l)$. The partial function $\psi(f)$ will be called the associated partial function. The martingale $f$ is in $\mathcal{P}$ if and only if $\psi(f)$ is total, in which case $\psi(f)$ is called the associated function. Using the work in [B-Ka-L] and [K2] about spaces of continuous functions, we prove the following:

Theorem 1.8 (a) The set of sequences of everywhere converging martingales whose associated functions converge pointwise is $\boldsymbol{\Pi}_{1}^{1}$-complete.
(b) The set of sequences of everywhere converging martingales whose associated functions converge pointwise to zero is $\boldsymbol{\Pi}_{1}^{1}$-complete.
(c) The set of sequences of everywhere converging martingales having a subsequence whose associated functions converge pointwise to zero is $\boldsymbol{\Sigma}_{2}^{1}$-complete.

## $2 \Sigma_{3}^{0}$ sets of measure zero

Notation. In the sequel, $B$ will be a Borel subset of $2^{\omega}$, and $M$ will be a $\lambda$-measurable subset of $2^{\omega}$. If $\beta \in 2^{\omega}$, then the density of $M$ at $\beta$ is the number $d(M, \beta):=\lim _{l \rightarrow \infty} \frac{\lambda\left(M \cap N_{\beta \mid l}\right)}{\lambda\left(N_{\beta \mid l}\right)}$ when it is defined. Note that $d(B, \beta)=1$ if $\beta \in B$ and $B$ is open. We first recall the Lebesgue density theorem (see 17.9 in [K2]).

Theorem 2.1 (Lebesgue) The equality $\lambda(M)=\lambda(\{\beta \in M \mid d(M, \beta)=1\})$ holds for any $\lambda$ measurable subset $M$ of $2^{\omega}$.

The reader should see [C] for the next lemma. We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.2 (Lusin-Menchoff) Let $F$ be a closed subset of $2^{\omega}$, and $M \supseteq F$ be a $\lambda$-measurable subset of $2^{\omega}$ such that $d(M, \beta)=1$ for each $\beta \in F$. Then there is a closed subset $C$ of $2^{\omega}$ such that
(1) $F \subseteq C \subseteq M$,
(2) $d(M, \beta)=1$ for each $\beta \in C$,
(3) $d(C, \beta)=1$ for each $\beta \in F$.

Proof. If $F$ is $2^{\omega}$, then we can take $C:=F$. So we may assume that $F$ is not $2^{\omega}$. We set $s^{-}:=s \mid(|s|-1)$ if $\emptyset \neq s \in 2^{<\omega}$. Note that $\neg F$ is the disjoint union of the elements of a sequence $\left(N_{s_{n}}\right)_{n \in \omega}$, where $N_{s_{n}^{-}} \cap F \neq \emptyset$ for each $n \in \omega$. Fix $n \in \omega$. By Theorem 2.1.

$$
\lambda\left(M \cap N_{s_{n}}\right)=\lambda\left(\left\{\beta \in M \cap N_{s_{n}} \mid d\left(M \cap N_{s_{n}}, \beta\right)=1\right\}\right) .
$$

The regularity of $\lambda$ gives a closed subset $F_{n}$ of $2^{\omega}$ contained in $\left\{\beta \in M \cap N_{s_{n}} \mid d\left(M \cap N_{s_{n}}, \beta\right)=1\right\}$ such that $\lambda\left(F_{n}\right) \geq\left(1-2^{-n}\right) \lambda\left(M \cap N_{s_{n}}\right)$. We set $C:=F \cup \bigcup_{n \in \omega} F_{n}$, which is closed since $\left|s_{n}\right| \rightarrow \infty$.

As Conditions (1) and (2) are clearly satisfied, pick $\beta \in F$. Note that

$$
\begin{aligned}
\lambda\left(N_{\beta \mid l} \backslash C\right) & =\Sigma_{s_{n} \supseteq \beta \mid l} \lambda\left(N_{s_{n}} \backslash C\right) \\
& \leq \Sigma_{s_{n} \supseteq \beta \mid l} \lambda\left(N_{s_{n}} \backslash F_{n}\right) \\
& \leq \Sigma_{s_{n} \supseteq \beta \mid l} 2^{-n} \lambda\left(M \cap N_{s_{n}}\right)+\Sigma_{s_{n} \supseteq \beta \mid l} \lambda\left(N_{s_{n}} \backslash M\right) \\
& \leq \Sigma_{s_{n} \supseteq \beta \mid l} 2^{-n} \lambda\left(N_{s_{n}}\right)+\lambda\left(N_{\beta \mid l} \backslash M\right) .
\end{aligned}
$$

This implies that the limit of $\frac{\lambda\left(N_{\beta \mid \lambda} \backslash C\right)}{\lambda\left(N_{\beta \mid l}\right)}$ is zero since $d(M, \beta)=1$.
The next topology is considered in [Lu-Ma-Z], see Chapter 6.
Definition 2.3 The $\tau$-topology on $2^{\omega}$ is generated by

$$
\mathcal{F}:=\left\{M \subseteq 2^{\omega} \mid M \text { is } \lambda \text {-measurable } \wedge \forall \beta \in M \quad d(M, \beta)=1\right\} .
$$

The next result is proved in [Lu-Ma-Z], but in a much more abstract way. This is the reason why we include a much more direct proof here, since it is not too long.

Lemma 2.4 The family $\mathcal{F}$ is a topology. In particular, any $\tau$-open set is $\lambda$-measurable.
Proof. Note first that $\mathcal{F}$ is closed under finite intersections, so that it is a basis for the $\tau$-topology. Indeed, let $M, M^{\prime}$ be in $\mathcal{F}$, and $\beta \in M \cap M^{\prime}$. Then we use the facts that

$$
\lambda\left(M \cap M^{\prime} \cap N_{\beta \mid l}\right)=\lambda\left(M \cap N_{\beta \mid l}\right)-\lambda\left(\left(M \cap N_{\beta \mid l}\right) \backslash M^{\prime}\right)
$$

and $\lambda\left(\left(M \cap N_{\beta \mid l}\right) \backslash M^{\prime}\right) \leq \lambda\left(N_{\beta \mid l} \backslash M^{\prime}\right)$.
Let $\mathcal{H}$ be a subfamily of $\mathcal{F}$, and $H:=\cup \mathcal{H}$. We claim that there is a countable subfamily $\mathcal{C}$ of $\mathcal{H}$ such that $m:=\sup \{\lambda(\cup \mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{H}$ countable $\}=\lambda(\cup \mathcal{C})$. Indeed, for each $n \in \omega$ there is $\mathcal{D}_{n} \subseteq \mathcal{H}$ countable such that $\lambda\left(\cup \mathcal{D}_{n}\right)>m-2^{-n}$, and $\mathcal{C}:=\bigcup_{n \in \omega} \mathcal{D}_{n}$ is suitable. Let $C:=\cup \mathcal{C}$.

Let $\beta \in H$, and $M$ in $\mathcal{H}$ with $\beta \in M$. Note that $\lambda(M \cup C)=\lambda(C)$ (consider the family $\mathcal{C} \cup\{M\}$ ). Thus $\lambda(M \backslash C)=0$. As $d(M, \beta)=1$, the equality $d(M \cap C, \beta)=1$ holds, and $d(\neg C, \beta)=0$. This implies that $H \backslash C$ is contained in $\{\beta \notin C \mid d(\neg C, \beta)<1\}$, which has $\lambda$-measure zero by Theorem 2.1. Therefore $H \backslash C$ has $\lambda$-measure zero and $H=C \cup(H \backslash C)$ is $\lambda$-measurable.

Pick $\beta \in H$, and $M \in \mathcal{H}$ with $\beta \in M$. Then $d(M, \beta)=1$, and thus $d(H, \beta)=1$. Therefore $H \in \mathcal{F}$. This finishes the proof.

The next lemma is in the style of Urysohn's theorem (see [Lu] for its version on the real line). We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.5 Let $C$ be a closed subset of $2^{\omega}$, and $G$ be a $G_{\delta}$ subset of $2^{\omega}$ disjoint from $C$ such that $\lambda(G)=0$. Then there is a $\tau$-continuous map $h: 2^{\omega} \rightarrow[0,1]$ such that $h_{\mid C} \equiv 0$ and $h_{\mid G} \equiv 1$.

Proof. Let $\left(F_{n}\right)_{n \in \omega}$ be an increasing sequence of closed subsets of $2^{\omega}$ with union $\neg G$ and $F_{0}=C$. We first construct a sequence $\left(C_{\frac{1}{2^{n}}}\right)_{n \in \omega}$ of closed subsets of $2^{\omega}$ with $F_{n} \subseteq C_{\frac{1}{2^{n}}} \subseteq \neg G, C_{\frac{1}{2^{n}}} \subseteq C_{\frac{1}{2^{n+1}}}$, and $d\left(C_{\frac{1}{2 n+1}}, \beta\right)=1$ for each $\beta \in C_{\frac{1}{2^{n}}}$. We first apply Lemma 2.2 to $F:=F_{0}$ and $M:=\neg G$, which gives $F_{0} \subseteq C_{1} \subseteq \neg G$. Then, inductively, we apply Lemma 2.2 to $F:=C_{\frac{1}{2^{n}}} \cup F_{n+1}$ and $M:=\neg G$, which gives $C_{\frac{1}{2^{n}}} \cup F_{n+1} \subseteq C_{\frac{1}{2^{n+1}}} \subseteq \neg G$ such that $d\left(C_{\frac{1}{2^{n+1}}}, \beta\right)=1$ for each $\beta \in C_{\frac{1}{2^{n}}}$.

Then we construct $C_{\frac{2 k+1}{2^{n}}}$, for $0<k<2^{n-1}$ and $n \geq 2$. This will give us a family $\left(C_{\frac{k}{2^{n}}}\right)_{n \in \omega, 0<k \leq 2^{n}}$ of closed subsets of $2^{\omega}$. We want to ensure that $C_{\zeta} \subseteq C_{\zeta^{\prime}}$ and $d\left(C_{\zeta^{\prime}}, \beta\right)=1$ for each $\beta \in C_{\zeta}$ if $\zeta^{\prime}<\zeta$. We proceed by induction on $n$. We apply Lemma 2.2 to $F:=C_{\frac{k+1}{2^{n-1}}}$ and $M:=C_{\frac{k}{2^{n-1}}}$, which gives $C_{\frac{2 k+1}{2^{n}}}$ such that $C_{\frac{k+1}{2^{n-1}}} \subseteq C_{\frac{2 k+1}{2^{n}}} \subseteq C_{\frac{k}{2^{n-1}}}, d\left(C_{\frac{k}{2^{n-1}}}, \beta\right)=1$ for each $\beta \in C_{\frac{2 k+1}{2^{n}}}$, and $d\left(C_{\frac{2 k+1}{2^{n}}}, \beta\right)=1$ for each $\beta \in C_{\frac{k+1}{2^{n-1}}}$. This allows us to define $\tilde{h}$ by

$$
\tilde{h}(\beta):=\left\{\begin{array}{l}
0 \text { if } \beta \in G, \\
\sup \left\{\zeta \mid \beta \in C_{\zeta}\right\} \text { if } \beta \notin G .
\end{array}\right.
$$

It remains to see that $\tilde{h}$ is $\tau$-continuous (and then we will set $h(\beta):=1-\tilde{h}(\beta)$ ). So let $b \in(0,1]$, and $\beta \in 2^{\omega}$ with $\tilde{h}(\beta)<b$. Note that there is $\zeta<b$ with $\tilde{h}(\beta)<\zeta$, so that $\beta \notin C_{\zeta}$. If $\gamma \notin C_{\zeta}$, then $\tilde{h}(\gamma) \leq \zeta<b$, so that $\neg C_{\zeta}$ is an open (and thus $\tau$-open since the $\tau$-topology is finer than the usual one) neighborhood of $\beta$ on which $\tilde{h}<b$. In particular, $\tilde{h}$ is Borel.

Now let $a \in[0,1)$. It is enough to see that $B:=\left\{\gamma \in 2^{\omega} \mid \tilde{h}(\gamma)>a\right\}$ is $\tau$-open. So assume that $\tilde{h}(\gamma)>a$. Note that there are $\zeta>\zeta^{\prime}>a$ with $\tilde{h}(\gamma)>\zeta$, so that $\gamma \in C_{\zeta} \subseteq C_{\zeta^{\prime}} \subseteq B$. Thus $d\left(C_{\zeta^{\prime}}, \gamma\right)=1$, by construction of the family. As $\tilde{h}$ is Borel, $B$ is Borel, $d(B, \gamma)$ is defined and equal to 1 .

Remark. We in fact proved that $h$ is lower semi-continuous.
Notation. If $h: 2^{\omega} \rightarrow[0,1]$ is a $\lambda$-measurable map and $s \in 2^{<\omega}$, then we set $f_{N_{s}} h d \lambda:=\frac{\int_{N_{s}} h d \lambda}{\lambda\left(N_{s}\right)}$.
Lemma 2.6 Let $h: 2^{\omega} \rightarrow[0,1]$ be a $\tau$-continuous map, and $\beta \in 2^{\omega}$. Then

$$
\lim _{l \rightarrow \infty} f_{N_{\beta \mid l}} h d \lambda=h(\beta) .
$$

Proof. Let $\varepsilon>0$, and $\beta \in M:=h^{-1}(B(h(\beta), \varepsilon))$. Note that $d(M, \gamma)=1$ for each $\gamma \in M$ since $h$ is $\tau$-continuous. As $h$ is $\lambda$-measurable, we can write

$$
\int_{N_{\beta \mid l}} h d \lambda=\int_{M \cap N_{\beta \mid l}} h d \lambda+\int_{N_{\beta \mid \downarrow} \backslash M} h d \lambda .
$$

Note that $0 \leq \int_{N_{\beta \mid \downarrow} \backslash M} h d \lambda \leq \lambda\left(N_{\beta \mid l} \backslash M\right)$, so that $0 \leq f_{N_{\beta \mid l} \backslash M} h d \lambda \leq \frac{\lambda\left(N_{\beta \mid} \backslash M\right)}{\lambda\left(N_{\beta \mid l}\right)} \rightarrow 0$. Similarly,

$$
f_{M \cap N_{\beta \mid l}} h d \lambda \in\left[(h(\beta)-\varepsilon) \frac{\lambda\left(M \cap N_{\beta \mid l}\right)}{\lambda\left(N_{\beta \mid l}\right)},(h(\beta)+\varepsilon) \frac{\lambda\left(M \cap N_{\beta \mid l}\right)}{\lambda\left(N_{\beta \mid l}\right)}\right],
$$

and we are done since $\frac{\lambda\left(M \cap N_{\beta \mid l}\right)}{\lambda\left(N_{\beta \mid l}\right)}$ tends to 1 as $l$ tends to $\infty$.
Now we come to our main lemma, inspired by Zahorski (see [Za]).
Lemma 2.7 Let $G$ be a $G_{\delta}$ subset of $2^{\omega}$ with $\lambda$-measure zero. Then there is a martingale $f$ with $G=D(f)$ and $\left\{o s c(f, \beta) \mid \beta \in 2^{\omega}\right\} \subseteq\{0\} \cup\left[\frac{1}{2}, 1\right]$.
Proof. Let $\left(G_{n}\right)_{n \in \omega}$ be a decreasing sequence of open subsets of $2^{\omega}$ with intersection $G$ and $G_{0}=2^{\omega}$.

- We construct $g_{n}: 2^{\omega} \rightarrow[0,1]$, open subsets $G_{n}^{*}, G_{n}^{* *}$ of $2^{\omega}$, and a sequence $\left(s_{j}^{n}\right)_{j \in I_{n}}$ of pairwise incompatible finite binary sequences, by induction on $n \in \omega$, such that, if $S_{n}:=\Sigma_{j \leq n}(-1)^{j} g_{j}$,
(1) $G \subseteq G_{n+1}^{*} \subseteq G_{n}^{* *}=\bigcup_{j \in I_{n}} N_{s_{j}^{n}} \subseteq G_{n}^{*} \subseteq G_{n} \wedge G_{0}^{*}=2^{\omega}$,
(2) $g_{n \mid G} \equiv 1 \wedge g_{n \mid \neg G_{n}^{*}} \equiv 0$,
(3) $g_{n}$ is $\tau$-continuous,
(4) $g_{n+1} \leq g_{n}$,
(5) $\lambda\left(G_{n+1}^{*} \cap N_{s_{j}^{n}}\right)<2^{-n-3} \lambda\left(N_{s_{j}^{n}}\right)$,
(6) $\left|f_{N_{s_{j}^{n}}} S_{n} d \lambda-S_{n}(\beta)\right|<2^{-3}$ if $\beta \in G \cap N_{s_{j}^{n}}$.

We set $g_{0}: \equiv 1, G_{0}^{*}, G_{0}^{* *}:=2^{\omega}, I_{0}:=\{0\}$ and $s_{0}^{0}:=\emptyset$. Assume that our objects are constructed up to $n$. We first construct an open subset $G_{n+1}^{*}$ of $2^{\omega}$ with $G \subseteq G_{n+1}^{*} \subseteq G_{n}^{* *} \cap G_{n+1}$ such that

$$
\lambda\left(G_{n+1}^{*} \cap N_{s_{j}^{n}}\right)<2^{-n-3} \lambda\left(N_{s_{j}^{n}}\right)
$$

if $j \in I_{n}$. For each $j \in I_{n}$, there is an open set $O_{j}$ with $G \cap N_{s_{j}^{n}} \subseteq O_{j} \subseteq G_{n+1} \cap N_{s_{j}^{n}}$ such that $\lambda\left(O_{j}\right)<2^{-n-3} \lambda\left(N_{s_{j}^{n}}\right)$. We then set $G_{n+1}^{*}:=\bigcup_{j \in I_{n}} O_{j}$.

We now apply Lemma 2.5 to $C:=\neg G_{n+1}^{*}$ and $G$, which gives a $\tau$-continuous map $h: 2^{\omega} \rightarrow[0,1]$ with $h_{\mid \neg G_{n+1}^{*}} \equiv 0$ and $h_{\mid G} \equiv 1$. We set $g_{n+1}:=\min \left(g_{n}, h\right)$, so that $g_{n+1}$ satisfies (2)-(4).

By Lemma 2.6, $\lim _{l \rightarrow \infty} f_{N_{\beta \mid l}} S_{n+1} d \lambda=S_{n+1}(\beta)$ for each $\beta \in G$. This gives $l(\beta) \in \omega$ minimal with $\left|f_{N_{\beta \mid l(\beta)}} S_{n+1} d \lambda-S_{n+1}(\beta)\right|<2^{-3}$ and $N_{\beta \mid l(\beta)} \subseteq G_{n+1}^{*}$. The set $G_{n+1}^{* *}$ is the union of the $N_{\beta \mid l(\beta)}$ 's, which defines $I_{n+1}$ and $\left(s_{j}^{n+1}\right)_{j \in I_{n+1}}\left(S_{n+1}(\beta)\right.$ is 0 if $n$ is even and 1 otherwise when $\beta \in G)$.

- We then define a partial map $f_{\infty}: 2^{\omega} \rightarrow[0,1]$ by $f_{\infty}:=\Sigma_{j \in \omega}(-1)^{j} g_{j}$. If $\beta \in G$, then $S_{n}(\beta)$ takes alternatively the values 1 and 0 , depending on the parity of $n$, so that $f_{\infty}(\beta)$ is not defined. If $\beta \notin G$, then there is $n$ such that $\beta \in \neg G_{n+1}^{*} \subseteq \neg G_{n+2}^{*} \subseteq \ldots$ This implies that $f_{\infty}(\beta)$ is defined and equal to $S_{n}(\beta)$. As $0 \leq \Sigma_{p \leq q}\left(g_{2 p}-g_{2 p+1}\right)=S_{2 q+1} \leq S_{2 q}=g_{0}+\Sigma_{1 \leq p \leq q}\left(g_{2 p}-g_{2 p-1}\right) \leq g_{0}, f_{\infty}$ takes values in $[0,1]$. So $f_{\infty}$ is a partial $\lambda$-measurable map defined $\lambda$-almost everywhere since $\lambda(G)=0$ (we use Lemma 2.4).
- This allows us to define $f: 2^{<\omega} \rightarrow[0,1]$ by $f(s):=f_{N_{s}} f_{\infty} d \lambda$. As $\lambda\left(N_{s}\right)=2 \lambda\left(N_{s \varepsilon}\right)$ for each $\varepsilon \in 2$, $f(s)=f_{N_{s}} f_{\infty} d \lambda=\frac{\int_{N_{s 0}} f_{\infty} d \lambda+\int_{N_{s 1}} f_{\infty} d \lambda}{\lambda\left(N_{s}\right)}=\frac{f(s 0)}{2}+\frac{f(s 1)}{2}$ and $f$ is a martingale.
- If $\beta \notin G$, then there is $n$ with $\beta \in G_{n}^{*} \backslash G_{n+1}^{*}$, so that $f_{\infty}(\beta)=S_{n}(\beta)$. By Lemma 2.6, $k \geq n$ implies that $\lim _{l \rightarrow \infty} f_{N_{\beta \mid l}} S_{k+1} d \lambda=S_{k+1}(\beta)=S_{n}(\beta)$ since $S_{k+1}$ is $\tau$-continuous. Note that, for each $k \geq n$,

$$
\begin{aligned}
\left|\int_{N_{\beta \mid l}}\left(f_{\infty}-S_{k+1}\right) d \lambda\right| & \leq \lambda\left(G_{k+2}^{*} \cap N_{\beta \mid l}\right) \\
& \leq \Sigma_{\beta \mid l \subseteq s_{j}^{k+1}} \lambda\left(G_{k+2}^{*} \cap N_{s_{j}^{k+1}}\right) \\
& \leq \Sigma_{\beta \mid l \subseteq s_{j}^{k+1}} 2^{-k-4} \lambda\left(N_{s_{j}^{k+1}}\right) \\
& \leq \lambda\left(N_{\beta \mid l}\right) 2^{-k-4} .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\left|f(\beta \mid l)-f_{\infty}(\beta)\right|=\left|f_{N_{\beta \mid l}} f_{\infty} d \lambda-f_{\infty}(\beta)\right| & =\left|f_{N_{\beta \mid l}}\left(f_{\infty}-S_{k+1}\right) d \lambda+f_{N_{\beta \mid l}} S_{k+1} d \lambda-S_{k+1}(\beta)\right| \\
& \leq 2^{-k-4}+\left|f_{N_{\beta \mid l}} S_{k+1} d \lambda-S_{k+1}(\beta)\right|
\end{aligned}
$$

so that $\lim _{l \rightarrow \infty} f(\beta \mid l)=f_{\infty}(\beta), \operatorname{osc}(f, \beta)=0$ and $\beta \notin D(f)$.

- If $\beta \in G$ and $n \in \omega$, then there is $j \in \omega$ with $\beta \in N_{s_{j}^{n}}$. Note that

$$
f\left(s_{j}^{n}\right)=f_{N_{s_{j}^{n}}} f_{\infty} d \lambda=f_{N_{s_{j}^{n}}} S_{n} d \lambda+f_{N_{s_{j}^{n}}}\left(f_{\infty}-S_{n}\right) d \lambda
$$

and $\left|\int_{N_{s_{j}^{n}}}\left(f_{\infty}-S_{n}\right) d \lambda\right| \leq \lambda\left(G_{n+1}^{*} \cap N_{s_{j}^{n}}\right)<\frac{1}{8} \lambda\left(N_{s_{j}^{n}}\right)$, so that $\left|f_{N_{s_{j}^{n}}}\left(f_{\infty}-S_{n}\right) d \lambda\right|<\frac{1}{8}$. By (6), $\left|f\left(s_{j}^{n}\right)-S_{n}(\beta)\right|<\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$. As $S_{n}(\beta)$ takes infinitely often the values 1 and $0, \operatorname{osc}(f, \beta) \geq \frac{1}{2}$ and $\beta \in D(f)$.

The main result will be a consequence of the main lemma and the following.

Lemma 2.8 Let $\left(f_{n}\right)_{n \in \omega}$ be a sequence of martingales such that

$$
\left\{o s c\left(f_{n}, \beta\right) \mid(n, \beta) \in \omega \times 2^{\omega}\right\} \subseteq\{0\} \cup\left[\frac{1}{2}, 1\right] .
$$

Then there is a martingale $f$ with $D(f)=\bigcup_{n \in \omega} D\left(f_{n}\right)$.
Proof. We first observe the following facts. Let $g, h: 2^{<\omega} \rightarrow \mathbb{R}$ be bounded, $\beta \in 2^{\omega}$ and $a \in \mathbb{R}$.
(1) $\operatorname{osc}(g+h, \beta) \leq \operatorname{osc}(g, \beta)+\operatorname{osc}(h, \beta)$.

This comes from the triangle inequality.
(2) $\operatorname{osc}(a g, \beta)=|a| \cdot \operatorname{osc}(g, \beta)$.
(3) $\operatorname{osc}(g+h, \beta)=\operatorname{osc}(h, \beta)$ if $\operatorname{osc}(g, \beta)=0$.

By (1), $\operatorname{osc}(h, \beta) \leq \operatorname{osc}(g+h, \beta)+\operatorname{osc}(-g, \beta)=\operatorname{osc}(g+h, \beta) \leq \operatorname{osc}(g, \beta)+\operatorname{osc}(h, \beta)=\operatorname{osc}(h, \beta)$, so that $\operatorname{osc}(h, \beta)=\operatorname{osc}(g+h, \beta)$.
(4) $\operatorname{osc}(g, \beta) \leq a$ if $g(\beta \mid l) \in[0, a]$ for each $l \in \omega$.

- We set $D_{n}:=D\left(f_{n}\right)$ for each $n \in \omega$, and $f:=\Sigma_{n \in \omega} 4^{-n} f_{n}$. Note that $f$ is defined and a martingale.
- If $\beta \notin \bigcup_{n \in \omega} D_{n}$, then $\operatorname{osc}\left(f_{n}, \beta\right)=0$ for each $n \in \omega$. In particular, osc $\left(4^{-n} f_{n}, \beta\right)=0$ for each $n \in \omega$, by (2). Let $\varepsilon>0$, and $M \in \omega$ with $\Sigma_{n>M} 4^{-n} \leq \varepsilon$. By (1), osc $\left(\Sigma_{n \leq M} 4^{-n} f_{n}, \beta\right)=0$. By (3) and (4), $\operatorname{osc}(f, \beta)=\operatorname{osc}\left(\Sigma_{n>M} 4^{-n} f_{n}, \beta\right) \leq \Sigma_{n>M} 4^{-n} \leq \varepsilon$. As $\varepsilon$ is arbitrary, $\operatorname{osc}(f, \beta)=0, \beta \notin D(f)$, which shows that $D(f) \subseteq \bigcup_{n \in \omega} D_{n}$.
- If $\beta \in \bigcup_{n \in \omega} D_{n}$, then let $m$ be minimal such that $\beta \in D_{m}$. Note that

$$
f=\Sigma_{n<m} 4^{-n} f_{n}+4^{-m} f_{m}+\Sigma_{n>m} 4^{-n} f_{n} .
$$

By (2) and (3), $\operatorname{osc}(f, \beta)=\operatorname{osc}\left(4^{-m} f_{m}+\Sigma_{n>m} 4^{-n} f_{n}, \beta\right)$. By (1), (2) and (4),

$$
\operatorname{osc}(f, \beta) \geq \operatorname{osc}\left(4^{-m} f_{m}, \beta\right)-\operatorname{osc}\left(\Sigma_{n>m} 4^{-n} f_{n}, \beta\right) \geq 4^{-m} \frac{1}{2}-4^{-m} \frac{1}{3}>0 .
$$

Thus $\beta \in D(f)$.

## 3 Effectivity and uniformity

- We refer to [M] for the basic notions of effective descriptive set theory. We first recall some material present in it.
- Let $\left(p_{n}\right)_{n \in \omega}$ be the sequence of prime numbers $2,3, \ldots$
- If $l \in \omega$ and $s \in \omega^{l}$, then $\bar{s}:=<s(0), \ldots, s(l-1)>:=p_{0}^{s(0)+1} \ldots p_{l-1}^{s(l-1)+1} \in \omega$ codes $s$ (if $l=0$, then $<>$ : $=1$ ).
- If $\alpha \in \omega^{\omega}$ and $l \in \omega$, then $\bar{\alpha}(l):=<\alpha(0), \ldots, \alpha(l-1)>\in \omega$ codes $\alpha \mid l \in \omega^{l}$, and $\alpha^{*}$ is defined by removing the first coordinate: $\alpha^{*}:=(\alpha(1), \alpha(2), \ldots)$.
- If $\kappa \in\{2, \omega\}$, then $<., .>:\left(\kappa^{\omega}\right)^{2} \rightarrow \kappa^{\omega}$ is a recursive homeomorphism with inverse map $\alpha \mapsto\left((\alpha)_{0},(\alpha)_{1}\right)$ defined for example by $(\alpha)_{\varepsilon}(n):=\alpha(2 n+\varepsilon)$ if $(n, \varepsilon) \in \omega \times 2$ (we will also consider recursive homeomorphisms $\langle., .,\rangle:.\left(\kappa^{\omega}\right)^{3} \rightarrow \kappa^{\omega}$ and $\left.\langle., ., \ldots\rangle:\left(\kappa^{\omega}\right)^{\omega} \rightarrow \kappa^{\omega}\right)$.
- If $u \in \omega$, then $\operatorname{Seq}(u)$ means that there are $l \in \omega$ and $s \in \omega^{l}$ (denoted by $s(u)$ ) such that $u=<s(0), \ldots, s(l-1)>$. The natural number $(u)_{i}$ is $s(i)$ if $i<l$, and 0 otherwise. The number $l$ is the length of $u$ and is denoted by $\operatorname{lh}(u)$. If $k \leq l$, then $\underline{u}(k):=<s(0), \ldots, s(k-1)>$, so that $\underline{u}(l)=u$. The standard basic clopen set is $N^{u}:=\left\{\beta \in 2^{\omega} \mid \forall i<\operatorname{lh}(u) \quad \beta(i)=(u)_{i}\right\}$. We set $u^{-}:=<(u)_{0}, \ldots,(u)_{\operatorname{lh}(u)-2}>\left(u^{-}:=<>\right.$if $\left.\operatorname{lh}(u) \leq 1\right)$.
- Let $X$ be a recursively presented Polish space. Then we will consider the effective basic open set $N(X, u)=B_{X}\left(r_{\left((u)_{1}\right)_{0}}, \frac{\left((u)_{1}\right)_{1}}{\left((u)_{1}\right)_{2}+1}\right)$.
- Let $n \geq 1$ be a natural number. A subset $T$ of $\omega^{n}$ is a tree if $\operatorname{Seq}\left(u_{i}\right)$ and $\operatorname{lh}\left(u_{i}\right)=\operatorname{lh}\left(u_{0}\right)$ for each $\left(u_{0}, \ldots, u_{n-1}\right) \in T$ and each $i<n$, and $\left(\underline{u_{0}}(k), \ldots, \underline{u_{n-1}}(k)\right) \in T$ if $\left(u_{0}, \ldots, u_{n-1}\right) \in T$ and $k \leq \operatorname{lh}\left(u_{0}\right)$.
- The next result is a part of 4A. 1 in [M].

Theorem 3.1 Let $m \geq 1$ be a natural number, and $B \in \Sigma_{1}^{0}\left(\omega^{\omega} \times\left(\omega^{\omega}\right)^{m}\right)$. Then there is a recursive subset $T$ of $\omega^{\omega} \times \omega^{m}$ such that $\left(\alpha, \alpha_{1}, \ldots, \alpha_{m}\right) \in B \Leftrightarrow \exists l \in \omega \quad\left(\alpha, \underline{\alpha_{1}}(l), \ldots, \underline{\alpha_{m}}(l)\right) \notin T$, and $T_{\alpha}:=\left\{\left(u_{0}, \ldots, u_{m-1}\right) \in \omega^{m} \mid\left(\alpha, u_{0}, \ldots, u_{m-1}\right) \in T\right\}$ is a tree for each $\alpha \in \omega^{\omega}$.

- The next result is a part of $4 \mathrm{~A} .7 \mathrm{in}[\mathrm{M}]$.

Theorem 3.2 Let $X$ be a recursively presented Polish space and $B \in \Delta_{1}^{1}(X)$. Then we can find a recursive function $\pi: \omega^{\omega} \rightarrow X$ and $C \in \Pi_{1}^{0}\left(\omega^{\omega}\right)$ such that $\pi$ is injective on $C$ and $\pi[C]=B$.

- We then recall some material from [L].

Notation. Let $X$ be a recursively presented Polish space. Recall that there is a pair $\left(\mathcal{W}^{X}, \mathcal{C}^{X}\right)$ such that

- $\mathcal{W}^{X} \subseteq \omega$ is a $\Pi_{1}^{1}$ set of codes for the $\Delta_{1}^{1}$ subsets of $X$,
- $\mathcal{C}^{X} \subseteq \omega \times X$ is $\Pi_{1}^{1}$ and $\Delta_{1}^{1}(X)=\left\{\mathcal{C}_{n}^{X} \mid n \in \mathcal{W}^{X}\right\}$, which means that $\mathcal{C}^{X}$ is "universal" for the $\Delta_{1}^{1}$ subsets of $X$,
- the relation " $n \in \mathcal{W}^{X} \wedge(n, x) \notin \mathcal{C}^{X}$ " is $\Pi_{1}^{1}$ in $(n, x)$.

If $X=\omega^{\omega} \times 2^{\omega}$, then we simply write $(\mathcal{W}, \mathcal{C}):=\left(\mathcal{W}^{X}, \mathcal{C}^{X}\right)$.
The next result will be extremely useful in the sequel.
The uniformization lemma. Let $X, Y$ be recursively presented Polish spaces, and $P \in \Pi_{1}^{1}(X \times Y)$. Then the set $P^{+}:=\left\{x \in X \mid \exists y \in \Delta_{1}^{1}(x) \quad(x, y) \in P\right\}$ is $\Pi_{1}^{1}$, and there is a partial $\Pi_{1}^{1}$-recursive map $f: X \rightarrow Y$ such that $(x, f(x)) \in P$ for each $x \in P^{+}$. If moreover $S \subseteq P^{+}$is a $\Sigma_{1}^{1}$ subset of $X$, then there is a total $\Delta_{1}^{1}$-recursive map $g: X \rightarrow Y$ such that $(x, g(x)) \in P$ for each $x \in S$.

- The following definition is inspired by 3 H .1 in [M].

Definition 3.3 (a) Let $\Gamma$ be a class of subsets of recursively presented Polish spaces, and $\boldsymbol{\Gamma}$ be the associated boldface class. A system of sets $\mathcal{U}^{X} \in \Gamma\left(\omega^{\omega} \times X\right)$, where is $X$ is a recursively presented Polish space, is a nice parametrization in $\Gamma$ for $\boldsymbol{\Gamma}$ if the following hold:
(1) $\boldsymbol{\Gamma}(X)=\left\{\mathcal{U}_{\alpha}^{X} \mid \alpha \in \omega^{\omega}\right\}$,
(2) $\Gamma(X)=\left\{\mathcal{U}_{\alpha}^{X} \mid \alpha \in \omega^{\omega}\right.$ recursive $\}$,
(3) if $X$ is a recursively presented Polish space, then there is $\mathcal{R}: \omega^{\omega} \times \omega^{\omega} \rightarrow \omega^{\omega}$ recursive such that $(\alpha, \gamma, x) \in \mathcal{U}^{\omega^{\omega} \times X} \Leftrightarrow(\mathcal{R}(\alpha, \gamma), x) \in \mathcal{U}^{X}$ if $(\alpha, \gamma, x) \in \omega^{\omega} \times \omega^{\omega} \times X$.
(b) If $\mathcal{U}$ belongs to a nice parametrization, then we will say that $\mathcal{U}$ is a good universal set .
(c) If $\mathcal{U}$ satisfies all these properties except maybe (3), then we will say that $\mathcal{U}$ is a suitable universal set .

By 3E.2, 3F. 6 and 3 H .1 in $[\mathrm{M}]$, there is a nice parametrization in $\Pi_{n}^{1}$ for $\Pi_{n}^{1}$, for each natural number $n \geq 1$.

- We now recall two results that can essentially be found in [K1]. The first one is Theorem 2.2.3.(a) (see also [T1]).

Theorem 3.4 (Tanaka) Let $U \in \Sigma_{1}^{1}\left(\omega^{\omega} \times \omega^{\omega}\right)$ be $\omega^{\omega}$-universal for the analytic subsets of $\omega^{\omega}$. Then $L(U):=\left\{(\alpha, p) \in \omega^{\omega} \times \omega \left\lvert\, \lambda\left(U_{\alpha} \cap 2^{\omega}\right)>\frac{(p)_{0}}{(p)_{1}+1}\right.\right\}$ is $\Sigma_{1}^{1}$.

Corollary 3.5 Let $B \in \Delta_{1}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$.
(a) The map $\lambda_{B}: \omega^{\omega} \rightarrow \mathbb{R}$ defined by $\lambda_{B}(\alpha):=\lambda\left(B_{\alpha}\right)$ is $\Delta_{1}^{1}$-recursive, and the partial function $(n, \alpha) \mapsto \lambda\left(\mathcal{C}_{n, \alpha}\right)$ is $\Pi_{1}^{1}$-recursive on its domain $\mathcal{W} \times \omega^{\omega}$.
(b) Let $D \subseteq \omega, O_{0} \in \Sigma_{1}^{1}\left(\omega \times \omega^{\omega} \times 2^{\omega}\right)$, and $O_{1} \in \Pi_{1}^{1}\left(\omega \times \omega^{\omega} \times 2^{\omega}\right)$ be such that $\lambda\left(\left(O_{0}\right)_{n, \alpha}\right)=\lambda\left(\left(O_{1}\right)_{n, \alpha}\right)$ if $n \in D$. Then the partial map $\lambda_{O}: D \times \omega^{\omega} \rightarrow \mathbb{R}$ defined by $\lambda_{O}(n, \alpha):=\lambda\left(\left(O_{0}\right)_{n, \alpha}\right)$ is $\Sigma_{1}^{1}$-recursive and $\Pi_{1}^{1}$-recursive on its domain.
(c) The partial map $d_{B}: \omega^{\omega} \times 2^{\omega} \rightarrow \mathbb{R}$ defined by $d_{B}(\alpha, \beta):=d\left(B_{\alpha}, \beta\right)$ is $\Delta_{1}^{1}$-recursive, and the partial map $(n, \alpha, \beta) \mapsto d\left(\mathcal{C}_{n, \alpha}, \beta\right)$ is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain

$$
\left\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^{\omega} \times 2^{\omega} \mid d\left(\mathcal{C}_{n, \alpha}, \beta\right) \text { exists }\right\}
$$

(d) Let $h: \omega^{\omega} \times 2^{\omega} \rightarrow \mathbb{R}$ be $\Delta_{1}^{1}$-recursive taking values in $[0,1]$. Then the partial map $i_{h}: \omega^{\omega} \times \omega \rightarrow \mathbb{R}$ defined by $i_{h}(\alpha, u):=\int_{N^{u}} h(\alpha,). d \lambda$ is $\Delta_{1}^{1}$-recursive on its $\Delta_{1}^{0}$ domain $\omega^{\omega} \times\{u \in \omega \mid \operatorname{Seq}(u)\}$.

Proof. (a) It is enough to see that the relations $P_{B}(\alpha, p) \Leftrightarrow \lambda\left(B_{\alpha}\right)>r_{p}:=(-1)^{(p)_{0}} \cdot \frac{(p)_{1}}{(p)_{2}+1}$ and

$$
Q_{B}(\alpha, p) \Leftrightarrow \lambda\left(B_{\alpha}\right)<r_{p}
$$

are $\Delta_{1}^{1}$ to see that $\lambda_{B}$ is $\Delta_{1}^{1}$-recursive. Note that there is $\phi: \omega^{2} \rightarrow \omega$ recursive with $r_{\phi(p, l)}=r_{p}-\frac{1}{l+1}$. Thus

$$
\begin{aligned}
Q_{B}(\alpha, p) & \Leftrightarrow \exists l \in \omega \lambda\left(B_{\alpha}\right) \leq r_{p}-\frac{1}{l+1} \\
& \Leftrightarrow \exists l \in \omega \neg\left(\lambda\left(B_{\alpha}\right)>r_{p}-\frac{1}{l+1}\right) \\
& \Leftrightarrow \exists l \in \omega \neg P_{B}(\alpha, \phi(p, l)),
\end{aligned}
$$

so that it is enough to see that $P_{B}$ is $\Delta_{1}^{1}$.

- Now let $S \in \Sigma_{1}^{1}\left(\omega^{\omega} \times\left(\omega^{\omega}\right)^{2}\right)$ be a good $\omega^{\omega}$-universal for the analytic subsets of $\left(\omega^{\omega}\right)^{2}$. We set

$$
U(\alpha, \gamma) \Leftrightarrow S\left((\alpha)_{0},(\alpha)_{1}, \gamma\right)
$$

so that $U \in \Sigma_{1}^{1}\left(\omega^{\omega} \times \omega^{\omega}\right)$ is $\omega^{\omega}$-universal for the analytic subsets of $\omega^{\omega}$. Let $A$ be a $\Sigma_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$. Then there is $\alpha_{0} \in \omega^{\omega}$ recursive with $A=S_{\alpha_{0}}$, so that

$$
\gamma \in A_{\alpha} \Leftrightarrow\left(\alpha_{0}, \alpha, \gamma\right) \in S \Leftrightarrow\left(<\alpha_{0}, \alpha>, \gamma\right) \in U .
$$

This implies that the relation $R_{A}(\alpha, p) \Leftrightarrow \lambda\left(A_{\alpha}\right)>r_{p}$, equivalent to

$$
\left((p)_{0} \text { is odd } \wedge(p)_{1}>0\right) \vee\left((p)_{0} \text { is even } \wedge\left(<\alpha_{0}, \alpha>,<(p)_{1},(p)_{2}>\right) \in L(U)\right)
$$

is $\Sigma_{1}^{1}$, by Theorem 3.4

- In particular, this applies to $A:=B$, so that $P_{B}$ is $\Sigma_{1}^{1}$. Now note that

$$
P_{B}(\alpha, p) \Leftrightarrow \lambda\left((\neg B)_{\alpha}\right)<1-r_{p} \Leftrightarrow Q_{\neg B}\left(\alpha, \phi^{\prime}(p)\right),
$$

for some $\phi^{\prime}: \omega \rightarrow \omega$ is recursive, so that $P_{B}$ is $\Pi_{1}^{1}$ by the previous computation.

- We set $\mathcal{C}^{\prime}:=\left\{(\gamma, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \gamma(0) \in \mathcal{W} \wedge\left(\gamma(0), \gamma^{*}, \beta\right) \in \mathcal{C}\right\}$. As $\mathcal{C}^{\prime}$ is $\Pi_{1}^{1}$,

$$
\mathcal{A}:=\left\{(\alpha, p) \in \omega^{\omega} \times \omega \mid \lambda\left(\left(\neg \mathcal{C}^{\prime}\right)_{\alpha}\right)>r_{p}\right\}
$$

is $\Sigma_{1}^{1}$, by the previous discussion. So let $n \in \mathcal{W}$. Note that

$$
\begin{aligned}
\lambda\left(\mathcal{C}_{n, \alpha}\right)>r_{p} & \Leftrightarrow \lambda\left(\neg \mathcal{C}_{n, \alpha}\right)<1-r_{p} \Leftrightarrow \lambda\left(\left(\neg \mathcal{C}^{\prime}\right)_{n \alpha}\right)<1-r_{p} \\
& \Leftrightarrow \exists l \in \omega \lambda\left(\left(\neg \mathcal{C}^{\prime}\right)_{n \alpha}\right) \leq 1-r_{p}-\frac{1}{l+1} \Leftrightarrow \exists l \in \omega\left(n \alpha, \phi^{\prime \prime}(p, l)\right) \notin \mathcal{A},
\end{aligned}
$$

for some recursive $\phi^{\prime \prime}: \omega^{2} \rightarrow \omega$. Similarly, the relation " $\lambda\left(\mathcal{C}_{n, \alpha}\right)<r_{p}$ " is $\Pi_{1}^{1}$ in $(n, \alpha, p)$ since the relation " $n \in \mathcal{W} \wedge(n, \alpha, \beta) \notin \mathcal{C}$ " is $\Pi_{1}^{1}$, so that $(n, \alpha) \mapsto \lambda\left(\mathcal{C}_{n, \alpha}\right)$ is $\Pi_{1}^{1}$-recursive on $\mathcal{W} \times \omega^{\omega}$.
(b) Let $A:=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid\left(\alpha(0), \alpha^{*}, \beta\right) \in O_{0}\right\}$. Note that $A$ is $\Sigma_{1}^{1}$. By (a), the relation $R_{A}(\alpha, p) \Leftrightarrow \lambda\left(A_{\alpha}\right)>r_{p}$ is $\Sigma_{1}^{1}$. Therefore the relation $R_{O_{0}}(n, \alpha, p) \Leftrightarrow R_{A}(n \alpha, p)$ is $\Sigma_{1}^{1}$ too. Moreover, $R_{O_{0}}(n, \alpha, p) \Leftrightarrow \lambda\left(\left(O_{0}\right)_{n, \alpha}\right)>r_{p} \Leftrightarrow \lambda_{O}(n, \alpha)>r_{p}$.

- Assume now that $n \in D$. Then as above there is $\phi^{\prime \prime}: \omega^{2} \rightarrow \omega$ recursive such that

$$
\begin{aligned}
\lambda_{O}(n, \alpha)>r_{p} & \Leftrightarrow \lambda\left(\left(O_{1}\right)_{n, \alpha}\right)>r_{p} \Leftrightarrow \lambda\left(\left(\neg O_{1}\right)_{n, \alpha}\right)<1-r_{p} \\
& \Leftrightarrow \exists l \in \omega \lambda \lambda\left(\left(\neg O_{1}\right)_{n, \alpha}\right) \leq 1-r_{p}-\frac{1}{l+1} \Leftrightarrow \exists l \in \omega \neg\left(\lambda\left(\left(\neg O_{1}\right)_{n, \alpha}\right)>r_{\phi^{\prime \prime}(p, l)}\right) \\
& \Leftrightarrow \exists l \in \omega \neg R_{\neg O_{1}}\left(n, \alpha, \phi^{\prime \prime}(p, l)\right),
\end{aligned}
$$

which shows the existence of $R_{O_{0}}^{\prime} \in \Pi_{1}^{1}$ such that $\lambda_{O}(n, \alpha)>r_{p} \Leftrightarrow R_{O_{0}}^{\prime}(n, \alpha, p)$ if $n \in D$.

- Assume that $n \in D$. Then there is $\phi^{\prime}: \omega \rightarrow \omega$ recursive such that

$$
\lambda_{O}(n, \alpha)<r_{q} \Leftrightarrow \lambda\left(\left(O_{1}\right)_{n, \alpha}\right)<r_{q} \Leftrightarrow \lambda\left(\left(\neg O_{1}\right)_{n, \alpha}\right)>1-r_{q} \Leftrightarrow R_{\neg O_{1}}\left(n, \alpha, \phi^{\prime}(q)\right),
$$

which shows the existence of $R_{O_{0}}^{\prime \prime} \in \Sigma_{1}^{1}$ such that $\lambda_{O}(n, \alpha)<r_{q} \Leftrightarrow R_{O_{0}}^{\prime \prime}(n, \alpha, q)$ if $n \in D$.

- Assume that $n \in D$. Then there is $\phi^{\prime \prime}: \omega^{2} \rightarrow \omega$ recursive such that

$$
\begin{aligned}
\lambda_{O}(n, \alpha)<r_{q} & \Leftrightarrow \lambda\left(\left(O_{0}\right)_{n, \alpha}\right)<r_{q} \Leftrightarrow \exists l \in \omega \lambda\left(\left(O_{0}\right)_{n, \alpha}\right) \leq 1-r_{q}-\frac{1}{l+1} \\
& \Leftrightarrow \exists l \in \omega \neg\left(\lambda\left(\left(O_{0}\right)_{n, \alpha}\right)>r_{\phi^{\prime \prime}(q, l)}\right) \Leftrightarrow \exists l \in \omega \neg R_{O_{0}}\left(n, \alpha, \phi^{\prime \prime}(q, l)\right),
\end{aligned}
$$

which shows the existence of $R_{O_{0}}^{\prime \prime \prime} \in \Pi_{1}^{1}$ such that $\lambda_{O}(n, \alpha)<r_{q} \Leftrightarrow R_{O_{0}}^{\prime \prime \prime}(n, \alpha, q)$ if $n \in D$.

- Finally, $r_{p}<\lambda_{O}(n, \alpha)<r_{q} \Leftrightarrow R_{O_{0}}(n, \alpha, p) \wedge R_{O_{0}}^{\prime \prime}(n, \alpha, q)$ and

$$
r_{p}<\lambda_{O}(n, \alpha)<r_{q} \Leftrightarrow R_{O_{0}}^{\prime}(n, \alpha, p) \wedge R_{O_{0}}^{\prime \prime \prime}(n, \alpha, q)
$$

if $n \in D$, which shows that $\lambda_{O}$ is $\Sigma_{1}^{1}$-recursive and $\Pi_{1}^{1}$-recursive on $D \times \omega$.
(c) We first prove the following. Let $X, Y$ be a recursively presented Polish spaces and $g: X \times \omega \rightarrow Y$ be a $\Delta_{1}^{1}$-recursive map. Then the partial map $h: X \rightarrow Y$ defined by

$$
h(x):=\lim _{l \rightarrow \infty} g(x, l)
$$

when this limit exists is $\Delta_{1}^{1}$-recursive.
Indeed, the domain $D$ of $h$ is $\left\{x \in X \mid \forall r \in \omega \quad \exists L \in \omega \quad \forall k, l \geq L d_{Y}(g(x, k), g(x, l))<2^{-r}\right\}$, so that $D$ is $\Delta_{1}^{1}$. If $x \in D$, then $h(x) \in N(Y, u)$ is equivalent to

$$
\exists p, q \in \omega \frac{p}{q+1}<\frac{\left((u)_{1}\right)_{1}}{\left((u)_{1}\right)_{2}+1} \wedge \exists L \in \omega \forall l \geq L g(x, l) \in N\left(Y,\left\langle 0,<\left((u)_{1}\right)_{0}, p, q>\right\rangle\right)
$$

and we are done.

- We set $B^{\prime}:=\left\{(\alpha, \gamma) \in \omega^{\omega} \times 2^{\omega} \mid\left((\alpha)_{0}, \gamma\right) \in B \wedge \gamma \in N_{(\alpha)_{1}^{*} \mid(\alpha)_{1}(0)}\right\}$, so that $B_{\alpha} \cap N_{\beta \mid l}=B_{<\alpha, l \beta>}^{\prime}$ and $B^{\prime}$ is $\Delta_{1}^{1}$. By (a), the map $g: \omega^{\omega} \times 2^{\omega} \times \omega \rightarrow[0,1]$ defined by $g(\alpha, \beta, l):=2^{-l} \lambda\left(B_{\alpha} \cap N_{\beta \mid l}\right)$ is $\Delta_{1}^{1}$-recursive. By the previous point, the partial map $h: \omega^{\omega} \times 2^{\omega} \rightarrow[0,1]$ defined by

$$
h(\alpha, \beta):=\lim _{l \rightarrow \infty} 2^{-l} \lambda\left(B_{\alpha} \cap N_{\beta \mid l}\right)
$$

when it exists is also $\Delta_{1}^{1}$-recursive. But $h=d_{B}$.

- Fix $n \in \mathcal{W}$. Then there is $q(n) \in \mathcal{W}$ such that

$$
\mathcal{C}_{q(n)}=\left\{(\gamma, \delta) \in \omega^{\omega} \times 2^{\omega}\left|\left(n,(\gamma)_{0}, \delta\right) \in \mathcal{C} \wedge(\gamma)_{1}^{*}\right|(\gamma)_{1}(0) \subseteq \delta\right\}
$$

Moreover, we may assume that $q$ is $\Pi_{1}^{1}$-recursive on $\mathcal{W}$, by the uniformization lemma. As $\Pi_{1}^{1}$ has the substitution property, the map $g^{\prime}:(n, \alpha, \beta, l) \mapsto 2^{-l} \lambda\left(\mathcal{C}_{q(n),<\alpha, l \beta>}\right)=2^{-l} \lambda\left(\mathcal{C}_{n, \alpha} \cap N_{\beta \mid l}\right)$ is $\Pi_{1}^{1}-$ recursive on $\mathcal{W} \times \omega^{\omega} \times 2^{\omega} \times \omega$. As above, the map

$$
h^{\prime}:(n, \alpha, \beta) \mapsto \lim _{l \rightarrow \infty} 2^{-l} \lambda\left(\mathcal{C}_{n, \alpha} \cap N_{\beta \mid l}\right)=d\left(\mathcal{C}_{n, \alpha}, \beta\right)
$$

is $\Pi_{1}^{1}$-recursive on the $\Pi_{1}^{1}$ set $\left\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^{\omega} \times 2^{\omega} \mid d\left(\mathcal{C}_{n, \alpha}, \beta\right)\right.$ exists $\}$.
(d) The argument here is partly similar to 11.6 and 17.25 in [K2]. We set, for $(k, l) \in \omega^{2}$,

$$
A_{k, l}:=h^{-1}\left(\left[\frac{k}{2^{l}}, \frac{k+1}{2^{l}}\right)\right)
$$

and define $h_{l}: \omega^{\omega} \times 2^{\omega} \rightarrow[0,1]$ by $h_{l}=\Sigma_{k \leq 2^{l}} \frac{k}{2^{l}} \chi_{A_{k, l}}$. We also define $R \subseteq \omega^{\omega} \times 2^{\omega} \times \omega^{3}$ by

$$
R(\alpha, \beta, u, k, l) \Leftrightarrow \frac{k}{2^{l}} \leq h(\alpha, \beta)<\frac{k+1}{2^{l}} \wedge \operatorname{Seq}(u) \wedge \beta \in N^{u}
$$

so that $R$ is $\Delta_{1}^{1}$. Then we define $O \subseteq \omega^{\omega} \times 2^{\omega}$ by

$$
O(\alpha, \beta) \Leftrightarrow \operatorname{Seq}(\alpha(0)) \wedge \operatorname{lh}(\alpha(0))=3 \wedge R\left(\alpha^{*}, \beta,(\alpha(0))_{0},(\alpha(0))_{1},(\alpha(0))_{2}\right)
$$

so that $O$ is $\Delta_{1}^{1}$.
Note that $\left(h_{l}\right)$ is a sequence of Borel functions pointwise converging to $h$. By Lebesgue's dominated convergence theorem, $\int_{N^{u}} h(\alpha,). d \lambda=\lim _{l \rightarrow \infty} \int_{N^{u}} h_{l}(\alpha,). d \lambda$ if $\operatorname{Seq}(u)$. Note that

$$
\begin{aligned}
\int_{N^{u}} h_{l}(\alpha, .) d \lambda & =\int_{N^{u}} \Sigma_{k \leq 2^{l}} \frac{k}{2^{l}} \chi_{A_{k, l}}(\alpha, .) d \lambda=\Sigma_{k \leq 2^{l}} \frac{k}{2^{l}} \lambda\left(\left(A_{k, l}\right)_{\alpha} \cap N^{u}\right) \\
& =\Sigma_{k \leq 2^{l}} \frac{k}{2^{l}} \lambda\left(R_{\alpha, u, k, l}\right)=\Sigma_{k \leq 2^{l}} \frac{k}{2^{l}} \lambda\left(O_{<u, k, l>\alpha}\right) .
\end{aligned}
$$

Using (a), this implies that the map $(\alpha, u, l) \mapsto \int_{N^{u}} h_{l}(\alpha,). d \lambda$ is $\Delta_{1}^{1}$-recursive on its $\Delta_{1}^{0}$ domain $\omega^{\omega} \times\{u \in \omega \mid \operatorname{Seq}(u)\} \times \omega$. As in the proof of (c), $i_{h}$ is $\Delta_{1}^{1}$-recursive on its domain.

We now prove a uniform version of Theorem 4.3.2 in [K1] (due to Tanaka, see [T2]).
Theorem 3.6 Let $B \in \Delta_{1}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$, and $\epsilon: \omega^{\omega} \rightarrow \mathbb{R}$ be $\Delta_{1}^{1}$-recursive such that $\epsilon(\alpha) \in(0,1]$ for each $\alpha \in \omega^{\omega}$. Then there is $T \in \Delta_{1}^{1}\left(\omega^{\omega} \times \omega\right)$ such that
(a) $T_{\alpha}$ is a tree for each $\alpha \in \omega^{\omega}$,
(b) if $K=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega \quad(\alpha, \bar{\beta}(l)) \in T\right\}$, then $K_{\alpha} \subseteq B_{\alpha}$ and $\lambda\left(K_{\alpha}\right) \geq \lambda\left(B_{\alpha}\right)-\epsilon(\alpha)$ for each $\alpha \in \omega^{\omega}$.

Proof. Theorem 3.2 gives $\pi: \omega^{\omega} \rightarrow \omega^{\omega} \times 2^{\omega}$ recursive and $C \in \Pi_{1}^{0}\left(\omega^{\omega}\right)$ such that $\pi$ is injective on $C$ and $\pi[C]=B$. We set $Q:=\left\{(\alpha, \beta, \gamma) \in\left(\omega^{\omega}\right)^{3} \mid \gamma \in C \wedge \pi(\gamma)=(\alpha, \beta)\right\}$. As $Q \in \Pi_{1}^{0}$, Theorem 3.1 gives a recursive subset $\bar{T}$ of $\omega^{\omega} \times \omega^{2}$ such that $(\alpha, \beta, \gamma) \in Q \Leftrightarrow \forall l \in \omega(\alpha, \bar{\beta}(l), \bar{\gamma}(l)) \in \bar{T}$ and $\bar{T}_{\alpha}$ is a tree for each $\alpha \in \omega^{\omega}$.

- We set, for $u, v \in \omega$,

$$
u \leq^{a} v \Leftrightarrow \operatorname{Seq}(u), \operatorname{Seq}(v) \wedge \operatorname{lh}(u)=\operatorname{lh}(v) \wedge \forall i<\operatorname{lh}(u)(u)_{i} \leq(v)_{i} .
$$

- Then we set, for $u \in \omega$ with $\operatorname{Seq}(u)$ and $\alpha \in \omega^{\omega}$,

$$
B_{\alpha}^{u}:=\left\{\beta \in 2^{\omega} \mid \exists \gamma \in \omega^{\omega} \bar{\gamma}(\operatorname{lh}(u)) \leq^{a} u \wedge \forall l \in \omega(\alpha, \bar{\beta}(l), \bar{\gamma}(l)) \in \bar{T}\right\}
$$

and $B^{\prime}:=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \operatorname{Seq}(\alpha(0)) \wedge \beta \in B_{\alpha^{*}}^{\alpha(0)}\right\}$. Note that $B^{\prime}$ is $\Sigma_{1}^{1}$. In fact, $B^{\prime}$ is $\Delta_{1}^{1}$ by uniqueness of the witness $\gamma$.

- We now define $\delta_{\alpha} \in \omega^{\omega}$ as follows. We define $\delta_{\alpha}(i)$ by induction on $i$. We first set

$$
\delta_{\alpha}(0):=\min \left\{k \in \omega \left\lvert\, \lambda\left(B_{\alpha}^{<k>}\right)>\lambda\left(B_{\alpha}\right)-\frac{\epsilon(\alpha)}{2}\right.\right\} .
$$

This number exists since $B_{\alpha}$ is the increasing union of the $B_{\alpha}^{<k>}$ 's. Then

$$
\delta_{\alpha}(i+1):=\min \left\{k \in \omega \left\lvert\, \lambda\left(B_{\alpha}^{<\delta_{\alpha}(0), \ldots, \delta_{\alpha}(i), k>}\right)>\lambda\left(B_{\alpha}\right)-\frac{\epsilon(\alpha)}{2}-\ldots-\frac{\epsilon(\alpha)}{2^{i+2}}\right.\right\}
$$

Note that $\delta_{\alpha} \in \Delta_{1}^{1}(\alpha)$, by Corollary 3.5(a).

- We set $T:=\left\{(\alpha, v) \in \omega^{\omega} \times \omega \mid \operatorname{Seq}(v) \wedge \exists u \leq^{a} \overline{\delta_{\alpha}}(\operatorname{lh}(v))(\alpha, v, u) \in \bar{T}\right\}$, so that $T \in \Delta_{1}^{1}\left(\omega^{\omega} \times \omega\right)$ and $T_{\alpha}$ is a tree for each $\alpha \in \omega^{\omega}$.
- We set $K:=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega \quad \beta \in B_{\alpha}^{\overline{\delta_{\alpha}}(l)}\right\}$, so that $K_{\alpha} \subseteq B_{\alpha}$ and

$$
\lambda\left(K_{\alpha}\right)=\lim _{l \rightarrow \infty} \lambda\left(B_{\alpha}^{\overline{\delta_{\alpha}}(l)}\right) \geq \lambda\left(B_{\alpha}\right)-\epsilon(\alpha)
$$

for each $\alpha \in \omega^{\omega}$ since $\left(B_{\alpha}^{\overline{\delta_{\alpha}}(l)}\right)_{l \in \omega}$ is decreasing. It remains to apply König's lemma to see that $K=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega(\alpha, \bar{\beta}(l)) \in T\right\}$ since

$$
\left\{s \in \omega^{<\omega} \mid<s(0), \ldots, s(|s|-1)>\leq^{a} \overline{\delta_{\alpha}}(|s|) \wedge(\alpha, \bar{\beta}(|s|),<s(0), \ldots, s(|s|-1)>) \in \bar{T}\right\}
$$

is a finitely splitting tree.

- We want to prove an effective and uniform version of the Lusin-Menchoff lemma. We first need the following result, which slightly and uniformly refines Theorem A in [L] at the first level of the Borel hierarchy.

Lemma 3.7 Let O be a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$ with open vertical sections. Then there is a $\Delta_{1}^{1}$-recursive map $f: \omega^{\omega} \rightarrow \omega^{\omega}$ such that $O_{\alpha}$ is the disjoint union $\bigcup\left\{N^{f(\alpha)(u)} \mid u \in \omega \wedge \operatorname{Seq}(f(\alpha)(u))\right\}$, for each $\alpha \in \omega^{\omega}$.

Proof. Let $P:=\left\{(\alpha, u) \in \omega^{\omega} \times \omega \mid \operatorname{Seq}(u) \wedge\left(\operatorname{lh}(u)=0 \vee\left(N^{u} \subseteq O_{\alpha} \wedge N^{u^{-}} \nsubseteq O_{\alpha}\right)\right)\right\}$. Note that $P$ is $\Pi_{1}^{1}$, since a nonempty $\Delta_{1}^{1}(\alpha)$ closed subset of $2^{\omega}$ contains a $\Delta_{1}^{1}(\alpha)$ point, by 4 F .15 in [M]. We then define a relation $R$ on $\omega^{\omega} \times 2^{\omega} \times \omega$ by $R(\alpha, \beta, u) \Leftrightarrow P(\alpha, u) \wedge \beta \in N^{u}$, so that $R$ is $\Pi_{1}^{1}$. Note that, for each $(\alpha, \beta) \in O$ there is $u$ with $R(\alpha, \beta, u)$. By 4B. 5 in [M], there is a $\Delta_{1}^{1}$-recursive map $g: \omega^{\omega} \times 2^{\omega} \rightarrow \omega$ such that $R(\alpha, \beta, g(\alpha, \beta))$ for each $(\alpha, \beta) \in O$. Fix $\alpha \in \omega^{\omega}$. Note that $S^{\alpha}:=\left\{g(\alpha, \beta) \mid \beta \in O_{\alpha}\right\}$ is a $\Sigma_{1}^{1}(\alpha)$ subset of $\omega$ contained in the $\Pi_{1}^{1}(\alpha)$ set $P_{\alpha}$. By 4B. 11 and 4 C in [M], there is $D^{\alpha} \in \Delta_{1}^{1}(\alpha)$ with $S^{\alpha} \subseteq D^{\alpha} \subseteq P_{\alpha}$. Note that $O_{\alpha} \subseteq \bigcup_{u \in D^{\alpha}} N^{u} \subseteq O_{\alpha}$, so that $O_{\alpha}$ is the disjoint union of the sequence $\left(N^{u}\right)_{u \in D^{\alpha}}$. We define $\delta_{\alpha} \in \omega^{\omega}$ by

$$
\delta_{\alpha}(u):=\left\{\begin{array}{l}
u \text { if } u \in D_{\alpha}, \\
0 \text { otherwise. }
\end{array}\right.
$$

Note that $\delta_{\alpha} \in \Delta_{1}^{1}(\alpha)$ and $O_{\alpha}$ is the disjoint union $\bigcup\left\{N^{\delta_{\alpha}(u)} \mid u \in \omega \wedge \operatorname{Seq}\left(\delta_{\alpha}(u)\right)\right\}$. As the set $\left\{(\alpha, \delta) \in \omega^{\omega} \times \omega^{\omega} \mid \delta \in \Delta_{1}^{1}(\alpha) \wedge O_{\alpha}\right.$ is the disjoint union $\left.\bigcup\left\{N^{\delta(u)} \mid u \in \omega \wedge \operatorname{Seq}(\delta(u))\right\}\right\}$ is $\Pi_{1}^{1}$, it remains to apply the uniformization lemma to get the desired map $f$.

Notation. We set $\mathcal{W}_{1}:=\left\{n \in \mathcal{W} \mid \forall \alpha \in \omega^{\omega} \exists \gamma_{n} \in \Delta_{1}^{1}(\alpha) \mathcal{C}_{n, \alpha}=\bigcup\left\{N^{\gamma_{n}(u)} \mid u \in \omega \wedge \operatorname{Seq}\left(\gamma_{n}(u)\right)\right\}\right.$, so that, by Lemma 3.7, $\mathcal{W}_{1}$ is a $\Pi_{1}^{1}$ set of codes for the $\Delta_{1}^{1}$ subsets of $\omega^{\omega} \times 2^{\omega}$ with open vertical sections.

Lemma 3.8 Let $F$ be a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$ with closed vertical sections, and $B$ be a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$ such that $B \supseteq F$ and $d\left(B_{\alpha}, \beta\right)=1$ for each $(\alpha, \beta) \in F$. Then there is a $\Delta_{1}^{1}$ subset $C$ of $\omega^{\omega} \times 2^{\omega}$ with closed vertical sections such that
(1) $F \subseteq C \subseteq B$,
(2) $d\left(B_{\alpha}, \beta\right)=1$ for each $(\alpha, \beta) \in C$,
(3) $d\left(C_{\alpha}, \beta\right)=1$ for each $(\alpha, \beta) \in F$.

Proof. Lemma 3.7 gives a $\Delta_{1}^{1}$-recursive map $f: \omega^{\omega} \rightarrow \omega^{\omega}$ such that $(\neg F)_{\alpha}$ is the disjoint union $\bigcup\left\{N^{f(\alpha)(u)} \mid u \in \omega \wedge \operatorname{Seq}(f(\alpha)(u))\right\}$, for each $\alpha \in \omega^{\omega}$. We set

$$
B^{\prime}:=\left\{(\alpha, \gamma) \in \omega^{\omega} \times 2^{\omega} \mid\left((\alpha)_{0}, \gamma\right) \in B \wedge \operatorname{Seq}\left(f\left((\alpha)_{0}\right)\left((\alpha)_{1}(0)\right)\right) \wedge \gamma \in N^{f\left((\alpha)_{0}\right)\left((\alpha)_{1}(0)\right)}\right\}
$$

so that $B^{\prime}$ is $\Delta_{1}^{1}$ and $B_{\alpha} \cap N^{f(\alpha)(u)}=B_{\left.<\alpha, u^{\infty}\right\rangle}^{\prime}$ if $\operatorname{Seq}(f(\alpha)(u))$. By Corollary 3.5.(c), the partial map $(\alpha, \beta, u) \mapsto d\left(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta\right)$ is $\Delta_{1}^{1}$-recursive. We then set

$$
B^{\prime \prime}:=\left\{(\alpha, \gamma) \in B^{\prime} \mid d\left(B_{(\alpha)_{0}} \cap N^{f\left((\alpha)_{0}\right)\left((\alpha)_{1}(0)\right)}, \gamma\right)=1\right\},
$$

so that $B^{\prime \prime}$ is $\Delta_{1}^{1}$ and $\left\{\beta \in B_{\alpha} \cap N^{f(\alpha)(u)} \mid d\left(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta\right)=1\right\}=B_{\left.<\alpha, u^{\infty}\right\rangle}^{\prime \prime}$ if $\operatorname{Seq}(f(\alpha)(u))$. We define $\epsilon: \omega^{\omega} \rightarrow \mathbb{R}$ by

$$
\varepsilon(\alpha):=\left\{\begin{array}{l}
2^{-(\alpha)_{1}(0)} \lambda\left(B_{\alpha}^{\prime}\right) \text { if } \lambda\left(B_{\alpha}^{\prime}\right) \neq 0, \\
1 \text { otherwise }
\end{array}\right.
$$

so that $\epsilon$ is $\Delta_{1}^{1}$-recursive by Corollary 3.5(a), and $\epsilon(\alpha) \in(0,1]$ for each $\alpha \in \omega^{\omega}$. Theorem 3.6 gives $T \in \Delta_{1}^{1}\left(\omega^{\omega} \times \omega\right)$ such that
(a) $T_{\alpha}$ is a tree for each $\alpha \in \omega^{\omega}$,
(b) if $K=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega \quad(\alpha, \bar{\beta}(l)) \in T\right\}$, then $K_{\alpha} \subseteq B_{\alpha}^{\prime \prime}$ and $\lambda\left(K_{\alpha}\right) \geq \lambda\left(B_{\alpha}^{\prime \prime}\right)-\epsilon(\alpha)$ for each $\alpha \in \omega^{\omega}$.

We set, for $u \in \omega$,

$$
F^{u}:=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \operatorname{Seq}(f(\alpha)(u)) \wedge\left(<\alpha, u^{\infty}>, \beta\right) \in K \wedge \lambda\left(B_{\left.<\alpha, u^{\infty}\right\rangle}^{\prime}\right) \neq 0\right\}
$$

As $K$ is $\Delta_{1}^{1}$ with closed vertical sections, so is $F^{u}$. If $\operatorname{Seq}(f(\alpha)(u))$ and $\lambda\left(B_{\left.<\alpha, u^{\infty}\right\rangle}^{\prime}\right)=0$, then $\lambda\left(B_{\alpha} \cap N^{f(\alpha)(u)}\right)=0$ and $F_{\alpha}^{u}=\emptyset$, so that $F_{\alpha}^{u} \subseteq\left\{\beta \in B_{\alpha} \cap N^{f(\alpha)(u)} \mid d\left(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta\right)=1\right\}$ and $\lambda\left(F_{\alpha}^{u}\right) \geq\left(1-2^{-u}\right) \lambda\left(B_{\alpha} \cap N^{f(\alpha)(u)}\right)$. If $\operatorname{Seq}(f(\alpha)(u))$ and $\lambda\left(B_{\left.<\alpha, u^{\infty}\right\rangle}^{\prime}\right) \neq 0$, then

$$
F_{\alpha}^{u}=K_{\left.<\alpha, u^{\infty}\right\rangle} \subseteq B_{\left.<\alpha, u^{\infty}\right\rangle}^{\prime \prime}=\left\{\beta \in B_{\alpha} \cap N^{f(\alpha)(u)} \mid d\left(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta\right)=1\right\}
$$

Moreover,

$$
\begin{aligned}
\lambda\left(F_{\alpha}^{u}\right) & =\lambda\left(K_{<\alpha, u^{\infty}>}\right) \geq \lambda\left(B_{<\alpha, u^{\infty}>}^{\prime \prime}\right)-\epsilon\left(<\alpha, u^{\infty}>\right)=\lambda\left(B_{<\alpha, u^{\infty}>}^{\prime \prime}\right)-2^{-u} \lambda\left(B_{<\alpha, u^{\infty}>}^{\prime}\right) \\
& =\left(1-2^{-u}\right) \lambda\left(B_{\alpha} \cap N^{f(\alpha)(u)}\right)
\end{aligned}
$$

since $\lambda\left(B_{\alpha} \cap N^{f(\alpha)(u)}\right)=\lambda\left(\left\{\beta \in B_{\alpha} \cap N^{f(\alpha)(u)} \mid d\left(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta\right)=1\right\}\right)$, by Theorem 2.1 It remains to set $C:=F \cup \bigcup_{u \in \omega} F^{u}$. We conclude as in the proof of Lemma2.2,

- We now want to prove an effective and uniform version of Lemma 2.5

Lemma 3.9 Let $C$ be a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$ with closed vertical sections, $\mathcal{G}$ be a Borel subset of $2^{\omega}$ with $\lambda(\mathcal{G})=0$, and $G$ be a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$ with $G_{\delta}$ vertical sections, contained in $\omega^{\omega} \times \mathcal{G}$ and disjoint from $C$. Then there is a $\Delta_{1}^{1}$-recursive map $h: \omega^{\omega} \times 2^{\omega} \rightarrow \mathbb{R}$ such that $h(\alpha, \cdot): 2^{\omega} \rightarrow[0,1]$ is $\tau$-continuous for each $\alpha \in \omega^{\omega}, h_{\mid C} \equiv 0$ and $h_{\mid G} \equiv 1$.

Proof. By Theorem 3.5 in [L], there is a $\Delta_{1}^{1}$ subset $F$ of $\omega \times \omega^{\omega} \times 2^{\omega}$ such that $F_{n, \alpha}$ is closed for each $(n, \alpha) \in \omega \times \omega^{\omega}$ and $\neg G=\bigcup_{n \in \omega} F_{n}$. Moreover, we may assume that $\left(F_{n}\right)_{n \in \omega}$ is increasing and $F_{0}=C$.

- We will define, by primitive recursion, a partial map $f: \omega \rightarrow \omega$ which is $\Pi_{1}^{1}$-recursive on its domain such that $f(n)$ essentially codes the set $C_{\frac{1}{2^{n}}}$ constructed in the proof of Lemma 2.5. As this map will in fact be total, it will be $\Delta_{1}^{1}$-recursive by the uniformization lemma.

We first apply Lemma 3.8 to $F:=F_{0}$ and $B:=\neg G$. This is possible because $G_{\alpha} \subseteq \mathcal{G}$, so that $(\neg G)_{\alpha}$ has $\lambda$-measure one and therefore density one at any point of $2^{\omega}$, for each $\alpha \in \omega^{\omega}$. Lemma 3.8 gives $C_{1} \in \Delta_{1}^{1}$ with closed vertical sections such that $\neg G \supseteq C_{1} \supseteq F_{0}$. Let $f(0) \in \mathcal{W}_{1}$ with $\mathcal{C}_{f(0)}=\neg C_{1}$.

More generally, we will have $\mathcal{C}_{f(n)}=\neg C_{\frac{1}{2^{n}}}$. As mentioned above, $f$ will be defined by primitive recursion, which means that there will be a partial map $g: \omega^{2} \rightarrow \omega$ such that $f(n+1)=g(f(n), n)$. This partial map $g$ will be $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain $\left\{m \in \mathcal{W}_{1} \mid \neg \mathcal{C}_{m} \subseteq \neg G\right\} \times \omega$, so that $f$ will be $\Pi_{1}^{1}$-recursive on its domain by 7 A .5 in [M]. The map $g$ will take values in $\mathcal{W}_{1}$, and is constructed in such a way that, if $A:=\neg \mathcal{C}_{m} \subseteq \neg G$ and $A^{\prime}:=\neg \mathcal{C}_{g(m, n)}$, then
(1) $A \cup F_{n+1} \subseteq A^{\prime} \subseteq \neg G$,
(2) $\forall(\alpha, \beta) \in A^{\prime} \quad d\left((\neg G)_{\alpha}, \beta\right)=1$,
(3) $\forall(\alpha, \beta) \in A \cup F_{n+1} d\left(A_{\alpha}^{\prime}, \beta\right)=1$.

Lemma 3.8 ensures that such a $g(m, n) \in \omega$ exists if $(m, n) \in\left\{q \in \mathcal{W}_{1} \mid \neg \mathcal{C}_{q} \subseteq \neg G\right\} \times \omega$. As the properties (1)-(3) are $\Pi_{1}^{1}$ by Corollary 3.5, the uniformization lemma ensures the existence of $g$. So we constructed a $\Delta_{1}^{1}$-recursive map $f: \omega \rightarrow \omega$, taking values in $\mathcal{W}_{1}$, such that $C_{\frac{1}{2^{n}}}:=\neg \mathcal{C}_{f(n)}$ is a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$ with closed vertical sections, $F_{n} \subseteq C_{\frac{1}{2^{n}}} \subseteq \neg G, C_{\frac{1}{2^{n}}}^{\subseteq} C_{\frac{1}{2^{n+1}}}$, and

$$
d\left(\left(C_{\frac{1}{2^{n+1}}}\right)_{\alpha}, \beta\right)=1
$$

if $(\alpha, \beta) \in C_{\frac{1}{2^{n}}}$.

- Similarly, we construct a $\Delta_{1}^{1}$-recursive map $\tilde{F}: \omega \rightarrow \omega$ satisfying the following properties, if

$$
D:=\left\{p \in \omega \mid \operatorname{Seq}(p) \wedge \operatorname{lh}(p)=2 \wedge 0<(p)_{1} \leq 2^{(p)_{0}}\right\} .
$$

(a) $\tilde{F}(p) \in \mathcal{W}_{1}$ if $p \in D$, in which case we set $C_{p}:=\neg \mathcal{C}_{\tilde{F}(p)}$,

(c) $d\left(\left(C_{p^{\prime}}\right)_{\alpha}, \beta\right)=1$ if $p, p^{\prime} \in D \wedge \frac{\left(p^{\prime}\right)_{1}}{2^{\left(p^{\prime}\right)_{0}}}<\frac{(p)_{1}}{\left.2^{(p)}\right)_{0}} \wedge(\alpha, \beta) \in C_{p}$.

- This allows us to define $h$ by

$$
1-h(\alpha, \beta):=\left\{\begin{array}{l}
0 \text { if }(\alpha, \beta) \in G, \\
\sup \left\{\left.\frac{(p)_{1}}{2^{(p)}} \right\rvert\, p \in D \wedge(\alpha, \beta) \in C_{p}\right\} \text { if }(\alpha, \beta) \notin G .
\end{array}\right.
$$

Note that $h$ is $\Delta_{1}^{1}$-recursive since $D \in \Delta_{1}^{0}$, so that the relation " $p \in D \wedge(\alpha, \beta) \in C_{p}$ " is $\Delta_{1}^{1}$ in $(p, \alpha, \beta)$. We conclude as in the proof of Lemma 2.5

- We are now ready to prove the main lemma in this section. We equip the space $[0,1]^{2<\omega}$ with the distance defined by $d(f, g):=\Sigma_{u \in \omega, \operatorname{Seq}(u)} \frac{|f(s(u))-g(s(u))|}{2^{u+1}}$. We give a recursive presentation of $\left([0,1]^{2<\omega}, d\right)$. We set
$f_{n}(s):=\left\{\begin{array}{l}\frac{\left((n)_{\bar{s}}\right)_{0}}{((n))_{\bar{s}}+\left((n)_{\bar{s}}\right)+1} \text { if } \operatorname{Seq}(n) \wedge \forall k<\operatorname{lh}(n)\left(\operatorname{Seq}\left((n)_{k}\right) \wedge \operatorname{lh}\left((n)_{k}\right)=2\right) \wedge \bar{s}<\operatorname{lh}(n), \\ 0 \text { otherwise },\end{array}\right.$
so that $\left(f_{n}\right)$ is dense in $[0,1]^{2^{<\omega}}$. It is now routine to check that the relations " $d\left(f_{m}, f_{n}\right) \leq \frac{p}{q+1}$ " and " $d\left(f_{m}, f_{n}\right)<\frac{p}{q+1}$ " are recursive in $(m, n, p, q)$. It is also routine to check that $F: \omega^{\omega} \rightarrow[0,1]^{2<\omega}$ is $\Delta_{1}^{1}$-recursive if the map $F^{\prime}: \omega \times \omega^{\omega} \rightarrow \mathbb{R}$ defined by $F^{\prime}(u, \alpha):=F(\alpha)(s(u))$ if $\operatorname{Seq}(u), 0$ otherwise, is $\Delta_{1}^{1}$-recursive ( $s(u)$ was defined at the beginning of Section 3).

Lemma 3.10 Let $\mathcal{V}:=\left\{(f, \beta) \in \mathcal{M} \times 2^{\omega} \mid \operatorname{osc}(f, \beta)>0\right\}$, $\mathcal{G}$ be a nonempty $G_{\delta} \cap \Delta_{1}^{1}$ subset of $2^{\omega}$ with $\lambda(\mathcal{G})=0$, and $G$ be a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$, contained in $\omega^{\omega} \times \mathcal{G}$, and with $G_{\delta}$ vertical sections. Then there is a $\Delta_{1}^{1}$-recursive map $F: \omega^{\omega} \rightarrow[0,1]^{2^{<\omega}}$, taking values in $\mathcal{M}$, and such that $G_{\alpha}=\mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^{\omega}$.

Proof. We will define, by primitive recursion, $f: \omega \rightarrow \omega^{4} \operatorname{coding} g_{n}, S_{n}, G_{n}^{*}$, and $\left(s_{j}^{n}\right)_{j \in I_{n}}$ defining $G_{n}^{* *}$ considered in the proof of the Lemma2.7 We must find $r: \omega^{4} \times \omega \rightarrow \omega^{4}$ with $f(n+1)=r(f(n), n)$. In practice,
(1) $f_{0}(n) \in \mathcal{W}_{1}$ codes $G_{n}^{*} \subseteq \omega^{\omega} \times 2^{\omega}$,
(2) $f_{1}(n) \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ codes the graph of $g_{n}: \omega^{\omega} \times 2^{\omega} \rightarrow \mathbb{R}$,
(3) $f_{2}(n) \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ codes the graph of $S_{n}: \omega^{\omega} \times 2^{\omega} \rightarrow \mathbb{R}$,
(4) $f_{3}(n) \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ codes the graph of the function $\alpha \mapsto\left(s_{j}^{n, \alpha}\right)_{j \in I_{n, \alpha}}$.

- By Theorem 3.5 in [L], there is a $\Delta_{1}^{1}$ subset $O$ of $\omega \times \omega^{\omega} \times 2^{\omega}$ such that $O_{n, \alpha}$ is open for each $(n, \alpha) \in \omega \times \omega^{\omega}$ and $G=\bigcap_{n \in \omega} O_{n}$. Moreover, we may assume that $\left(O_{n}\right)_{n \in \omega}$ is decreasing and $O_{0}=\omega^{\omega} \times 2^{\omega}$.
$\bullet$ Let $n_{0} \in \mathcal{W}_{1}$ with $\mathcal{C}_{n_{0}}=\omega^{\omega} \times 2^{\omega}, n_{1} \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ with $\mathcal{C}_{n_{1}}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}=\left\{(\alpha, \beta, r) \in \omega^{\omega} \times 2^{\omega} \times \mathbb{R} \mid r=1\right\}$, and $n_{3} \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ with $\mathcal{C}_{n 3}^{\omega \omega} \times \omega^{\omega}=\left\{(\alpha, \gamma) \in \omega^{\omega} \times \omega^{\omega} \mid \gamma=10^{\infty}\right\}$. We set $f(0):=\left(n_{0}, n_{1}, n_{1}, n_{3}\right)$, so that $\mathcal{C}_{n_{0}}=G_{0}^{*}, \mathcal{C}_{n_{1}}^{\omega^{\omega}} \times 2^{\omega} \times \mathbb{R}=\operatorname{Gr}\left(g_{0}\right)=\operatorname{Gr}\left(S_{0}\right), \mathcal{C}_{n_{3}}^{\omega^{\omega}} \times \omega^{\omega}=\operatorname{Gr}\left(\alpha \mapsto 10^{\infty}\right)$,

$$
\left\{u \in \omega \mid \operatorname{Seq}\left(\left(10^{\infty}\right)(u)\right)\right\}=\{0\}=I_{0}
$$

and $\left(10^{\infty}\right)(0)=1=<>=s_{0}^{0}$. So $f(0)$ is as desired.

- We now study the induction step. This means that we must define $r\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right) \in \omega^{4}$.
(1) We first define $r_{0}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right)$ coding $G_{n+1}^{*}$. Fix $n_{3} \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ coding the graph of a $\Delta_{1}^{1}$-recursive function $\phi: \omega^{\omega} \rightarrow \omega^{\omega}$ such that the sequences $s(\phi(\alpha)(u))$ coded by the $u$ 's with $\operatorname{Seq}(\phi(\alpha)(u))$ are pairwise incompatible and $G_{\alpha} \subseteq \bigcup\left\{N^{\phi(\alpha)(u)} \mid u \in \omega \wedge \operatorname{Seq}(\phi(\alpha)(u))\right\}$ (we call $P_{3}$ the $\Pi_{1}^{1}$ set of such $n_{3}$ 's). Let $\alpha \in \omega^{\omega}$. Assume that $\operatorname{Seq}(\phi(\alpha)(u))$ (which intuitively means that $u \in I_{n, \alpha}$ and $s_{u}^{n, \alpha}$ is coded by $\left.\phi(\alpha)(u)\right)$. By continuity of $\lambda$,

$$
0=\lambda\left(G_{\alpha} \cap N^{\phi(\alpha)(u)}\right)=\lim _{j \rightarrow \infty} \lambda\left(O_{j, \alpha} \cap N^{\phi(\alpha)(u)}\right)
$$

This gives $j(n, \alpha, u)>n$ minimal with $\lambda\left(O_{j(n, \alpha, u), \alpha} \cap N^{\phi(\alpha)(u)}\right)<2^{-n-3-\operatorname{lh}(\phi(\alpha)(u))}$ (note that $2^{-\operatorname{lh}(\phi(\alpha)(u))}=\lambda\left(N^{\phi(\alpha)(u)}\right)$ ). Moreover, $G_{\alpha} \cap N^{\phi(\alpha)(u)} \subseteq O_{j(n, \alpha, u), \alpha} \cap N^{\phi(\alpha)(u)} \subseteq O_{n+1, \alpha} \cap N^{\phi(\alpha)(u)}$, so that $O_{j(n, \alpha, u), \alpha} \cap N^{\phi(\alpha)(u)}$ satisfies the properties of the set $O_{j}$ in the proof of Lemma 2.7 We will have $G_{n+1, \alpha}^{*}=\bigcup_{\operatorname{Seq}(\phi(\alpha)(u))} O_{j(n, \alpha, u), \alpha} \cap N^{\phi(\alpha)(u)}$. By Corollary 3.5 and the uniformization lemma, we may assume that the map $j$ is $\Delta_{1}^{1}$-recursive on its $\Delta_{1}^{1}$ domain

$$
\left\{(n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid \operatorname{Seq}(\phi(\alpha)(u))\right\} .
$$

Note that $G_{n+1}^{*}$ is a $\Delta_{1}^{1}$ subset of $\omega^{\omega} \times 2^{\omega}$ with open vertical sections, which gives $m \in \mathcal{W}_{1}$ such that $\mathcal{C}_{m}=G_{n+1}^{*}$. By incompatibility, $G_{n+1, \alpha}^{*} \cap N^{\phi(\alpha)(u)}=O_{j(n, \alpha, u), \alpha} \cap N^{\phi(\alpha)(u)}$. So we proved that, for each $\left(n_{3}, n\right) \in P_{3} \times \omega$, there is $m \in \mathcal{W}_{1}$ such that, for each $\alpha \in \omega^{\omega}$,

$$
\begin{aligned}
& \text { (1) } G_{\alpha} \subseteq \mathcal{C}_{m, \alpha} \subseteq O_{n+1, \alpha} \cap \bigcup\left\{N^{\phi(\alpha)(u)} \mid u \in \omega \wedge \operatorname{Seq}(\phi(\alpha)(u))\right\} \text {, } \\
& \text { (5) } \lambda\left(\mathcal{C}_{m, \alpha} \cap N^{\phi(\alpha)(u)}\right)<2^{-n-3-\ln (\phi(\alpha)(u))} \text { if } u \in \omega \wedge \operatorname{Seq}(\phi(\alpha)(u)) .
\end{aligned}
$$

By Corollary 3.5 and the uniformization lemma, we may assume that the map $\tilde{r}_{0}:\left(n_{3}, n\right) \mapsto m$ is $\Pi_{1}^{1}$-recursive on $P_{3} \times \omega$. We set $r_{0}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right):=\tilde{r}_{0}\left(n_{3}, n\right)$, which defines a partial map $r_{0}$ which is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain $\omega^{3} \times P_{3} \times \omega$.
(2) We now define $r_{1}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right)$ coding $g_{n+1}$. We use Lemma 3.9 and its proof. Note that $r_{0}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right) \in D_{0}:=\left\{m \in \mathcal{W}_{1} \mid G \subseteq \mathcal{C}_{m}\right\}$. The proof of Lemma 3.9 shows that for any $m \in D_{0}$ there is $\tilde{F}_{m} \in \omega^{\omega} \cap \Delta_{1}^{1}$ satisfying the conditions (a), (b), (c) and

$$
\text { (d) } \forall p \in D \neg\left(0<(p)_{1}=2^{(p)_{0}}\right) \vee \mathcal{C}_{\tilde{F}_{m}(p)} \subseteq \mathcal{C}_{m}
$$

The uniformization lemma shows that we may assume that the partial map $\tilde{F}: m \mapsto \tilde{F}_{m}$ is $\Pi_{1}^{1}$ recursive on $D_{0}$.

The definition of $h$ in the proof of Lemma 3.9 and the uniformization lemma show the existence of a partial map $\tilde{H}: \omega \rightarrow \omega$, which is $\Pi_{1}^{1}$-recursive on $D_{0}$, and such that $\tilde{H}(m)$ is in $\mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ and codes the graph of a $\Delta_{1}^{1}$-recurive map $h: \omega^{\omega} \times 2^{\omega} \rightarrow \mathbb{R}$ with

$$
1-h(\alpha, \beta):=\left\{\begin{array}{l}
0 \text { if }(\alpha, \beta) \in G \\
\sup \left\{\left.\frac{(p)_{1}}{2^{(p)_{0}}} \right\rvert\, p \in D \wedge(\alpha, \beta) \notin \mathcal{C}_{\tilde{F}(m)(p)}\right\} \text { if }(\alpha, \beta) \notin G
\end{array}\right.
$$

if $m \in D_{0}$. We set $P_{1}:=\left\{c \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} \mid \mathcal{C}_{c}\right.$ is the graph of a function $\left.\zeta_{c}\right\}$. It is routine to check that there is a $\Pi_{1}^{1}$-recursive partial map $I: \omega^{2} \rightarrow \omega$ on its domain $P_{1}^{2}$ such that $I\left(c, c^{\prime}\right) \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ is the graph of the function $\min \left(\zeta_{c}, \zeta_{c^{\prime}}\right)$ if $c, c^{\prime} \in P_{1}$. We set

$$
r_{1}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right):=I\left(n_{1}, \tilde{H}\left(r_{0}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right)\right)\right)
$$

so that $r_{1}$ is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain $\omega \times P_{1} \times \omega \times P_{3} \times \omega$.
(3) We now define $r_{2}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right)$ coding

$$
S_{n+1}=\left\{\begin{array}{l}
S_{n}+g_{n+1} \text { if } n \text { is odd } \\
S_{n}-g_{n+1} \text { if } n \text { is even }
\end{array}\right.
$$

It is routine to check that there is a $\Pi_{1}^{1}$-recursive partial map $S: \omega^{3} \rightarrow \omega$ on its domain $P_{1}^{2} \times \omega$ such that $S\left(c, c^{\prime}, n\right) \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ codes the graph of the function

$$
(\alpha, \beta) \mapsto\left\{\left\{\begin{array}{l}
\zeta_{c}(\alpha, \beta)+\zeta_{c^{\prime}}(\alpha, \beta) \text { if } n \text { is odd } \\
\zeta_{c}(\alpha, \beta)-\zeta_{c^{\prime}}(\alpha, \beta) \text { if } n \text { is even }
\end{array}\right.\right.
$$

if $\left(c, c^{\prime}, n\right) \in P_{1}^{2} \times \omega$. We set $r_{2}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right):=S\left(n_{2}, r_{1}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right), n\right)$, so that $r_{2}$ is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain $\omega \times P_{1}^{2} \times P_{3} \times \omega$.
(4) We now define $r_{3}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right)$ coding the graph of the function $\alpha \mapsto\left(s_{j}^{n+1, \alpha}\right)_{j \in I_{n+1, \alpha}}$. We want to ensure the two following conditions:
(1) $G_{\alpha} \subseteq \bigcup_{j \in I_{n+1, \alpha}} N_{s_{j}^{n+1, \alpha}} \subseteq G_{n+1, \alpha}^{*}$
(6) $\left|f_{N_{s_{j}^{n+1, \alpha}}} S_{n+1}(\alpha,). d \lambda-S_{n+1}(\alpha, \beta)\right|<2^{-3}$ if $j \in I_{n+1, \alpha} \wedge \beta \in G_{\alpha} \cap N_{s_{j}^{n+1, \alpha}}$

Note first that in practice

$$
S_{n+1}(\alpha, \beta)=\left\{\begin{array}{l}
0 \text { if } n \text { is even } \\
1 \text { if } n \text { is odd }
\end{array}\right.
$$

if $(\alpha, \beta) \in G$ since $g_{p}(\alpha, \beta)=1$ for each $p$ in this case. So there is $\psi: \omega \rightarrow \mathbb{R}^{2}$ recursive with

$$
\left|f_{N_{s_{j}^{n+1, \alpha}}} S_{n+1}(\alpha, .) d \lambda-S_{n+1}(\alpha, \beta)\right|<2^{-3} \Leftrightarrow \psi_{0}(n)<f_{N_{s_{j}^{n+1, \alpha}}} S_{n+1}(\alpha, .) d \lambda<\psi_{1}(n)
$$

if $(\alpha, \beta) \in G$. We use Corollary 3.5 and its proof. Note that $r_{2}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right) \in P_{1}$.

We first consider $n_{0}^{\prime} \in \mathcal{W}_{1}$ and $n_{2}^{\prime} \in P_{1}$ (coding $G_{n+1}^{*}$ and $S_{n+1}$ respectively) as variables. We define $R_{0}, R_{1} \subseteq \omega \times \omega^{\omega} \times 2^{\omega} \times \omega^{3}$ by

$$
\begin{aligned}
& R_{0}\left(n_{2}^{\prime}, \alpha, \beta, u, k, l\right) \Leftrightarrow \exists r \in \mathbb{R} \neg\left(n_{2}^{\prime} \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} \wedge\left(n_{2}^{\prime}, \alpha, \beta, r\right) \notin \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}\right) \wedge \\
& \left(\frac{k}{2^{l}} \leq r<\frac{k+1}{2^{l}} \wedge \operatorname{Seq}(u) \wedge \beta \in N^{u}\right) \\
& R_{1}\left(n_{2}^{\prime}, \alpha, \beta, u, k, l\right) \Leftrightarrow \quad \forall r \in \mathbb{R}\left(n_{2}^{\prime} \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} \wedge\left(n_{2}^{\prime}, \alpha, \beta, r\right) \notin \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}\right) \vee \\
& \left(\frac{k}{2^{t}} \leq r<\frac{k+1}{2^{l}} \wedge \operatorname{Seq}(u) \wedge \beta \in N^{u}\right),
\end{aligned}
$$

so that $R_{0}$ is $\Sigma_{1}^{1}, R_{1}$ is $\Pi_{1}^{1}$, and $R_{0}\left(n_{2}^{\prime}, \alpha, \beta, u, k, l\right) \Leftrightarrow R_{1}\left(n_{2}^{\prime}, \alpha, \beta, u, k, l\right)$ if $n_{2}^{\prime} \in P_{1}$. Then, as in the proof of Corollary 3.5.(d), we define $O_{0}, O_{1} \subseteq \omega \times \omega^{\omega} \times 2^{\omega}$ by

$$
O_{\varepsilon}\left(n_{2}^{\prime}, \alpha, \beta\right) \Leftrightarrow \operatorname{Seq}(\alpha(0)) \wedge \operatorname{lh}(\alpha(0))=3 \wedge R_{\varepsilon}\left(n_{2}^{\prime}, \alpha^{*}, \beta,(\alpha(0))_{0},(\alpha(0))_{1},(\alpha(0))_{2}\right)
$$

if $\varepsilon \in 2$, so that $O_{0}$ is $\Sigma_{1}^{1}, O_{1}$ is $\Pi_{1}^{1}$, and $O_{0}\left(n_{2}^{\prime}, \alpha, \beta\right) \Leftrightarrow O_{1}\left(n_{2}^{\prime}, \alpha, \beta\right)$ if $n_{2}^{\prime} \in P_{1}$. In particular, $n_{2}^{\prime} \in P_{1}$ and $\operatorname{Seq}(u)$ imply that

$$
\int_{N^{u}} S_{n+1}(\alpha, .) d \lambda=\lim _{l \rightarrow \infty} \Sigma_{k \leq 2^{l}} \frac{k}{2^{l}} \lambda\left(\left(O_{\varepsilon}\right)_{n_{2}^{\prime},<u, k, l>\alpha}\right)
$$

for each $\varepsilon \in 2$. Thus $a<\int_{N^{u}} S_{n+1}(\alpha,). d \lambda<b$ is in this case equivalent to

$$
\exists p_{0}, p_{1}, q_{0}, q_{1}, N \in \omega a<\frac{p_{0}}{p_{1}+1} \wedge \frac{q_{0}}{q_{1}+1}<b \wedge \forall l \geq N \frac{p_{0}}{p_{1}+1} \leq \Sigma_{k \leq 2^{2}} \frac{k}{2^{l}} \lambda\left(\left(O_{\varepsilon}\right)_{n_{2}^{\prime},<u, k, l>\alpha}\right) \leq \frac{q_{0}}{q_{1}+1} .
$$

By Corollary 3.5.(b) applied to $D:=P_{1}$, the partial map $\lambda_{O}: P_{1} \times \omega^{\omega} \rightarrow \mathbb{R}$ defined by

$$
\lambda_{O}\left(n_{2}^{\prime}, \alpha\right):=\lambda\left(\left(O_{0}\right)_{n_{2}^{\prime}, \alpha}\right)
$$

is $\Sigma_{1}^{1}$-recursive and $\Pi_{1}^{1}$-recursive on its domain. By 3E.2, 3G. 1 and 3G. 2 in [M], these two classes of functions are closed under composition. In particular, the partial map

$$
\left(n_{2}^{\prime}, \alpha, u, l\right) \mapsto \Sigma_{k \leq 2^{l}} \frac{k}{2^{l}} \lambda\left(\left(O_{\varepsilon}\right)_{n_{2}^{\prime},<u, k, l>\alpha}\right)
$$

is $\Sigma_{1}^{1}$-recursive and $\Pi_{1}^{1}$-recursive on $P_{1} \times \omega^{\omega} \times \omega^{2}$. This shows the existence of $Q_{0} \in \Sigma_{1}^{1}\left(\omega^{2} \times \omega^{\omega} \times \omega\right)$ and $Q_{1} \in \Pi_{1}^{1}\left(\omega^{2} \times \omega^{\omega} \times \omega\right)$ such that

$$
Q_{0}\left(n_{2}^{\prime}, n, \alpha, u\right) \Leftrightarrow Q_{1}\left(n_{2}^{\prime}, n, \alpha, u\right) \Leftrightarrow \operatorname{Seq}(u) \wedge \psi_{0}(n)<f_{N^{u}} S_{n+1}(\alpha, .) d \lambda<\psi_{1}(n)
$$

if $n_{2}^{\prime} \in P_{1}$. We now consider $n_{0}^{\prime} \in \mathcal{W}_{1}$ and $n_{2}^{\prime} \in P_{1}$ as parameters. We set

$$
\begin{aligned}
& P_{n_{0}^{\prime}, n_{2}^{\prime}}(n, \alpha, u) \Leftrightarrow \\
& \quad Q_{1}\left(n_{2}^{\prime}, n, \alpha, u\right) \wedge N^{u} \subseteq \mathcal{C}_{n_{0}^{\prime}, \alpha} \wedge \forall k<\operatorname{lh}(u)\left(\neg Q_{0}\left(n_{2}^{\prime}, n, \alpha, \underline{u}(k)\right) \vee N \underline{u}(k) \nsubseteq \mathcal{C}_{n_{0}^{\prime}, \alpha}\right) .
\end{aligned}
$$

Note that for each $(\alpha, \beta) \in G$ there is $l \in \omega$ minimal with the properties that $N_{\beta \mid l} \subseteq \mathcal{C}_{n_{0}^{\prime}, \alpha}$ and $Q_{1}\left(n_{2}^{\prime}, n, \alpha,<\beta(0), \ldots, \beta(l-1)>\right)$, so that $P_{n_{0}^{\prime}, n_{2}^{\prime}}(n, \alpha,<\beta(0), \ldots, \beta(l-1)>)$ since $n_{0}^{\prime} \in \mathcal{W}_{1}$ and $n_{2}^{\prime} \in P_{1}$. As $n_{0}^{\prime} \in \mathcal{W}_{1}, N \underline{\underline{u}}{ }^{(k)} \backslash \mathcal{C}_{n_{0}^{\prime}, \alpha}$ is a $\Delta_{1}^{1}(\alpha)$ compact subset of $2^{\omega}$, so that it contains a $\Delta_{1}^{1}(\alpha)$ point if it is not empty (see 4F. 15 in $[\mathrm{M}]$ ). This shows that $P_{n_{0}^{\prime}, n_{2}^{\prime}}$ is $\Pi_{1}^{1}$.

The uniformization lemma provides a $\Delta_{1}^{1}$-recursive map $L: \omega \times \omega^{\omega} \times 2^{\omega} \rightarrow \omega$ such that

$$
P_{n_{0}^{\prime}, n_{2}^{\prime}}(n, \alpha,<\beta(0), \ldots, \beta(L(n, \alpha, \beta)-1)>)
$$

if $(\alpha, \beta) \in G$. Note that the $\Sigma_{1}^{1}$ set

$$
\sigma:=\left\{(n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid \exists \beta \in G_{\alpha} \quad u=<\beta(0), \ldots, \beta(L(n, \alpha, \beta)-1)>\right\}
$$

is contained in the $\Pi_{1}^{1}$ set $\pi:=\left\{(n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid P_{n_{0}^{\prime}, n_{2}^{\prime}}(n, \alpha, u)\right\}$. By 7B. 3 in [M], there is a $\Delta_{1}^{1}$ subset $\delta$ of $\omega \times \omega^{\omega} \times \omega$ such that $\sigma \subseteq \delta \subseteq \pi$. We now also consider $n$ as a parameter and define $\varphi: \omega^{\omega} \rightarrow \omega^{\omega}$ by

$$
\varphi(\alpha)(u):=\left\{\begin{array}{l}
u \text { if }(n, \alpha, u) \in \delta \\
0 \text { otherwise }
\end{array}\right.
$$

Note that $\varphi$ is $\Delta_{1}^{1}$-recursive, and that $\operatorname{Seq}(\varphi(\alpha)(u))$ is equivalent to $(n, \alpha, u) \in \delta$. In particular,
(1) $G_{\alpha} \subseteq \bigcup\left\{N^{\varphi(\alpha)(u)} \mid u \in \omega \wedge \operatorname{Seq}(\varphi(\alpha)(u))\right\} \subseteq \mathcal{C}_{n_{0}^{\prime}, \alpha}$
(6) $\left|f_{N^{\varphi(\alpha)(u)}} S_{n+1}(\alpha,). d \lambda-S_{n+1}(\alpha, \beta)\right|<2^{-3}$ if $\operatorname{Seq}(\varphi(\alpha)(u)) \wedge \beta \in G_{\alpha} \cap N^{\varphi(\alpha)(u)}$
for each $\alpha \in \omega^{\omega}$. Let $k \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ such that $\mathcal{C}_{k}^{\omega^{\omega} \times \omega^{\omega}}=\operatorname{Gr}(\varphi)$. We now consider $n_{0}^{\prime}, n_{2}^{\prime}$ and $n$ as variables again. Note that for each $\left(n_{0}^{\prime}, n_{2}^{\prime}, n\right) \in \mathcal{W}_{1} \times P_{1} \times \omega$ there is $k \in \omega$ such that

$$
R\left(n_{0}^{\prime}, n_{2}^{\prime}, n, k\right) \Leftrightarrow\left\{\begin{array}{l}
k \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}} \wedge \\
\left(\forall \alpha \in \omega^{\omega} \forall \gamma \in \omega^{\omega}\left(k \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}} \wedge \neg \mathcal{C}^{\omega^{\omega} \times \omega^{\omega}}(k, \alpha, \gamma)\right) \vee\right. \\
\left((1) G_{\alpha} \subseteq \bigcup\left\{N^{\gamma(u)} \mid u \in \omega \wedge \operatorname{Seq}(\gamma(u))\right\} \subseteq \mathcal{C}_{n_{0}^{\prime}, \alpha}\right. \\
\left.\left.\wedge(6) \forall u \in \omega \neg \operatorname{Seq}(\gamma(u)) \vee Q_{1}\left(n_{2}^{\prime}, n, \alpha, u\right)\right)\right)
\end{array}\right.
$$

Note that $R \in \Pi_{1}^{1}\left(\omega^{4}\right)$. The uniformization lemma provides a partial map $K: \omega^{3} \mapsto \omega$ which is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain $\mathcal{W}_{1} \times P_{1} \times \omega$, and $R\left(n_{0}^{\prime}, n_{2}^{\prime}, n, K\left(n_{0}^{\prime}, n_{2}^{\prime}, n\right)\right)$ if

$$
\left(n_{0}^{\prime}, n_{2}^{\prime}, n\right) \in \mathcal{W}_{1} \times P_{1} \times \omega
$$

It remains to set $r_{3}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right):=K\left(n_{0}^{\prime}, n_{2}^{\prime}, n\right)$ if $n_{0}^{\prime}=r_{0}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right)$ and

$$
n_{2}^{\prime}=r_{2}\left(n_{0}, n_{1}, n_{2}, n_{3}, n\right)
$$

so that $r_{3}$ is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain $\mathcal{W}_{1} \times P_{1}^{2} \times P_{3} \times \omega$.
Finally, $r$ is $\Pi_{1}^{1}$-recursive on $\mathcal{W}_{1} \times P_{1}^{2} \times P_{3} \times \omega, f$ is $\Pi_{1}^{1}$-recursive on $\omega$, and thus $f$ is $\Delta_{1}^{1}$-recursive by the uniformization lemma since it is total.

- We are now ready to define the dimension two versions of $G_{n}^{*}, g_{n}, S_{n}$, and $\left(s_{j}^{n}\right)_{j \in I_{n}}$ :
(1) $G_{n}^{*}:=\mathcal{C}_{f_{0}(n)}$,
(2) $g_{n}(\alpha, \beta)=\rho \Leftrightarrow\left(f_{1}(n), \alpha, \beta, \rho\right) \in \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$,
(3) $S_{n}(\alpha, \beta)=\rho \Leftrightarrow\left(f_{2}(n), \alpha, \beta, \rho\right) \in \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$,
(4) $\left\{\begin{array}{l}(i) j \in I_{n, \alpha} \Leftrightarrow \exists \delta \in \omega^{\omega} \quad\left(f_{3}(n), \alpha, \delta\right) \in \mathcal{C}^{\omega^{\omega} \times \omega^{\omega}} \wedge \operatorname{Seq}(\delta(j)), \\ (i i) s_{j}^{n, \alpha}=\delta(j) \text { if } j \in I_{n, \alpha} .\end{array}\right.$

By construction of $r$, these objects satisfy the conditions (1)-(6) of the proof of Lemma 2.7

- Consequently, the martingale $F(\alpha)$ will be defined in such a way that if $u \in \omega$ codes $s \in 2^{<\omega}$, then $F(\alpha)(s)=f_{N^{u}} f_{\infty}(\alpha,). d \lambda$. Note that $G=\bigcap_{n \in \omega} G_{n}^{*}$, so that $\neg G$ is the disjoint union of the $G_{n}^{*} \backslash G_{n+1}^{*}$ 's. Thus

$$
\begin{aligned}
\int_{N^{u}} f_{\infty}(\alpha, .) d \lambda & =\int_{N^{u} \backslash G_{\alpha}} f_{\infty}(\alpha, .) d \lambda=\Sigma_{n \in \omega} \int_{N^{u} \cap\left(G_{n}^{*}\right)_{\alpha} \backslash\left(G_{n+1}^{*}\right)_{\alpha}} f_{\infty}(\alpha, .) d \lambda \\
& =\Sigma_{n \in \omega} \Sigma_{j \leq n}(-1)^{j} \int_{N^{u} \cap\left(G_{n}^{*}\right)_{\alpha} \backslash\left(G_{n+1}^{*}\right)_{\alpha}} g_{j}(\alpha, .) d \lambda \\
& =\lim _{l \rightarrow \infty} \Sigma_{n \leq l} \Sigma_{j \leq n}(-1)^{j} \int_{N^{u} \cap\left(G_{n}^{*}\right)_{\alpha} \backslash\left(G_{n+1}^{*}\right)_{\alpha}} g_{j}(\alpha, .) d \lambda .
\end{aligned}
$$

Consequently, in order to prove that $F$ is $\Delta_{1}^{1}$-recursive, it is enough to check that the partial map $(u, \alpha, j, n) \mapsto \int_{N^{u} \cap\left(G_{n}^{*}\right)_{\alpha} \backslash\left(G_{n+1}^{*}\right)_{\alpha}} g_{j}(\alpha,). d \lambda$ is $\Delta_{1}^{1}$-recursive from $\{u \in \omega \mid \operatorname{Seq}(u)\} \times \omega^{\omega} \times \omega^{2}$ into $\mathbb{R}$. By Corollary 3.5, it is enough to check that the map $h: \omega^{\omega} \times 2^{\omega} \rightarrow \mathbb{R}$ defined by

$$
h(\alpha, \beta):=\left\{\begin{array}{l}
g_{(\alpha(0))_{0}}\left(\alpha^{*}, \beta\right) \text { if } \operatorname{Seq}(\alpha(0)) \wedge \operatorname{lh}(\alpha(0))=2 \wedge\left(\alpha^{*}, \beta\right) \in G_{(\alpha(0))_{1}}^{*} \backslash G_{(\alpha(0))_{1}+1}^{*}, \\
0 \text { otherwise },
\end{array}\right.
$$

is $\Delta_{1}^{1}$-recursive. This comes from the facts that

$$
(\alpha, \beta) \in G_{n}^{*} \Leftrightarrow\left(f_{0}(n), \alpha, \beta\right) \in \mathcal{C} \Leftrightarrow \neg\left(f_{0}(n) \in \mathcal{W} \wedge\left(f_{0}(n), \alpha, \beta\right) \notin \mathcal{C}\right)
$$

is $\Delta_{1}^{1}$ in $(\alpha, \beta, n)$ and

$$
\begin{array}{r}
g_{n}(\alpha, \beta) \in N(\mathbb{R}, p) \Leftrightarrow \exists \rho \in \mathbb{R} \neg\left(f_{1}(n) \in \mathcal{W}^{\omega \times 2^{\omega} \times \mathbb{R}} \wedge\left(f_{1}(n), \alpha, \beta, \rho\right) \notin \mathcal{C}^{\omega \times 2^{\omega} \times \mathbb{R}}\right) \wedge \\
\rho \in N(\mathbb{R}, p) \\
\Leftrightarrow \forall \rho \in \mathbb{R}\left(f_{1}(n) \in \mathcal{W}^{\omega \times 2^{\omega} \times \mathbb{R}} \wedge\left(f_{1}(n), \alpha, \beta, \rho\right) \notin \mathcal{C}^{\omega \times 2^{\omega} \times \mathbb{R}}\right) \vee \\
\rho \in N(\mathbb{R}, p)
\end{array}
$$

is $\Delta_{1}^{1}$ in $(\alpha, \beta, n, p)$.

- Finally, the map $F$ is $\Delta_{1}^{1}$-recursive and is as required.


## 4 First consequences

## (A) Universal sets

- We first recall some material from [K2]. The first result can be found in Section 23.F (see also [Za]).

Theorem 4.1 (Zahorski) Let $B$ be a subset of $[0,1]$. The following are equivalent:
(a) there are $S \in \boldsymbol{\Sigma}_{2}^{0}$ and $P \in \boldsymbol{\Pi}_{3}^{0}$ with $m(P)=1$, where $m$ is the Lebesgue measure on $[0,1]$, such that $B=S \cap P$,
(b) there is $f \in C([0,1])$ with $B=\left\{x \in[0,1] \mid f^{\prime}(x)\right.$ exists $\}$ (we consider only one-sided derivatives at the endpoints).

The second result is 23.23 .

Theorem 4.2 Let $\mathcal{G}$ be a $G_{\delta}$ subset of $(0,1)$ with $m(\mathcal{G})=0$. Then

$$
\left\{(f, x) \in C([0,1]) \times \mathcal{G} \mid f^{\prime}(x) \text { exists }\right\}
$$

is $C([0,1])$-universal for $\boldsymbol{\Pi}_{3}^{0}(\mathcal{G})$.

- We prove results in that spirit here.

Theorem 4.3 Let $B$ be a subset of $2^{\omega}$. Then the following are equivalent:
(a) $B$ is $\boldsymbol{\Sigma}_{3}^{0}$ and has $\lambda$-measure zero,
(b) there is $f \in \mathcal{M}$ with $B=\left\{\beta \in 2^{\omega} \mid \operatorname{osc}(f, \beta)>0\right\}$.

Proof. (a) $\Rightarrow$ (b) Write $B=\bigcup_{n \in \omega} G_{n}$, where the $G_{n}$ 's are $G_{\delta}$. Lemma 2.7 gives, for each $n$, a martingale $f_{n}$ with $G_{n}=D\left(f_{n}\right)$ and $\left\{\operatorname{osc}\left(f_{n}, \beta\right) \mid \beta \in 2^{\omega}\right\} \subseteq\{0\} \cup\left[\frac{1}{2}, 1\right]$. Lemma 2.8 gives $f \in \mathcal{M}$ with $D(f)=B$.
(b) $\Rightarrow$ (a) We already noticed in the introduction that $B$ is $\boldsymbol{\Sigma}_{3}^{0}$. By Doob's theorem, $B$ has $\lambda$-measure zero (see [D]).

Corollary 4.4 Let $\mathcal{G}$ be a $G_{\delta}$ subset of $2^{\omega}$ with $\lambda(\mathcal{G})=0$. Then $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \operatorname{osc}(f, \beta)>0\}$ is $\mathcal{M}$-universal for $\Sigma_{3}^{0}(\mathcal{G})$.

For example, $\left\{\beta \in 2^{\omega} \mid \forall n \in \omega \beta(2 n)=0\right\}$ is a $\Pi_{1}^{0}$ copy of $2^{\omega}$ and has $\lambda$-measure zero.

## (B) Complete sets

- By 33.G in [K2], there is a uniform version of Zahorski's theorem, which allows to prove the following result

Theorem 4.5 (Mazurkiewicz) The set of differentiable functions in $C([0,1])$ is $\Pi_{1}^{1}$-complete.

- Here again, there is a result in that spirit.

Theorem 4.6 The set $\mathcal{P}:=\left\{f \in \mathcal{M} \mid \forall \beta \in 2^{\omega} \operatorname{osc}(f, \beta)=0\right\}$ is $\Pi_{1}^{1}$-complete.
Notation. Let $\mathcal{K}:=\left\{\beta \in 2^{\omega} \mid \forall n \in \omega \beta(2 n)=0\right\}$, which is a $\Pi_{1}^{0}$ copy of the Cantor space $2^{\omega}$ with $\lambda(\mathcal{K})=0$. In particular, $\mathcal{K}$ is a nonempty $G_{\delta} \cap \Delta_{1}^{1}$ subset of $2^{\omega}$.

Proof. Let $U \in \Pi_{1}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$ be $\omega^{\omega}$-universal for the co-analytic subsets of $2^{\omega}$, and

$$
\Pi:=\left\{\alpha \in \omega^{\omega} \mid\left((\alpha)_{0},(\alpha)_{1}\right) \in U\right\} .
$$

Note that $\Pi \in \Pi_{1}^{1}$. If $P \in \Pi_{1}^{1}\left(2^{\omega}\right)$, then $P=U_{\alpha}$ for some $\alpha \in \omega^{\omega}$, so that the map $\beta \mapsto<\alpha, \beta>$ is a continuous reduction of $P$ to $\Pi$ and $\Pi$ is $\Pi_{1}^{1}$-complete. Let $H \in \Pi_{2}^{0}\left(\omega^{\omega} \times 2^{\omega}\right)$ with $\neg \Pi=\Pi_{0}[H]$. We set $G:=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid\left(\alpha,(\beta)_{1}\right) \in H \wedge \beta \in \mathcal{K}\right\}$, so that $G \in \Delta_{1}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$, has $G_{\delta}$ vertical sections and $G \subseteq \omega^{\omega} \times \mathcal{K}$. Lemma3.10 gives $F: \omega^{\omega} \rightarrow \mathcal{M}$ Borel such that $G_{\alpha}=\mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^{\omega}$.

Thus

$$
\alpha \notin \Pi \Leftrightarrow \exists \beta \in 2^{\omega} \quad(\alpha, \beta) \in H \Leftrightarrow \exists \beta \in 2^{\omega} \quad(\alpha, \beta) \in G \Leftrightarrow \exists \beta \in 2^{\omega} \quad(F(\alpha), \beta) \in \mathcal{V} \Leftrightarrow F(\alpha) \notin \mathcal{P}
$$

Thus $\Pi=F^{-1}(\mathcal{P})$ and $\mathcal{P}$ is Borel $\Pi_{1}^{1}$-complete. By 26.C in [K2], $\mathcal{P}$ is $\Pi_{1}^{1}$-complete.

- We now prove Theorem 1.8 Let $X$ be a metrizable compact space and $Y$ be a Polish space. We equip $\mathcal{C}(X, Y)$ with the topology of uniform convergence, so that it is a Polish space (see 4.19 in [K2]). We use the map $\psi$ defined before Theorem 1.8

Theorem 4.7 (a) The set $\mathcal{P}_{1}:=\left\{\left(f_{k}\right)_{k \in \omega} \in \mathcal{P}^{\omega} \mid\left(\psi\left(f_{k}\right)\right)_{k \in \omega}\right.$ pointwise converges $\}$ is $\boldsymbol{\Pi}_{1}^{1}$-complete.
(b) The set $\mathcal{P}_{2}:=\left\{\left(f_{k}\right)_{k \in \omega} \in \mathcal{P}^{\omega} \mid\left(\psi\left(f_{k}\right)\right)_{k \in \omega}\right.$ pointwise converges to zero $\}$ is $\boldsymbol{\Pi}_{1}^{1}$-complete.
(c) The set $\mathcal{S}:=\left\{\left(f_{k}\right)_{k \in \omega} \in \mathcal{P}^{\omega} \mid \exists \gamma \in \omega^{\omega}\left(\psi\left(f_{\gamma(i)}\right)\right)_{i \in \omega}\right.$ pointwise converges to zero $\}$ is $\boldsymbol{\Sigma}_{2^{-}}^{1-}$ complete.

Proof. We define $\varphi: \mathcal{C}\left(2^{\omega},[0,1]\right) \rightarrow \mathcal{M}$ by $\varphi(h)(s):=f_{N_{s}} h d \lambda$. As in the proof of Lemma 2.7, $\varphi$ is well-defined. It is also continuous, and injective: if $h \neq h^{\prime}$, then we can find $q \in \omega$ and $s \in 2^{<\omega}$ such that $h(\beta)-h^{\prime}(\beta)>2^{-q}$ for each $\beta \in N_{s}$ or $h^{\prime}(\beta)-h(\beta)>2^{-q}$ for each $\beta \in N_{s}$, so that

$$
\left|\varphi(h)(s)-\varphi\left(h^{\prime}\right)(s)\right|=\frac{1}{\lambda\left(N_{s}\right)}\left|\int_{N_{s}} h d \lambda-\int_{N_{s}} h^{\prime} d \lambda\right| \geq 2^{-q}
$$

This implies that the range $\mathcal{R}$ of $\varphi$ is Borel and $\psi:=\varphi^{-1}: \mathcal{R} \rightarrow \mathcal{C}\left(2^{\omega},[0,1]\right)$ is Borel. As every continuous map $h: 2^{\omega} \rightarrow[0,1]$ is $\tau$-continuous,

$$
\lim _{l \rightarrow \infty} \varphi(h)(\beta \mid l)=\lim _{l \rightarrow \infty} f_{N_{\beta \mid l}} h d \lambda=h(\beta)
$$

for each $\beta \in 2^{\omega}$, by Lemma 2.6. This implies that $f \in \mathcal{P}$ and $\psi(f)(\beta)=\lim _{l \rightarrow \infty} f(\beta \mid l)$ for each $\beta \in 2^{\omega}$ if $f \in \mathcal{R}$.
(a) Note that the proof of 33.11 in [K2] shows that the set

$$
P_{1}:=\left\{\left(h_{k}\right)_{k \in \omega} \in\left(\mathcal{C}\left(2^{\omega},[0,1]\right)\right)^{\omega} \mid\left(h_{k}\right)_{k \in \omega} \text { pointwise converges }\right\}
$$

is $\Pi_{1}^{1}$-complete. As $\mathcal{E}:=\left\{\left(f_{k}\right)_{k \in \omega} \in \mathcal{R}^{\omega} \mid\left(\psi\left(f_{k}\right)\right)_{k \in \omega}\right.$ pointwise converges $\}=\left(\psi^{\omega}\right)^{-1}\left(P_{1}\right)$, the equalities $P_{1}=\left(\varphi^{\omega}\right)^{-1}(\mathcal{E})=\left(\varphi^{\omega}\right)^{-1}\left(\mathcal{P}_{1}\right)$ hold and $\mathcal{P}_{1}$ is $\Pi_{1}^{1}$-complete.
(b) We argue as in (a).
(c) As in [B-Ka-L], the set

$$
S:=\left\{\left(h_{k}\right)_{k \in \omega} \in\left(\mathcal{C}\left(2^{\omega},[0,1]\right)\right)^{\omega} \mid \exists \gamma \in \omega^{\omega}\left(h_{\gamma(i)}\right)_{i \in \omega} \text { pointwise converges to zero }\right\}
$$

is $\boldsymbol{\Sigma}_{2}^{1}$-complete. Indeed, fix $Q \in \boldsymbol{\Sigma}_{2}^{1}\left(2^{\omega}\right)$.

Lemma 2.2 in [B-Ka-L] gives $\left(g_{k}\right)_{k \in \omega} \in\left(\mathcal{C}\left(2^{\omega} \times 2^{\omega}, 2\right)\right)^{\omega}$ such that, for each $\delta \in 2^{\omega}$, the following are equivalent:
(i) $\delta \in Q$,
(ii) $\exists \gamma \in \omega^{\omega} \quad \forall \beta \in 2^{\omega} \lim _{i \rightarrow \infty} g_{\gamma(i)}(\delta, \beta)=0$.

We define, $g: 2^{\omega} \rightarrow\left(\mathcal{C}\left(2^{\omega},[0,1]\right)\right)^{\omega}$ by $g(\delta)(k)(\beta):=g_{k}(\delta, \beta)$. Then $g$ is continuous and reduces $Q$ to $S$. As
$\mathcal{E}^{\prime}:=\left\{\left(f_{k}\right)_{k \in \omega} \in \mathcal{R}^{\omega} \mid \exists \gamma \in \omega^{\omega}\left(\psi\left(f_{\gamma(i)}\right)\right)_{i \in \omega}\right.$ pointwise converges to zero $\}=\left(\psi^{\omega}\right)^{-1}(S)$, $S=\left(\varphi^{\omega}\right)^{-1}\left(\mathcal{E}^{\prime}\right)=\left(\varphi^{\omega}\right)^{-1}(\mathcal{S})$ and $\mathcal{S}$ is $\boldsymbol{\Sigma}_{2}^{1}$-complete.

## 5 Universal and complete sets in the spaces $\mathcal{C}\left(2^{\omega}, X\right)$

- It is known that if $\boldsymbol{\Gamma}$ is a self-dual Wadge class and $X$ is a Polish space, then there is no set which is $X$-universal for the subsets of $X$ in $\boldsymbol{\Gamma}$ (see 22.7 in [K2]). This is no longer the case if the space of codes is different from the space of coded sets.

Proposition 5.1 Let $X$ be a Polish space, $\boldsymbol{\Gamma}$ be a Wadge class with complete set $C \in \boldsymbol{\Gamma}(X)$, and $\mathcal{U}^{\Gamma}:=\left\{(h, \beta) \in \mathcal{C}\left(2^{\omega}, X\right) \times 2^{\omega} \mid h(\beta) \in C\right\}$. Then $\mathcal{U}^{\boldsymbol{\Gamma}}$ is $\mathcal{C}\left(2^{\omega}, X\right)$-universal for the $\boldsymbol{\Gamma}$ subsets of $2^{\omega}$.

Proof. As the evaluation map $(h, \beta) \mapsto h(\beta)$ is continuous, $\mathcal{U}^{\boldsymbol{\Gamma}} \in \boldsymbol{\Gamma}$. If $A \in \boldsymbol{\Gamma}\left(2^{\omega}\right)$, then $A=h^{-1}(C)$ for some $h \in \mathcal{C}\left(2^{\omega}, X\right)$, so that $A=\mathcal{U}_{h}^{\Gamma}$.

We will partially strengthen this result to get our uniform universal sets.

- Recall that it is proved in [K3] that a Borel $\Pi_{1}^{1}$-complete set is actually $\Pi_{1}^{1}$-complete. In fact, Kechris's proof shows the result for the classes $\boldsymbol{\Pi}_{n}^{1}$. Our main tool is a uniform version of this. Kechris's result has recently been strengthened in $[\mathrm{P}]$ as follows.

Theorem 5.2 (Pawlikowski) Let $n \geq 1$ be a natural number, and $C \subseteq X \subseteq 2^{\omega}$. If Borel functions from $2^{\omega}$ into $X$ give as preimages of $C$ all $\Pi_{n}^{1}$ subsets of $2^{\omega}$, then so do continuous injections.

The main tool mentioned above is the following:
Theorem 5.3 Let $n \geq 1$ be a natural number, $\mathcal{U}^{\Pi_{n}^{1}, 2^{\omega}}$ be a suitable $\omega^{\omega}$-universal set for the $\boldsymbol{\Pi}_{n}^{1}$ subsets of $2^{\omega}, X$ be a recursively presented Polish space, $C \in \Pi_{n}^{1}(X), \mathcal{R}: \omega^{\omega} \times \omega^{\omega} \rightarrow \omega^{\omega}$ be a recursive map, and $b: \omega^{\omega} \rightarrow X$ be a $\Delta_{1}^{1}$-recursive map such that

$$
(\alpha, \beta) \in \mathcal{U}^{\Pi_{n}^{1}, 2^{\omega}} \Leftrightarrow b(\mathcal{R}(\alpha, \beta)) \in C
$$

for each $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Then there is a $\Delta_{1}^{1}$-recursive map $f: \omega^{\omega} \rightarrow \mathcal{C}\left(2^{\omega}, X\right)$ such that

$$
(\alpha, \beta) \in \mathcal{U}^{\boldsymbol{\Pi}_{n}^{1}, 2^{\omega}} \Leftrightarrow f(\alpha)(\beta) \in C
$$

for each $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$.

- We first recall some material from [K3].

Definition 5.4 (a) A coding system for nonempty perfect binary trees is a pair $(\mathcal{D}, \mathcal{O})$, where $\mathcal{D} \subseteq 2^{\omega}$ and $\mathcal{O}: \mathcal{D} \rightarrow\left\{T \in 2^{2^{<\omega}} \mid T\right.$ is a nonempty perfect binary tree $\}$ is onto.
(b) A coding system $(\mathcal{D}, \mathcal{O})$ is nice if
(i) for any $\alpha \in \omega^{\omega}$ and any $\Delta_{1}^{1}(\alpha)$-recursive map $H: 2^{\omega} \times 2^{\omega} \rightarrow \omega$, we can find $\beta \in \mathcal{D} \cap \Delta_{1}^{1}(\alpha)$ and $k \in \omega$ such that $H(\beta, \delta)=k$ for each $\delta$ in the body $[\mathcal{O}(\beta)]$ of $\mathcal{O}(\beta)$,
(ii) $\mathcal{D}$ is $\Pi_{1}^{1}$ and, for $\beta \in \mathcal{D}$, the relation

$$
R(m, \beta) \Leftrightarrow \operatorname{Seq}(m) \wedge\left((m)_{0}, \ldots,(m)_{l h(m)-1}\right) \in \mathcal{O}(\beta)
$$

is $\Delta_{1}^{1}$, i.e., there are $\Pi_{1}^{1}$ relations $\Pi_{0}, \Pi_{1}$ such that $R(m, \beta) \Leftrightarrow \Pi_{0}(m, \beta) \Leftrightarrow \neg \Pi_{1}(m, \beta)$ if $\beta \in \mathcal{D}$.
Nice coding systems exist. If $\beta \in \mathcal{D}$, then there is a canonical homeomorphism $\beta^{*}$ from $[\mathcal{O}(\beta)]$ onto $2^{\omega}$. We now check that the construction of $\beta^{*}$ is effective.
Lemma 5.5 (a) The partial function $e:(\beta, \delta) \mapsto \beta^{*}(\delta)$ is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain

$$
\operatorname{Domain}(e):=\left\{(\beta, \delta) \in \mathcal{D} \times 2^{\omega} \mid \delta \in[\mathcal{O}(\beta)]\right\} .
$$

(b) The partial function $\iota:(\beta, \gamma) \mapsto$ the unique $\delta \in[\mathcal{O}(\beta)]$ with $\beta^{*}(\delta)=\gamma$ is $\Pi_{1}^{1}$-recursive on its $\Pi_{1}^{1}$ domain $\mathcal{D} \times 2^{\omega}$.

Proof. (a) We define a $\Pi_{1}^{1}$ relation $\mathcal{Q}$ on $\omega^{2} \times\left(2^{\omega}\right)^{2}$ by

$$
\mathcal{Q}\left(p, p^{\prime}, \beta, \delta\right) \Leftrightarrow\left(\left(\forall \varepsilon \in 2 \Pi_{0}\left(\overline{\left(\delta \mid p^{\prime}\right) \varepsilon}, \beta\right)\right) \wedge\left(\forall p \leq p^{\prime \prime}<p^{\prime} \exists \varepsilon \in 2 \Pi_{1}\left(\left(\overline{\left.\delta \mid p^{\prime \prime}\right) \varepsilon}, \beta\right)\right)\right) .\right.
$$

Note that

$$
\beta^{*}(\delta)(n)=\varepsilon \Leftrightarrow\left\{\begin{array}{l}
\exists l \in \omega \operatorname{Seq}(l) \wedge \operatorname{lh}(l)=n+1 \wedge \delta\left((l)_{n}\right)=\varepsilon \wedge \mathcal{Q}\left(0,(l)_{0}, \beta, \delta\right) \wedge \\
\forall m<n(l)_{m}<(l)_{m+1} \wedge \mathcal{Q}\left((l)_{m}+1,(l)_{m+1}, \beta, \delta\right)
\end{array}\right.
$$

if $\beta \in \mathcal{D}$. The proof of (b) is similar.

- Let $X$ be a recursively presented Polish space, and $d_{X}$ and $\left(r_{n}^{X}\right)_{n \in \omega}$ be respectively a distance function and a recursive presentation of $X$. We now give a recursive presentation of $\mathcal{C}\left(2^{\omega}, X\right)$, equipped with the usual distance defined by

$$
d\left(h, h^{\prime}\right):=\sup _{\beta \in 2^{\omega}} d_{X}\left(h(\beta), h^{\prime}(\beta)\right),
$$

since this is not present in $[\mathrm{M}]$. We define, by primitive recursion, a recursive map $\nu: \omega \rightarrow \omega$ such that $\nu(i)$ enumerates $\left\{s \in 2^{<\omega}| | s \mid=i\right\}$. We first set $\nu(0):=1=<>$. Then

$$
\nu(i+1)=k \Leftrightarrow \operatorname{Seq}(k) \wedge \operatorname{lh}(k)=2^{i+1} \wedge \forall l<2^{i} \forall \varepsilon \in 2(k)_{\varepsilon 2^{i}+l}=\overline{s\left((\nu(i))_{l}\right) \varepsilon} .
$$

If $\operatorname{Seq}(n)$ and $\operatorname{lh}(n)=2^{i}$ for some $i(<n)$, then we define $h_{n}: 2^{\omega} \rightarrow X$ by $h_{n}(\beta):=r_{(n) t}^{X}$ if

$$
\beta \mid i=s_{l}^{i}:=s\left((\nu(i))_{l}\right) .
$$

If $\neg \operatorname{Seq}(n)$ or $\operatorname{lh}(n) \neq 2^{i}$ for each $i$, then we define $h_{n}: 2^{\omega} \rightarrow X$ by $h_{n}(\beta):=r_{0}^{X}$ if $\beta \in 2^{\omega}$. In any case, $h_{n} \in \mathcal{C}\left(2^{\omega}, X\right)$ and takes finitely many values.

Lemma 5.6 Let $X$ be a recursively presented Polish space. Then the sequence $\left(h_{n}\right)_{n \in \omega}$ is a recursive presentation of $\mathcal{C}\left(2^{\omega}, X\right)$, equipped with $d$.

Proof. We have to see that $\left(h_{n}\right)$ is dense in $\mathcal{C}\left(2^{\omega}, X\right)$. So let $h \in \mathcal{C}\left(2^{\omega}, X\right), \epsilon>0$ and $m \in \omega$ with $2^{-m}<\frac{\epsilon}{2}$. As $h$ is uniformly continuous, there is $i \in \omega$ such that $d_{X}(h(\beta), h(\delta))<2^{-m}$ if $\beta|i=\delta| i$. We choose, for each $l<2^{i}, n_{l} \in \omega$ such that $d_{X}\left(r_{n_{l}}^{X}, h\left(s_{l}^{i} 0^{\infty}\right)\right)<2^{-m}$. We set $n:=<n_{0}, \ldots, n_{2^{i}-1}>$. If $\beta \in 2^{\omega}$ and $\beta \mid i=s_{l}^{i}$, then $d_{X}\left(h(\beta), h_{n}(\beta)\right) \leq d_{X}\left(h(\beta), h\left(s_{l}^{i} 0^{\infty}\right)\right)+d_{X}\left(h\left(s_{l}^{i} 0^{\infty}\right), r_{n_{l}}^{X}\right) \leq 2^{-m}+2^{-m}$, so that $d\left(h, h_{n}\right)<\epsilon$. It is routine to check that the relations " $d\left(h_{m}, h_{n}\right) \leq \frac{p}{q+1}$ " and " $d\left(h_{m}, h_{n}\right)<\frac{p}{q+1}$ " are recursive in ( $m, n, p, q$ ).

We saw in the proof of Proposition 5.1 that the evaluation map $(h, \beta) \mapsto h(\beta)$ is continuous from $\mathcal{C}\left(2^{\omega}, X\right) \times 2^{\omega}$ into $X$. We can say more if $X$ is recursively presented.
Lemma 5.7 Let $X$ be a recursively presented Polish space. Then the evaluation map is recursive.
Proof. Note that

$$
\begin{aligned}
h(\beta) \in N(X, n) & \Leftrightarrow d_{X}\left(h(\beta), r_{\left((n)_{1}\right)_{0}}^{X}\right)<\frac{\left((n)_{1}\right)_{1}}{\left((n)_{1}+1\right.} \\
& \Leftrightarrow \exists m, i, l \in \omega \stackrel{\operatorname{Seq}(m) \stackrel{1}{\ln } \operatorname{lh}(m)=2^{i} \wedge \beta \mid i=s_{l}^{i} \wedge(m)_{l}=\left((n)_{1}\right)_{0} \wedge}{ } \quad d\left(h, h_{m}\right)<\frac{\left((n)_{1}\right)_{1}}{\left((n)_{1}\right)_{2}+1},
\end{aligned}
$$

which gives the result.

- We then strengthen 7A. 3 in $[\mathrm{M}]$ about primitive recursion as follows. If $Z, Y$ are recursively presented Polish spaces, $g: Z \rightarrow Y$ and $h: Y \times \omega \times Z \rightarrow Y$ are $\Pi_{1}^{1}$-recursive and $f: \omega \times Z \rightarrow Y$ is defined by

$$
\left\{\begin{array}{l}
f(0, z):=g(z), \\
f(n+1, z):=h(f(n, z), n, z)
\end{array}\right.
$$

then $f$ is also $\Pi_{1}^{1}$-recursive. If $m: Z \rightarrow Z$ is $\Pi_{1}^{1}$-recursive, then the proof of 7 A .3 in $[\mathrm{M}]$ shows that the map $f^{\prime}: \omega \times Z \rightarrow Y$ defined by

$$
\left\{\begin{array}{l}
f^{\prime}(0, z):=g(z), \\
f^{\prime}(n+1, z):=h\left(f^{\prime}(n, m(z)), n, z\right),
\end{array}\right.
$$

is also $\Pi_{1}^{1}$-recursive. As in 7 A .5 in $[\mathrm{M}]$, this can be extended to partial functions which are $\Pi_{1}^{1}$ recursive on their domain.

- We are ready for the proof of our main tool.

Proof of Theorem 5.3, 3E.6 in [M] provides $\pi: \omega^{\omega} \rightarrow X$ recursive, $\mathcal{F} \in \Pi_{1}^{0}\left(\omega^{\omega}\right)$ and a $\Delta_{1}^{1}$-recursive injection $\rho: X \rightarrow \omega^{\omega}$ such that $\pi_{\mid \mathcal{F}}$ is injective, $\pi[\mathcal{F}]=X$ and $\rho$ is the inverse of $\pi_{\mid \mathcal{F}}$. Let us show that the map $\mu: h \mapsto \pi \circ h$ is $\Delta_{1}^{1}$-recursive from $\mathcal{C}\left(2^{\omega}, \omega^{\omega}\right)$ into $\mathcal{C}\left(2^{\omega}, X\right)$. More generally, let $Y$ be a recursively presented Polish space, and $\psi: Y \rightarrow \mathcal{C}\left(2^{\omega}, X\right)$. Note that

$$
\begin{aligned}
\psi(y) \in N\left(\mathcal{C}\left(2^{\omega}, X\right), n\right) & \Leftrightarrow d\left(\psi(y), h_{\left((n)_{1}\right)_{0}}\right)<\frac{\left((n)_{1}\right)_{1}}{\left((n)_{1}\right)_{2}+1} \\
& \Leftrightarrow \exists m \in \omega \sup _{\beta \in 2^{\omega}} d_{X}\left(\psi(y)(\beta), h_{\left((n)_{1}\right)_{0}}(\beta)\right)<\frac{\left((m)_{1}\right)_{1}}{\left(()_{1}\right)_{2}+1}<\frac{\left((n)_{1}\right)_{1}}{((n))_{1}+1} \\
& \Leftrightarrow \exists m \in \omega \forall \beta \in 2^{\omega} d_{X}\left(\psi(y)(\beta), h_{\left((n)_{1}\right)_{0}}(\beta)\right)<\frac{\left((m)_{1}\right)_{1}}{\left((m)_{1}\right)_{2}+1}<\frac{\left((n)_{1}\right)_{1}}{\left((n)_{1}\right)_{2}+1}
\end{aligned}
$$

and $h_{\left((n)_{1}\right)_{0}}(\beta)=r_{g(n, \beta)}^{X}$ for some recursive map $g: \omega \times 2^{\omega} \rightarrow \omega$.

In the present case, $Y=\mathcal{C}\left(2^{\omega}, \omega^{\omega}\right)$ and $\psi(y)(\beta)=\pi(y(\beta))$. Thus

$$
\begin{aligned}
d_{X}\left(\psi(y)(\beta), h_{\left((n)_{1}\right)_{0}}(\beta)\right)<\frac{\left((m)_{1}\right)_{1}}{\left((m)_{1}\right)_{2}+1} & \Leftrightarrow d_{X}\left(\pi(y(\beta)), r_{g(n, \beta)}^{X}\right)<\frac{\left((m)_{1}\right)_{1}}{\left((m)_{1}\right)_{2}+1} \\
& \Leftrightarrow \pi(y(\beta)) \in N\left(X,\left\langle 0,<g(n, \beta),\left((m)_{1}\right)_{1},\left((m)_{1}\right)_{2}>\right\rangle\right) \\
& \Leftrightarrow\left(y(\beta),\left\langle 0,<g(n, \beta),\left((m)_{1}\right)_{1},\left((m)_{1}\right)_{2}>\right\rangle\right) \in G^{\pi},
\end{aligned}
$$

where $G^{\pi}$ is the $\Sigma_{1}^{0}$ neighborhood diagram of $\pi$. As the evaluation map is recursive, $h \mapsto \pi \circ h$ is $\Pi_{1}^{1}$-recursive and total, and thus $\Delta_{1}^{1}$-recursive.

- Let us show that there is a $\Delta_{1}^{1}$-recursive map $f: \omega^{\omega} \rightarrow \mathcal{C}\left(2^{\omega}, X\right)$ such that $\mathcal{U}_{\alpha}^{\boldsymbol{\Pi}_{n}^{1}, 2^{\omega}}=(f(\alpha))^{-1}(C)$ for each $\alpha \in \omega^{\omega}$. We adapt the proof of the main result in [K3]. We set $A:=\pi^{-1}(C)$. As $C \in \Pi_{n}^{1}(X)$, $A \in \Pi_{n}^{1}\left(\omega^{\omega}\right)$. If $<\beta^{0}, \delta^{0}>\in 2^{\omega}$, then we inductively define, for $i \in \omega, m_{i}, \beta^{i+1}, \delta^{i+1}$ as follows. If $\left(\beta^{i}, \delta^{i}\right)$ is given and in $\operatorname{Domain}(e)$, then $\left(\beta^{i}\right)^{*}\left(\delta^{i}\right)=<x_{i}, \beta^{i+1}, \delta^{i+1}>$ and

$$
m_{i}:=\left\{\begin{array}{l}
\text { the location of the first } 0 \text { in } x_{i} \text { if it exists, } \\
2 \text { otherwise. }
\end{array}\right.
$$

We then set $Q:=\left\{\left(\alpha,<\beta^{0}, \delta^{0}>\right) \in \omega^{\omega} \times 2^{\omega} \mid \forall i \in \omega\left(\beta^{i}, \delta^{i}\right) \in \operatorname{Domain}(e) \wedge\left(\alpha,\left(m_{i}\right)\right) \in \mathcal{U}^{\Pi_{n}^{1}, 2^{\omega}}\right\}$ and $B^{*}:=Q_{\alpha}$, so that $Q \in \Pi_{n}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$ and $\beta \in B^{*} \Leftrightarrow(\alpha, \beta) \in Q$ for each $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$ (note that $B^{*}$ depends on $\alpha$, but we denote it like this to keep the notation of [K3]). We define $I: \omega^{\omega} \rightarrow 2^{\omega}$ by $I(\alpha):=0^{\alpha(0)} 10^{\alpha(1)} 1 \ldots$ Note that $I$ a $\Delta_{1}^{1}$-recursive injection onto the $\Pi_{2}^{0}$ set

$$
\mathbb{P}_{\infty}:=\left\{\beta \in 2^{\omega} \mid \forall p \in \omega \quad \exists q \geq p \quad \beta(q)=1\right\}
$$

so that there is a $\Delta_{1}^{1}$-recursive map $\phi: 2^{\omega} \rightarrow \omega^{\omega}$ which is the inverse of $I$ on $\mathbb{P}_{\infty}$. We set

$$
Q^{\prime}:=\left\{\delta \in 2^{\omega} \mid(\delta)_{0} \in \mathbb{P}_{\infty} \wedge\left(\phi\left((\delta)_{0}\right),(\delta)_{1}\right) \in Q\right\}
$$

so that $Q^{\prime} \in \Pi_{n}^{1}\left(2^{\omega}\right)$. As $\mathcal{U}^{\Pi_{n}^{1}, 2^{\omega}}$ is suitable, there is $\alpha_{Q} \in \omega^{\omega}$ recursive with $Q^{\prime}=\mathcal{U}_{\alpha_{Q}}^{\Pi_{n}^{1}, 2^{\omega}}$. Note that

$$
\begin{aligned}
\beta \in B^{*} & \Leftrightarrow(\alpha, \beta) \in Q \Leftrightarrow<I(\alpha), \beta>\in Q^{\prime} \Leftrightarrow\left(\alpha_{Q},<I(\alpha), \beta>\right) \in \mathcal{U}^{\Pi_{n}^{1}, 2^{\omega}} \\
& \Leftrightarrow b\left(\mathcal{R}\left(\alpha_{Q},<I(\alpha), \beta>\right)\right) \in C \Leftrightarrow \rho\left(b\left(\mathcal{R}\left(\alpha_{Q},<I(\alpha), \beta>\right)\right)\right) \in A .
\end{aligned}
$$

We set $G:=\rho\left(b\left(\mathcal{R}\left(\alpha_{Q},<I(\alpha), .>\right)\right)\right)$, so that $G: 2^{\omega} \rightarrow \omega^{\omega}$ is $\Delta_{1}^{1}(\alpha)$-recursive and $<\beta^{0}, \delta^{0}>$ is in $B^{*}$ if and only if $G\left(<\beta^{0}, \delta^{0}>\right) \in A$.

- As in [K3], we can find $F: 2^{<\omega} \rightarrow\left(2^{\omega} \times \omega\right)^{<\omega}$ satisfying the following properties:
(1) $t \subseteq t^{\prime} \Rightarrow F(t) \subseteq F\left(t^{\prime}\right)$
(2) $|F(t)|=|t|+1$
(3) (i) if $F(\emptyset)=\left(\beta^{0}, k_{0}\right)$, then $\beta^{0} \in \mathcal{D} \wedge \forall \delta^{0} \in\left[\mathcal{O}\left(\beta^{0}\right)\right] G\left(<\beta^{0}, \delta^{0}>\right)(0)=k_{0}$
(ii) if $F\left(\varepsilon_{0}, \ldots, \varepsilon_{n}\right)=\left(\beta^{0}, k_{0}, \beta^{1}, k_{1}, \ldots, \beta^{n+1}, k_{n+1}\right)$, then
(a) $\forall i \leq n+1 \quad \beta^{i} \in \mathcal{D}$
(b) for all $\delta^{n+1} \in\left[\mathcal{O}\left(\beta^{n+1}\right)\right]$, if $\delta^{n}, \ldots, \delta^{0}$ are the uniquely determined members of $\left[\mathcal{O}\left(\beta^{n}\right)\right], \ldots,\left[\mathcal{O}\left(\beta^{0}\right)\right]$ such that $\forall i \leq n\left(\beta^{i}\right)^{*}\left(\delta^{i}\right)=<\overline{\varepsilon_{i}}, \beta^{i+1}, \delta^{i+1}>$, where $\overline{\varepsilon_{i}}=1^{\varepsilon_{i}} 01^{\infty}$, then $\forall i \leq n+1 \quad G\left(<\beta^{0}, \delta^{0}>\right)(i)=k_{i}$.

We will need an effective version of this, so that we give the details of the construction of $F$. In fact, the $\beta^{i}$ 's involved in the definition of $F$ can be $\Delta_{1}^{1}(\alpha)$. In order to see this, we first define

$$
H_{0}: 2^{\omega} \times 2^{\omega} \rightarrow \omega
$$

by $H_{0}(\beta, \delta):=G(<\beta, \delta>)(0)$. As $G$ is $\Delta_{1}^{1}(\alpha)$-recursive, $H_{0}$ too, and the niceness of the coding system gives $\beta^{0} \in \mathcal{D} \cap \Delta_{1}^{1}(\alpha)$ and $k_{0} \in \omega$ such that $G\left(<\beta^{0}, \delta^{0}>\right)(0)=k_{0}$ for each $\delta^{0} \in\left[\mathcal{O}\left(\beta^{0}\right)\right]$. Now suppose that $n \in \omega,\left(\varepsilon_{0}, \ldots, \varepsilon_{n}\right)$ and $F\left(\varepsilon_{0}, \ldots, \varepsilon_{n-1}\right)=\left(\beta^{0}, k_{0}, \ldots, \beta^{n}, k_{n}\right)$ are given. We define

$$
H_{n+1}: 2^{\omega} \times 2^{\omega} \rightarrow \omega
$$

as follows. Given $(\beta, \delta) \in 2^{\omega} \times 2^{\omega}$, let $\delta^{n}, \ldots, \delta^{0}$ be the uniquely determined members of $\left[\mathcal{O}\left(\beta^{n}\right)\right], \ldots$, $\left[\mathcal{O}\left(\beta^{0}\right)\right]$ resp., such that $\left(\beta^{n}\right)^{*}\left(\delta^{n}\right)=<\overline{\varepsilon_{n}}, \beta, \delta>$, and $\left(\beta^{i}\right)^{*}\left(\delta^{i}\right)=<\overline{\varepsilon_{i}}, \beta^{i+1}, \delta^{i+1}>$ if $i<n$. Put $H_{n+1}(\beta, \delta):=G\left(<\beta^{0}, \delta^{0}>\right)(n+1)$. As $H_{n+1}$ is $\Delta_{1}^{1}(\alpha)$ (it is total and $\Pi_{1}^{1}(\alpha)$-recursive since $\iota$ is $\Pi_{1}^{1}$-recursive), the niceness of the coding system gives $\beta^{n+1} \in \mathcal{D} \cap \Delta_{1}^{1}(\alpha)$ and $k_{n+1} \in \omega$ such that $G\left(<\beta^{0}, \delta^{0}>\right)(n+1)=k_{n+1}$ for each $\delta^{n+1} \in\left[\mathcal{O}\left(\beta^{n+1}\right)\right]$. Then

$$
F\left(\varepsilon_{0}, \ldots, \varepsilon_{n}\right):=\left(\beta^{0}, k_{0}, \ldots, \beta^{n+1}, k_{n+1}\right)
$$

so that $F$ is as desired. So we can assume that the $\beta^{i}$, s are $\Delta_{1}^{1}(\alpha)$ in the conditions required for $F$.

- By [K3] again, the map $h_{\alpha}:\left(\varepsilon_{i}\right) \mapsto\left(k_{i}\right)$ is continuous and $\mathcal{U}_{\alpha}^{\Pi_{n}^{1}, 2^{\omega}}=h_{\alpha}^{-1}(A)$. As this is not too long to prove, we give the details for completeness. The map $h_{\alpha}$ is in fact more than continuous: it is Lipschitz, by definition. Fix $\left(\varepsilon_{i}\right)$. We apply $F$ to the initial segments of $\left(\varepsilon_{i}\right)$, which gives $\left(\beta^{i}\right)$. For each $n$, we define perfect sets $C_{0}^{n}, C_{1}^{n}, \ldots, C_{n}^{n} \subseteq 2^{\omega}$ with $C_{i}^{n} \subseteq\left[\mathcal{O}\left(\beta^{i}\right)\right]$ if $i \leq n$, as follows:

$$
\begin{aligned}
& C_{n}^{n}:=\left\{\delta^{n} \in\left[\mathcal{O}\left(\beta^{n}\right)\right] \mid \exists \delta^{n+1} \in 2^{\omega}\left(\beta^{n}\right)^{*}\left(\delta^{n}\right)=<\overline{\varepsilon_{n}}, \beta^{n+1}, \delta^{n+1}>\right\} \\
& C_{n-1}^{n}:=\left\{\delta^{n-1} \in\left[\mathcal{O}\left(\beta^{n-1}\right)\right] \mid \exists \delta^{n} \in C_{n}^{n}\left(\beta^{n-1}\right)^{*}\left(\delta^{n-1}\right)=<\overline{\varepsilon_{n-1}}, \beta^{n}, \delta^{n}>\right\} \\
& \ldots \\
& C_{0}^{n}:=\left\{\delta^{0} \in\left[\mathcal{O}\left(\beta^{0}\right)\right] \mid \exists \delta^{1} \in C_{1}^{n}\left(\beta^{0}\right)^{*}\left(\delta^{0}\right)=<\overline{\varepsilon_{0}}, \beta^{1}, \delta^{1}>\right\}
\end{aligned}
$$

Note that
(4) $\delta^{0} \in C_{0}^{n} \Rightarrow<\beta^{i}, \delta^{i}>\in \operatorname{Domain}(e)$ for each $i \leq n$, where $\delta^{1}, \ldots, \delta^{n}$ are computed according to the formula in (3). $(i i) \cdot(b)$,
(5) $n^{\prime} \geq n \Rightarrow \forall i \leq n \quad C_{i}^{n^{\prime}} \subseteq C_{i}^{n}$.

This implies that $\left[\mathcal{O}\left(\beta^{0}\right)\right] \supseteq C_{0}^{0} \supseteq C_{0}^{1} \supseteq C_{0}^{2} \supseteq \ldots$ and $\bigcap_{n \in \omega} C_{0}^{n}$ contains some $\delta^{0}$. Note that $<\beta^{i}, \delta^{i}>$ is in Domain $(e)$, and $\left(\beta^{i}\right)^{*}\left(\delta^{i}\right)=<\overline{\varepsilon_{i}}, \beta^{i+1}, \delta^{i+1}>$ for each $i \in \omega$. By (3).(ii).(b),

$$
G\left(<\beta^{0}, \delta^{0}>\right)=k_{i}
$$

for each $i \in \omega$. As $<\beta^{0}, \delta^{0}>\in B^{*} \Leftrightarrow G\left(<\beta^{0}, \delta^{0}>\right) \in A$,

$$
\left(\forall i \in \omega<\beta^{i}, \delta^{i}>\in \operatorname{Domain}(e) \wedge\left(\varepsilon_{i}\right) \in \mathcal{U}_{\alpha}^{\Pi_{n}^{1}, 2^{\omega}}\right) \Leftrightarrow\left(k_{i}\right) \in A
$$

As $<\beta^{i}, \delta^{i}>$ is in Domain $(e)$ for each $i \in \omega,\left(\varepsilon_{i}\right) \in \mathcal{U}_{\alpha}^{\boldsymbol{\Pi}_{n}^{1}, 2^{\omega}} \Leftrightarrow h_{\alpha}\left(\left(\varepsilon_{i}\right)\right)=\left(k_{i}\right) \in A$.

- So we found, for each $\alpha \in \omega^{\omega}, h_{\alpha} \in \mathcal{C}\left(2^{\omega}, \omega^{\omega}\right)$ such that $\mathcal{U}_{\alpha}^{\Pi_{n}^{1}}, 2^{\omega}=\left(\pi \circ h_{\alpha}\right)^{-1}(C)=\left(\mu\left(h_{\alpha}\right)\right)^{-1}(C)$. It remains to see that the map $\psi: \alpha \mapsto h_{\alpha}$, from $\omega^{\omega}$ into $\mathcal{C}\left(2^{\omega}, \omega^{\omega}\right)$, can be $\Delta_{1}^{1}$-recursive (then $f$ will be $\mu \circ \psi$ ). By the previous discussion, it is enough to see that the relation " $k_{i}=k$ " is $\Delta_{1}^{1}$ in $\left(\alpha,\left(\varepsilon_{i}\right), i, k\right) \in \omega^{\omega} \times 2^{\omega} \times \omega^{2}$.
- We will define, by primitive recursion, a $\Delta_{1}^{1}$-recursive map $\tilde{f}: \omega \times \omega^{\omega} \times 2^{\omega} \rightarrow 2^{\omega} \times \omega$ such that $\tilde{f}\left(n, \alpha,\left(\varepsilon_{i}\right)\right)$ will be of the form $\left(<\tilde{\beta}^{0}, \ldots, \tilde{\beta}^{n}, \tilde{\beta}^{n}, \ldots>,<\tilde{k}^{0}, \ldots, \tilde{k}^{n}>\right)$ and can play the role of $F\left(\varepsilon_{0}, \ldots, \varepsilon_{n-1}\right)$. We first set

$$
\begin{aligned}
& P:=\left\{\left(\alpha,\left(\varepsilon_{i}\right), \beta, k\right) \in \omega^{\omega} \times\left(2^{\omega}\right)^{2} \times \omega \mid\right. \\
&\left.\forall i \in \omega(\beta)_{i}=(\beta)_{0} \in \mathcal{D} \cap \Delta_{1}^{1}(\alpha) \wedge \forall \delta \in\left[\mathcal{O}\left((\beta)_{0}\right)\right] G\left(<(\beta)_{0}, \delta>\right)(0)=k\right\} .
\end{aligned}
$$

Note that $P$ is $\Pi_{1}^{1}$ and for any $\left(\alpha,\left(\varepsilon_{i}\right)\right) \in \omega^{\omega} \times 2^{\omega}$ there is $(\beta, k) \in 2^{\omega} \times \omega$ such that $\left(\alpha,\left(\varepsilon_{i}\right), \beta, k\right) \in P$. The uniformization lemma gives a $\Delta_{1}^{1}$-recursive map $\tilde{g}: \omega^{\omega} \times 2^{\omega} \rightarrow 2^{\omega} \times \omega$ such that

$$
\left(\alpha,\left(\varepsilon_{i}\right), \tilde{g}\left(\alpha,\left(\varepsilon_{i}\right)\right)\right) \in P
$$

for each $\left(\alpha,\left(\varepsilon_{i}\right)\right) \in \omega^{\omega} \times 2^{\omega}$. Then we set

$$
D:=\left\{\left(\beta, p, n, \alpha,\left(\varepsilon_{i}\right)\right) \in 2^{\omega} \times \omega^{2} \times \omega^{\omega} \times 2^{\omega} \mid \operatorname{Seq}(p) \wedge \operatorname{lh}(p)=n+1 \wedge \forall q \in \omega(\beta)_{q} \in \mathcal{D} \cap \Delta_{1}^{1}(\alpha)\right\} .
$$

Note that $D$ is $\Pi_{1}^{1}$, as well as

$$
\begin{array}{r}
R:=\left\{\left(\beta, p, n, \alpha,\left(\varepsilon_{i}\right), \beta^{\prime}, k^{\prime}\right) \in D \times 2^{\omega} \times \omega \mid \forall i>n\left(\beta^{\prime}\right)_{i}=\left(\beta^{\prime}\right)_{n+1} \in \mathcal{D} \cap \Delta_{1}^{1}(\alpha) \wedge\right. \\
\operatorname{Seq}\left(k^{\prime}\right) \wedge \operatorname{lh}\left(k^{\prime}\right)=n+2 \wedge \forall i \leq n\left(\beta^{\prime}\right)_{i}=(\beta)_{i} \wedge\left(k^{\prime}\right)_{i}=(p)_{i} \wedge \\
\forall \delta \in 2^{\omega}\left(\exists i \leq n+1(\delta)_{i} \notin\left[\mathcal{O}\left(\left(\beta^{\prime}\right)_{i}\right)\right] \vee \exists i \leq n\left(\beta^{\prime}\right)_{i}^{*}\left((\delta)_{i}\right) \neq<\overline{\varepsilon_{i}},\left(\beta^{\prime}\right)_{i+1},(\delta)_{i+1}>\vee\right. \\
\left.\left.\forall i \leq n+1 G\left(<\left(\beta^{\prime}\right)_{0},(\delta)_{0}>\right)(i)=\left(k^{\prime}\right)_{i}\right)\right\} .
\end{array}
$$

Moreover, for each $\left(\beta, p, n, \alpha,\left(\varepsilon_{i}\right)\right) \in D=\Pi_{2^{\omega} \times \omega^{2} \times \omega^{\omega} \times 2^{\omega}}[R]$ there is $\left(\beta^{\prime}, k^{\prime}\right) \in\left(2^{\omega} \cap \Delta_{1}^{1}(\alpha)\right) \times \omega$ such that $\left(\beta, p, n, \alpha,\left(\varepsilon_{i}\right), \beta^{\prime}, k^{\prime}\right) \in R$. The uniformization lemma gives a partial map

$$
\tilde{h}: 2^{\omega} \times \omega^{2} \times \omega^{\omega} \times 2^{\omega} \rightarrow 2^{\omega} \times \omega
$$

which is $\Pi_{1}^{1}$-recursive on its domain $D$, and such that $\left(\beta, p, n, \alpha,\left(\varepsilon_{i}\right), \tilde{h}\left(\beta, p, n, \alpha,\left(\varepsilon_{i}\right)\right)\right) \in R$ if $\left(\beta, p, n, \alpha,\left(\varepsilon_{i}\right)\right) \in D$. This implies that the partial map $\tilde{f}$ defined by

$$
\left\{\begin{array}{l}
\tilde{f}\left(0, \alpha,\left(\varepsilon_{i}\right)\right):=\tilde{g}\left(\alpha,\left(\varepsilon_{i}\right)\right), \\
\tilde{f}\left(n+1, \alpha,\left(\varepsilon_{i}\right)\right):=\tilde{h}\left(\tilde{f}\left(n, \alpha,\left(\varepsilon_{i}\right)\right), n, \alpha,\left(\varepsilon_{i}\right)\right),
\end{array}\right.
$$

is $\Pi_{1}^{1}$-recursive.

Moreover, an induction shows that $\left(\tilde{f}\left(n, \alpha,\left(\varepsilon_{i}\right)\right), n, \alpha,\left(\varepsilon_{i}\right)\right) \in D$ for each $\left(n, \alpha,\left(\varepsilon_{i}\right)\right)$, so that $\tilde{f}$ is in fact total, and thus $\Delta_{1}^{1}$-recursive. More precisely, $\tilde{f}\left(n, \alpha,\left(\varepsilon_{i}\right)\right)$ is of the form

$$
\left(<\beta^{0}, \ldots, \beta^{n}, \beta^{n}, \ldots>,<k_{0}, \ldots, k_{n}>\right),
$$

where $\left(\varepsilon_{0}, \ldots, \varepsilon_{n-1}\right) \mapsto\left(\beta^{0}, k_{0}, \ldots, \beta^{n}, k_{n}\right)$ satisfies the properties (1)-(3) of $F$. It remains to note that $k_{i}=\tilde{f}\left(i, \alpha,\left(\varepsilon_{i}\right)\right)(1)(i)$.

- We now prove the consequences of our main tool.

Definition 5.8 Let $\Gamma$ be a class of subsets of recursively presented Polish spaces, $\boldsymbol{\Gamma}$ be the corresponding boldface class, $X, Y$ be recursively presented Polish spaces, and $\mathcal{U} \in \Gamma(Y \times X)$. We say that $\mathcal{U}$ is effectively uniformly $Y$-universal for the $\Gamma$ subsets of $X$ if the following hold:
(1) $\boldsymbol{\Gamma}(X)=\left\{\mathcal{U}_{y} \mid y \in Y\right\}$,
(2) $\Gamma(X)=\left\{\mathcal{U}_{y} \mid y \in Y \Delta_{1}^{1}\right.$-recursive $\}$,
(3) for each $S \in \boldsymbol{\Gamma}\left(\omega^{\omega} \times X\right)$, there is a Borel map $b: \omega^{\omega} \rightarrow Y$ such that $S_{\alpha}=\mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$,
(4) for each $S \in \Gamma\left(\omega^{\omega} \times X\right)$, there is a $\Delta_{1}^{1}$-recursive map $b: \omega^{\omega} \rightarrow Y$ such that $S_{\alpha}=\mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$.

Notation. Let $\mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} \in \Pi_{1}^{1}$ be a good $\omega^{\omega}$-universal for the $\Pi_{1}^{1}$ subsets of $2^{\omega}, X_{1}$ be a recursively presented Polish space, and $\mathcal{C}_{1}$ be a $\Pi_{1}^{1}$ subset of $X_{1}$ for which there is a $\Delta_{1}^{1}$-recursive map $b: \omega^{\omega} \rightarrow X_{1}$ such that

$$
(\alpha, \beta) \in \mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} \Leftrightarrow b(<\alpha, \beta>) \in \mathcal{C}_{1}
$$

if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. We define, for each natural number $n \geq 1$,

- $X_{n+1}:=\mathcal{C}\left(2^{\omega}, X_{n}\right)$ (inductively),
- $\mathcal{C}_{n+1}:=\left\{h \in X_{n+1} \mid \forall \beta \in 2^{\omega} \quad h(\beta) \notin \mathcal{C}_{n}\right\}$ (inductively),
- $\mathcal{U}_{n}:=\left\{(h, \beta) \in X_{n+1} \times 2^{\omega} \mid h(\beta) \in \mathcal{C}_{n}\right\}$.

Theorem 5.9 Let $n \geq 1$ be a natural number. Then
(a) the set $\mathcal{U}_{n}$ is effectively uniformly $X_{n+1}$-universal for the $\Pi_{n}^{1}$ subsets of $2^{\omega}$,
(b) the set $\mathcal{C}_{n}$ is $\Pi_{n}^{1}$-complete.

Proof. We argue by induction on $n$.
(a) Assume first that $n=1$, and fix $S \in \Pi_{1}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$. Our assumption gives $b_{1}: \omega^{\omega} \rightarrow X_{1}$. As $\mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} \in \Pi_{1}^{1}$ is a good $\omega^{\omega}$-universal for the $\Pi_{1}^{1}$ subsets of $2^{\omega}$, there is by Theorem 5.3] a $\Delta_{1}^{1}$-recursive map $f_{1}: \omega^{\omega} \rightarrow \mathcal{C}\left(2^{\omega}, X_{1}\right)$ such that $(\alpha, \beta) \in \mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} \Leftrightarrow f_{1}(\alpha)(\beta) \in \mathcal{C}_{1}$ if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Let $\alpha_{S} \in \omega^{\omega}$ with $S=\mathcal{U}_{\alpha_{S}}^{\Pi_{1}^{1}, \omega^{\omega} \times 2^{\omega}}$. Note that

$$
(\alpha, \beta) \in S \Leftrightarrow\left(\mathcal{R}\left(\alpha_{S}, \alpha\right), \beta\right) \in \mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} \Leftrightarrow f_{1}\left(\mathcal{R}\left(\alpha_{S}, \alpha\right)\right)(\beta) \in \mathcal{C}_{1} \Leftrightarrow\left(f_{1}\left(\mathcal{R}\left(\alpha_{S}, \alpha\right)\right), \beta\right) \in \mathcal{U}_{1} .
$$

As $\mathcal{C}_{1}$ is $\Pi_{1}^{1}, \mathcal{U}_{1}$ too. If $A \in \Pi_{1}^{1}\left(2^{\omega}\right)$, then $A=\mathcal{U}_{\alpha}^{\Pi_{1}^{1}, 2^{\omega}}$ for some $\alpha \in \omega^{\omega}$. Applying the previous discussion to $S:=\mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}}$, we get $A=\left(\mathcal{U}_{1}\right)_{f_{1}\left(\mathcal{R}\left(\alpha_{S}, \alpha\right)\right)}$, so that $\mathcal{U}_{1}$ is $X_{2}$-universal for the $\Pi_{1}^{1}$ subsets of $2^{\omega}$, effectively and uniformly.

We now study $\mathcal{U}_{n+1}$. Fix $S \in \boldsymbol{\Pi}_{n+1}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$. Let $\mathcal{U}^{\boldsymbol{\Pi}_{n}^{1}, 2^{\omega}}$ be a good $\omega^{\omega}$-universal for the $\boldsymbol{\Pi}_{n}^{1}$ subsets of $2^{\omega}$. We set $\mathcal{V}^{\Pi_{n+1}^{1}, 2^{\omega}}:=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall \delta \in 2^{\omega} \quad(\mathcal{R}(\alpha, \beta), \delta) \notin \mathcal{U}^{\Pi_{n}^{1}, 2^{\omega}}\right\}$, so that $\mathcal{V}^{\Pi_{n+1}^{1}, 2^{\omega}}$ is a suitable $\omega^{\omega}$-universal for the $\Pi_{n+1}^{1}$ subsets of $2^{\omega}$. Moreover, the induction assumption gives a $\Delta_{1}^{1}$-recursive map $b_{n+1}: \omega^{\omega} \rightarrow X_{n+1}$ such that

$$
\begin{aligned}
(\alpha, \beta) \in \mathcal{V}^{\Pi_{n+1}^{1}, 2^{\omega}} & \Leftrightarrow \forall \delta \in 2^{\omega} \quad(\mathcal{R}(\alpha, \beta), \delta) \notin \mathcal{U}_{n}^{\Pi_{n}^{1}, 2^{\omega}} \Leftrightarrow \forall \delta \in 2^{\omega} \quad\left(b_{n+1}(\mathcal{R}(\alpha, \beta)), \delta\right) \notin \mathcal{U}_{n} \\
& \Leftrightarrow \forall \delta \in 2^{\omega} \quad b_{n+1}(\mathcal{R}(\alpha, \beta))(\delta) \notin \mathcal{C}_{n} \Leftrightarrow b_{n+1}(\mathcal{R}(\alpha, \beta)) \in \mathcal{C}_{n+1}
\end{aligned}
$$

Theorem 5.3 gives a $\Delta_{1}^{1}$-recursive map $f_{n+1}$ such that $(\alpha, \beta) \in \mathcal{V}^{\Pi_{n+1}^{1}, 2^{\omega}} \Leftrightarrow f_{n+1}(\alpha)(\beta) \in \mathcal{C}_{n+1}$ if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Let

$$
Q \in \boldsymbol{\Pi}_{n}^{1}\left(\omega^{\omega} \times 2^{\omega} \times 2^{\omega}\right) \subseteq \boldsymbol{\Pi}_{n}^{1}\left(\omega^{\omega} \times \omega^{\omega} \times 2^{\omega}\right)
$$

such that $(\alpha, \beta) \in S \Leftrightarrow \forall \delta \in 2^{\omega} \quad(\alpha, \beta, \delta) \notin Q$, and $\alpha_{Q} \in \omega^{\omega}$ such that $Q=\mathcal{U}_{\alpha_{Q}}^{\Pi_{n}^{1}, \omega^{\omega} \times \omega^{\omega} \times 2^{\omega}}$. Note that

$$
\begin{aligned}
(\alpha, \beta) \in S & \Leftrightarrow \forall \delta \in 2^{\omega}\left(\mathcal{R}\left(\mathcal{R}^{\prime}\left(\alpha_{Q}, \alpha\right), \beta\right), \delta\right) \notin \mathcal{U}_{n}^{\Pi_{n}^{1}, 2^{\omega}} \Leftrightarrow\left(\mathcal{R}^{\prime}\left(\alpha_{Q}, \alpha\right), \beta\right) \in \mathcal{V}^{\Pi_{n+1}^{1}, 2^{\omega}} \\
& \Leftrightarrow f_{n+1}\left(\mathcal{R}^{\prime}\left(\alpha_{Q}, \alpha\right)\right)(\beta) \in \mathcal{C}_{n+1} \Leftrightarrow\left(f_{n+1}\left(\mathcal{R}^{\prime}\left(\alpha_{Q}, \alpha\right)\right), \beta\right) \in \mathcal{U}_{n+1} .
\end{aligned}
$$

As $\mathcal{C}_{n} \in \Pi_{n}^{1}, \mathcal{C}_{n+1} \in \Pi_{n+1}^{1}$ and $\mathcal{U}_{n+1} \in \Pi_{n+1}^{1}$. If $A \in \boldsymbol{\Pi}_{n+1}^{1}\left(2^{\omega}\right)$, then $A=\mathcal{U}_{\alpha}^{\Pi_{n+1}^{1}, 2^{\omega}}$ for some $\alpha \in \omega^{\omega}$. Applying the previous discussion to $S:=\mathcal{U}^{\boldsymbol{\Pi}_{n+1}^{1}, 2^{\omega}}$, we get $A=\left(\mathcal{U}_{n+1}\right)_{f_{n+1}\left(\mathcal{R}^{\prime}\left(\alpha_{Q}, \alpha\right)\right)}$, so that $\mathcal{U}_{n+1}$ is $X_{n+2}$-universal for the analytic subsets of $2^{\omega}$, effectively and uniformly.
(b) By definition, $\mathcal{C}_{1} \in \Pi_{1}^{1}$, and $\mathcal{C}_{n+1} \in \Pi_{n+1}^{1}$ if $\mathcal{C}_{n} \in \Pi_{n}^{1}$. Assume first that $E \in \Pi_{n}^{1}\left(2^{\omega}\right)$. Then $E=\left(\mathcal{U}_{n}\right)_{h}$ for some $h \in \mathcal{C}\left(2^{\omega}, X_{n}\right)$, by (a). Thus $E=h^{-1}\left(\mathcal{C}_{n}\right)$. If $Z$ is a zero-dimensional Polish space and $D \in \boldsymbol{\Pi}_{n}^{1}(Z)$, then we may assume that $Z$ is a $G_{\delta}$ subset of $2^{\omega}$ by 7.8 in [K2], so that $D \in \boldsymbol{\Pi}_{n}^{1}\left(2^{\omega}\right)$. The previous discussion gives $g \in \mathcal{C}\left(2^{\omega}, X_{n}\right)$ with $D=g^{-1}\left(\mathcal{C}_{n}\right)$. Thus $D=\left(g_{\mid Z}\right)^{-1}\left(\mathcal{C}_{n}\right)$ and $\mathcal{C}_{n}$ is $\boldsymbol{\Pi}_{n}^{1}$-complete.

Proof of Theorem 1.7, By Theorem 5.9] it is enough to show that if $\mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} \in \Pi_{1}^{1}$ is a good $\omega^{\omega}$ universal set for the $\Pi_{1}^{1}$ subsets of $2^{\omega}$, then there is a $\Delta_{1}^{1}$-recursive map $b: \omega^{\omega} \rightarrow[0,1]^{2<\omega}$ such that $(\alpha, \beta) \in \mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} \Leftrightarrow b(<\alpha, \beta>) \in \mathcal{P}$ if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Let $H \in \Pi_{2}^{0}\left(\omega^{\omega} \times 2^{\omega} \times 2^{\omega}\right)$ such that $\neg \mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}}=\Pi_{\omega^{\omega} \times 2^{\omega}}[H]$. We set $G:=\left\{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid\left((\alpha)_{0},(\alpha)_{1},(\beta)_{1}\right) \in H \wedge \beta \in \mathcal{K}\right\}$, so that $G \in \Delta_{1}^{1}\left(\omega^{\omega} \times 2^{\omega}\right)$, has $G_{\delta}$ vertical sections and $G \subseteq \omega^{\omega} \times \mathcal{K}$. Lemma3.10 gives a $\Delta_{1}^{1}$-recursive map $F: \omega^{\omega} \rightarrow[0,1]^{2<\omega}$, taking values in $\mathcal{M}$, and such that $G_{\alpha}=\mathcal{V}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$. If $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$, then

$$
\begin{aligned}
(\alpha, \beta) \notin \mathcal{U}^{\Pi_{1}^{1}, 2^{\omega}} & \Leftrightarrow \exists \delta \in 2^{\omega} \quad(\alpha, \beta, \delta) \in H \Leftrightarrow \exists \delta \in 2^{\omega} \quad(<\alpha, \beta>, \delta) \in G \\
& \Leftrightarrow \exists \delta \in 2^{\omega}(b(<\alpha, \beta>), \delta) \in \mathcal{V} \Leftrightarrow b(<\alpha, \beta>) \notin \mathcal{P} .
\end{aligned}
$$

This finishes the proof.
Questions. Let $U$ be a $\Pi_{2}^{0}$ subset of $\omega^{\omega} \times 2^{\omega}$ which is universal for $\Pi_{2}^{0}\left(2^{\omega}\right)$. We set

$$
G:=\left\{(\alpha, \beta) \in \omega^{\omega} \times \mathcal{K} \mid\left(\alpha,(\beta)_{1}\right) \in U\right\} .
$$

Note that $G$ is a $\Pi_{2}^{0}$ subset of $\omega^{\omega} \times 2^{\omega}$ contained in $\omega^{\omega} \times \mathcal{K}$ which is universal for $\Pi_{2}^{0}(\mathcal{K})$. Indeed, fix $H \in \boldsymbol{\Pi}_{2}^{0}(\mathcal{K})$. Then $H^{\prime}:=\left\{\gamma \in 2^{\omega} \mid<0^{\infty}, \gamma>\in H\right\}$ is $\boldsymbol{\Pi}_{2}^{0}$, which gives $\alpha_{0} \in \omega^{\omega}$ with $H^{\prime}=U_{\alpha_{0}}$. Then $H=G_{\alpha_{0}}$.

Let $\alpha \mapsto\left((\alpha)_{k}\right)_{k \in \omega}$ be a homeomorphism between $\omega^{\omega}$ and $\left(\omega^{\omega}\right)^{\omega}$, with inverse map

$$
\left(\alpha_{k}\right)_{k \in \omega} \mapsto<\alpha_{0}, \alpha_{1}, \ldots>
$$

We set $S^{\prime}:=\left\{\alpha \in \omega^{\omega} \mid \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad \forall \beta \in 2^{\omega} \quad \beta \notin G_{(\alpha)_{\gamma(i)}}\right\}$. Note that $S^{\prime}$ is $\Sigma_{2}^{1}$.
(1) Is $S^{\prime}$ a Borel $\boldsymbol{\Sigma}_{2}^{1}$-complete set?

Assume that this is the case. Then the set $\mathcal{S}_{2}:=\left\{\left(f_{k}\right)_{k \in \omega} \in \mathcal{M}^{\omega} \mid \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad f_{\gamma(i)} \in \mathcal{P}\right\}$ of sequences of martingales having a subsequence made of everywhere converging martingales is Borel $\boldsymbol{\Sigma}_{2}^{1}$-complete. Indeed, Lemma 3.10 gives a Borel map $F: \omega^{\omega} \rightarrow \mathcal{M}$ such that $G_{\alpha}=\mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^{\omega}$. The map $\tilde{F}: \omega^{\omega} \rightarrow \mathcal{M}^{\omega}$ defined by $\tilde{F}(\alpha)(k):=F\left((\alpha)_{k}\right)$ is Borel. Moreover,

$$
\begin{aligned}
\tilde{F}(\alpha) \in \mathcal{S}_{2} & \Leftrightarrow \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad \forall \beta \in 2^{\omega} \quad \beta \notin D\left(F\left((\alpha)_{\gamma(i)}\right)\right) \\
& \Leftrightarrow \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad \forall \beta \in 2^{\omega} \quad \beta \notin \mathcal{V}_{F\left((\alpha)_{\gamma(i)}\right)} \\
& \Leftrightarrow \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad \forall \beta \in 2^{\omega} \quad \beta \notin G_{(\alpha)_{\gamma(i)}} \\
& \Leftrightarrow \alpha \in S^{\prime},
\end{aligned}
$$

so that $S^{\prime}=\tilde{F}^{-1}\left(\mathcal{S}_{2}\right)$.
(2) Is there a Borel map $f: \mathcal{C}\left(2^{\omega},[0,1]\right) \rightarrow \omega^{\omega}$ such that, for each $\left(h_{k}\right)_{k \in \omega} \in\left(\mathcal{C}\left(2^{\omega},[0,1]\right)\right)^{\omega}$ and each $\beta \in 2^{\omega}$, the following are equivalent:
(a) $\lim _{k \rightarrow \infty} h_{k}(\beta)=0$,
(b) $\forall k \in \omega \quad \beta \notin G_{f\left(h_{k}\right)}$ ?

Assume that this is the case. Then $S^{\prime}$ (and therefore $\mathcal{S}_{2}$ ) is Borel $\boldsymbol{\Sigma}_{2}^{1}$-complete, and thus $\boldsymbol{\Sigma}_{2}^{1}$ complete (see [P]). We define $F:\left(\mathcal{C}\left(2^{\omega},[0,1]\right)\right)^{\omega} \rightarrow \omega^{\omega}$ by $F\left(\left(h_{k}\right)_{k \in \omega}\right):=<f\left(h_{0}\right), f\left(h_{1}\right), \ldots>$, so that $F$ is Borel. Note that

$$
\begin{aligned}
F\left(\left(h_{k}\right)_{k \in \omega}\right) \in S^{\prime} & \Leftrightarrow \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad \forall \beta \in 2^{\omega} \quad \beta \notin G_{f\left(h_{\gamma(i)}\right)} \\
& \Leftrightarrow \exists \gamma \in \omega^{\omega} \forall \beta \in 2^{\omega} \lim _{i \rightarrow \infty} h_{\gamma(i)}(\beta)=0 \\
& \Leftrightarrow\left(h_{k}\right)_{k \in \omega} \in S,
\end{aligned}
$$

so that $S=F^{-1}\left(S^{\prime}\right)$.

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