Universal and complete sets in martingale theory

Dominique LECOMTE and Miroslav ZELENÝ¹

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 Université Paris 6, Institut de Mathématiques de Jussieu, Projet Analyse Fonctionnelle Couloir 16-26, 4ème étage, Case 247, 4, place Jussieu, 75 252 Paris Cedex 05, France dominique.lecomte@upmc.fr

> • Université de Picardie, I.U.T. de l'Oise, site de Creil, 13, allée de la faïencerie, 60 107 Creil, France

 ¹ Charles University, Faculty of Mathematics and Physics, Department of Mathematical Analysis Sokolovská 83, 186 75 Prague, Czech Republic zeleny@karlin.mff.cuni.cz

Abstract. The Doob convergence theorem implies that the set of divergence of any martingale has measure zero. We prove that, conversely, any $G_{\delta\sigma}$ subset of the Cantor space with Lebesgue-measure zero can be represented as the set of divergence of some martingale. In fact, this is effective and uniform. A consequence of this is that the set of everywhere converging martingales is Π_1^1 -complete, in a uniform way. We derive from this some universal and complete sets for the whole projective hierarchy, via a general method. We provide some other complete sets for the classes Π_1^1 and Σ_2^1 in the theory of martingales.

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1 Introduction

The reader should see [K2] for the notation used in this paper.

Definition 1.1 We say that a map $f: 2^{<\omega} \to [0,1]$ is a martingale if $f(s) = \frac{f(s0)+f(s1)}{2}$ for each $s \in 2^{<\omega}$. The set of martingales is denoted by \mathcal{M} and is a compact subset of $[0,1]^{2^{<\omega}}$ (equipped with the usual product topology).

This terminology is not the standard one, but the set \mathcal{M} can be interpreted as the set of all discrete martingales (in the classical sense) taking values in [0,1], as follows. If $s \in 2^{<\omega}$, then

$$N_s := \{\beta \in 2^\omega \mid s \subseteq \beta\}$$

is the usual basic clopen set. Let $f \in \mathcal{M}$. If $n \in \omega$, then let \mathcal{S}_n be the σ -algebra on 2^{ω} generated by $\{N_s \mid s \in 2^n\}$, and $f_n : 2^{\omega} \to [0, 1]$ be defined by $f_n(\beta) := f(\beta|n)$. Then the sequence $(f_n)_{n \in \omega}$ is a discrete martingale taking values in [0,1] with respect to the sequence of σ -algebras $(\mathcal{S}_n)_{n \in \omega}$ and the usual Lebesgue product measure λ on 2^{ω} . Conversely, if $(f_n)_{n \in \omega}$ is any such martingale, it can be viewed as an element of \mathcal{M} by setting $f(s) := f_{|s|}(\alpha)$ if $\alpha \in N_s$. This definition is correct because $f_{|s|}$, as a function measurable with respect to $\mathcal{S}_{|s|}$, has a constant value on N_s .

Definition 1.2 Let f be a martingale and $\beta \in 2^{\omega}$. The oscillation of f at β is the number

$$osc(f,\beta) := inf_{N \in \omega} sup_{p,q > N} |f(\beta|p) - f(\beta|q)|.$$

The set of divergence of f is $D(f) := \{\beta \in 2^{\omega} \mid osc(f, \beta) > 0\}.$

By definition, if f is a martingale, then

$$\beta \in D(f) \Leftrightarrow \exists r \in \omega \ \forall N \in \omega \ \exists p, q \ge N \ |f(\beta|p) - f(\beta|q)| > 2^{-r}$$

This shows that $D(f) \in \Sigma_3^0$. Moreover, D(f) has λ -measure zero, by Doob's convergence theorem (see Chapter XI, Section 14 in [D]). So it is natural to ask whether any Σ_3^0 subset of 2^{ω} with λ -measure zero is the set of divergence of some martingale (this question was asked by Louveau). We answer positively:

Theorem 1.3 Let B be a subset of 2^{ω} . Then the following are equivalent:

- (a) B is Σ_3^0 and has λ -measure zero,
- (b) there is a martingale f with B = D(f).

Definition 1.4 Let Γ be a class of subsets of Polish spaces, X, Y be Polish spaces, and $U \subseteq Y \times X$.

(a) We say that \mathcal{U} is Y-universal for the Γ subsets of X if $\mathcal{U} \in \Gamma(Y \times X)$ and $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y\}$.

(b) We say that \mathcal{U} is uniformly Y-universal for the Γ subsets of X if \mathcal{U} is Y-universal for the Γ subsets of X and, for each $S \in \Gamma(\omega^{\omega} \times X)$, there is a Borel map $b : \omega^{\omega} \to Y$ such that $S_{\alpha} = \mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$.

Corollary 1.5 Let \mathcal{G} be a G_{δ} subset of 2^{ω} with $\lambda(\mathcal{G}) = 0$. Then the set $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid \beta \in D(f)\}$ is \mathcal{M} -universal for the Σ_3^0 subsets of \mathcal{G} .

In fact, we prove an effective and uniform version of the implication (a) \Rightarrow (b) in Theorem 1.3. In particular, we can associate, via a Borel map F, a martingale to a code α of an arbitrary G_{δ} subset G of \mathcal{G} (as in the previous corollary), in such a way that $G = D(F(\alpha))$. A consequence of this is the following:

Theorem 1.6 The set \mathcal{P} of everywhere converging martingales is Π_1^1 -complete.

These statements are in the spirit of some results concerning the differentiability of functions due to Zahorski and Mazurkiewicz (see Section 4 for details). In fact, \mathcal{P} is Π_1^1 -complete in a uniform way, which allows to derive some universal and complete sets for the whole projective hierarchy, in spaces of continous functions, starting from \mathcal{P} . More precisely, let $P_1 := [0, 1]^{2^{<\omega}}$ and $C_1 := \mathcal{P}$. We define, for each natural number $n \ge 1$,

• the space $P_{n+1} := \mathcal{C}(2^{\omega}, P_n)$ of continuous functions from 2^{ω} into P_n , equipped with the topology of uniform convergence (inductively),

- $C_{n+1} := \{h \in P_{n+1} \mid \forall \beta \in 2^{\omega} \ h(\beta) \notin C_n\}$ (inductively),
- $U_n := \{(h, \beta) \in P_{n+1} \times 2^{\omega} \mid h(\beta) \in C_n\}.$

We prove the following:

Theorem 1.7 Let $n \ge 1$ be a natural number. Then (a) the set U_n is uniformly P_{n+1} -universal for the Π_n^1 subsets of 2^{ω} ,

(b) the set C_n is Π^1_n -complete.

In fact, our method is more general and works if we start with a Π_1^1 set which is complete in a uniform way.

Let f be a martingale. As D(f) has λ -measure zero, we can associate to f the partial function $\psi(f)$ defined λ -almost everywhere by $\psi(f)(\beta) := \lim_{l \to \infty} f(\beta|l)$. The partial function $\psi(f)$ will be called the **associated partial function**. The martingale f is in \mathcal{P} if and only if $\psi(f)$ is total, in which case $\psi(f)$ is called the **associated function**. Using the work in [B-Ka-L] and [K2] about spaces of continuous functions, we prove the following:

Theorem 1.8 (a) The set of sequences of everywhere converging martingales whose associated functions converge pointwise is Π_1^1 -complete.

(b) The set of sequences of everywhere converging martingales whose associated functions converge pointwise to zero is Π_1^1 -complete.

(c) The set of sequences of everywhere converging martingales having a subsequence whose associated functions converge pointwise to zero is Σ_2^1 -complete.

2 Σ_3^0 sets of measure zero

Notation. In the sequel, B will be a Borel subset of 2^{ω} , and M will be a λ -measurable subset of 2^{ω} . If $\beta \in 2^{\omega}$, then the **density of** M **at** β is the number $d(M, \beta) := \lim_{l \to \infty} \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}$ when it is defined. Note that $d(B, \beta) = 1$ if $\beta \in B$ and B is open. We first recall the Lebesgue density theorem (see 17.9 in [K2]).

Theorem 2.1 (Lebesgue) The equality $\lambda(M) = \lambda(\{\beta \in M \mid d(M,\beta) = 1\})$ holds for any λ -measurable subset M of 2^{ω} .

The reader should see [C] for the next lemma. We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.2 (*Lusin-Menchoff*) Let F be a closed subset of 2^{ω} , and $M \supseteq F$ be a λ -measurable subset of 2^{ω} such that $d(M, \beta) = 1$ for each $\beta \in F$. Then there is a closed subset C of 2^{ω} such that

(1) $F \subseteq C \subseteq M$, (2) $d(M, \beta) = 1$ for each $\beta \in C$, (3) $d(C, \beta) = 1$ for each $\beta \in F$.

Proof. If F is 2^{ω} , then we can take C := F. So we may assume that F is not 2^{ω} . We set $s^- := s|(|s|-1)$ if $\emptyset \neq s \in 2^{<\omega}$. Note that $\neg F$ is the disjoint union of the elements of a sequence $(N_{s_n})_{n \in \omega}$, where $N_{s_n} \cap F \neq \emptyset$ for each $n \in \omega$. Fix $n \in \omega$. By Theorem 2.1,

$$\lambda(M \cap N_{s_n}) = \lambda(\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}).$$

The regularity of λ gives a closed subset F_n of 2^{ω} contained in $\{\beta \in M \cap N_{s_n} \mid d(M \cap N_{s_n}, \beta) = 1\}$ such that $\lambda(F_n) \ge (1-2^{-n})\lambda(M \cap N_{s_n})$. We set $C := F \cup \bigcup_{n \in \omega} F_n$, which is closed since $|s_n| \to \infty$.

As Conditions (1) and (2) are clearly satisfied, pick $\beta \in F$. Note that

$$\lambda(N_{\beta|l} \setminus C) = \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus C)$$

$$\leq \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus F_n)$$

$$\leq \sum_{s_n \supseteq \beta|l} 2^{-n} \lambda(M \cap N_{s_n}) + \sum_{s_n \supseteq \beta|l} \lambda(N_{s_n} \setminus M)$$

$$\leq \sum_{s_n \supset \beta|l} 2^{-n} \lambda(N_{s_n}) + \lambda(N_{\beta|l} \setminus M).$$

This implies that the limit of $\frac{\lambda(N_{\beta|l}\setminus C)}{\lambda(N_{\beta|l})}$ is zero since $d(M,\beta) = 1$.

The next topology is considered in [Lu-Ma-Z], see Chapter 6.

Definition 2.3 The τ -topology on 2^{ω} is generated by

$$\mathcal{F} := \{ M \subseteq 2^{\omega} \mid M \text{ is } \lambda \text{-measurable} \land \forall \beta \in M \ d(M, \beta) = 1 \}.$$

The next result is proved in [Lu-Ma-Z], but in a much more abstract way. This is the reason why we include a much more direct proof here, since it is not too long.

Lemma 2.4 The family \mathcal{F} is a topology. In particular, any τ -open set is λ -measurable.

Proof. Note first that \mathcal{F} is closed under finite intersections, so that it is a basis for the τ -topology. Indeed, let M, M' be in \mathcal{F} , and $\beta \in M \cap M'$. Then we use the facts that

$$\lambda(M \cap M' \cap N_{\beta|l}) = \lambda(M \cap N_{\beta|l}) - \lambda((M \cap N_{\beta|l}) \setminus M')$$

and $\lambda((M \cap N_{\beta|l}) \setminus M') \leq \lambda(N_{\beta|l} \setminus M').$

Let \mathcal{H} be a subfamily of \mathcal{F} , and $H := \cup \mathcal{H}$. We claim that there is a countable subfamily \mathcal{C} of \mathcal{H} such that $m := \sup\{\lambda(\cup \mathcal{D}) \mid \mathcal{D} \subseteq \mathcal{H} \text{ countable}\} = \lambda(\cup \mathcal{C})$. Indeed, for each $n \in \omega$ there is $\mathcal{D}_n \subseteq \mathcal{H}$ countable such that $\lambda(\cup \mathcal{D}_n) > m - 2^{-n}$, and $\mathcal{C} := \bigcup_{n \in \omega} \mathcal{D}_n$ is suitable. Let $C := \cup \mathcal{C}$.

Let $\beta \in H$, and M in \mathcal{H} with $\beta \in M$. Note that $\lambda(M \cup C) = \lambda(C)$ (consider the family $\mathcal{C} \cup \{M\}$). Thus $\lambda(M \setminus C) = 0$. As $d(M, \beta) = 1$, the equality $d(M \cap C, \beta) = 1$ holds, and $d(\neg C, \beta) = 0$. This implies that $H \setminus C$ is contained in $\{\beta \notin C \mid d(\neg C, \beta) < 1\}$, which has λ -measure zero by Theorem 2.1. Therefore $H \setminus C$ has λ -measure zero and $H = C \cup (H \setminus C)$ is λ -measurable.

Pick $\beta \in H$, and $M \in \mathcal{H}$ with $\beta \in M$. Then $d(M, \beta) = 1$, and thus $d(H, \beta) = 1$. Therefore $H \in \mathcal{F}$. This finishes the proof.

The next lemma is in the style of Urysohn's theorem (see [Lu] for its version on the real line). We include a proof to be self-contained and also because we will prove an effective and uniform version of it later.

Lemma 2.5 Let C be a closed subset of 2^{ω} , and G be a G_{δ} subset of 2^{ω} disjoint from C such that $\lambda(G) = 0$. Then there is a τ -continuous map $h: 2^{\omega} \to [0, 1]$ such that $h_{|C} \equiv 0$ and $h_{|G} \equiv 1$.

Proof. Let $(F_n)_{n\in\omega}$ be an increasing sequence of closed subsets of 2^{ω} with union $\neg G$ and $F_0 = C$. We first construct a sequence $(C_{\frac{1}{2^n}})_{n\in\omega}$ of closed subsets of 2^{ω} with $F_n \subseteq C_{\frac{1}{2^n}} \subseteq \neg G$, $C_{\frac{1}{2^n}} \subseteq C_{\frac{1}{2^{n+1}}}$, and $d(C_{\frac{1}{2^{n+1}}}, \beta) = 1$ for each $\beta \in C_{\frac{1}{2^n}}$. We first apply Lemma 2.2 to $F := F_0$ and $M := \neg G$, which gives $F_0 \subseteq C_1 \subseteq \neg G$. Then, inductively, we apply Lemma 2.2 to $F := C_{\frac{1}{2^n}} \cup F_{n+1}$ and $M := \neg G$, which gives $C_{\frac{1}{2^n}} \cup F_{n+1} \subseteq C_{\frac{1}{2^{n+1}}} \subseteq \neg G$ such that $d(C_{\frac{1}{2^{n+1}}}, \beta) = 1$ for each $\beta \in C_{\frac{1}{2^n}}$.

Then we construct $C_{\frac{2k+1}{2^n}}$, for $0 < k < 2^{n-1}$ and $n \ge 2$. This will give us a family $(C_{\frac{k}{2^n}})_{n \in \omega, 0 < k \le 2^n}$ of closed subsets of 2^{ω} . We want to ensure that $C_{\zeta} \subseteq C_{\zeta'}$ and $d(C_{\zeta'}, \beta) = 1$ for each $\beta \in C_{\zeta}$ if $\zeta' < \zeta$. We proceed by induction on n. We apply Lemma 2.2 to $F := C_{\frac{k+1}{2^{n-1}}}$ and $M := C_{\frac{k}{2^{n-1}}}$, which gives $C_{\frac{2k+1}{2^n}}$ such that $C_{\frac{k+1}{2^{n-1}}} \subseteq C_{\frac{k}{2^n-1}}$, $d(C_{\frac{k}{2^{n-1}}}, \beta) = 1$ for each $\beta \in C_{\frac{2k+1}{2^n}}$, and $d(C_{\frac{2k+1}{2^n}}, \beta) = 1$ for each $\beta \in C_{\frac{k+1}{2^n}}$. This allows us to define \tilde{h} by

$$\tilde{h}(\beta) := \begin{cases} 0 \text{ if } \beta \in G,\\ \sup\{\zeta \mid \beta \in C_{\zeta}\} \text{ if } \beta \notin G. \end{cases}$$

It remains to see that \tilde{h} is τ -continuous (and then we will set $h(\beta) := 1 - \tilde{h}(\beta)$). So let $b \in (0, 1]$, and $\beta \in 2^{\omega}$ with $\tilde{h}(\beta) < b$. Note that there is $\zeta < b$ with $\tilde{h}(\beta) < \zeta$, so that $\beta \notin C_{\zeta}$. If $\gamma \notin C_{\zeta}$, then $\tilde{h}(\gamma) \le \zeta < b$, so that $\neg C_{\zeta}$ is an open (and thus τ -open since the τ -topology is finer than the usual one) neighborhood of β on which $\tilde{h} < b$. In particular, \tilde{h} is Borel. Now let $a \in [0, 1)$. It is enough to see that $B := \{\gamma \in 2^{\omega} \mid \tilde{h}(\gamma) > a\}$ is τ -open. So assume that $\tilde{h}(\gamma) > a$. Note that there are $\zeta > \zeta' > a$ with $\tilde{h}(\gamma) > \zeta$, so that $\gamma \in C_{\zeta} \subseteq C_{\zeta'} \subseteq B$. Thus $d(C_{\zeta'}, \gamma) = 1$, by construction of the family. As \tilde{h} is Borel, B is Borel, $d(B, \gamma)$ is defined and equal to 1.

Remark. We in fact proved that *h* is lower semi-continuous.

Notation. If $h: 2^{\omega} \to [0, 1]$ is a λ -measurable map and $s \in 2^{<\omega}$, then we set $\oint_{N_s} h \ d\lambda := \frac{\int_{N_s} h \ d\lambda}{\lambda(N_s)}$. Lemma 2.6 Let $h: 2^{\omega} \to [0, 1]$ be a τ -continuous map, and $\beta \in 2^{\omega}$. Then

$$\lim_{l\to\infty} \int_{N_{\beta|l}} h \, d\lambda = h(\beta).$$

Proof. Let $\varepsilon > 0$, and $\beta \in M := h^{-1} (B(h(\beta), \varepsilon))$. Note that $d(M, \gamma) = 1$ for each $\gamma \in M$ since h is τ -continuous. As h is λ -measurable, we can write

$$\int_{N_{\beta|l}} h \, d\lambda = \int_{M \cap N_{\beta|l}} h \, d\lambda + \int_{N_{\beta|l} \setminus M} h \, d\lambda$$

Note that $0 \leq \int_{N_{\beta|l} \setminus M} h \ d\lambda \leq \lambda(N_{\beta|l} \setminus M)$, so that $0 \leq \int_{N_{\beta|l} \setminus M} h \ d\lambda \leq \frac{\lambda(N_{\beta|l} \setminus M)}{\lambda(N_{\beta|l})} \to 0$. Similarly,

$$f_{M \cap N_{\beta|l}} h \, d\lambda \in \left[\left(h(\beta) - \varepsilon \right) \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}, \left(h(\beta) + \varepsilon \right) \frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})} \right],$$

and we are done since $\frac{\lambda(M \cap N_{\beta|l})}{\lambda(N_{\beta|l})}$ tends to 1 as l tends to ∞ .

Now we come to our main lemma, inspired by Zahorski (see [Za]).

Lemma 2.7 Let G be a G_{δ} subset of 2^{ω} with λ -measure zero. Then there is a martingale f with G = D(f) and $\{osc(f,\beta) \mid \beta \in 2^{\omega}\} \subseteq \{0\} \cup [\frac{1}{2}, 1].$

Proof. Let $(G_n)_{n \in \omega}$ be a decreasing sequence of open subsets of 2^{ω} with intersection G and $G_0 = 2^{\omega}$.

• We construct $g_n : 2^{\omega} \to [0, 1]$, open subsets G_n^*, G_n^{**} of 2^{ω} , and a sequence $(s_j^n)_{j \in I_n}$ of pairwise incompatible finite binary sequences, by induction on $n \in \omega$, such that, if $S_n := \sum_{j \leq n} (-1)^j g_j$,

$$\begin{array}{l} (1) \ G \subseteq G_{n+1}^* \subseteq G_n^{**} = \bigcup_{j \in I_n} \ N_{s_j^n} \subseteq G_n^* \subseteq G_n \ \land \ G_0^* = 2^{\omega}, \\ (2) \ g_{n|G} \equiv 1 \ \land \ g_{n|\neg G_n^*} \equiv 0, \\ (3) \ g_n \ \text{is } \tau \text{-continuous,} \\ (4) \ g_{n+1} \leq g_n, \\ (5) \ \lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3} \lambda(N_{s_j^n}), \\ (6) \ | \ \int_{N_{s_j^n}} S_n \ d\lambda - S_n(\beta) | < 2^{-3} \ \text{if } \beta \in G \cap N_{s_j^n}. \end{array}$$

We set $g_0 :\equiv 1, G_0^*, G_0^{**} := 2^{\omega}, I_0 := \{0\}$ and $s_0^0 := \emptyset$. Assume that our objects are constructed up to n. We first construct an open subset G_{n+1}^* of 2^{ω} with $G \subseteq G_{n+1}^* \subseteq G_n^{**} \cap G_{n+1}$ such that

$$\lambda(G_{n+1}^* \cap N_{s_j^n}) < 2^{-n-3}\lambda(N_{s_j^n})$$

if $j \in I_n$. For each $j \in I_n$, there is an open set O_j with $G \cap N_{s_j^n} \subseteq O_j \subseteq G_{n+1} \cap N_{s_j^n}$ such that $\lambda(O_j) < 2^{-n-3}\lambda(N_{s_j^n})$. We then set $G_{n+1}^* := \bigcup_{j \in I_n} O_j$.

We now apply Lemma 2.5 to $C := \neg G_{n+1}^*$ and G, which gives a τ -continuous map $h: 2^{\omega} \to [0, 1]$ with $h_{|\neg G_{n+1}^*} \equiv 0$ and $h_{|G} \equiv 1$. We set $g_{n+1} := \min(g_n, h)$, so that g_{n+1} satisfies (2)-(4).

By Lemma 2.6, $\lim_{l\to\infty} \int_{N_{\beta|l}} S_{n+1} d\lambda = S_{n+1}(\beta)$ for each $\beta \in G$. This gives $l(\beta) \in \omega$ minimal with $|\int_{N_{\beta|l(\beta)}} S_{n+1} d\lambda - S_{n+1}(\beta)| < 2^{-3}$ and $N_{\beta|l(\beta)} \subseteq G_{n+1}^*$. The set G_{n+1}^{**} is the union of the $N_{\beta|l(\beta)}$'s, which defines I_{n+1} and $(s_j^{n+1})_{j\in I_{n+1}}$ $(S_{n+1}(\beta)$ is 0 if n is even and 1 otherwise when $\beta \in G$).

• We then define a partial map $f_{\infty}: 2^{\omega} \to [0, 1]$ by $f_{\infty}:= \sum_{j \in \omega} (-1)^j g_j$. If $\beta \in G$, then $S_n(\beta)$ takes alternatively the values 1 and 0, depending on the parity of n, so that $f_{\infty}(\beta)$ is not defined. If $\beta \notin G$, then there is n such that $\beta \in \neg G_{n+1}^* \subseteq \neg G_{n+2}^* \subseteq ...$ This implies that $f_{\infty}(\beta)$ is defined and equal to $S_n(\beta)$. As $0 \leq \sum_{p \leq q} (g_{2p} - g_{2p+1}) = S_{2q+1} \leq S_{2q} = g_0 + \sum_{1 \leq p \leq q} (g_{2p} - g_{2p-1}) \leq g_0$, f_{∞} takes values in [0, 1]. So f_{∞} is a partial λ -measurable map defined λ -almost everywhere since $\lambda(G) = 0$ (we use Lemma 2.4).

• This allows us to define $f: 2^{<\omega} \to [0, 1]$ by $f(s) := \int_{N_s} f_{\infty} d\lambda$. As $\lambda(N_s) = 2\lambda(N_{s\varepsilon})$ for each $\varepsilon \in 2$, $f(s) = \int_{N_s} f_{\infty} d\lambda = \frac{\int_{N_{s0}} f_{\infty} d\lambda + \int_{N_{s1}} f_{\infty} d\lambda}{\lambda(N_s)} = \frac{f(s0)}{2} + \frac{f(s1)}{2}$ and f is a martingale.

• If $\beta \notin G$, then there is n with $\beta \in G_n^* \setminus G_{n+1}^*$, so that $f_\infty(\beta) = S_n(\beta)$. By Lemma 2.6, $k \ge n$ implies that $\lim_{l \to \infty} f_{N_{\beta l l}} S_{k+1} d\lambda = S_{k+1}(\beta) = S_n(\beta)$ since S_{k+1} is τ -continuous. Note that, for each $k \ge n$,

$$\begin{split} \left| \int_{N_{\beta|l}} (f_{\infty} - S_{k+1}) d\lambda \right| &\leq \lambda (G_{k+2}^* \cap N_{\beta|l}) \\ &\leq \Sigma_{\beta|l \subseteq s_j^{k+1}} \lambda (G_{k+2}^* \cap N_{s_j^{k+1}}) \\ &\leq \Sigma_{\beta|l \subseteq s_j^{k+1}} 2^{-k-4} \lambda (N_{s_j^{k+1}}) \\ &\leq \lambda (N_{\beta|l}) 2^{-k-4}. \end{split}$$

Moreover,

$$\begin{split} |f(\beta|l) - f_{\infty}(\beta)| &= |f_{N_{\beta|l}} f_{\infty} d\lambda - f_{\infty}(\beta)| = |f_{N_{\beta|l}} \left(f_{\infty} - S_{k+1} \right) d\lambda + f_{N_{\beta|l}} S_{k+1} d\lambda - S_{k+1}(\beta)| \\ &\leq 2^{-k-4} + |f_{N_{\beta|l}} S_{k+1} d\lambda - S_{k+1}(\beta)|, \end{split}$$

so that $\lim_{l\to\infty} f(\beta|l) = f_{\infty}(\beta)$, $\operatorname{osc}(f,\beta) = 0$ and $\beta \notin D(f)$.

• If $\beta \in G$ and $n \in \omega$, then there is $j \in \omega$ with $\beta \in N_{s_i^n}$. Note that

$$f(s_j^n) = \oint_{N_{s_j^n}} f_\infty \, d\lambda = \oint_{N_{s_j^n}} S_n \, d\lambda + \oint_{N_{s_j^n}} (f_\infty - S_n) \, d\lambda$$

and $|\int_{N_{s_j^n}} (f_\infty - S_n) d\lambda| \leq \lambda(G_{n+1}^* \cap N_{s_j^n}) < \frac{1}{8}\lambda(N_{s_j^n})$, so that $|\int_{N_{s_j^n}} (f_\infty - S_n) d\lambda| < \frac{1}{8}$. By (6), $|f(s_j^n) - S_n(\beta)| < \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$. As $S_n(\beta)$ takes infinitely often the values 1 and 0, $\operatorname{osc}(f, \beta) \geq \frac{1}{2}$ and $\beta \in D(f)$.

The main result will be a consequence of the main lemma and the following.

Lemma 2.8 Let $(f_n)_{n \in \omega}$ be a sequence of martingales such that

$$\{osc(f_n,\beta) \mid (n,\beta) \in \omega \times 2^{\omega}\} \subseteq \{0\} \cup [\frac{1}{2},1].$$

Then there is a martingale f with $D(f) = \bigcup_{n \in \omega} D(f_n)$.

Proof. We first observe the following facts. Let $g, h: 2^{<\omega} \to \mathbb{R}$ be bounded, $\beta \in 2^{\omega}$ and $a \in \mathbb{R}$.

(1)
$$\operatorname{osc}(g+h,\beta) \leq \operatorname{osc}(g,\beta) + \operatorname{osc}(h,\beta)$$
.

This comes from the triangle inequality.

(2) $\operatorname{osc}(ag,\beta) = |a| \cdot \operatorname{osc}(g,\beta).$

(3) $\operatorname{osc}(g+h,\beta) = \operatorname{osc}(h,\beta)$ if $\operatorname{osc}(g,\beta) = 0$.

By (1), $\operatorname{osc}(h,\beta) \leq \operatorname{osc}(g+h,\beta) + \operatorname{osc}(-g,\beta) = \operatorname{osc}(g+h,\beta) \leq \operatorname{osc}(g,\beta) + \operatorname{osc}(h,\beta) = \operatorname{osc}(h,\beta)$, so that $\operatorname{osc}(h,\beta) = \operatorname{osc}(g+h,\beta)$.

(4) $\operatorname{osc}(g,\beta) \leq a$ if $g(\beta|l) \in [0,a]$ for each $l \in \omega$.

• We set $D_n := D(f_n)$ for each $n \in \omega$, and $f := \sum_{n \in \omega} 4^{-n} f_n$. Note that f is defined and a martingale.

• If $\beta \notin \bigcup_{n \in \omega} D_n$, then $\operatorname{osc}(f_n, \beta) = 0$ for each $n \in \omega$. In particular, $\operatorname{osc}(4^{-n}f_n, \beta) = 0$ for each $n \in \omega$, by (2). Let $\varepsilon > 0$, and $M \in \omega$ with $\Sigma_{n > M} 4^{-n} \le \varepsilon$. By (1), $\operatorname{osc}(\Sigma_{n \le M} 4^{-n}f_n, \beta) = 0$. By (3) and (4), $\operatorname{osc}(f, \beta) = \operatorname{osc}(\Sigma_{n > M} 4^{-n}f_n, \beta) \le \Sigma_{n > M} 4^{-n} \le \varepsilon$. As ε is arbitrary, $\operatorname{osc}(f, \beta) = 0$, $\beta \notin D(f)$, which shows that $D(f) \subseteq \bigcup_{n \in \omega} D_n$.

• If $\beta \in \bigcup_{n \in \omega} D_n$, then let m be minimal such that $\beta \in D_m$. Note that

$$f = \sum_{n < m} 4^{-n} f_n + 4^{-m} f_m + \sum_{n > m} 4^{-n} f_n.$$

By (2) and (3), $\operatorname{osc}(f,\beta) = \operatorname{osc}(4^{-m}f_m + \sum_{n>m} 4^{-n}f_n,\beta)$. By (1), (2) and (4),

$$\operatorname{osc}(f,\beta) \ge \operatorname{osc}(4^{-m}f_m,\beta) - \operatorname{osc}(\Sigma_{n>m} 4^{-n}f_n,\beta) \ge 4^{-m}\frac{1}{2} - 4^{-m}\frac{1}{3} > 0.$$

Thus $\beta \in D(f)$.

3 Effectivity and uniformity

- We refer to [M] for the basic notions of effective descriptive set theory. We first recall some material present in it.

• Let $(p_n)_{n \in \omega}$ be the sequence of prime numbers 2, 3, ...

• If $l \in \omega$ and $s \in \omega^{l}$, then $\overline{s} := \langle s(0), ..., s(l-1) \rangle := p_{0}^{s(0)+1} ... p_{l-1}^{s(l-1)+1} \in \omega$ codes s (if l = 0, then $\langle \rangle := 1$).

• If $\alpha \in \omega^{\omega}$ and $l \in \omega$, then $\overline{\alpha}(l) := < \alpha(0), ..., \alpha(l-1) > \in \omega$ codes $\alpha | l \in \omega^l$, and α^* is defined by removing the first coordinate: $\alpha^* := (\alpha(1), \alpha(2), ...)$.

• If $\kappa \in \{2, \omega\}$, then $< ., . >: (\kappa^{\omega})^2 \to \kappa^{\omega}$ is a recursive homeomorphism with inverse map $\alpha \mapsto ((\alpha)_0, (\alpha)_1)$ defined for example by $(\alpha)_{\varepsilon}(n) := \alpha(2n+\varepsilon)$ if $(n, \varepsilon) \in \omega \times 2$ (we will also consider recursive homeomorphisms $< ., ., . >: (\kappa^{\omega})^3 \to \kappa^{\omega}$ and $< ., ., .. >: (\kappa^{\omega})^{\omega} \to \kappa^{\omega}$).

• If $u \in \omega$, then Seq(u) means that there are $l \in \omega$ and $s \in \omega^l$ (denoted by s(u)) such that $u = \langle s(0), ..., s(l-1) \rangle$. The natural number $(u)_i$ is s(i) if i < l, and 0 otherwise. The number l is the **length** of u and is denoted by h(u). If $k \leq l$, then $\underline{u}(k) := \langle s(0), ..., s(k-1) \rangle$, so that $\underline{u}(l) = u$. The standard basic clopen set is $N^u := \{\beta \in 2^\omega \mid \forall i < \ln(u) \ \beta(i) = (u)_i\}$. We set $u^- := \langle (u)_0, ..., (u)_{\lfloor h(u) - 2} \rangle (u^- := \langle \rangle \text{ if } h(u) \leq 1)$.

• Let X be a recursively presented Polish space. Then we will consider the effective basic open set $N(X, u) = B_X(r_{((u)_1)_0}, \frac{((u)_1)_1}{((u)_1)_2+1}).$

• Let $n \ge 1$ be a natural number. A subset T of ω^n is a **tree** if $Seq(u_i)$ and $lh(u_i) = lh(u_0)$ for each $(u_0, ..., u_{n-1}) \in T$ and each i < n, and $(\underline{u_0}(k), ..., u_{n-1}(k)) \in T$ if $(u_0, ..., u_{n-1}) \in T$ and $k \le lh(u_0)$.

• The next result is a part of 4A.1 in [M].

Theorem 3.1 Let $m \ge 1$ be a natural number, and $B \in \Sigma_1^0(\omega^{\omega} \times (\omega^{\omega})^m)$. Then there is a recursive subset T of $\omega^{\omega} \times \omega^m$ such that $(\alpha, \alpha_1, ..., \alpha_m) \in B \Leftrightarrow \exists l \in \omega \quad (\alpha, \underline{\alpha_1}(l), ..., \underline{\alpha_m}(l)) \notin T$, and $T_{\alpha} := \{(u_0, ..., u_{m-1}) \in \omega^m \mid (\alpha, u_0, ..., u_{m-1}) \in T\}$ is a tree for each $\alpha \in \omega^{\omega}$.

• The next result is a part of 4A.7 in [M].

Theorem 3.2 Let X be a recursively presented Polish space and $B \in \Delta_1^1(X)$. Then we can find a recursive function $\pi: \omega^{\omega} \to X$ and $C \in \Pi_1^0(\omega^{\omega})$ such that π is injective on C and $\pi[C] = B$.

- We then recall some material from [L].

Notation. Let X be a recursively presented Polish space. Recall that there is a pair $(\mathcal{W}^X, \mathcal{C}^X)$ such that

• $\mathcal{W}^X \subseteq \omega$ is a Π_1^1 set of codes for the Δ_1^1 subsets of X,

• $\mathcal{C}^X \subseteq \omega \times X$ is Π_1^1 and $\Delta_1^1(X) = \{\mathcal{C}_n^X \mid n \in \mathcal{W}^X\}$, which means that \mathcal{C}^X is "universal" for the Δ_1^1 subsets of X,

• the relation " $n \in \mathcal{W}^X \land (n, x) \notin \mathcal{C}^X$ " is Π_1^1 in (n, x).

If $X = \omega^{\omega} \times 2^{\omega}$, then we simply write $(\mathcal{W}, \mathcal{C}) := (\mathcal{W}^X, \mathcal{C}^X)$.

The next result will be extremely useful in the sequel.

The uniformization lemma. Let X, Y be recursively presented Polish spaces, and $P \in \Pi_1^1(X \times Y)$. Then the set $P^+ := \{x \in X \mid \exists y \in \Delta_1^1(x) \ (x, y) \in P\}$ is Π_1^1 , and there is a partial Π_1^1 -recursive map $f: X \to Y$ such that $(x, f(x)) \in P$ for each $x \in P^+$. If moreover $S \subseteq P^+$ is a Σ_1^1 subset of X, then there is a total Δ_1^1 -recursive map $g: X \to Y$ such that $(x, g(x)) \in P$ for each $x \in S$.

- The following definition is inspired by 3H.1 in [M].

Definition 3.3 (a) Let Γ be a class of subsets of recursively presented Polish spaces, and Γ be the associated boldface class. A system of sets $\mathcal{U}^X \in \Gamma(\omega^{\omega} \times X)$, where is X is a recursively presented Polish space, is a **nice parametrization** in Γ for Γ if the following hold:

- (1) $\Gamma(X) = \{\mathcal{U}^X_\alpha \mid \alpha \in \omega^\omega\},\$
- (2) $\Gamma(X) = \{ \mathcal{U}^X_{\alpha} \mid \alpha \in \omega^{\omega} \text{ recursive} \},$

(3) if X is a recursively presented Polish space, then there is $\mathcal{R}: \omega^{\omega} \times \omega^{\omega} \to \omega^{\omega}$ recursive such that $(\alpha, \gamma, x) \in \mathcal{U}^{\omega^{\omega} \times X} \Leftrightarrow (\mathcal{R}(\alpha, \gamma), x) \in \mathcal{U}^X$ if $(\alpha, \gamma, x) \in \omega^{\omega} \times \omega^{\omega} \times X$.

(b) If \mathcal{U} belongs to a nice parametrization, then we will say that \mathcal{U} is a good universal set .

(c) If U satisfies all these properties except maybe (3), then we will say that U is a suitable universal set .

By 3E.2, 3F.6 and 3H.1 in [M], there is a nice parametrization in Π_n^1 for Π_n^1 , for each natural number $n \ge 1$.

- We now recall two results that can essentially be found in [K1]. The first one is Theorem 2.2.3.(a) (see also [T1]).

Theorem 3.4 (Tanaka) Let $U \in \Sigma_1^1(\omega^{\omega} \times \omega^{\omega})$ be ω^{ω} -universal for the analytic subsets of ω^{ω} . Then $L(U) := \{(\alpha, p) \in \omega^{\omega} \times \omega \mid \lambda(U_{\alpha} \cap 2^{\omega}) > \frac{(p)_0}{(p)_1 + 1}\}$ is Σ_1^1 .

Corollary 3.5 Let $B \in \Delta_1^1(\omega^{\omega} \times 2^{\omega})$.

(a) The map $\lambda_B : \omega^{\omega} \to \mathbb{R}$ defined by $\lambda_B(\alpha) := \lambda(B_\alpha)$ is Δ_1^1 -recursive, and the partial function $(n, \alpha) \mapsto \lambda(\mathcal{C}_{n,\alpha})$ is Π_1^1 -recursive on its domain $\mathcal{W} \times \omega^{\omega}$.

(b) Let $D \subseteq \omega$, $O_0 \in \Sigma_1^1(\omega \times \omega^{\omega} \times 2^{\omega})$, and $O_1 \in \Pi_1^1(\omega \times \omega^{\omega} \times 2^{\omega})$ be such that $\lambda((O_0)_{n,\alpha}) = \lambda((O_1)_{n,\alpha})$ if $n \in D$. Then the partial map $\lambda_O : D \times \omega^{\omega} \to \mathbb{R}$ defined by $\lambda_O(n, \alpha) := \lambda((O_0)_{n,\alpha})$ is Σ_1^1 -recursive and Π_1^1 -recursive on its domain.

(c) The partial map $d_B: \omega^{\omega} \times 2^{\omega} \to \mathbb{R}$ defined by $d_B(\alpha, \beta) := d(B_{\alpha}, \beta)$ is Δ_1^1 -recursive, and the partial map $(n, \alpha, \beta) \mapsto d(\mathcal{C}_{n,\alpha}, \beta)$ is Π_1^1 -recursive on its Π_1^1 domain

$$\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^{\omega} \times 2^{\omega} \mid d(\mathcal{C}_{n,\alpha}, \beta) \text{ exists}\}.$$

(d) Let $h: \omega^{\omega} \times 2^{\omega} \to \mathbb{R}$ be Δ_1^1 -recursive taking values in [0, 1]. Then the partial map $i_h: \omega^{\omega} \times \omega \to \mathbb{R}$ defined by $i_h(\alpha, u):=\int_{N^u} h(\alpha, .) \ d\lambda$ is Δ_1^1 -recursive on its Δ_1^0 domain $\omega^{\omega} \times \{u \in \omega \mid Seq(u)\}$.

Proof. (a) It is enough to see that the relations $P_B(\alpha, p) \Leftrightarrow \lambda(B_\alpha) > r_p := (-1)^{(p)_0} \cdot \frac{(p)_1}{(p)_2 + 1}$ and

$$Q_B(\alpha, p) \Leftrightarrow \lambda(B_\alpha) < r_p$$

are Δ_1^1 to see that λ_B is Δ_1^1 -recursive. Note that there is $\phi: \omega^2 \to \omega$ recursive with $r_{\phi(p,l)} = r_p - \frac{1}{l+1}$. Thus

$$Q_B(\alpha, p) \Leftrightarrow \exists l \in \omega \ \lambda(B_\alpha) \le r_p - \frac{1}{l+1} \\ \Leftrightarrow \exists l \in \omega \ \neg (\lambda(B_\alpha) > r_p - \frac{1}{l+1}) \\ \Leftrightarrow \exists l \in \omega \ \neg P_B(\alpha, \phi(p, l)),$$

so that it is enough to see that P_B is Δ_1^1 .

• Now let $S \in \Sigma_1^1(\omega^\omega \times (\omega^\omega)^2)$ be a good ω^ω -universal for the analytic subsets of $(\omega^\omega)^2$. We set

$$U(\alpha, \gamma) \Leftrightarrow S((\alpha)_0, (\alpha)_1, \gamma),$$

so that $U \in \Sigma_1^1(\omega^\omega \times \omega^\omega)$ is ω^ω -universal for the analytic subsets of ω^ω . Let A be a Σ_1^1 subset of $\omega^\omega \times 2^\omega$. Then there is $\alpha_0 \in \omega^\omega$ recursive with $A = S_{\alpha_0}$, so that

$$\gamma \in A_{\alpha} \Leftrightarrow (\alpha_0, \alpha, \gamma) \in S \Leftrightarrow (<\alpha_0, \alpha >, \gamma) \in U.$$

This implies that the relation $R_A(\alpha, p) \Leftrightarrow \lambda(A_\alpha) > r_p$, equivalent to

$$((p)_0 \text{ is odd } \land (p)_1 > 0) \lor ((p)_0 \text{ is even } \land (< \alpha_0, \alpha >, < (p)_1, (p)_2 >) \in L(U)),$$

is Σ_1^1 , by Theorem 3.4.

• In particular, this applies to A := B, so that P_B is Σ_1^1 . Now note that

$$P_B(\alpha, p) \Leftrightarrow \lambda \big((\neg B)_\alpha \big) < 1 - r_p \Leftrightarrow Q_{\neg B} \big(\alpha, \phi'(p) \big),$$

for some $\phi': \omega \to \omega$ is recursive, so that P_B is Π^1_1 by the previous computation.

• We set
$$\mathcal{C}' := \{(\gamma, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \gamma(0) \in \mathcal{W} \land (\gamma(0), \gamma^*, \beta) \in \mathcal{C}\}$$
. As \mathcal{C}' is Π_1^1 ,
$$\mathcal{A} := \{(\alpha, p) \in \omega^{\omega} \times \omega \mid \lambda((\neg \mathcal{C}')_{\alpha}) > r_p\}$$

is Σ_1^1 , by the previous discussion. So let $n \in \mathcal{W}$. Note that

$$\begin{split} \lambda(\mathcal{C}_{n,\alpha}) > r_p &\Leftrightarrow \lambda(\neg \mathcal{C}_{n,\alpha}) < 1 - r_p \Leftrightarrow \lambda\big((\neg \mathcal{C}')_{n\alpha}\big) < 1 - r_p \\ &\Leftrightarrow \exists l \in \omega \ \lambda\big((\neg \mathcal{C}')_{n\alpha}\big) \le 1 - r_p - \frac{1}{l+1} \Leftrightarrow \exists l \in \omega \ \big(n\alpha, \phi''(p, l)\big) \notin \mathcal{A}, \end{split}$$

for some recursive $\phi'': \omega^2 \to \omega$. Similarly, the relation " $\lambda(\mathcal{C}_{n,\alpha}) < r_p$ " is Π_1^1 in (n, α, p) since the relation " $n \in \mathcal{W} \land (n, \alpha, \beta) \notin \mathcal{C}$ " is Π_1^1 , so that $(n, \alpha) \mapsto \lambda(\mathcal{C}_{n,\alpha})$ is Π_1^1 -recursive on $\mathcal{W} \times \omega^{\omega}$.

(b) Let $A := \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid (\alpha(0), \alpha^*, \beta) \in O_0\}$. Note that A is Σ_1^1 . By (a), the relation $R_A(\alpha, p) \Leftrightarrow \lambda(A_\alpha) > r_p$ is Σ_1^1 . Therefore the relation $R_{O_0}(n, \alpha, p) \Leftrightarrow R_A(n\alpha, p)$ is Σ_1^1 too. Moreover, $R_{O_0}(n, \alpha, p) \Leftrightarrow \lambda((O_0)_{n,\alpha}) > r_p \Leftrightarrow \lambda_O(n, \alpha) > r_p$.

• Assume now that $n \in D$. Then as above there is $\phi'' : \omega^2 \to \omega$ recursive such that

$$\begin{split} \lambda_O(n,\alpha) > r_p &\Leftrightarrow \lambda\big((O_1)_{n,\alpha}\big) > r_p \Leftrightarrow \lambda\big((\neg O_1)_{n,\alpha}\big) < 1 - r_p \\ &\Leftrightarrow \exists l \in \omega \ \lambda\big((\neg O_1)_{n,\alpha}\big) \le 1 - r_p - \frac{1}{l+1} \Leftrightarrow \exists l \in \omega \ \neg\Big(\lambda\big((\neg O_1)_{n,\alpha}\big) > r_{\phi''(p,l)}\Big) \\ &\Leftrightarrow \exists l \in \omega \ \neg R_{\neg O_1}\big(n,\alpha,\phi''(p,l)\big), \end{split}$$

which shows the existence of $R'_{O_0} \in \Pi^1_1$ such that $\lambda_O(n, \alpha) > r_p \Leftrightarrow R'_{O_0}(n, \alpha, p)$ if $n \in D$.

• Assume that $n \in D$. Then there is $\phi' : \omega \to \omega$ recursive such that

$$\lambda_O(n,\alpha) < r_q \Leftrightarrow \lambda\big((O_1)_{n,\alpha}\big) < r_q \Leftrightarrow \lambda\big((\neg O_1)_{n,\alpha}\big) > 1 - r_q \Leftrightarrow R_{\neg O_1}\big(n,\alpha,\phi'(q)\big),$$

which shows the existence of $R_{O_0}'' \in \Sigma_1^1$ such that $\lambda_O(n, \alpha) < r_q \Leftrightarrow R_{O_0}''(n, \alpha, q)$ if $n \in D$.

• Assume that $n \in D$. Then there is $\phi'' : \omega^2 \to \omega$ recursive such that

$$\lambda_{O}(n,\alpha) < r_{q} \Leftrightarrow \lambda\big((O_{0})_{n,\alpha}\big) < r_{q} \Leftrightarrow \exists l \in \omega \ \lambda\big((O_{0})_{n,\alpha}\big) \le 1 - r_{q} - \frac{1}{l+1} \\ \Leftrightarrow \exists l \in \omega \ \neg\Big(\lambda\big((O_{0})_{n,\alpha}\big) > r_{\phi''(q,l)}\Big) \Leftrightarrow \exists l \in \omega \ \neg R_{O_{0}}\big(n,\alpha,\phi''(q,l)\big),$$

which shows the existence of $R_{O_0}^{\prime\prime\prime} \in \Pi_1^1$ such that $\lambda_O(n, \alpha) < r_q \Leftrightarrow R_{O_0}^{\prime\prime\prime}(n, \alpha, q)$ if $n \in D$.

• Finally, $r_p < \lambda_O(n, \alpha) < r_q \Leftrightarrow R_{O_0}(n, \alpha, p) \land R''_{O_0}(n, \alpha, q)$ and

$$r_p < \lambda_O(n, \alpha) < r_q \Leftrightarrow R'_{O_0}(n, \alpha, p) \land R'''_{O_0}(n, \alpha, q)$$

if $n \in D$, which shows that λ_O is Σ_1^1 -recursive and Π_1^1 -recursive on $D \times \omega$.

(c) We first prove the following. Let X, Y be a recursively presented Polish spaces and $g: X \times \omega \to Y$ be a Δ_1^1 -recursive map. Then the partial map $h: X \to Y$ defined by

$$h(x) := \lim_{l \to \infty} g(x, l)$$

when this limit exists is Δ_1^1 -recursive.

Indeed, the domain D of h is $\{x \in X \mid \forall r \in \omega \; \exists L \in \omega \; \forall k, l \geq L \; d_Y(g(x,k),g(x,l)) < 2^{-r}\}$, so that D is Δ_1^1 . If $x \in D$, then $h(x) \in N(Y, u)$ is equivalent to

$$\exists p,q \in \omega \quad \frac{p}{q+1} < \frac{\left((u)_1\right)_1}{\left((u)_1\right)_2 + 1} \quad \land \ \exists L \in \omega \quad \forall l \ge L \quad g(x,l) \in N\left(Y, \left\langle 0, < \left((u)_1\right)_0, p, q > \right\rangle\right),$$

and we are done.

• We set $B' := \{(\alpha, \gamma) \in \omega^{\omega} \times 2^{\omega} \mid ((\alpha)_0, \gamma) \in B \land \gamma \in N_{(\alpha)_1^*|(\alpha)_1(0)}\}$, so that $B_{\alpha} \cap N_{\beta|l} = B'_{<\alpha,l\beta>}$ and B' is Δ_1^1 . By (a), the map $g : \omega^{\omega} \times 2^{\omega} \times \omega \to [0, 1]$ defined by $g(\alpha, \beta, l) := 2^{-l}\lambda(B_{\alpha} \cap N_{\beta|l})$ is Δ_1^1 -recursive. By the previous point, the partial map $h : \omega^{\omega} \times 2^{\omega} \to [0, 1]$ defined by

$$h(\alpha,\beta) := \lim_{l \to \infty} 2^{-l} \lambda(B_{\alpha} \cap N_{\beta|l})$$

when it exists is also Δ_1^1 -recursive. But $h = d_B$.

• Fix $n \in \mathcal{W}$. Then there is $q(n) \in \mathcal{W}$ such that

$$\mathcal{C}_{q(n)} \!=\! \left\{ (\gamma, \delta) \!\in\! \omega^{\omega} \!\times\! 2^{\omega} \mid \left(n, (\gamma)_0, \delta \right) \!\in\! \mathcal{C} \land (\gamma)_1^* \!\mid\! (\gamma)_1(0) \!\subseteq\! \delta \right\}$$

Moreover, we may assume that q is Π_1^1 -recursive on \mathcal{W} , by the uniformization lemma. As Π_1^1 has the substitution property, the map $g': (n, \alpha, \beta, l) \mapsto 2^{-l}\lambda(\mathcal{C}_{q(n), <\alpha, l\beta>}) = 2^{-l}\lambda(\mathcal{C}_{n,\alpha} \cap N_{\beta|l})$ is Π_1^1 -recursive on $\mathcal{W} \times \omega^{\omega} \times 2^{\omega} \times \omega$. As above, the map

$$h':(n,\alpha,\beta)\mapsto \lim_{l\to\infty} 2^{-l}\lambda(\mathcal{C}_{n,\alpha}\cap N_{\beta|l})=d(\mathcal{C}_{n,\alpha},\beta)$$

is Π_1^1 -recursive on the Π_1^1 set $\{(n, \alpha, \beta) \in \mathcal{W} \times \omega^{\omega} \times 2^{\omega} \mid d(\mathcal{C}_{n, \alpha}, \beta) \text{ exists}\}.$

(d) The argument here is partly similar to 11.6 and 17.25 in [K2]. We set, for $(k, l) \in \omega^2$,

$$A_{k,l} := h^{-1} \left([\frac{k}{2^l}, \frac{k+1}{2^l}) \right)$$

and define $h_l: \omega^\omega \times 2^\omega \to [0,1]$ by $h_l = \sum_{k \leq 2^l} \frac{k}{2^l} \chi_{A_{k,l}}$. We also define $R \subseteq \omega^\omega \times 2^\omega \times \omega^3$ by

$$R(\alpha,\beta,u,k,l) \Leftrightarrow \frac{k}{2^l} \leq h(\alpha,\beta) < \frac{k\!+\!1}{2^l} \wedge \operatorname{Seq}(u) \wedge \beta \in N^u,$$

so that R is Δ_1^1 . Then we define $O \subseteq \omega^{\omega} \times 2^{\omega}$ by

$$O(\alpha,\beta) \Leftrightarrow \operatorname{Seq}(\alpha(0)) \wedge \operatorname{lh}(\alpha(0)) = 3 \wedge R(\alpha^*,\beta,(\alpha(0))_0,(\alpha(0))_1,(\alpha(0))_2),$$

so that O is Δ_1^1 .

Note that (h_l) is a sequence of Borel functions pointwise converging to h. By Lebesgue's dominated convergence theorem, $\int_{N^u} h(\alpha, .) d\lambda = \lim_{l \to \infty} \int_{N^u} h_l(\alpha, .) d\lambda$ if Seq(u). Note that

$$\begin{split} \int_{N^u} h_l(\alpha, .) \ d\lambda &= \int_{N^u} \Sigma_{k \le 2^l} \ \frac{k}{2^l} \chi_{A_{k,l}}(\alpha, .) \ d\lambda &= \Sigma_{k \le 2^l} \ \frac{k}{2^l} \lambda \big((A_{k,l})_\alpha \cap N^u \big) \\ &= \Sigma_{k \le 2^l} \ \frac{k}{2^l} \lambda (R_{\alpha, u, k, l}) = \Sigma_{k \le 2^l} \ \frac{k}{2^l} \lambda (O_{< u, k, l > \alpha}). \end{split}$$

Using (a), this implies that the map $(\alpha, u, l) \mapsto \int_{N^u} h_l(\alpha, .) d\lambda$ is Δ_1^1 -recursive on its Δ_1^0 domain $\omega^{\omega} \times \{u \in \omega \mid \text{Seq}(u)\} \times \omega$. As in the proof of (c), i_h is Δ_1^1 -recursive on its domain.

We now prove a uniform version of Theorem 4.3.2 in [K1] (due to Tanaka, see [T2]).

Theorem 3.6 Let $B \in \Delta_1^1(\omega^{\omega} \times 2^{\omega})$, and $\epsilon : \omega^{\omega} \to \mathbb{R}$ be Δ_1^1 -recursive such that $\epsilon(\alpha) \in (0, 1]$ for each $\alpha \in \omega^{\omega}$. Then there is $T \in \Delta_1^1(\omega^{\omega} \times \omega)$ such that (a) T_{α} is a tree for each $\alpha \in \omega^{\omega}$, (b) if $K = \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega \ (\alpha, \overline{\beta}(l)) \in T\}$, then $K_{\alpha} \subseteq B_{\alpha}$ and $\lambda(K_{\alpha}) \ge \lambda(B_{\alpha}) - \epsilon(\alpha)$ for each $\alpha \in \omega^{\omega}$.

Proof. Theorem 3.2 gives $\pi: \omega^{\omega} \to \omega^{\omega} \times 2^{\omega}$ recursive and $C \in \Pi_1^0(\omega^{\omega})$ such that π is injective on C and $\pi[C] = B$. We set $Q := \{(\alpha, \beta, \gamma) \in (\omega^{\omega})^3 \mid \gamma \in C \land \pi(\gamma) = (\alpha, \beta)\}$. As $Q \in \Pi_1^0$, Theorem 3.1 gives a recursive subset \overline{T} of $\omega^{\omega} \times \omega^2$ such that $(\alpha, \beta, \gamma) \in Q \Leftrightarrow \forall l \in \omega \ (\alpha, \overline{\beta}(l), \overline{\gamma}(l)) \in \overline{T}$ and \overline{T}_{α} is a tree for each $\alpha \in \omega^{\omega}$.

• We set, for $u, v \in \omega$,

 $u \leq^{a} v \Leftrightarrow \operatorname{Seq}(u), \operatorname{Seq}(v) \wedge \operatorname{lh}(u) = \operatorname{lh}(v) \wedge \forall i < \operatorname{lh}(u) \ (u)_{i} \leq (v)_{i}.$

• Then we set, for $u \in \omega$ with Seq(u) and $\alpha \in \omega^{\omega}$,

$$B^{u}_{\alpha} := \left\{ \beta \in 2^{\omega} \mid \exists \gamma \in \omega^{\omega} \ \overline{\gamma} \big(\mathrm{lh}(u) \big) \leq^{a} u \ \land \ \forall l \in \omega \ \left(\alpha, \overline{\beta}(l), \overline{\gamma}(l) \right) \in \overline{T} \right\}$$

and $B' := \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \text{Seq}(\alpha(0)) \land \beta \in B^{\alpha(0)}_{\alpha^*}\}$. Note that B' is Σ^1_1 . In fact, B' is Δ^1_1 by uniqueness of the witness γ .

• We now define $\delta_{\alpha} \in \omega^{\omega}$ as follows. We define $\delta_{\alpha}(i)$ by induction on *i*. We first set

$$\delta_{\alpha}(0) := \min\{k \in \omega \mid \lambda(B_{\alpha}^{}) > \lambda(B_{\alpha}) - \frac{\epsilon(\alpha)}{2}\}.$$

This number exists since B_{α} is the increasing union of the $B_{\alpha}^{\langle k \rangle}$'s. Then

$$\delta_{\alpha}(i+1) := \min\{k \in \omega \mid \lambda(B_{\alpha}^{<\delta_{\alpha}(0),\dots,\delta_{\alpha}(i),k>}) > \lambda(B_{\alpha}) - \frac{\epsilon(\alpha)}{2} - \dots - \frac{\epsilon(\alpha)}{2^{i+2}}\}.$$

Note that $\delta_{\alpha} \in \Delta_1^1(\alpha)$, by Corollary 3.5.(a).

• We set $T := \{(\alpha, v) \in \omega^{\omega} \times \omega \mid \text{Seq}(v) \land \exists u \leq^{a} \overline{\delta_{\alpha}}(\ln(v)) \ (\alpha, v, u) \in \overline{T}\}$, so that $T \in \Delta_{1}^{1}(\omega^{\omega} \times \omega)$ and T_{α} is a tree for each $\alpha \in \omega^{\omega}$.

• We set
$$K := \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega \ \beta \in B_{\alpha}^{\overline{\delta_{\alpha}}(l)}\}$$
, so that $K_{\alpha} \subseteq B_{\alpha}$ and
 $\lambda(K_{\alpha}) = \lim_{l \to \infty} \lambda(B_{\alpha}^{\overline{\delta_{\alpha}}(l)}) \ge \lambda(B_{\alpha}) - \epsilon(\alpha)$

for each $\alpha \in \omega^{\omega}$ since $(B_{\alpha}^{\overline{\delta_{\alpha}}(l)})_{l \in \omega}$ is decreasing. It remains to apply König's lemma to see that $K = \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega \ (\alpha, \overline{\beta}(l)) \in T\}$ since

$$\left\{s \in \omega^{<\omega} \mid < s(0), ..., s(|s|-1) > \leq^a \overline{\delta_{\alpha}}(|s|) \land (\alpha, \overline{\beta}(|s|), < s(0), ..., s(|s|-1) >) \in \overline{T}\right\}$$

is a finitely splitting tree.

- We want to prove an effective and uniform version of the Lusin-Menchoff lemma. We first need the following result, which slightly and uniformly refines Theorem A in [L] at the first level of the Borel hierarchy.

Lemma 3.7 Let O be a Δ_1^1 subset of $\omega^{\omega} \times 2^{\omega}$ with open vertical sections. Then there is a Δ_1^1 -recursive map $f: \omega^{\omega} \to \omega^{\omega}$ such that O_{α} is the disjoint union $\bigcup \{ N^{f(\alpha)(u)} \mid u \in \omega \land Seq(f(\alpha)(u)) \}$, for each $\alpha \in \omega^{\omega}$.

Proof. Let $P := \{(\alpha, u) \in \omega^{\omega} \times \omega \mid \text{Seq}(u) \land (\ln(u) = 0 \lor (N^u \subseteq O_\alpha \land N^{u^-} \not\subseteq O_\alpha))\}$. Note that P is Π_1^1 , since a nonempty $\Delta_1^1(\alpha)$ closed subset of 2^{ω} contains a $\Delta_1^1(\alpha)$ point, by 4F.15 in [M]. We then define a relation R on $\omega^{\omega} \times 2^{\omega} \times \omega$ by $R(\alpha, \beta, u) \Leftrightarrow P(\alpha, u) \land \beta \in N^u$, so that R is Π_1^1 . Note that, for each $(\alpha, \beta) \in O$ there is u with $R(\alpha, \beta, u)$. By 4B.5 in [M], there is a Δ_1^1 -recursive map $g : \omega^{\omega} \times 2^{\omega} \to \omega$ such that $R(\alpha, \beta, g(\alpha, \beta))$ for each $(\alpha, \beta) \in O$. Fix $\alpha \in \omega^{\omega}$. Note that $S^{\alpha} := \{g(\alpha, \beta) \mid \beta \in O_{\alpha}\}$ is a $\Sigma_1^1(\alpha)$ subset of ω contained in the $\Pi_1^1(\alpha)$ set P_α . By 4B.11 and 4C in [M], there is $D^{\alpha} \in \Delta_1^1(\alpha)$ with $S^{\alpha} \subseteq D^{\alpha} \subseteq P_{\alpha}$. Note that $O_{\alpha} \subseteq \bigcup_{u \in D^{\alpha}} N^u \subseteq O_{\alpha}$, so that O_{α} is the disjoint union of the sequence $(N^u)_{u \in D^{\alpha}}$. We define $\delta_{\alpha} \in \omega^{\omega}$ by

$$\delta_{\alpha}(u) := \begin{cases} u \text{ if } u \in D_{\alpha}, \\ 0 \text{ otherwise.} \end{cases}$$

Note that $\delta_{\alpha} \in \Delta_{1}^{1}(\alpha)$ and O_{α} is the disjoint union $\bigcup \{ N^{\delta_{\alpha}(u)} \mid u \in \omega \land \operatorname{Seq}(\delta_{\alpha}(u)) \}$. As the set $\{ (\alpha, \delta) \in \omega^{\omega} \times \omega^{\omega} \mid \delta \in \Delta_{1}^{1}(\alpha) \land O_{\alpha} \text{ is the disjoint union } \bigcup \{ N^{\delta(u)} \mid u \in \omega \land \operatorname{Seq}(\delta(u)) \} \}$ is Π_{1}^{1} , it remains to apply the uniformization lemma to get the desired map f. \Box

Notation. We set $\mathcal{W}_1 := \{n \in \mathcal{W} \mid \forall \alpha \in \omega^{\omega} \exists \gamma_n \in \Delta_1^1(\alpha) \ \mathcal{C}_{n,\alpha} = \bigcup \{N^{\gamma_n(u)} \mid u \in \omega \land \operatorname{Seq}(\gamma_n(u))\},\$ so that, by Lemma 3.7, \mathcal{W}_1 is a Π_1^1 set of codes for the Δ_1^1 subsets of $\omega^{\omega} \times 2^{\omega}$ with open vertical sections.

Lemma 3.8 Let F be a Δ_1^1 subset of $\omega^{\omega} \times 2^{\omega}$ with closed vertical sections, and B be a Δ_1^1 subset of $\omega^{\omega} \times 2^{\omega}$ such that $B \supseteq F$ and $d(B_{\alpha}, \beta) = 1$ for each $(\alpha, \beta) \in F$. Then there is a Δ_1^1 subset C of $\omega^{\omega} \times 2^{\omega}$ with closed vertical sections such that

- (1) $F \subseteq C \subseteq B$,
- (2) $d(B_{\alpha},\beta) = 1$ for each $(\alpha,\beta) \in C$,
- (3) $d(C_{\alpha}, \beta) = 1$ for each $(\alpha, \beta) \in F$.

Proof. Lemma 3.7 gives a Δ_1^1 -recursive map $f: \omega^{\omega} \to \omega^{\omega}$ such that $(\neg F)_{\alpha}$ is the disjoint union $\bigcup \{N^{f(\alpha)(u)} \mid u \in \omega \land \text{Seq}(f(\alpha)(u))\}$, for each $\alpha \in \omega^{\omega}$. We set

$$B' := \Big\{ (\alpha, \gamma) \in \omega^{\omega} \times 2^{\omega} \mid \big((\alpha)_0, \gamma \big) \in B \land \operatorname{Seq} \Big(f \big((\alpha)_0 \big) \big((\alpha)_1(0) \big) \Big) \land \gamma \in N^{f((\alpha)_0)((\alpha)_1(0))} \Big\},$$

so that B' is Δ_1^1 and $B_{\alpha} \cap N^{f(\alpha)(u)} = B'_{<\alpha,u^{\infty}>}$ if Seq $(f(\alpha)(u))$. By Corollary 3.5.(c), the partial map $(\alpha, \beta, u) \mapsto d(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta)$ is Δ_1^1 -recursive. We then set

$$B'' := \{ (\alpha, \gamma) \in B' \mid d(B_{(\alpha)_0} \cap N^{f((\alpha)_0)((\alpha)_1(0))}, \gamma) = 1 \},$$

so that B'' is Δ_1^1 and $\{\beta \in B_\alpha \cap N^{f(\alpha)(u)} \mid d(B_\alpha \cap N^{f(\alpha)(u)}, \beta) = 1\} = B''_{<\alpha, u^\infty>}$ if Seq $(f(\alpha)(u))$. We define $\epsilon : \omega^\omega \to \mathbb{R}$ by

$$\varepsilon(\alpha) := \begin{cases} 2^{-(\alpha)_1(0)} \lambda(B'_{\alpha}) \text{ if } \lambda(B'_{\alpha}) \neq 0, \\ 1 \text{ otherwise,} \end{cases}$$

so that ϵ is Δ_1^1 -recursive by Corollary 3.5.(a), and $\epsilon(\alpha) \in (0, 1]$ for each $\alpha \in \omega^{\omega}$. Theorem 3.6 gives $T \in \Delta_1^1(\omega^{\omega} \times \omega)$ such that

(a) T_{α} is a tree for each $\alpha \in \omega^{\omega}$,

(b) if $K = \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall l \in \omega \ (\alpha, \overline{\beta}(l)) \in T\}$, then $K_{\alpha} \subseteq B_{\alpha}''$ and $\lambda(K_{\alpha}) \ge \lambda(B_{\alpha}'') - \epsilon(\alpha)$ for each $\alpha \in \omega^{\omega}$.

We set, for $u \in \omega$,

$$F^{u} := \left\{ (\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \operatorname{Seq}(f(\alpha)(u)) \land (< \alpha, u^{\infty} >, \beta) \in K \land \lambda(B'_{<\alpha, u^{\infty} >}) \neq 0 \right\}.$$

As K is Δ_1^1 with closed vertical sections, so is F^u . If $\operatorname{Seq}(f(\alpha)(u))$ and $\lambda(B'_{<\alpha,u^{\infty}>}) = 0$, then $\lambda(B_{\alpha} \cap N^{f(\alpha)(u)}) = 0$ and $F_{\alpha}^u = \emptyset$, so that $F_{\alpha}^u \subseteq \{\beta \in B_{\alpha} \cap N^{f(\alpha)(u)} \mid d(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta) = 1\}$ and $\lambda(F_{\alpha}^u) \ge (1-2^{-u})\lambda(B_{\alpha} \cap N^{f(\alpha)(u)})$. If $\operatorname{Seq}(f(\alpha)(u))$ and $\lambda(B'_{<\alpha,u^{\infty}>}) \ne 0$, then

$$F_{\alpha}^{u} = K_{\langle \alpha, u^{\infty} \rangle} \subseteq B_{\langle \alpha, u^{\infty} \rangle}^{\prime\prime} = \{\beta \in B_{\alpha} \cap N^{f(\alpha)(u)} \mid d(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta) = 1\}.$$

Moreover,

$$\begin{split} \lambda(F^u_{\alpha}) &= \lambda(K_{<\alpha,u^{\infty}>}) \geq \lambda(B''_{<\alpha,u^{\infty}>}) - \epsilon(<\alpha,u^{\infty}>) = \lambda(B''_{<\alpha,u^{\infty}>}) - 2^{-u}\lambda(B'_{<\alpha,u^{\infty}>}) \\ &= (1 - 2^{-u})\lambda(B_{\alpha} \cap N^{f(\alpha)(u)}) \end{split}$$

since $\lambda(B_{\alpha} \cap N^{f(\alpha)(u)}) = \lambda(\{\beta \in B_{\alpha} \cap N^{f(\alpha)(u)} \mid d(B_{\alpha} \cap N^{f(\alpha)(u)}, \beta) = 1\})$, by Theorem 2.1. It remains to set $C := F \cup \bigcup_{u \in \omega} F^u$. We conclude as in the proof of Lemma 2.2.

- We now want to prove an effective and uniform version of Lemma 2.5.

Lemma 3.9 Let C be a Δ_1^1 subset of $\omega^{\omega} \times 2^{\omega}$ with closed vertical sections, G be a Borel subset of 2^{ω} with $\lambda(\mathcal{G}) = 0$, and G be a Δ_1^1 subset of $\omega^{\omega} \times 2^{\omega}$ with G_{δ} vertical sections, contained in $\omega^{\omega} \times \mathcal{G}$ and disjoint from C. Then there is a Δ_1^1 -recursive map $h: \omega^{\omega} \times 2^{\omega} \to \mathbb{R}$ such that $h(\alpha, \cdot): 2^{\omega} \to [0, 1]$ is τ -continuous for each $\alpha \in \omega^{\omega}$, $h_{|C} \equiv 0$ and $h_{|G} \equiv 1$.

Proof. By Theorem 3.5 in [L], there is a Δ_1^1 subset F of $\omega \times \omega^{\omega} \times 2^{\omega}$ such that $F_{n,\alpha}$ is closed for each $(n, \alpha) \in \omega \times \omega^{\omega}$ and $\neg G = \bigcup_{n \in \omega} F_n$. Moreover, we may assume that $(F_n)_{n \in \omega}$ is increasing and $F_0 = C$.

• We will define, by primitive recursion, a partial map $f: \omega \to \omega$ which is Π_1^1 -recursive on its domain such that f(n) essentially codes the set $C_{\frac{1}{2^n}}$ constructed in the proof of Lemma 2.5. As this map will in fact be total, it will be Δ_1^1 -recursive by the uniformization lemma.

We first apply Lemma 3.8 to $F := F_0$ and $B := \neg G$. This is possible because $G_\alpha \subseteq \mathcal{G}$, so that $(\neg G)_\alpha$ has λ -measure one and therefore density one at any point of 2^ω , for each $\alpha \in \omega^\omega$. Lemma 3.8 gives $C_1 \in \Delta_1^1$ with closed vertical sections such that $\neg G \supseteq C_1 \supseteq F_0$. Let $f(0) \in \mathcal{W}_1$ with $\mathcal{C}_{f(0)} = \neg C_1$.

More generally, we will have $C_{f(n)} = \neg C_{\frac{1}{2^n}}$. As mentioned above, f will be defined by primitive recursion, which means that there will be a partial map $g: \omega^2 \to \omega$ such that f(n+1) = g(f(n), n). This partial map g will be Π_1^1 -recursive on its Π_1^1 domain $\{m \in \mathcal{W}_1 \mid \neg \mathcal{C}_m \subseteq \neg G\} \times \omega$, so that f will be Π_1^1 -recursive on its domain by 7A.5 in [M]. The map g will take values in \mathcal{W}_1 , and is constructed in such a way that, if $A := \neg \mathcal{C}_m \subseteq \neg G$ and $A' := \neg \mathcal{C}_{g(m,n)}$, then

(1)
$$A \cup F_{n+1} \subseteq A' \subseteq \neg G$$
,
(2) $\forall (\alpha, \beta) \in A' \ d((\neg G)_{\alpha}, \beta) = 1$,
(3) $\forall (\alpha, \beta) \in A \cup F_{n+1} \ d(A'_{\alpha}, \beta) = 1$

Lemma 3.8 ensures that such a $g(m,n) \in \omega$ exists if $(m,n) \in \{q \in W_1 \mid \neg C_q \subseteq \neg G\} \times \omega$. As the properties (1)-(3) are Π_1^1 by Corollary 3.5, the uniformization lemma ensures the existence of g. So we constructed a Δ_1^1 -recursive map $f: \omega \to \omega$, taking values in W_1 , such that $C_{\frac{1}{2^n}} := \neg C_{f(n)}$ is a Δ_1^1 subset of $\omega^\omega \times 2^\omega$ with closed vertical sections, $F_n \subseteq C_{\frac{1}{2^n}} \subseteq \neg G$, $C_{\frac{1}{2^n}} \subseteq C_{\frac{1}{2^{n+1}}}$, and

$$d\big((C_{\frac{1}{2^{n+1}}})_{\alpha},\beta) \!=\! 1$$

if $(\alpha, \beta) \in C_{\frac{1}{2n}}$.

• Similarly, we construct a Δ_1^1 -recursive map $\tilde{F}: \omega \to \omega$ satisfying the following properties, if

$$D := \{ p \in \omega \mid \text{Seq}(p) \land \ln(p) = 2 \land 0 < (p)_1 \le 2^{(p)_0} \}.$$
(a) $\tilde{F}(p) \in \mathcal{W}_1$ if $p \in D$, in which case we set $C_p := \neg \mathcal{C}_{\tilde{F}(p)}$,
(b) $C_p \subseteq C_{p'}$ if $p, p' \in D \land \frac{(p')_1}{2^{(p')_0}} \le \frac{(p)_1}{2^{(p)_0}},$
(c) $d((C_{p'})_{\alpha}, \beta) = 1$ if $p, p' \in D \land \frac{(p')_1}{2^{(p')_0}} < \frac{(p)_1}{2^{(p)_0}} \land (\alpha, \beta) \in C_p.$

• This allows us to define h by

$$1 - h(\alpha, \beta) := \begin{cases} 0 \text{ if } (\alpha, \beta) \in G, \\ \sup\{\frac{(p)_1}{2^{(p)_0}} \mid p \in D \land (\alpha, \beta) \in C_p\} \text{ if } (\alpha, \beta) \notin G. \end{cases}$$

Note that h is Δ_1^1 -recursive since $D \in \Delta_1^0$, so that the relation " $p \in D \land (\alpha, \beta) \in C_p$ " is Δ_1^1 in (p, α, β) . We conclude as in the proof of Lemma 2.5.

- We are now ready to prove the main lemma in this section. We equip the space $[0,1]^{2^{<\omega}}$ with the distance defined by $d(f,g) := \sum_{u \in \omega, \text{Seq}(u)} \frac{|f(s(u)) - g(s(u))|}{2^{u+1}}$. We give a recursive presentation of $([0,1]^{2^{<\omega}}, d)$. We set

$$f_n(s) := \begin{cases} \frac{((n)\overline{s})_0}{((n)\overline{s})_0 + ((n)\overline{s})_1 + 1} \text{ if } \operatorname{Seq}(n) \land \forall k < \operatorname{lh}(n) \ \left(\operatorname{Seq}((n)_k\right) \land \operatorname{lh}((n)_k) = 2 \right) \land \overline{s} < \operatorname{lh}(n), \\ 0 \text{ otherwise,} \end{cases}$$

so that (f_n) is dense in $[0,1]^{2^{<\omega}}$. It is now routine to check that the relations " $d(f_m, f_n) \leq \frac{p}{q+1}$ " and " $d(f_m, f_n) < \frac{p}{q+1}$ " are recursive in (m, n, p, q). It is also routine to check that $F : \omega^{\omega} \to [0, 1]^{2^{<\omega}}$ is Δ_1^1 -recursive if the map $F' : \omega \times \omega^{\omega} \to \mathbb{R}$ defined by $F'(u, \alpha) := F(\alpha)(s(u))$ if Seq(u), 0 otherwise, is Δ_1^1 -recursive (s(u)) was defined at the beginning of Section 3).

Lemma 3.10 Let $\mathcal{V} := \{(f,\beta) \in \mathcal{M} \times 2^{\omega} \mid osc(f,\beta) > 0\}$, \mathcal{G} be a nonempty $G_{\delta} \cap \Delta_{1}^{1}$ subset of 2^{ω} with $\lambda(\mathcal{G}) = 0$, and G be a Δ_{1}^{1} subset of $\omega^{\omega} \times 2^{\omega}$, contained in $\omega^{\omega} \times \mathcal{G}$, and with G_{δ} vertical sections. Then there is a Δ_{1}^{1} -recursive map $F : \omega^{\omega} \to [0,1]^{2^{<\omega}}$, taking values in \mathcal{M} , and such that $G_{\alpha} = \mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^{\omega}$.

Proof. We will define, by primitive recursion, $f: \omega \to \omega^4$ coding g_n , S_n , G_n^* , and $(s_j^n)_{j \in I_n}$ defining G_n^{**} considered in the proof of the Lemma 2.7. We must find $r: \omega^4 \times \omega \to \omega^4$ with f(n+1) = r(f(n), n). In practice,

- (1) $f_0(n) \in \mathcal{W}_1$ codes $G_n^* \subseteq \omega^\omega \times 2^\omega$,
- (2) $f_1(n) \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ codes the graph of $g_n : \omega^{\omega} \times 2^{\omega} \to \mathbb{R}$,
- (3) $f_2(n) \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ codes the graph of $S_n : \omega^{\omega} \times 2^{\omega} \to \mathbb{R}$,
- (4) $f_3(n) \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ codes the graph of the function $\alpha \mapsto (s_i^{n,\alpha})_{j \in I_{n,\alpha}}$.

• By Theorem 3.5 in [L], there is a Δ_1^1 subset O of $\omega \times \omega^{\omega} \times 2^{\omega}$ such that $O_{n,\alpha}$ is open for each $(n,\alpha) \in \omega \times \omega^{\omega}$ and $G = \bigcap_{n \in \omega} O_n$. Moreover, we may assume that $(O_n)_{n \in \omega}$ is decreasing and $O_0 = \omega^{\omega} \times 2^{\omega}$.

• Let $n_0 \in \mathcal{W}_1$ with $\mathcal{C}_{n_0} = \omega^{\omega} \times 2^{\omega}$, $n_1 \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ with $\mathcal{C}_{n_1}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} = \{(\alpha, \beta, r) \in \omega^{\omega} \times 2^{\omega} \times \mathbb{R} \mid r = 1\}$, and $n_3 \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ with $\mathcal{C}_{n_3}^{\omega^{\omega} \times \omega^{\omega}} = \{(\alpha, \gamma) \in \omega^{\omega} \times \omega^{\omega} \mid \gamma = 10^{\infty}\}$. We set $f(0) := (n_0, n_1, n_1, n_3)$, so that $\mathcal{C}_{n_0} = G_0^*$, $\mathcal{C}_{n_1}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} = \operatorname{Gr}(g_0) = \operatorname{Gr}(S_0)$, $\mathcal{C}_{n_3}^{\omega^{\omega} \times \omega^{\omega}} = \operatorname{Gr}(\alpha \mapsto 10^{\infty})$,

$$\{u \in \omega \mid \text{Seq}((10^{\infty})(u))\} = \{0\} = I_0$$

and $(10^{\infty})(0) = 1 = <> = s_0^0$. So f(0) is as desired.

• We now study the induction step. This means that we must define $r(n_0, n_1, n_2, n_3, n) \in \omega^4$.

(1) We first define $r_0(n_0, n_1, n_2, n_3, n)$ coding G_{n+1}^* . Fix $n_3 \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ coding the graph of a Δ_1^1 -recursive function $\phi : \omega^{\omega} \to \omega^{\omega}$ such that the sequences $s(\phi(\alpha)(u))$ coded by the *u*'s with $\operatorname{Seq}(\phi(\alpha)(u))$ are pairwise incompatible and $G_{\alpha} \subseteq \bigcup \{N^{\phi(\alpha)(u)} \mid u \in \omega \land \operatorname{Seq}(\phi(\alpha)(u))\}$ (we call P_3 the Π_1^1 set of such n_3 's). Let $\alpha \in \omega^{\omega}$. Assume that $\operatorname{Seq}(\phi(\alpha)(u))$ (which intuitively means that $u \in I_{n,\alpha}$ and $s_u^{n,\alpha}$ is coded by $\phi(\alpha)(u)$). By continuity of λ ,

$$0 = \lambda(G_{\alpha} \cap N^{\phi(\alpha)(u)}) = \lim_{i \to \infty} \lambda(O_{i,\alpha} \cap N^{\phi(\alpha)(u)}).$$

This gives $j(n, \alpha, u) > n$ minimal with $\lambda(O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}) < 2^{-n-3-\operatorname{lh}(\phi(\alpha)(u))}$ (note that $2^{-\operatorname{lh}(\phi(\alpha)(u))} = \lambda(N^{\phi(\alpha)(u)})$). Moreover, $G_{\alpha} \cap N^{\phi(\alpha)(u)} \subseteq O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)} \subseteq O_{n+1,\alpha} \cap N^{\phi(\alpha)(u)}$, so that $O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$ satisfies the properties of the set O_j in the proof of Lemma 2.7. We will have $G_{n+1,\alpha}^* = \bigcup_{\operatorname{Seq}(\phi(\alpha)(u))} O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$. By Corollary 3.5 and the uniformization lemma, we may assume that the map j is Δ_1^1 -recursive on its Δ_1^1 domain

$$\{(n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid \operatorname{Seq}(\phi(\alpha)(u))\}.$$

Note that G_{n+1}^* is a Δ_1^1 subset of $\omega^{\omega} \times 2^{\omega}$ with open vertical sections, which gives $m \in \mathcal{W}_1$ such that $\mathcal{C}_m = G_{n+1}^*$. By incompatibility, $G_{n+1,\alpha}^* \cap N^{\phi(\alpha)(u)} = O_{j(n,\alpha,u),\alpha} \cap N^{\phi(\alpha)(u)}$. So we proved that, for each $(n_3, n) \in P_3 \times \omega$, there is $m \in \mathcal{W}_1$ such that, for each $\alpha \in \omega^{\omega}$,

(1)
$$G_{\alpha} \subseteq \mathcal{C}_{m,\alpha} \subseteq O_{n+1,\alpha} \cap \bigcup \left\{ N^{\phi(\alpha)(u)} \mid u \in \omega \land \operatorname{Seq}(\phi(\alpha)(u)) \right\},$$

(5) $\lambda(\mathcal{C}_{m,\alpha} \cap N^{\phi(\alpha)(u)}) < 2^{-n-3-\operatorname{lh}(\phi(\alpha)(u))} \text{ if } u \in \omega \land \operatorname{Seq}(\phi(\alpha)(u)).$

By Corollary 3.5 and the uniformization lemma, we may assume that the map $\tilde{r}_0 : (n_3, n) \mapsto m$ is Π_1^1 -recursive on $P_3 \times \omega$. We set $r_0(n_0, n_1, n_2, n_3, n) := \tilde{r}_0(n_3, n)$, which defines a partial map r_0 which is Π_1^1 -recursive on its Π_1^1 domain $\omega^3 \times P_3 \times \omega$.

(2) We now define $r_1(n_0, n_1, n_2, n_3, n)$ coding g_{n+1} . We use Lemma 3.9 and its proof. Note that $r_0(n_0, n_1, n_2, n_3, n) \in D_0 := \{m \in \mathcal{W}_1 \mid G \subseteq \mathcal{C}_m\}$. The proof of Lemma 3.9 shows that for any $m \in D_0$ there is $\tilde{F}_m \in \omega^{\omega} \cap \Delta_1^1$ satisfying the conditions (a), (b), (c) and

$$(d) \forall p \in D \neg (0 < (p)_1 = 2^{(p)_0}) \lor \mathcal{C}_{\tilde{F}_m(p)} \subseteq \mathcal{C}_m.$$

The uniformization lemma shows that we may assume that the partial map $\tilde{F}: m \mapsto \tilde{F}_m$ is Π_1^1 -recursive on D_0 .

The definition of h in the proof of Lemma 3.9 and the uniformization lemma show the existence of a partial map $\tilde{H}: \omega \to \omega$, which is Π_1^1 -recursive on D_0 , and such that $\tilde{H}(m)$ is in $\mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ and codes the graph of a Δ_1^1 -recurive map $h: \omega^{\omega} \times 2^{\omega} \to \mathbb{R}$ with

$$1 - h(\alpha, \beta) := \begin{cases} 0 \text{ if } (\alpha, \beta) \in G \\ \sup\{\frac{(p)_1}{2^{(p)_0}} \mid p \in D \ \land \ (\alpha, \beta) \notin \mathcal{C}_{\tilde{F}(m)(p)} \} \text{ if } (\alpha, \beta) \notin G \end{cases}$$

if $m \in D_0$. We set $P_1 := \{c \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} \mid C_c \text{ is the graph of a function } \zeta_c\}$. It is routine to check that there is a Π_1^1 -recursive partial map $I : \omega^2 \to \omega$ on its domain P_1^2 such that $I(c, c') \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}$ is the graph of the function $\min(\zeta_c, \zeta_{c'})$ if $c, c' \in P_1$. We set

$$r_1(n_0, n_1, n_2, n_3, n) := I\Big(n_1, \tilde{H}\big(r_0(n_0, n_1, n_2, n_3, n)\big)\Big),$$

so that r_1 is Π_1^1 -recursive on its Π_1^1 domain $\omega \times P_1 \times \omega \times P_3 \times \omega$.

(3) We now define $r_2(n_0, n_1, n_2, n_3, n)$ coding

$$S_{n+1} = \begin{cases} S_n + g_{n+1} \text{ if } n \text{ is odd,} \\ S_n - g_{n+1} \text{ if } n \text{ is even.} \end{cases}$$

It is routine to check that there is a Π_1^1 -recursive partial map $S: \omega^3 \to \omega$ on its domain $P_1^2 \times \omega$ such that $S(c, c', n) \in \mathcal{W}^{\omega^\omega \times 2^\omega \times \mathbb{R}}$ codes the graph of the function

$$(\alpha,\beta) \mapsto \begin{cases} \zeta_c(\alpha,\beta) + \zeta_{c'}(\alpha,\beta) \text{ if } n \text{ is odd} \\ \zeta_c(\alpha,\beta) - \zeta_{c'}(\alpha,\beta) \text{ if } n \text{ is even} \end{cases}$$

if $(c, c', n) \in P_1^2 \times \omega$. We set $r_2(n_0, n_1, n_2, n_3, n) := S(n_2, r_1(n_0, n_1, n_2, n_3, n), n)$, so that r_2 is Π_1^1 -recursive on its Π_1^1 domain $\omega \times P_1^2 \times P_3 \times \omega$.

(4) We now define $r_3(n_0, n_1, n_2, n_3, n)$ coding the graph of the function $\alpha \mapsto (s_j^{n+1,\alpha})_{j \in I_{n+1,\alpha}}$. We want to ensure the two following conditions:

$$\begin{array}{l} (1) \ G_{\alpha} \subseteq \bigcup_{j \in I_{n+1,\alpha}} \ N_{s_{j}^{n+1,\alpha}} \subseteq G_{n+1,\alpha}^{*} \\ (6) \ | \ f_{N_{s_{j}^{n+1,\alpha}}} \ S_{n+1}(\alpha,.) \ d\lambda - S_{n+1}(\alpha,\beta) | < 2^{-3} \ \text{if} \ j \in I_{n+1,\alpha} \ \land \ \beta \in G_{\alpha} \cap N_{s_{j}^{n+1,\alpha}} \\ \end{array}$$

Note first that in practice

$$S_{n+1}(\alpha,\beta) = \begin{cases} 0 \text{ if } n \text{ is even} \\ 1 \text{ if } n \text{ is odd} \end{cases}$$

if $(\alpha, \beta) \in G$ since $g_p(\alpha, \beta) = 1$ for each p in this case. So there is $\psi : \omega \to \mathbb{R}^2$ recursive with

$$| \int_{N_{s_{j}^{n+1,\alpha}}} S_{n+1}(\alpha,.) \, d\lambda - S_{n+1}(\alpha,\beta) | < 2^{-3} \Leftrightarrow \psi_{0}(n) < \int_{N_{s_{j}^{n+1,\alpha}}} S_{n+1}(\alpha,.) \, d\lambda < \psi_{1}(n)$$

if $(\alpha, \beta) \in G$. We use Corollary 3.5 and its proof. Note that $r_2(n_0, n_1, n_2, n_3, n) \in P_1$.

We first consider $n'_0 \in W_1$ and $n'_2 \in P_1$ (coding G^*_{n+1} and S_{n+1} respectively) as variables. We define $R_0, R_1 \subseteq \omega \times \omega^{\omega} \times 2^{\omega} \times \omega^3$ by

$$\begin{split} R_{0}(n'_{2},\alpha,\beta,u,k,l) \Leftrightarrow & \exists r \in \mathbb{R} \ \neg \left(n'_{2} \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} \ \land \ (n'_{2},\alpha,\beta,r) \notin \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}\right) \land \\ & \left(\frac{k}{2^{l}} \leq r < \frac{k+1}{2^{l}} \ \land \ \operatorname{Seq}(u) \ \land \ \beta \in N^{u}\right) \\ R_{1}(n'_{2},\alpha,\beta,u,k,l) \Leftrightarrow & \forall r \in \mathbb{R} \ \left(n'_{2} \in \mathcal{W}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}} \ \land \ (n'_{2},\alpha,\beta,r) \notin \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}}\right) \lor \\ & \left(\frac{k}{2^{l}} \leq r < \frac{k+1}{2^{l}} \ \land \ \operatorname{Seq}(u) \ \land \ \beta \in N^{u}\right), \end{split}$$

so that R_0 is Σ_1^1 , R_1 is Π_1^1 , and $R_0(n'_2, \alpha, \beta, u, k, l) \Leftrightarrow R_1(n'_2, \alpha, \beta, u, k, l)$ if $n'_2 \in P_1$. Then, as in the proof of Corollary 3.5.(d), we define $O_0, O_1 \subseteq \omega \times \omega^{\omega} \times 2^{\omega}$ by

$$O_{\varepsilon}(n_{2}',\alpha,\beta) \Leftrightarrow \operatorname{Seq}(\alpha(0)) \wedge \operatorname{lh}(\alpha(0)) = 3 \wedge R_{\varepsilon}(n_{2}',\alpha^{*},\beta,(\alpha(0))_{0},(\alpha(0))_{1},(\alpha(0))_{2})$$

if $\varepsilon \in 2$, so that O_0 is Σ_1^1 , O_1 is Π_1^1 , and $O_0(n'_2, \alpha, \beta) \Leftrightarrow O_1(n'_2, \alpha, \beta)$ if $n'_2 \in P_1$. In particular, $n'_2 \in P_1$ and Seq(u) imply that

$$\int_{N^u} S_{n+1}(\alpha, .) \ d\lambda = \lim_{l \to \infty} \sum_{k \le 2^l} \frac{k}{2^l} \lambda \big((O_{\varepsilon})_{n'_2, < u, k, l > \alpha} \big)$$

for each $\varepsilon \in 2$. Thus $a < \int_{N^u} S_{n+1}(\alpha, .) d\lambda < b$ is in this case equivalent to

$$\exists p_0, p_1, q_0, q_1, N \in \omega \ a < \frac{p_0}{p_1 + 1} \land \frac{q_0}{q_1 + 1} < b \land \forall l \ge N \ \frac{p_0}{p_1 + 1} \le \sum_{k \le 2^l} \frac{k}{2^l} \lambda \left((O_{\varepsilon})_{n'_2, < u, k, l > \alpha} \right) \le \frac{q_0}{q_1 + 1}$$

By Corollary 3.5.(b) applied to $D := P_1$, the partial map $\lambda_O : P_1 \times \omega^{\omega} \to \mathbb{R}$ defined by

$$\lambda_O(n_2',\alpha) := \lambda \big((O_0)_{n_2',\alpha} \big)$$

is Σ_1^1 -recursive and Π_1^1 -recursive on its domain. By 3E.2, 3G.1 and 3G.2 in [M], these two classes of functions are closed under composition. In particular, the partial map

$$(n_2', \alpha, u, l) \mapsto \sum_{k \le 2^l} \frac{k}{2^l} \lambda \left((O_{\varepsilon})_{n_2', < u, k, l > \alpha} \right)$$

is Σ_1^1 -recursive and Π_1^1 -recursive on $P_1 \times \omega^{\omega} \times \omega^2$. This shows the existence of $Q_0 \in \Sigma_1^1(\omega^2 \times \omega^{\omega} \times \omega)$ and $Q_1 \in \Pi_1^1(\omega^2 \times \omega^{\omega} \times \omega)$ such that

$$Q_0(n'_2, n, \alpha, u) \Leftrightarrow Q_1(n'_2, n, \alpha, u) \Leftrightarrow \operatorname{Seq}(u) \land \psi_0(n) < \int_{N^u} S_{n+1}(\alpha, .) \ d\lambda < \psi_1(n)$$

if $n'_2 \in P_1$. We now consider $n'_0 \in W_1$ and $n'_2 \in P_1$ as parameters. We set

$$\begin{split} P_{n'_{0},n'_{2}}(n,\alpha,u) \Leftrightarrow \\ Q_{1}(n'_{2},n,\alpha,u) \ \land \ N^{u} \subseteq \mathcal{C}_{n'_{0},\alpha} \ \land \ \forall k < \mathrm{lh}(u) \ \left(\neg Q_{0}\left(n'_{2},n,\alpha,\underline{u}(k)\right) \ \lor \ N^{\underline{u}(k)} \not\subseteq \mathcal{C}_{n'_{0},\alpha}\right). \end{split}$$

Note that for each $(\alpha, \beta) \in G$ there is $l \in \omega$ minimal with the properties that $N_{\beta|l} \subseteq C_{n'_0,\alpha}$ and $Q_1(n'_2, n, \alpha, < \beta(0), ..., \beta(l-1) >)$, so that $P_{n'_0,n'_2}(n, \alpha, < \beta(0), ..., \beta(l-1) >)$ since $n'_0 \in \mathcal{W}_1$ and $n'_2 \in P_1$. As $n'_0 \in \mathcal{W}_1$, $N^{\underline{u}(k)} \setminus C_{n'_0,\alpha}$ is a $\Delta_1^1(\alpha)$ compact subset of 2^{ω} , so that it contains a $\Delta_1^1(\alpha)$ point if it is not empty (see 4F.15 in [M]). This shows that $P_{n'_0,n'_2}$ is Π_1^1 .

The uniformization lemma provides a Δ_1^1 -recursive map $L: \omega \times \omega^\omega \times 2^\omega \to \omega$ such that

$$P_{n'_0,n'_2}\Big(n,\alpha,<\beta(0),...,\beta\big(L(n,\alpha,\beta)-1\big)>\Big)$$

if $(\alpha, \beta) \in G$. Note that the Σ_1^1 set

$$\sigma := \left\{ (n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid \exists \beta \in G_{\alpha} \ u = <\beta(0), ..., \beta \left(L(n, \alpha, \beta) - 1 \right) > \right\}$$

is contained in the Π_1^1 set $\pi := \{(n, \alpha, u) \in \omega \times \omega^{\omega} \times \omega \mid P_{n'_0, n'_2}(n, \alpha, u)\}$. By 7B.3 in [M], there is a Δ_1^1 subset δ of $\omega \times \omega^{\omega} \times \omega$ such that $\sigma \subseteq \delta \subseteq \pi$. We now also consider n as a parameter and define $\varphi: \omega^{\omega} \to \omega^{\omega}$ by

$$\varphi(\alpha)(u) := \begin{cases} u \text{ if } (n, \alpha, u) \in \delta, \\ 0 \text{ otherwise.} \end{cases}$$

Note that φ is Δ_1^1 -recursive, and that Seq $(\varphi(\alpha)(u))$ is equivalent to $(n, \alpha, u) \in \delta$. In particular,

(1) $G_{\alpha} \subseteq \bigcup \left\{ N^{\varphi(\alpha)(u)} \mid u \in \omega \land \operatorname{Seq}(\varphi(\alpha)(u)) \right\} \subseteq \mathcal{C}_{n'_{0},\alpha}$ (6) $| \int_{N^{\varphi(\alpha)(u)}} S_{n+1}(\alpha, .) d\lambda - S_{n+1}(\alpha, \beta) | < 2^{-3} \text{ if } \operatorname{Seq}(\varphi(\alpha)(u)) \land \beta \in G_{\alpha} \cap N^{\varphi(\alpha)(u)}$

for each $\alpha \in \omega^{\omega}$. Let $k \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}}$ such that $\mathcal{C}_{k}^{\omega^{\omega} \times \omega^{\omega}} = \operatorname{Gr}(\varphi)$. We now consider n'_{0} , n'_{2} and n as variables again. Note that for each $(n'_{0}, n'_{2}, n) \in \mathcal{W}_{1} \times P_{1} \times \omega$ there is $k \in \omega$ such that

$$R(n'_{0},n'_{2},n,k) \Leftrightarrow \begin{cases} k \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}} \land \\ \left(\forall \alpha \in \omega^{\omega} \ \forall \gamma \in \omega^{\omega} \ \left(k \in \mathcal{W}^{\omega^{\omega} \times \omega^{\omega}} \land \neg \mathcal{C}^{\omega^{\omega} \times \omega^{\omega}}(k,\alpha,\gamma) \right) \lor \\ \left((1) \ G_{\alpha} \subseteq \bigcup \left\{ N^{\gamma(u)} \mid u \in \omega \land \operatorname{Seq}(\gamma(u)) \right\} \subseteq \mathcal{C}_{n'_{0},\alpha} \\ \land (6) \ \forall u \in \omega \ \neg \operatorname{Seq}(\gamma(u)) \lor Q_{1}(n'_{2},n,\alpha,u) \right) \end{cases}$$

Note that $R \in \Pi_1^1(\omega^4)$. The uniformization lemma provides a partial map $K : \omega^3 \mapsto \omega$ which is Π_1^1 -recursive on its Π_1^1 domain $\mathcal{W}_1 \times P_1 \times \omega$, and $R(n'_0, n'_2, n, K(n'_0, n'_2, n))$ if

 $(n_0', n_2', n) \in \mathcal{W}_1 \times P_1 \times \omega.$

It remains to set $r_3(n_0, n_1, n_2, n_3, n) := K(n'_0, n'_2, n)$ if $n'_0 = r_0(n_0, n_1, n_2, n_3, n)$ and

$$n_2' = r_2(n_0, n_1, n_2, n_3, n),$$

so that r_3 is Π_1^1 -recursive on its Π_1^1 domain $\mathcal{W}_1 \times P_1^2 \times P_3 \times \omega$.

Finally, r is Π_1^1 -recursive on $W_1 \times P_1^2 \times P_3 \times \omega$, f is Π_1^1 -recursive on ω , and thus f is Δ_1^1 -recursive by the uniformization lemma since it is total.

• We are now ready to define the dimension two versions of G_n^* , g_n , S_n , and $(s_j^n)_{j \in I_n}$:

$$(1) G_n^* := \mathcal{C}_{f_0(n)},$$

$$(2) g_n(\alpha, \beta) = \rho \Leftrightarrow (f_1(n), \alpha, \beta, \rho) \in \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}},$$

$$(3) S_n(\alpha, \beta) = \rho \Leftrightarrow (f_2(n), \alpha, \beta, \rho) \in \mathcal{C}^{\omega^{\omega} \times 2^{\omega} \times \mathbb{R}},$$

$$(4) \begin{cases} (i) j \in I_{n,\alpha} \Leftrightarrow \exists \delta \in \omega^{\omega} \ (f_3(n), \alpha, \delta) \in \mathcal{C}^{\omega^{\omega} \times \omega^{\omega}} \land \operatorname{Seq}(\delta(j)), \\ (ii) s_j^{n,\alpha} = \delta(j) \text{ if } j \in I_{n,\alpha}. \end{cases}$$

By construction of r, these objects satisfy the conditions (1)-(6) of the proof of Lemma 2.7.

• Consequently, the martingale $F(\alpha)$ will be defined in such a way that if $u \in \omega$ codes $s \in 2^{<\omega}$, then $F(\alpha)(s) = \int_{N^u} f_{\infty}(\alpha, .) d\lambda$. Note that $G = \bigcap_{n \in \omega} G_n^*$, so that $\neg G$ is the disjoint union of the $G_n^* \setminus G_{n+1}^*$'s. Thus

$$\begin{split} \int_{N^{u}} f_{\infty}(\alpha, .) \ d\lambda &= \int_{N^{u} \setminus G_{\alpha}} f_{\infty}(\alpha, .) \ d\lambda = \sum_{n \in \omega} \int_{N^{u} \cap (G_{n}^{*})_{\alpha} \setminus (G_{n+1}^{*})_{\alpha}} f_{\infty}(\alpha, .) \ d\lambda \\ &= \sum_{n \in \omega} \sum_{j \leq n} \ (-1)^{j} \int_{N^{u} \cap (G_{n}^{*})_{\alpha} \setminus (G_{n+1}^{*})_{\alpha}} g_{j}(\alpha, .) \ d\lambda \\ &= \lim_{l \to \infty} \sum_{n \leq l} \sum_{j \leq n} \ (-1)^{j} \int_{N^{u} \cap (G_{n}^{*})_{\alpha} \setminus (G_{n+1}^{*})_{\alpha}} g_{j}(\alpha, .) \ d\lambda. \end{split}$$

Consequently, in order to prove that F is Δ_1^1 -recursive, it is enough to check that the partial map $(u, \alpha, j, n) \mapsto \int_{N^u \cap (G_n^*)_\alpha \setminus (G_{n+1}^*)_\alpha} g_j(\alpha, .) d\lambda$ is Δ_1^1 -recursive from $\{u \in \omega \mid \text{Seq}(u)\} \times \omega^\omega \times \omega^2$ into \mathbb{R} . By Corollary 3.5, it is enough to check that the map $h : \omega^\omega \times 2^\omega \to \mathbb{R}$ defined by

$$h(\alpha,\beta) := \begin{cases} g_{(\alpha(0))_0}(\alpha^*,\beta) \text{ if } \operatorname{Seq}(\alpha(0)) \land \operatorname{lh}(\alpha(0)) = 2 \land (\alpha^*,\beta) \in G^*_{(\alpha(0))_1} \backslash G^*_{(\alpha(0))_1+1}, \\ 0 \text{ otherwise,} \end{cases}$$

is Δ_1^1 -recursive. This comes from the facts that

$$(\alpha,\beta) \in G_n^* \Leftrightarrow (f_0(n),\alpha,\beta) \in \mathcal{C} \Leftrightarrow \neg (f_0(n) \in \mathcal{W} \land (f_0(n),\alpha,\beta) \notin \mathcal{C})$$

is Δ_1^1 in (α, β, n) and

$$g_{n}(\alpha,\beta) \in N(\mathbb{R},p) \Leftrightarrow \exists \rho \in \mathbb{R} \neg \left(f_{1}(n) \in \mathcal{W}^{\omega \times 2^{\omega} \times \mathbb{R}} \land \left(f_{1}(n), \alpha, \beta, \rho \right) \notin \mathcal{C}^{\omega \times 2^{\omega} \times \mathbb{R}} \right) \land$$
$$\rho \in N(\mathbb{R},p)$$
$$\Leftrightarrow \forall \rho \in \mathbb{R} \left(f_{1}(n) \in \mathcal{W}^{\omega \times 2^{\omega} \times \mathbb{R}} \land \left(f_{1}(n), \alpha, \beta, \rho \right) \notin \mathcal{C}^{\omega \times 2^{\omega} \times \mathbb{R}} \right) \lor$$
$$\rho \in N(\mathbb{R},p)$$

is Δ_1^1 in (α, β, n, p) .

• Finally, the map F is Δ_1^1 -recursive and is as required.

4 First consequences

(A) Universal sets

- We first recall some material from [K2]. The first result can be found in Section 23.F (see also [Za]).

Theorem 4.1 (*Zahorski*) Let *B* be a subset of [0, 1]. The following are equivalent: (a) there are $S \in \Sigma_2^0$ and $P \in \Pi_3^0$ with m(P) = 1, where *m* is the Lebesgue measure on [0, 1], such that $B = S \cap P$, (b) there is $f \in C([0, 1])$ with $B = \{x \in [0, 1] \mid f'(x) \text{ exists}\}$ (we consider only one-sided derivatives at the endpoints).

The second result is 23.23.

Theorem 4.2 Let \mathcal{G} be a G_{δ} subset of (0,1) with $m(\mathcal{G})=0$. Then

$$\{(f, x) \in C([0, 1]) \times \mathcal{G} \mid f'(x) \text{ exists}\}$$

is C([0, 1])-universal for $\Pi_3^0(\mathcal{G})$.

- We prove results in that spirit here.

Theorem 4.3 Let B be a subset of 2^{ω} . Then the following are equivalent:

(a) *B* is Σ_3^0 and has λ -measure zero, (b) there is $f \in \mathcal{M}$ with $B = \{\beta \in 2^{\omega} \mid osc(f, \beta) > 0\}$.

Proof. (a) \Rightarrow (b) Write $B = \bigcup_{n \in \omega} G_n$, where the G_n 's are G_{δ} . Lemma 2.7 gives, for each n, a martingale f_n with $G_n = D(f_n)$ and $\{ \operatorname{osc}(f_n, \beta) \mid \beta \in 2^{\omega} \} \subseteq \{0\} \cup [\frac{1}{2}, 1]$. Lemma 2.8 gives $f \in \mathcal{M}$ with D(f) = B.

(b) \Rightarrow (a) We already noticed in the introduction that *B* is Σ_3^0 . By Doob's theorem, *B* has λ -measure zero (see [D]).

Corollary 4.4 Let \mathcal{G} be a G_{δ} subset of 2^{ω} with $\lambda(\mathcal{G}) = 0$. Then $\{(f, \beta) \in \mathcal{M} \times \mathcal{G} \mid osc(f, \beta) > 0\}$ is \mathcal{M} -universal for $\Sigma_3^0(\mathcal{G})$.

For example, $\{\beta \in 2^{\omega} \mid \forall n \in \omega \ \beta(2n) = 0\}$ is a Π_1^0 copy of 2^{ω} and has λ -measure zero.

(B) Complete sets

- By 33.G in [K2], there is a uniform version of Zahorski's theorem, which allows to prove the following result

Theorem 4.5 (*Mazurkiewicz*) The set of differentiable functions in C([0, 1]) is Π_1^1 -complete.

- Here again, there is a result in that spirit.

Theorem 4.6 The set $\mathcal{P} := \{f \in \mathcal{M} \mid \forall \beta \in 2^{\omega} \text{ osc}(f, \beta) = 0\}$ is Π_1^1 -complete.

Notation. Let $\mathcal{K} := \{\beta \in 2^{\omega} \mid \forall n \in \omega \ \beta(2n) = 0\}$, which is a Π_1^0 copy of the Cantor space 2^{ω} with $\lambda(\mathcal{K}) = 0$. In particular, \mathcal{K} is a nonempty $G_{\delta} \cap \Delta_1^1$ subset of 2^{ω} .

Proof. Let $U \in \Pi_1^1(\omega^{\omega} \times 2^{\omega})$ be ω^{ω} -universal for the co-analytic subsets of 2^{ω} , and

$$\Pi := \{ \alpha \in \omega^{\omega} \mid ((\alpha)_0, (\alpha)_1) \in U \}$$

Note that $\Pi \in \Pi_1^1$. If $P \in \Pi_1^1(2^{\omega})$, then $P = U_{\alpha}$ for some $\alpha \in \omega^{\omega}$, so that the map $\beta \mapsto \langle \alpha, \beta \rangle$ is a continuous reduction of P to Π and Π is Π_1^1 -complete. Let $H \in \Pi_2^0(\omega^{\omega} \times 2^{\omega})$ with $\neg \Pi = \Pi_0[H]$. We set $G := \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid (\alpha, (\beta)_1) \in H \land \beta \in \mathcal{K}\}$, so that $G \in \Delta_1^1(\omega^{\omega} \times 2^{\omega})$, has G_{δ} vertical sections and $G \subseteq \omega^{\omega} \times \mathcal{K}$. Lemma 3.10 gives $F : \omega^{\omega} \to \mathcal{M}$ Borel such that $G_{\alpha} = \mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^{\omega}$. Thus

$$\alpha \notin \Pi \Leftrightarrow \exists \beta \in 2^{\omega} \ (\alpha, \beta) \in H \Leftrightarrow \exists \beta \in 2^{\omega} \ (\alpha, \beta) \in G \Leftrightarrow \exists \beta \in 2^{\omega} \ (F(\alpha), \beta) \in \mathcal{V} \Leftrightarrow F(\alpha) \notin \mathcal{P}.$$

Thus $\Pi = F^{-1}(\mathcal{P})$ and \mathcal{P} is Borel Π_1^1 -complete. By 26.C in [K2], \mathcal{P} is Π_1^1 -complete.

- We now prove Theorem 1.8. Let X be a metrizable compact space and Y be a Polish space. We equip C(X, Y) with the topology of uniform convergence, so that it is a Polish space (see 4.19 in [K2]). We use the map ψ defined before Theorem 1.8.

Theorem 4.7 (a) The set $\mathcal{P}_1 := \{(f_k)_{k \in \omega} \in \mathcal{P}^{\omega} \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges} \}$ is Π_1^1 -complete. (b) The set $\mathcal{P}_2 := \{(f_k)_{k \in \omega} \in \mathcal{P}^{\omega} \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges to zero} \}$ is Π_1^1 -complete. (c) The set $\mathcal{S} := \{(f_k)_{k \in \omega} \in \mathcal{P}^{\omega} \mid \exists \gamma \in \omega^{\omega} \ (\psi(f_{\gamma(i)}))_{i \in \omega} \text{ pointwise converges to zero} \}$ is Σ_2^1 -complete.

Proof. We define $\varphi: \mathcal{C}(2^{\omega}, [0, 1]) \to \mathcal{M}$ by $\varphi(h)(s) := \int_{N_s} h \ d\lambda$. As in the proof of Lemma 2.7, φ is well-defined. It is also continuous, and injective: if $h \neq h'$, then we can find $q \in \omega$ and $s \in 2^{<\omega}$ such that $h(\beta) - h'(\beta) > 2^{-q}$ for each $\beta \in N_s$ or $h'(\beta) - h(\beta) > 2^{-q}$ for each $\beta \in N_s$, so that

$$|\varphi(h)(s) - \varphi(h')(s)| = \frac{1}{\lambda(N_s)} |\int_{N_s} h \, d\lambda - \int_{N_s} h' \, d\lambda| \ge 2^{-q}.$$

This implies that the range \mathcal{R} of φ is Borel and $\psi := \varphi^{-1} : \mathcal{R} \to \mathcal{C}(2^{\omega}, [0, 1])$ is Borel. As every continuous map $h: 2^{\omega} \to [0, 1]$ is τ -continuous,

$$\lim_{l\to\infty}\varphi(h)(\beta|l) = \lim_{l\to\infty} \oint_{N_{\beta|l}} h \ d\lambda = h(\beta)$$

for each $\beta \in 2^{\omega}$, by Lemma 2.6. This implies that $f \in \mathcal{P}$ and $\psi(f)(\beta) = \lim_{l \to \infty} f(\beta|l)$ for each $\beta \in 2^{\omega}$ if $f \in \mathcal{R}$.

(a) Note that the proof of 33.11 in [K2] shows that the set

$$P_1 := \left\{ (h_k)_{k \in \omega} \in \left(\mathcal{C}(2^{\omega}, [0, 1]) \right)^{\omega} \mid (h_k)_{k \in \omega} \text{ pointwise converges} \right\}$$

is Π_1^1 -complete. As $\mathcal{E} := \{(f_k)_{k \in \omega} \in \mathcal{R}^{\omega} \mid (\psi(f_k))_{k \in \omega} \text{ pointwise converges}\} = (\psi^{\omega})^{-1}(P_1)$, the equalities $P_1 = (\varphi^{\omega})^{-1}(\mathcal{E}) = (\varphi^{\omega})^{-1}(\mathcal{P}_1)$ hold and \mathcal{P}_1 is Π_1^1 -complete.

(b) We argue as in (a).

(c) As in [B-Ka-L], the set

$$S := \{(h_k)_{k \in \omega} \in (\mathcal{C}(2^{\omega}, [0, 1]))^{\omega} \mid \exists \gamma \in \omega^{\omega} \ (h_{\gamma(i)})_{i \in \omega} \text{ pointwise converges to zero}\},\$$

is Σ_2^1 -complete. Indeed, fix $Q \in \Sigma_2^1(2^{\omega})$.

Lemma 2.2 in [B-Ka-L] gives $(g_k)_{k\in\omega} \in (\mathcal{C}(2^{\omega}\times 2^{\omega}, 2))^{\omega}$ such that, for each $\delta \in 2^{\omega}$, the following are equivalent:

(i)
$$\delta \in Q$$
,
(ii) $\exists \gamma \in \omega^{\omega} \ \forall \beta \in 2^{\omega} \ \lim_{i \to \infty} g_{\gamma(i)}(\delta, \beta) = 0$.

We define, $g: 2^{\omega} \to (\mathcal{C}(2^{\omega}, [0, 1]))^{\omega}$ by $g(\delta)(k)(\beta) := g_k(\delta, \beta)$. Then g is continuous and reduces Q to S. As

$$\mathcal{E}' := \left\{ (f_k)_{k \in \omega} \in \mathcal{R}^{\omega} \mid \exists \gamma \in \omega^{\omega} \; \left(\psi(f_{\gamma(i)}) \right)_{i \in \omega} \text{ pointwise converges to zero} \right\} = (\psi^{\omega})^{-1}(S),$$

 $S\!=\!(\varphi^\omega)^{-1}(\mathcal{E}')\!=\!(\varphi^\omega)^{-1}(\mathcal{S})$ and \mathcal{S} is $\mathbf{\Sigma}_2^1\text{-complete.}$

5 Universal and complete sets in the spaces $\mathcal{C}(2^{\omega}, X)$

- It is known that if Γ is a self-dual Wadge class and X is a Polish space, then there is no set which is X-universal for the subsets of X in Γ (see 22.7 in [K2]). This is no longer the case if the space of codes is different from the space of coded sets.

Proposition 5.1 Let X be a Polish space, Γ be a Wadge class with complete set $C \in \Gamma(X)$, and $\mathcal{U}^{\Gamma} := \{(h, \beta) \in \mathcal{C}(2^{\omega}, X) \times 2^{\omega} \mid h(\beta) \in C\}$. Then \mathcal{U}^{Γ} is $\mathcal{C}(2^{\omega}, X)$ -universal for the Γ subsets of 2^{ω} .

Proof. As the evaluation map $(h, \beta) \mapsto h(\beta)$ is continuous, $\mathcal{U}^{\Gamma} \in \Gamma$. If $A \in \Gamma(2^{\omega})$, then $A = h^{-1}(C)$ for some $h \in \mathcal{C}(2^{\omega}, X)$, so that $A = \mathcal{U}_h^{\Gamma}$.

We will partially strengthen this result to get our uniform universal sets.

- Recall that it is proved in [K3] that a Borel Π_1^1 -complete set is actually Π_1^1 -complete. In fact, Kechris's proof shows the result for the classes Π_n^1 . Our main tool is a uniform version of this. Kechris's result has recently been strengthened in [P] as follows.

Theorem 5.2 (Pawlikowski) Let $n \ge 1$ be a natural number, and $C \subseteq X \subseteq 2^{\omega}$. If Borel functions from 2^{ω} into X give as preimages of C all $\mathbf{\Pi}_n^1$ subsets of 2^{ω} , then so do continuous injections.

The main tool mentioned above is the following:

Theorem 5.3 Let $n \ge 1$ be a natural number, $\mathcal{U}^{\Pi_n^1, 2^{\omega}}$ be a suitable ω^{ω} -universal set for the Π_n^1 subsets of 2^{ω} , X be a recursively presented Polish space, $C \in \Pi_n^1(X)$, $\mathcal{R} : \omega^{\omega} \times \omega^{\omega} \to \omega^{\omega}$ be a recursive map, and $b: \omega^{\omega} \to X$ be a Δ_1^1 -recursive map such that

$$(\alpha,\beta) \in \mathcal{U}^{\mathbf{\Pi}_n^1,2^\omega} \Leftrightarrow b(\mathcal{R}(\alpha,\beta)) \in \mathcal{C}$$

for each $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Then there is a Δ_1^1 -recursive map $f: \omega^{\omega} \to \mathcal{C}(2^{\omega}, X)$ such that

$$(\alpha,\beta) \in \mathcal{U}^{\mathbf{\Pi}_n^1,2^\omega} \Leftrightarrow f(\alpha)(\beta) \in C$$

for each $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$.

- We first recall some material from [K3].

Definition 5.4 (a) A coding system for nonempty perfect binary trees is a pair $(\mathcal{D}, \mathcal{O})$, where $\mathcal{D} \subseteq 2^{\omega}$ and $\mathcal{O}: \mathcal{D} \rightarrow \{T \in 2^{2^{<\omega}} \mid T \text{ is a nonempty perfect binary tree}\}$ is onto.

(b) A coding system $(\mathcal{D}, \mathcal{O})$ is nice if

(i) for any $\alpha \in \omega^{\omega}$ and any $\Delta_1^1(\alpha)$ -recursive map $H: 2^{\omega} \times 2^{\omega} \to \omega$, we can find $\beta \in \mathcal{D} \cap \Delta_1^1(\alpha)$ and $k \in \omega$ such that $H(\beta, \delta) = k$ for each δ in the body $[\mathcal{O}(\beta)]$ of $\mathcal{O}(\beta)$,

(ii) \mathcal{D} is Π_1^1 and, for $\beta \in \mathcal{D}$, the relation

$$R(m,\beta) \Leftrightarrow Seq(m) \land ((m)_0,...,(m)_{lh(m)-1}) \in \mathcal{O}(\beta)$$

is Δ_1^1 , i.e., there are Π_1^1 relations Π_0, Π_1 such that $R(m, \beta) \Leftrightarrow \Pi_0(m, \beta) \Leftrightarrow \neg \Pi_1(m, \beta)$ if $\beta \in \mathcal{D}$.

Nice coding systems exist. If $\beta \in \mathcal{D}$, then there is a canonical homeomorphism β^* from $[\mathcal{O}(\beta)]$ onto 2^{ω} . We now check that the construction of β^* is effective.

Lemma 5.5 (a) The partial function $e:(\beta, \delta) \mapsto \beta^*(\delta)$ is Π_1^1 -recursive on its Π_1^1 domain

$$Domain(e) := \{ (\beta, \delta) \in \mathcal{D} \times 2^{\omega} \mid \delta \in [\mathcal{O}(\beta)] \}.$$

(b) The partial function $\iota: (\beta, \gamma) \mapsto$ the unique $\delta \in [\mathcal{O}(\beta)]$ with $\beta^*(\delta) = \gamma$ is Π_1^1 -recursive on its Π_1^1 domain $\mathcal{D} \times 2^{\omega}$.

Proof. (a) We define a Π_1^1 relation \mathcal{Q} on $\omega^2 \times (2^\omega)^2$ by

$$\mathcal{Q}(p,p',\beta,\delta) \Leftrightarrow \Big(\big(\forall \varepsilon \in 2 \ \Pi_0(\overline{(\delta|p')\varepsilon},\beta) \big) \land \big(\forall p \leq p'' < p' \ \exists \varepsilon \in 2 \ \Pi_1(\overline{(\delta|p'')\varepsilon},\beta) \big) \Big).$$

Note that

$$\beta^*(\delta)(n) = \varepsilon \Leftrightarrow \begin{cases} \exists l \in \omega \; \operatorname{Seq}(l) \land \operatorname{lh}(l) = n+1 \land \delta((l)_n) = \varepsilon \land \mathcal{Q}(0, (l)_0, \beta, \delta) \land \\ \forall m < n \; (l)_m < (l)_{m+1} \land \mathcal{Q}((l)_m + 1, (l)_{m+1}, \beta, \delta) \end{cases}$$

if $\beta \in \mathcal{D}$. The proof of (b) is similar.

- Let X be a recursively presented Polish space, and d_X and $(r_n^X)_{n\in\omega}$ be respectively a distance function and a recursive presentation of X. We now give a **recursive presentation of** $C(2^{\omega}, X)$, equipped with the usual distance defined by

$$d(h,h') := \sup_{\beta \in 2^{\omega}} d_X(h(\beta),h'(\beta)),$$

since this is not present in [M]. We define, by primitive recursion, a recursive map $\nu: \omega \to \omega$ such that $\nu(i)$ enumerates $\{s \in 2^{<\omega} \mid |s|=i\}$. We first set $\nu(0):=1=<>$. Then

$$\nu(i+1) = k \Leftrightarrow \operatorname{Seq}(k) \land \operatorname{lh}(k) = 2^{i+1} \land \forall l < 2^i \ \forall \varepsilon \in 2 \ (k)_{\varepsilon 2^i+l} = s\Big(\big(\nu(i)\big)_l\Big)\varepsilon.$$

If Seq(n) and $h(n) = 2^i$ for some i (< n), then we define $h_n : 2^\omega \to X$ by $h_n(\beta) := r_{(n)_l}^X$ if

$$\beta |i = s_l^i := s\Big(\big(\nu(i)\big)_l\Big).$$

If \neg Seq(n) or $h(n) \neq 2^i$ for each i, then we define $h_n : 2^{\omega} \to X$ by $h_n(\beta) := r_0^X$ if $\beta \in 2^{\omega}$. In any case, $h_n \in \mathcal{C}(2^{\omega}, X)$ and takes finitely many values.

Lemma 5.6 Let X be a recursively presented Polish space. Then the sequence $(h_n)_{n \in \omega}$ is a recursive presentation of $C(2^{\omega}, X)$, equipped with d.

Proof. We have to see that (h_n) is dense in $\mathcal{C}(2^{\omega}, X)$. So let $h \in \mathcal{C}(2^{\omega}, X)$, $\epsilon > 0$ and $m \in \omega$ with $2^{-m} < \frac{\epsilon}{2}$. As h is uniformly continuous, there is $i \in \omega$ such that $d_X(h(\beta), h(\delta)) < 2^{-m}$ if $\beta | i = \delta | i$. We choose, for each $l < 2^i$, $n_l \in \omega$ such that $d_X(r_{n_l}^X, h(s_l^i 0^{\infty})) < 2^{-m}$. We set $n := < n_0, ..., n_{2^i-1} >$. If $\beta \in 2^{\omega}$ and $\beta | i = s_l^i$, then $d_X(h(\beta), h_n(\beta)) \le d_X(h(\beta), h(s_l^i 0^{\infty})) + d_X(h(s_l^i 0^{\infty}), r_{n_l}^X) \le 2^{-m} + 2^{-m}$, so that $d(h, h_n) < \epsilon$. It is routine to check that the relations " $d(h_m, h_n) \le \frac{p}{q+1}$ " and " $d(h_m, h_n) < \frac{p}{q+1}$ " are recursive in (m, n, p, q).

We saw in the proof of Proposition 5.1 that the evaluation map $(h, \beta) \mapsto h(\beta)$ is continuous from $C(2^{\omega}, X) \times 2^{\omega}$ into X. We can say more if X is recursively presented.

Lemma 5.7 Let X be a recursively presented Polish space. Then the evaluation map is recursive.

Proof. Note that

which gives the result.

- We then strengthen 7A.3 in [M] about **primitive recursion** as follows. If Z, Y are recursively presented Polish spaces, $g: Z \to Y$ and $h: Y \times \omega \times Z \to Y$ are Π_1^1 -recursive and $f: \omega \times Z \to Y$ is defined by

$$\begin{cases} f(0,z)\!:=\!g(z),\\ f(n\!+\!1,z)\!:=\!h\bigl(f(n,z),n,z\bigr), \end{cases}$$

then f is also Π_1^1 -recursive. If $m: Z \to Z$ is Π_1^1 -recursive, then the proof of 7A.3 in [M] shows that the map $f': \omega \times Z \to Y$ defined by

$$\begin{cases} f'(0,z) := g(z), \\ f'(n+1,z) := h\Big(f'\big(n,m(z)\big), n, z\Big), \end{cases}$$

is also Π_1^1 -recursive. As in 7A.5 in [M], this can be extended to partial functions which are Π_1^1 -recursive on their domain.

- We are ready for the proof of our main tool.

Proof of Theorem 5.3. 3E.6 in [M] provides $\pi: \omega^{\omega} \to X$ recursive, $\mathcal{F} \in \Pi_1^0(\omega^{\omega})$ and a Δ_1^1 -recursive injection $\rho: X \to \omega^{\omega}$ such that $\pi|_{\mathcal{F}}$ is injective, $\pi[\mathcal{F}] = X$ and ρ is the inverse of $\pi|_{\mathcal{F}}$. Let us show that the map $\mu: h \mapsto \pi \circ h$ is Δ_1^1 -recursive from $\mathcal{C}(2^{\omega}, \omega^{\omega})$ into $\mathcal{C}(2^{\omega}, X)$. More generally, let Y be a recursively presented Polish space, and $\psi: Y \to \mathcal{C}(2^{\omega}, X)$. Note that

$$\begin{split} \psi(y) \in N\left(\mathcal{C}(2^{\omega}, X), n\right) &\Leftrightarrow d\left(\psi(y), h_{((n)_{1})_{0}}\right) < \frac{((n)_{1})_{1}}{((n)_{1})_{2}+1} \\ &\Leftrightarrow \exists m \in \omega \ \sup_{\beta \in 2^{\omega}} \ d_{X}\left(\psi(y)(\beta), h_{((n)_{1})_{0}}(\beta)\right) < \frac{((m)_{1})_{1}}{((m)_{1})_{2}+1} < \frac{((n)_{1})_{1}}{((n)_{1})_{2}+1} \\ &\Leftrightarrow \exists m \in \omega \ \forall \beta \in 2^{\omega} \ d_{X}\left(\psi(y)(\beta), h_{((n)_{1})_{0}}(\beta)\right) < \frac{((m)_{1})_{1}}{((m)_{1})_{2}+1} < \frac{((n)_{1})_{1}}{((n)_{1})_{2}+1} \end{split}$$

and $h_{((n)_1)_0}(\beta) = r_{g(n,\beta)}^X$ for some recursive map $g : \omega \times 2^\omega \to \omega$.

In the present case, $Y = \mathcal{C}(2^{\omega}, \omega^{\omega})$ and $\psi(y)(\beta) = \pi(y(\beta))$. Thus

$$\begin{aligned} d_X \big(\psi(y)(\beta), h_{((n)_1)_0}(\beta) \big) < &\frac{((m)_1)_1}{((m)_1)_2 + 1} \Leftrightarrow d_X \Big(\pi \big(y(\beta) \big), r_{g(n,\beta)}^X \Big) < &\frac{((m)_1)_1}{((m)_1)_2 + 1} \\ \Leftrightarrow \pi \big(y(\beta) \big) \in N \big(X, \big\langle 0, < g(n,\beta), \big((m)_1 \big)_1, \big((m)_1 \big)_2 > \big\rangle \big) \\ \Leftrightarrow \big(y(\beta), \big\langle 0, < g(n,\beta), \big((m)_1 \big)_1, \big((m)_1 \big)_2 > \big\rangle \big) \in G^{\pi}, \end{aligned}$$

where G^{π} is the Σ_1^0 neighborhood diagram of π . As the evaluation map is recursive, $h \mapsto \pi \circ h$ is Π_1^1 -recursive and total, and thus Δ_1^1 -recursive.

• Let us show that there is a Δ_1^1 -recursive map $f: \omega^{\omega} \to \mathcal{C}(2^{\omega}, X)$ such that $\mathcal{U}_{\alpha}^{\Pi_n^1, 2^{\omega}} = (f(\alpha))^{-1}(C)$ for each $\alpha \in \omega^{\omega}$. We adapt the proof of the main result in [K3]. We set $A := \pi^{-1}(C)$. As $C \in \Pi_n^1(X)$, $A \in \Pi_n^1(\omega^{\omega})$. If $\langle \beta^0, \delta^0 \rangle \in 2^{\omega}$, then we inductively define, for $i \in \omega$, $m_i, \beta^{i+1}, \delta^{i+1}$ as follows. If (β^i, δ^i) is given and in Domain(e), then $(\beta^i)^*(\delta^i) = \langle x_i, \beta^{i+1}, \delta^{i+1} \rangle$ and

 $m_i := \begin{cases} \text{ the location of the first 0 in } x_i \text{ if it exists,} \\ 2 \text{ otherwise.} \end{cases}$

We then set $Q := \{(\alpha, < \beta^0, \delta^0 >) \in \omega^{\omega} \times 2^{\omega} \mid \forall i \in \omega \ (\beta^i, \delta^i) \in \text{Domain}(e) \land (\alpha, (m_i)) \in \mathcal{U}^{\Pi_n^1, 2^{\omega}}\}$ and $B^* := Q_{\alpha}$, so that $Q \in \Pi_n^1(\omega^{\omega} \times 2^{\omega})$ and $\beta \in B^* \Leftrightarrow (\alpha, \beta) \in Q$ for each $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$ (note that B^* depends on α , but we denote it like this to keep the notation of [K3]). We define $I : \omega^{\omega} \to 2^{\omega}$ by $I(\alpha) := 0^{\alpha(0)} 10^{\alpha(1)} 1...$ Note that I a Δ_1^1 -recursive injection onto the Π_2^0 set

$$\mathbb{P}_{\infty} := \{ \beta \! \in \! 2^{\omega} \mid \forall p \! \in \! \omega \; \exists q \! \geq \! p \; \beta(q) \! = \! 1 \}$$

so that there is a Δ_1^1 -recursive map $\phi: 2^\omega \to \omega^\omega$ which is the inverse of I on \mathbb{P}_∞ . We set

$$Q' := \Big\{ \delta \in 2^{\omega} \mid (\delta)_0 \in \mathbb{P}_{\infty} \land \Big(\phi \big((\delta)_0 \big), (\delta)_1 \Big) \in Q \Big\},$$

so that $Q' \in \Pi_n^1(2^\omega)$. As $\mathcal{U}^{\Pi_n^1, 2^\omega}$ is suitable, there is $\alpha_Q \in \omega^\omega$ recursive with $Q' = \mathcal{U}^{\Pi_n^1, 2^\omega}_{\alpha_Q}$. Note that

$$\beta \in B^* \Leftrightarrow (\alpha, \beta) \in Q \Leftrightarrow < I(\alpha), \beta > \in Q' \Leftrightarrow (\alpha_Q, < I(\alpha), \beta >) \in \mathcal{U}^{\Pi_n^1, 2^{\omega}}$$
$$\Leftrightarrow b\big(\mathcal{R}(\alpha_Q, < I(\alpha), \beta >)\big) \in C \Leftrightarrow \rho\big(b\big(\mathcal{R}(\alpha_Q, < I(\alpha), \beta >)\big)\big) \in A.$$

We set $G := \rho \Big(b \big(\mathcal{R}(\alpha_Q, \langle I(\alpha), . \rangle) \big) \Big)$, so that $G : 2^{\omega} \to \omega^{\omega}$ is $\Delta_1^1(\alpha)$ -recursive and $\langle \beta^0, \delta^0 \rangle$ is in B^* if and only if $G(\langle \beta^0, \delta^0 \rangle) \in A$.

• As in [K3], we can find $F: 2^{<\omega} \to (2^{\omega} \times \omega)^{<\omega}$ satisfying the following properties:

$$\begin{array}{l} (1) \ t \subseteq t' \Rightarrow F(t) \subseteq F(t') \\ (2) \ |F(t)| = |t| + 1 \\ (3) \ (i) \ \text{if} \ F(\emptyset) = (\beta^0, k_0), \ \text{then} \ \beta^0 \in \mathcal{D} \ \land \ \forall \delta^0 \in [\mathcal{O}(\beta^0)] \ G(<\beta^0, \delta^0 >)(0) = k_0 \\ (ii) \ \text{if} \ F(\varepsilon_0, ..., \varepsilon_n) = (\beta^0, k_0, \beta^1, k_1, ..., \beta^{n+1}, k_{n+1}), \ \text{then} \\ (a) \ \forall i \leq n+1 \ \beta^i \in \mathcal{D} \\ (b) \ \text{for all} \ \delta^{n+1} \in [\mathcal{O}(\beta^{n+1})], \ \text{if} \ \delta^n, ..., \delta^0 \ \text{are the uniquely determined members of} \\ [\mathcal{O}(\beta^n)], ..., [\mathcal{O}(\beta^0)] \ \text{such that} \ \forall i \leq n \ (\beta^i)^*(\delta^i) = <\overline{\varepsilon_i}, \beta^{i+1}, \delta^{i+1} > , \ \text{where} \\ \overline{\varepsilon_i} = 1^{\varepsilon_i} 01^{\infty}, \ \text{then} \ \forall i \leq n+1 \ G(<\beta^0, \delta^0 >)(i) = k_i. \end{array}$$

We will need an effective version of this, so that we give the details of the construction of F. In fact, the β^i 's involved in the definition of F can be $\Delta_1^1(\alpha)$. In order to see this, we first define

$$H_0: 2^\omega \times 2^\omega \to \omega$$

by $H_0(\beta, \delta) := G(\langle \beta, \delta \rangle)(0)$. As G is $\Delta_1^1(\alpha)$ -recursive, H_0 too, and the niceness of the coding system gives $\beta^0 \in \mathcal{D} \cap \Delta_1^1(\alpha)$ and $k_0 \in \omega$ such that $G(\langle \beta^0, \delta^0 \rangle)(0) = k_0$ for each $\delta^0 \in [\mathcal{O}(\beta^0)]$. Now suppose that $n \in \omega$, $(\varepsilon_0, ..., \varepsilon_n)$ and $F(\varepsilon_0, ..., \varepsilon_{n-1}) = (\beta^0, k_0, ..., \beta^n, k_n)$ are given. We define

$$H_{n+1}: 2^{\omega} \times 2^{\omega} \to \omega$$

as follows. Given $(\beta, \delta) \in 2^{\omega} \times 2^{\omega}$, let $\delta^n, ..., \delta^0$ be the uniquely determined members of $[\mathcal{O}(\beta^n)], ..., [\mathcal{O}(\beta^0)]$ resp., such that $(\beta^n)^*(\delta^n) = \langle \overline{\varepsilon_n}, \beta, \delta \rangle$, and $(\beta^i)^*(\delta^i) = \langle \overline{\varepsilon_i}, \beta^{i+1}, \delta^{i+1} \rangle$ if i < n. Put $H_{n+1}(\beta, \delta) := G(\langle \beta^0, \delta^0 \rangle)(n+1)$. As H_{n+1} is $\Delta_1^1(\alpha)$ (it is total and $\Pi_1^1(\alpha)$ -recursive since ι is Π_1^1 -recursive), the niceness of the coding system gives $\beta^{n+1} \in \mathcal{D} \cap \Delta_1^1(\alpha)$ and $k_{n+1} \in \omega$ such that $G(\langle \beta^0, \delta^0 \rangle)(n+1) = k_{n+1}$ for each $\delta^{n+1} \in [\mathcal{O}(\beta^{n+1})]$. Then

$$F(\varepsilon_0, ..., \varepsilon_n) := (\beta^0, k_0, ..., \beta^{n+1}, k_{n+1}),$$

so that F is as desired. So we can assume that the β^{i} 's are $\Delta_1^1(\alpha)$ in the conditions required for F.

• By [K3] again, the map $h_{\alpha} : (\varepsilon_i) \mapsto (k_i)$ is continuous and $\mathcal{U}_{\alpha}^{\Pi_n^1, 2^{\omega}} = h_{\alpha}^{-1}(A)$. As this is not too long to prove, we give the details for completeness. The map h_{α} is in fact more than continuous: it is Lipschitz, by definition. Fix (ε_i) . We apply F to the initial segments of (ε_i) , which gives (β^i) . For each n, we define perfect sets $C_0^n, C_1^n, ..., C_n^n \subseteq 2^{\omega}$ with $C_i^n \subseteq [\mathcal{O}(\beta^i)]$ if $i \le n$, as follows:

$$\begin{split} C_n^n &:= \{\delta^n \in [\mathcal{O}(\beta^n)] \mid \exists \delta^{n+1} \in 2^{\omega} \quad (\beta^n)^* (\delta^n) = <\overline{\varepsilon_n}, \beta^{n+1}, \delta^{n+1} > \}, \\ C_{n-1}^n &:= \{\delta^{n-1} \in [\mathcal{O}(\beta^{n-1})] \mid \exists \delta^n \in C_n^n \quad (\beta^{n-1})^* (\delta^{n-1}) = <\overline{\varepsilon_{n-1}}, \beta^n, \delta^n > \}, \\ \dots \\ C_0^n &:= \{\delta^0 \in [\mathcal{O}(\beta^0)] \mid \exists \delta^1 \in C_1^n \quad (\beta^0)^* (\delta^0) = <\overline{\varepsilon_0}, \beta^1, \delta^1 > \}. \end{split}$$

Note that

(4) $\delta^0 \in C_0^n \Rightarrow \langle \beta^i, \delta^i \rangle \in \text{Domain}(e)$ for each $i \leq n$, where $\delta^1, ..., \delta^n$ are computed according to the formula in (3).(*ii*).(*b*),

(5) $n' \ge n \Rightarrow \forall i \le n \ C_i^{n'} \subseteq C_i^n$.

This implies that $[\mathcal{O}(\beta^0)] \supseteq C_0^0 \supseteq C_0^1 \supseteq C_0^2 \supseteq \dots$ and $\bigcap_{n \in \omega} C_0^n$ contains some δ^0 . Note that $\langle \beta^i, \delta^i \rangle$ is in Domain(e), and $(\beta^i)^*(\delta^i) = \langle \overline{\varepsilon_i}, \beta^{i+1}, \delta^{i+1} \rangle$ for each $i \in \omega$. By (3).(ii).(b),

$$G(<\beta^0,\delta^0>)=k_i$$

 $\text{for each } i \!\in\! \omega. \text{ As } < \beta^0, \delta^0 > \!\!\in\! B^* \Leftrightarrow G(<\beta^0, \delta^0 >) \!\in\! A,$

$$\left(\forall i \in \omega < \beta^i, \delta^i > \in \text{Domain}(e) \land (\varepsilon_i) \in \mathcal{U}^{\Pi^1_n, 2^\omega}_{\alpha}\right) \Leftrightarrow (k_i) \in A.$$

 $As < \beta^i, \delta^i > \text{is in Domain}(e) \text{ for each } i \in \omega, (\varepsilon_i) \in \mathcal{U}_{\alpha}^{\mathbf{\Pi}_n^1, 2^{\omega}} \Leftrightarrow h_{\alpha}\big((\varepsilon_i)\big) = (k_i) \in A.$

• So we found, for each $\alpha \in \omega^{\omega}$, $h_{\alpha} \in \mathcal{C}(2^{\omega}, \omega^{\omega})$ such that $\mathcal{U}_{\alpha}^{\Pi_{n}^{1}, 2^{\omega}} = (\pi \circ h_{\alpha})^{-1}(C) = (\mu(h_{\alpha}))^{-1}(C)$. It remains to see that the map $\psi : \alpha \mapsto h_{\alpha}$, from ω^{ω} into $\mathcal{C}(2^{\omega}, \omega^{\omega})$, can be Δ_{1}^{1} -recursive (then f will be $\mu \circ \psi$). By the previous discussion, it is enough to see that the relation " $k_{i} = k$ " is Δ_{1}^{1} in $(\alpha, (\varepsilon_{i}), i, k) \in \omega^{\omega} \times 2^{\omega} \times \omega^{2}$.

• We will define, by primitive recursion, a Δ_1^1 -recursive map $\tilde{f}: \omega \times \omega^{\omega} \times 2^{\omega} \to 2^{\omega} \times \omega$ such that $\tilde{f}(n, \alpha, (\varepsilon_i))$ will be of the form $(\langle \tilde{\beta}^0, ..., \tilde{\beta}^n, \tilde{\beta}^n, ... \rangle, \langle \tilde{k}^0, ..., \tilde{k}^n \rangle)$ and can play the role of $F(\varepsilon_0, ..., \varepsilon_{n-1})$. We first set

$$\begin{split} P := & \Big\{ \big(\alpha, (\varepsilon_i), \beta, k \big) \in \omega^{\omega} \times (2^{\omega})^2 \times \omega \mid \\ & \forall i \in \omega \ (\beta)_i = (\beta)_0 \in \mathcal{D} \cap \Delta^1_1(\alpha) \land \forall \delta \in \big[\mathcal{O}\big((\beta)_0 \big) \big] \ G(<(\beta)_0, \delta >)(0) = k \Big\}. \end{split}$$

Note that P is Π_1^1 and for any $(\alpha, (\varepsilon_i)) \in \omega^{\omega} \times 2^{\omega}$ there is $(\beta, k) \in 2^{\omega} \times \omega$ such that $(\alpha, (\varepsilon_i), \beta, k) \in P$. The uniformization lemma gives a Δ_1^1 -recursive map $\tilde{g} : \omega^{\omega} \times 2^{\omega} \to 2^{\omega} \times \omega$ such that

$$\left(\alpha, (\varepsilon_i), \tilde{g}(\alpha, (\varepsilon_i))\right) \in P$$

for each $(\alpha, (\varepsilon_i)) \in \omega^{\omega} \times 2^{\omega}$. Then we set

$$D := \Big\{ \big(\beta, p, n, \alpha, (\varepsilon_i)\big) \in 2^{\omega} \times \omega^2 \times \omega^{\omega} \times 2^{\omega} \mid \operatorname{Seq}(p) \wedge \operatorname{lh}(p) = n + 1 \wedge \forall q \in \omega \ (\beta)_q \in \mathcal{D} \cap \Delta^1_1(\alpha) \Big\}.$$

Note that D is Π_1^1 , as well as

$$\begin{split} R &:= \Big\{ \left(\beta, p, n, \alpha, (\varepsilon_i), \beta', k'\right) \in D \times 2^{\omega} \times \omega \ | \ \forall i > n \ (\beta')_i = (\beta')_{n+1} \in \mathcal{D} \cap \Delta_1^1(\alpha) \land \\ & \operatorname{Seq}(k') \wedge \operatorname{lh}(k') = n + 2 \land \forall i \le n \ (\beta')_i = (\beta)_i \land (k')_i = (p)_i \land \\ & \forall \delta \in 2^{\omega} \ \left(\exists i \le n+1 \ (\delta)_i \notin \left[\mathcal{O}\big((\beta')_i\big) \right] \lor \exists i \le n \ (\beta')_i^* \big((\delta)_i\big) \neq <\overline{\varepsilon_i}, (\beta')_{i+1}, (\delta)_{i+1} > \lor \\ & \forall i \le n+1 \ G\big(< (\beta')_0, (\delta)_0 > \big)(i) = (k')_i \big) \Big\}. \end{split}$$

Moreover, for each $(\beta, p, n, \alpha, (\varepsilon_i)) \in D = \prod_{2^{\omega} \times \omega^2 \times \omega^{\omega} \times 2^{\omega}} [R]$ there is $(\beta', k') \in (2^{\omega} \cap \Delta_1^1(\alpha)) \times \omega$ such that $(\beta, p, n, \alpha, (\varepsilon_i), \beta', k') \in R$. The uniformization lemma gives a partial map

$$\tilde{h}: 2^{\omega} \times \omega^2 \times \omega^{\omega} \times 2^{\omega} \to 2^{\omega} \times \omega$$

which is Π_1^1 -recursive on its domain D, and such that $(\beta, p, n, \alpha, (\varepsilon_i), \tilde{h}(\beta, p, n, \alpha, (\varepsilon_i))) \in R$ if $(\beta, p, n, \alpha, (\varepsilon_i)) \in D$. This implies that the partial map \tilde{f} defined by

$$\begin{cases} \tilde{f}(0,\alpha,(\varepsilon_i)) := \tilde{g}(\alpha,(\varepsilon_i)), \\ \tilde{f}(n+1,\alpha,(\varepsilon_i)) := \tilde{h}(\tilde{f}(n,\alpha,(\varepsilon_i)), n,\alpha,(\varepsilon_i)), \end{cases}$$

is Π_1^1 -recursive.

Moreover, an induction shows that $(\tilde{f}(n, \alpha, (\varepsilon_i)), n, \alpha, (\varepsilon_i)) \in D$ for each $(n, \alpha, (\varepsilon_i))$, so that \tilde{f} is in fact total, and thus Δ_1^1 -recursive. More precisely, $\tilde{f}(n, \alpha, (\varepsilon_i))$ is of the form

$$(<\beta^0, ..., \beta^n, \beta^n, ... >, < k_0, ..., k_n >),$$

where $(\varepsilon_0, ..., \varepsilon_{n-1}) \mapsto (\beta^0, k_0, ..., \beta^n, k_n)$ satisfies the properties (1)-(3) of *F*. It remains to note that $k_i = \tilde{f}(i, \alpha, (\varepsilon_i))(1)(i)$.

- We now prove the consequences of our main tool.

Definition 5.8 Let Γ be a class of subsets of recursively presented Polish spaces, Γ be the corresponding boldface class, X, Y be recursively presented Polish spaces, and $\mathcal{U} \in \Gamma(Y \times X)$. We say that \mathcal{U} is effectively uniformly Y-universal for the Γ subsets of X if the following hold:

- (1) $\Gamma(X) = \{\mathcal{U}_y \mid y \in Y\},\$
- (2) $\Gamma(X) = \{ \mathcal{U}_y \mid y \in Y \ \Delta_1^1 \text{-recursive} \},$

(3) for each $S \in \Gamma(\omega^{\omega} \times X)$, there is a Borel map $b: \omega^{\omega} \to Y$ such that $S_{\alpha} = \mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$, (4) for each $S \in \Gamma(\omega^{\omega} \times X)$, there is a Δ_1^1 -recursive map $b: \omega^{\omega} \to Y$ such that $S_{\alpha} = \mathcal{U}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$.

Notation. Let $\mathcal{U}^{\Pi_1^1, 2^{\omega}} \in \Pi_1^1$ be a good ω^{ω} -universal for the Π_1^1 subsets of 2^{ω} , X_1 be a recursively presented Polish space, and \mathcal{C}_1 be a Π_1^1 subset of X_1 for which there is a Δ_1^1 -recursive map $b: \omega^{\omega} \to X_1$ such that

$$(\alpha,\beta) \in \mathcal{U}^{\Pi_1,2^{\omega}} \Leftrightarrow b(<\alpha,\beta>) \in \mathcal{C}_1$$

if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. We define, for each natural number $n \ge 1$,

- $X_{n+1} := \mathcal{C}(2^{\omega}, X_n)$ (inductively),
- $\mathcal{C}_{n+1} := \{h \in X_{n+1} \mid \forall \beta \in 2^{\omega} \ h(\beta) \notin \mathcal{C}_n\}$ (inductively),
- $\mathcal{U}_n := \{(h, \beta) \in X_{n+1} \times 2^{\omega} \mid h(\beta) \in \mathcal{C}_n\}.$

Theorem 5.9 Let $n \ge 1$ be a natural number. Then

(a) the set U_n is effectively uniformly X_{n+1}-universal for the Π¹_n subsets of 2^ω,
(b) the set C_n is Π¹_n-complete.

Proof. We argue by induction on n.

(a) Assume first that n = 1, and fix $S \in \Pi_1^1(\omega^{\omega} \times 2^{\omega})$. Our assumption gives $b_1 : \omega^{\omega} \to X_1$. As $\mathcal{U}^{\Pi_1^1, 2^{\omega}} \in \Pi_1^1$ is a good ω^{ω} -universal for the Π_1^1 subsets of 2^{ω} , there is by Theorem 5.3 a Δ_1^1 -recursive map $f_1 : \omega^{\omega} \to \mathcal{C}(2^{\omega}, X_1)$ such that $(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^{\omega}} \Leftrightarrow f_1(\alpha)(\beta) \in \mathcal{C}_1$ if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Let $\alpha_S \in \omega^{\omega}$ with $S = \mathcal{U}_{\alpha_S}^{\Pi_1^1, \omega^{\omega} \times 2^{\omega}}$. Note that

$$(\alpha,\beta) \in S \Leftrightarrow \left(\mathcal{R}(\alpha_S,\alpha),\beta \right) \in \mathcal{U}^{\Pi_1^1,2^{\omega}} \Leftrightarrow f_1\left(\mathcal{R}(\alpha_S,\alpha) \right)(\beta) \in \mathcal{C}_1 \Leftrightarrow \left(f_1\left(\mathcal{R}(\alpha_S,\alpha) \right),\beta \right) \in \mathcal{U}_1.$$

As C_1 is Π_1^1 , \mathcal{U}_1 too. If $A \in \mathbf{\Pi}_1^1(2^{\omega})$, then $A = \mathcal{U}_{\alpha}^{\mathbf{\Pi}_1^1, 2^{\omega}}$ for some $\alpha \in \omega^{\omega}$. Applying the previous discussion to $S := \mathcal{U}^{\mathbf{\Pi}_1^1, 2^{\omega}}$, we get $A = (\mathcal{U}_1)_{f_1(\mathcal{R}(\alpha_S, \alpha))}$, so that \mathcal{U}_1 is X_2 -universal for the $\mathbf{\Pi}_1^1$ subsets of 2^{ω} , effectively and uniformly.

We now study \mathcal{U}_{n+1} . Fix $S \in \Pi^1_{n+1}(\omega^{\omega} \times 2^{\omega})$. Let $\mathcal{U}^{\Pi^1_n, 2^{\omega}}$ be a good ω^{ω} -universal for the Π^1_n subsets of 2^{ω} . We set $\mathcal{V}^{\Pi^1_{n+1}, 2^{\omega}} := \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid \forall \delta \in 2^{\omega} \ (\mathcal{R}(\alpha, \beta), \delta) \notin \mathcal{U}^{\Pi^1_n, 2^{\omega}}\}$, so that $\mathcal{V}^{\Pi^1_{n+1}, 2^{\omega}}$ is a suitable ω^{ω} -universal for the Π^1_{n+1} subsets of 2^{ω} . Moreover, the induction assumption gives a Δ^1_1 -recursive map $b_{n+1} : \omega^{\omega} \to X_{n+1}$ such that

$$(\alpha,\beta) \in \mathcal{V}^{\mathbf{\Pi}_{n+1}^{1},2^{\omega}} \Leftrightarrow \forall \delta \in 2^{\omega} \left(\mathcal{R}(\alpha,\beta),\delta \right) \notin \mathcal{U}^{\mathbf{\Pi}_{n}^{1},2^{\omega}} \Leftrightarrow \forall \delta \in 2^{\omega} \left(b_{n+1} \left(\mathcal{R}(\alpha,\beta) \right),\delta \right) \notin \mathcal{U}_{n} \\ \Leftrightarrow \forall \delta \in 2^{\omega} \ b_{n+1} \left(\mathcal{R}(\alpha,\beta) \right) (\delta) \notin \mathcal{C}_{n} \Leftrightarrow b_{n+1} \left(\mathcal{R}(\alpha,\beta) \right) \in \mathcal{C}_{n+1}$$

Theorem 5.3 gives a Δ_1^1 -recursive map f_{n+1} such that $(\alpha, \beta) \in \mathcal{V}^{\Pi_{n+1}^1, 2^{\omega}} \Leftrightarrow f_{n+1}(\alpha)(\beta) \in \mathcal{C}_{n+1}$ if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Let

$$Q\!\in\!\boldsymbol{\Pi}_n^1(\omega^\omega\!\times\!2^\omega\!\times\!2^\omega)\!\subseteq\!\boldsymbol{\Pi}_n^1(\omega^\omega\!\times\!\omega^\omega\!\times\!2^\omega)$$

such that $(\alpha, \beta) \in S \Leftrightarrow \forall \delta \in 2^{\omega} \ (\alpha, \beta, \delta) \notin Q$, and $\alpha_Q \in \omega^{\omega}$ such that $Q = \mathcal{U}_{\alpha_Q}^{\mathbf{\Pi}_n^1, \omega^{\omega} \times \omega^{\omega} \times 2^{\omega}}$. Note that

$$(\alpha,\beta) \in S \Leftrightarrow \forall \delta \in 2^{\omega} \left(\mathcal{R}(\mathcal{R}'(\alpha_Q,\alpha),\beta),\delta \right) \notin \mathcal{U}^{\mathbf{\Pi}_n^{1,2^{\omega}}} \Leftrightarrow \left(\mathcal{R}'(\alpha_Q,\alpha),\beta \right) \in \mathcal{V}^{\mathbf{\Pi}_{n+1}^{1,2^{\omega}}} \\ \Leftrightarrow f_{n+1} \big(\mathcal{R}'(\alpha_Q,\alpha) \big)(\beta) \in \mathcal{C}_{n+1} \Leftrightarrow \Big(f_{n+1} \big(\mathcal{R}'(\alpha_Q,\alpha) \big),\beta \Big) \in \mathcal{U}_{n+1}.$$

As $C_n \in \Pi_n^1$, $C_{n+1} \in \Pi_{n+1}^1$ and $\mathcal{U}_{n+1} \in \Pi_{n+1}^1$. If $A \in \Pi_{n+1}^1(2^{\omega})$, then $A = \mathcal{U}_{\alpha}^{\Pi_{n+1}^1, 2^{\omega}}$ for some $\alpha \in \omega^{\omega}$. Applying the previous discussion to $S := \mathcal{U}^{\Pi_{n+1}^1, 2^{\omega}}$, we get $A = (\mathcal{U}_{n+1})_{f_{n+1}(\mathcal{R}'(\alpha_Q, \alpha))}$, so that \mathcal{U}_{n+1} is X_{n+2} -universal for the analytic subsets of 2^{ω} , effectively and uniformly.

(b) By definition, $C_1 \in \Pi_1^1$, and $C_{n+1} \in \Pi_{n+1}^1$ if $C_n \in \Pi_n^1$. Assume first that $E \in \Pi_n^1(2^{\omega})$. Then $E = (\mathcal{U}_n)_h$ for some $h \in \mathcal{C}(2^{\omega}, X_n)$, by (a). Thus $E = h^{-1}(\mathcal{C}_n)$. If Z is a zero-dimensional Polish space and $D \in \Pi_n^1(Z)$, then we may assume that Z is a G_{δ} subset of 2^{ω} by 7.8 in [K2], so that $D \in \Pi_n^1(2^{\omega})$. The previous discussion gives $g \in \mathcal{C}(2^{\omega}, X_n)$ with $D = g^{-1}(\mathcal{C}_n)$. Thus $D = (g_{|Z})^{-1}(\mathcal{C}_n)$ and \mathcal{C}_n is Π_n^1 -complete.

Proof of Theorem 1.7. By Theorem 5.9, it is enough to show that if $\mathcal{U}^{\Pi_1^1, 2^{\omega}} \in \Pi_1^1$ is a good ω^{ω} universal set for the Π_1^1 subsets of 2^{ω} , then there is a Δ_1^1 -recursive map $b: \omega^{\omega} \to [0, 1]^{2^{<\omega}}$ such that $(\alpha, \beta) \in \mathcal{U}^{\Pi_1^1, 2^{\omega}} \Leftrightarrow b(<\alpha, \beta >) \in \mathcal{P}$ if $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$. Let $H \in \Pi_2^0(\omega^{\omega} \times 2^{\omega} \times 2^{\omega})$ such that $-\mathcal{U}^{\Pi_1^1, 2^{\omega}} = \Pi_{\omega^{\omega} \times 2^{\omega}}[H]$. We set $G := \{(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega} \mid ((\alpha)_0, (\alpha)_1, (\beta)_1) \in H \land \beta \in \mathcal{K}\}$, so that $G \in \Delta_1^1(\omega^{\omega} \times 2^{\omega})$, has G_{δ} vertical sections and $G \subseteq \omega^{\omega} \times \mathcal{K}$. Lemma 3.10 gives a Δ_1^1 -recursive map $F: \omega^{\omega} \to [0, 1]^{2^{<\omega}}$, taking values in \mathcal{M} , and such that $G_{\alpha} = \mathcal{V}_{b(\alpha)}$ for each $\alpha \in \omega^{\omega}$. If $(\alpha, \beta) \in \omega^{\omega} \times 2^{\omega}$, then

$$\begin{aligned} (\alpha,\beta) \notin \mathcal{U}^{\mathbf{\Pi}_{1}^{1},2^{\omega}} &\Leftrightarrow \exists \delta \in 2^{\omega} \ (\alpha,\beta,\delta) \in H \Leftrightarrow \exists \delta \in 2^{\omega} \ (<\alpha,\beta>,\delta) \in G \\ &\Leftrightarrow \exists \delta \in 2^{\omega} \ (b(<\alpha,\beta>),\delta) \in \mathcal{V} \Leftrightarrow b(<\alpha,\beta>) \notin \mathcal{P}. \end{aligned}$$

This finishes the proof.

Questions. Let U be a Π_2^0 subset of $\omega^{\omega} \times 2^{\omega}$ which is universal for $\Pi_2^0(2^{\omega})$. We set

$$G := \{ (\alpha, \beta) \in \omega^{\omega} \times \mathcal{K} \mid (\alpha, (\beta)_1) \in U \}.$$

Note that G is a Π_2^0 subset of $\omega^{\omega} \times 2^{\omega}$ contained in $\omega^{\omega} \times \mathcal{K}$ which is universal for $\Pi_2^0(\mathcal{K})$. Indeed, fix $H \in \Pi_2^0(\mathcal{K})$. Then $H' := \{\gamma \in 2^{\omega} \mid < 0^{\infty}, \gamma \rangle \in H\}$ is Π_2^0 , which gives $\alpha_0 \in \omega^{\omega}$ with $H' = U_{\alpha_0}$. Then $H = G_{\alpha_0}$.

Let $\alpha \mapsto ((\alpha)_k)_{k \in \omega}$ be a homeomorphism between ω^{ω} and $(\omega^{\omega})^{\omega}$, with inverse map

$$(\alpha_k)_{k\in\omega}\mapsto <\alpha_0,\alpha_1,\ldots>.$$

We set $S' := \{ \alpha \in \omega^{\omega} \mid \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad \forall \beta \in 2^{\omega} \quad \beta \notin G_{(\alpha)_{\gamma(i)}} \}$. Note that S' is Σ_2^1 .

(1) Is S' a Borel Σ_2^1 -complete set?

Assume that this is the case. Then the set $S_2 := \{(f_k)_{k \in \omega} \in \mathcal{M}^{\omega} \mid \exists \gamma \in \omega^{\omega} \quad \forall i \in \omega \quad f_{\gamma(i)} \in \mathcal{P}\}$ of sequences of martingales having a subsequence made of everywhere converging martingales is Borel Σ_2^1 -complete. Indeed, Lemma 3.10 gives a Borel map $F : \omega^{\omega} \to \mathcal{M}$ such that $G_{\alpha} = \mathcal{V}_{F(\alpha)}$ for each $\alpha \in \omega^{\omega}$. The map $\tilde{F} : \omega^{\omega} \to \mathcal{M}^{\omega}$ defined by $\tilde{F}(\alpha)(k) := F((\alpha)_k)$ is Borel. Moreover,

$$\begin{split} \tilde{F}(\alpha) &\in \mathcal{S}_2 \Leftrightarrow \exists \gamma \in \omega^{\omega} \ \forall i \in \omega \ \forall \beta \in 2^{\omega} \ \beta \notin D\Big(F\big((\alpha)_{\gamma(i)}\big)\Big) \\ &\Leftrightarrow \exists \gamma \in \omega^{\omega} \ \forall i \in \omega \ \forall \beta \in 2^{\omega} \ \beta \notin \mathcal{V}_{F((\alpha)_{\gamma(i)})} \\ &\Leftrightarrow \exists \gamma \in \omega^{\omega} \ \forall i \in \omega \ \forall \beta \in 2^{\omega} \ \beta \notin G_{(\alpha)_{\gamma(i)}} \\ &\Leftrightarrow \alpha \in S', \end{split}$$

so that $S' = \tilde{F}^{-1}(\mathcal{S}_2)$.

(2) Is there a Borel map $f: \mathcal{C}(2^{\omega}, [0, 1]) \to \omega^{\omega}$ such that, for each $(h_k)_{k \in \omega} \in (\mathcal{C}(2^{\omega}, [0, 1]))^{\omega}$ and each $\beta \in 2^{\omega}$, the following are equivalent:

(a) $\lim_{k\to\infty} h_k(\beta) = 0$, (b) $\forall k \in \omega \ \beta \notin G_{f(h_k)}$?

Assume that this is the case. Then S' (and therefore S_2) is Borel Σ_2^1 -complete, and thus Σ_2^1 -complete (see [P]). We define $F: (\mathcal{C}(2^{\omega}, [0, 1]))^{\omega} \to \omega^{\omega}$ by $F((h_k)_{k \in \omega}) := \langle f(h_0), f(h_1), \ldots \rangle$, so that F is Borel. Note that

$$\begin{split} F\big((h_k)_{k\in\omega}\big)\!\in\!S' &\Leftrightarrow \exists \gamma\!\in\!\omega^\omega \ \forall i\!\in\!\omega \ \forall \beta\!\in\!2^\omega \ \beta\!\notin\!G_{f(h_{\gamma(i)})} \\ &\Leftrightarrow \exists \gamma\!\in\!\omega^\omega \ \forall \beta\!\in\!2^\omega \ \lim_{i\to\infty} h_{\gamma(i)}(\beta)\!=\!0 \\ &\Leftrightarrow (h_k)_{k\in\omega}\!\in\!S, \end{split}$$

so that $S = F^{-1}(S')$.

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