THE STRONG TREE PROPERTY AND WEAK SQUARE

YAIR HAYUT AND SPENCER UNGER

ABSTRACT. We show that it is consistent, relative to ω many supercompact cardinals, that the super tree property holds at $\aleph_n$ for all $2 \leq n < \omega$ but there are weak square and a very good scale at $\aleph_\omega$.

1. INTRODUCTION

In this paper we consider the strong and super tree properties which characterize strong and super compactness at inaccessible cardinals. We show that certain consequences of supercompactness do not follow from the super tree property. We begin with some definitions. Let $\kappa \leq \lambda$ be cardinals with κ regular.

Definition 1. We define a (κ, λ)-tree to be a sequence $T = \langle T_x \mid x \in P_\kappa\lambda \rangle$ such that for all $x \in P_\kappa\lambda$:

(1) T_x is a nonempty set of functions from x to 2 and
(2) for all $y \subseteq x$ and all $f \in T_x$, $f \restriction y \in T_y$.

Definition 2. A (κ, λ)-tree T is *thin* if for all $x \in P_\kappa\lambda$, $|T_x| < \kappa$.

Definition 3. A function $b : \lambda \to 2$ is a *cofinal branch* through a (κ, λ)-tree T if for all $x \in P_\kappa\lambda$, $b \restriction x \in T_x$.

Definition 4. We define two reflection properties:

(1) $\mathrm{TP}(\kappa, \lambda)$ holds if every thin (κ, λ)-tree has a cofinal branch.
(2) $\mathrm{ITP}(\kappa, \lambda)$ holds if for every thin (κ, λ)-tree T and every sequence $\langle d_x \mid x \in P_\kappa\lambda \rangle$ such that for all x, $d_x \in T_x$, there is a cofinal branch b through T such that $\{x \mid b \restriction x = d_x\}$ is stationary.

Note that $\mathrm{TP}(\kappa, \kappa)$ is just the tree property at κ. We say that κ has the strong tree property if $\mathrm{TP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$ and κ has the super tree property if $\mathrm{ITP}(\kappa, \lambda)$ holds for all $\lambda \geq \kappa$. The notion of *thinness* was isolated by Weiss [9]. It allowed for the reformulation of previous results of Jech [3] and Magidor [5] characterizing strong and super compactness respectively. In particular an inaccessible cardinal κ is strongly compact if and only if it has the strong tree property and it is supercompact if and only if it has the super tree property.

In order to state our main theorems, we give some standard definitions. We start with the square principles $\square_{\mu,\lambda}$, which were first defined in [7].

Definition 5. Let $\lambda \leq \mu$ be cardinals. We define a $\square_{\mu,\lambda}$-sequence and say that $\square_{\mu,\lambda}$ holds if and only if there is a $\square_{\mu,\lambda}$-sequence. A sequence $\langle \mathcal{C}_\alpha \mid \alpha < \mu^+ \rangle$ is a $\square_{\mu,\lambda}$-sequence if

(1) for all $\alpha < \mu^+$, $1 \leq |\mathcal{C}_\alpha| \leq \lambda$,
(2) for all $\alpha < \mu^+$ and all $C \in \mathcal{C}_\alpha$, C is club in α and $\mathrm{otp}(C) \leq \mu$, and
(3) for all $\alpha < \mu^+$ and all $C \in \mathcal{C}_\alpha$, if $\beta \in \mathrm{acc}(C)$, then $C \cap \beta \in \mathcal{C}_\beta$.

The main ideas for this paper were conceived at the Workshop on High and Low forcing at the American Institute of Mathematics in January 2016.

Note that $\square_{\mu,\mu}$ is typically written $\square^*_\mu$ and by a theorem of Jensen [4] it is equivalent to the existence of a special μ^+-Aronszajn tree. We will sometimes write $\square_{\mu,<\lambda}$ with the obvious interpretation.

Next we give some definitions from PCF theory. Let μ be a singular cardinal of cofinality ω. Let $\langle \mu_i \mid i < \omega \rangle$ be an increasing sequence of regular cardinals cofinal in μ. We define an ordering on $\prod_{i<\omega} \mu_i$ as follows. Let $f, g \in \prod_{i<\omega} \mu_i$ and set $f <^* g$ if and only if there is a $j < \omega$ such that for all $i \geq j$, $f(i) < g(i)$. We say that $\langle f_\beta \mid \beta < \mu^+ \rangle$ is a *scale of length* μ^+ *in* $\prod_{i<\omega} \mu_i$ if it is increasing and cofinal in $\prod_{i<\omega} \mu_i$ under the $<^*$ ordering. A point $\alpha < \mu^+$ of uncountable cofinality is a *good point* (respectively *very good*) if there is an unbounded (respectively club) $A \subseteq \alpha$ such that $\langle f_\beta(i) \mid \beta \in A \rangle$ is strictly increasing for all large i. If α is not good, then we say that α is a *bad point*. A scale $\langle f_\beta \mid \beta < \mu^+ \rangle$ is *good* (respectively *very good*) if there is a club $F \subseteq \mu^+$ such that each $\alpha \in F$ with $\mathrm{cf}(\alpha) > \omega$ is good (respectively very good). *Bad scales* of length μ^+ are those which are not good. In particular a bad scale has stationarily many bad points.

The following theorem of Magidor and Shelah [6] shows that strongly compact cardinals have some accumulated affect on the universe.

Theorem 6. *Let λ be a singular limit of strongly compact cardinals. Then λ^+ has the tree property.*

Note that it is consistent that κ is supercompact but the tree property fails for every cardinal above κ [1]. This shows that the fact that λ^+ has the tree property could not be deduced only from the existence of a single strongly compact below it, but we had to use the full power of the cofinal sequence of the strongly compact cardinals.

The following theorem of Shelah shows that supercompact cardinals have an effect on the PCF structure above them.

Theorem 7. *If κ is supercompact and $\mu > \kappa$ is singular cardinal of cofinality ω, then all scales of length μ^+ are bad. In particular, there are no very good scales of length μ^+.*

In this paper, we show that it is not possible to replace the large cardinal assumptions in the above theorems with the super tree property.

Theorem 8. *It is consistent relative to ω many supercompact cardinals that for $2 \leq n < \omega$ the super tree property holds at $\aleph_n$ and $\square^*_{\aleph_\omega}$. Similarly it is consistent that the super tree property holds at each $\aleph_n$ for $2 \leq n < \omega$ and there is a very good scale of length $\aleph_{\omega+1}$.*

In fact we get the consistency of $\square_{\aleph_\omega,<\aleph_\omega}$ together with the super tree property at every $\aleph_n$ with $n \geq 2$. Note that this is the strongest possible square at this scenario, since the super tree property at $\aleph_n$ implies the failure of $\square_{\lambda,<\aleph_n}$ for all $\lambda \geq \aleph_n$ by a theorem of Weiss [9].

The theorems of this paper can be seen as extensions of work of the second author [8] who showed that the super tree property at the $\aleph_n$'s is consistent with the combinatorial principle $\aleph_{\omega+1} \in I[\aleph_{\omega+1}]$, which implies that all scales of length $\aleph_{\omega+1}$ are good and is a weakening of $\square^*_{\aleph_\omega}$.

Throughout the paper we work in ZFC. Any large cardinals assumption will be specified. Our terminology is mostly standard. We denote by $V[\mathbb{P}]$ the generic extension of the model V by a generic filter for $\mathbb{P}$. We write "$V[\mathbb{P}] \models \phi$" for the assertion "$V \models \Vdash_{\mathbb{P}} \phi$".

2. MAIN THEOREM

Towards the proof of the main theorem we need the following lemma.

Lemma 9. *Let $\langle \kappa_n \mid n < \omega \rangle$ be an increasing sequence of supercompact cardinals. There is a forcing extension in which for all $n < \omega$, $\kappa_n = \aleph_{n+2}$, the super tree property holds at $\aleph_{n+2}$ and it is indestructible under any $\aleph_{n+2}$-directed closed forcing.*

To prove this we repeat the argument from Theorem 7.5 of [8] in the presence of this extra $\aleph_n$-directed closed forcing. In particular the conclusion of the lemma holds in Cummings and Foreman's [2] model for the tree property at $\aleph_n$ for all $n \geq 2$. We will follow the notation of [8] closely. The interested reader is advised to have a copy of it on hand. The less interested reader can take the lemma as a black box.

Proof. Let $\mathbb{R}_\omega$ be the Cummings-Foreman iteration defined from the sequence $\langle \kappa_n \mid n < \omega \rangle$. In the extension by $\mathbb{R}_\omega$, let $\mathbb{X}$ be $\aleph_{n+2}$-directed closed. Working in $V[\mathbb{R}_{n+1}]$, we let $\mathbb{A}_\mathbb{X} = \mathcal{A}(\mathbb{X}, \mathbb{R}_\omega/\mathbb{R}_{n+1})$ be the forcing of $\mathbb{R}_\omega/\mathbb{R}_{n+1}$-terms for elements of $\mathbb{X}$. Clearly $\mathbb{A}_\mathbb{X}$ is κ_n-directed closed in $V[\mathbb{R}_{n+1}]$. By increasing the amount of supercompactness if necessary we can find a generic embedding with critical point κ_n and domain $V[\mathbb{R}_\omega][\mathbb{A}_\mathbb{X}]$ using the argument from Section 3 of [8]. We do this by incorporating $\mathbb{A}_\mathbb{X}$ into the name returned by $j(F)(\kappa_n)$ where F is the Laver function.

We fix a thin $(\aleph_{n+2}, \lambda)$-tree T and a sequence $\langle d_x \mid x \in P_\kappa \lambda \rangle$ such that for all x, $d_x \in T_x$. Using the generic embedding, we have a cofinal branch $b : \lambda \to 2$ through T such that the set $\{x \mid b \restriction x = d_x\}$ is stationary. By the analogs of Lemmas 4.1 and 4.2 for our embedding, we have that b is in the extension of $V[\mathbb{R}_\omega][\mathbb{X}]$ by the product of $\mathbb{S}_\mathbb{X} = \mathcal{S}(\mathbb{X}, \mathbb{R}_\omega/\mathbb{R}_{n+1})$ (a quotient forcing defined from $\mathbb{A}_\mathbb{X}$) and the forcing from Lemma 4.2 of [8].

It remains to show that this forcing cannot have added the branch. To do this we just incorporate $\mathbb{S}_\mathbb{X}$ with the other $\mathbb{S}$ forcings from Lemma 4.2 of [8]. In particular we show that $\mathbb{S}_X$ is $\aleph_{n+1}$-closed and $< \aleph_{n+2}$-distributive over $M[\mathbb{R}_\omega][\mathbb{X}]$. The closure is immediate from Lemma 2.12 of [8] and the fact that $\mathbb{R}_\omega/\mathbb{R}_{n+1}$ is $< \aleph_{n+1}$-distributive in $V[\mathbb{R}_{n+1}]$. The distributivity is immediate from the $\aleph_{n+2}$-directed closure of $\mathbb{A}_\mathbb{X}$ in $M_n = V[\mathbb{R}_{n+1}]$ and the end of the proof of Lemma 4.4 on [8]. This finishes the proof. □

Let us define next the forcing notions for adding and threading weak square as well as forcing notions for adding and threading a very good scale.

Definition 10. Let $\mathbb{S}$ be the forcing notion for adding $\square_{\mu,<\mu}$ using bounded approximations. A condition in $\mathbb{S}$ is a sequence of the form $\langle \mathcal{C}_\alpha \mid \alpha \leq \gamma \rangle$ where:

(1) $\gamma < \mu^+$.
(2) $0 < |\mathcal{C}_\alpha| < \mu$ for all limit $\alpha \leq \gamma$.
(3) Every $C \in \mathcal{C}_\alpha$ is closed unbounded subset of α with $\mathrm{otp}(C) \leq \mu$.
(4) If $\beta \in \mathrm{acc}\, C$, $C \in \mathcal{C}_\alpha$ then $C \cap \beta \in \mathcal{C}_\beta$.

We order $\mathbb{S}$ by end extension.

The generic filter for $\mathbb{S}$ is a $\square_{\mu,<\mu}$-sequence. For such sequences $\mathcal{C}$, we define the threading forcing $\mathbb{T}_\rho$. The elements of $\mathbb{T}$ are members of $\mathcal{C}_\alpha$ for some $\alpha < \mu^+$, with order type $< \rho$, ordered by end extension.

The following fact is standard:

Claim 11. *$\mathbb{S} * \mathbb{T}_\rho$ has a ρ-directed closed dense subset.*

It follows that forcing with $\mathbb{S}$ preserves cardinals up to μ^+. Let us define now a forcing for adding a very good scale at μ^+ and the corresponding threading forcing.

Definition 12. Let μ be a singular cardinal of countable cofinality, and let $\langle \mu_n \mid n < \omega \rangle$ be an increasing sequence of regular cardinals cofinal at μ.

The forcing notion $\mathbb{S}_{sc}$ is the forcing for adding a very good scale using bounded approximations. A condition in $\mathbb{S}_{sc}$ is a pair $\langle d, s \rangle$ where:

(1) $s = \langle g_\alpha \mid \alpha \leq \gamma \rangle$, $\gamma < \mu^+$, where $g_\alpha \in \prod_{n<\omega} \mu_n$, increasing modulo finite.
(2) $d \subseteq \gamma + 1$ is closed set of very good points for s.

We order $\mathbb{S}_{sc}$ by end extension.

It is straightforward to see that this forcing adds a very good scale of length μ^+. Similarly to the square forcing, there is a natural threading forcing for $\mathbb{S}_{sc}$. For $n < \omega$, let $\mathbb{T}_{sc,n}$ be the forcing notion for adding a club E in μ^+ such that for every $\alpha < \beta$ in E and $m \geq n$, $g_\alpha(m) < g_\beta(m)$ with approximations of ordertype at most μ_n ordered by end extension.

Lemma 13. *For $n < \omega$, $\mathbb{S}_{sc} * \mathbb{T}_{sc,n}$ has a μ_n-directed closed dense subset.*

Proof. Let us show first that $\mathbb{S}_{sc}$ is $< \mu^+$-distributive.

Claim 14. *$\mathbb{S}_{sc}$ is $< \mu$-strategically closed.*

Proof. Let us define for an ordinal $\rho < \mu$, a winning strategy for the generic game of length ρ. Let us pick n such that $\rho < \mu_n$. Assume that we played the first β steps in the game and let $\langle p_\alpha \mid \alpha < \beta \rangle$ be the play so far. Let us denote $p_\alpha = \langle d_\alpha, s_\alpha \rangle$ and let $\gamma_\alpha = \max \operatorname{dom} s_\alpha$, $\langle f_i \mid i < \sup_\alpha \gamma_\alpha \rangle = \bigcup_{\alpha<\beta} s_\alpha$, the scale constructed so far. Let $d = \bigcup_{\alpha<\beta} d_\alpha$.

The strategy will be to pick $p_\beta = \langle e, \langle f_i \mid i < \sup \gamma_\alpha \rangle^\frown \langle g \rangle \rangle$ where:

(1) $e = d$ if β is not a limit ordinal and otherwise $e = d \cup \{\gamma_\beta\}$, $\gamma_\beta = \sup_{\alpha<\beta} \gamma_\alpha$.
(2) g is an upper bound (modulo finite errors) of all f_i, $i < \sup \gamma_\alpha$ and for all $\alpha < \beta$, and $m \geq n$, $g(m) \geq f_{\gamma_\alpha}(m)$.

We need to verify that γ_β is indeed a good point whenever β is a limit ordinal. The club $\{\gamma_\alpha \mid \alpha < \beta\}$ witnesses that this is the case. □

Since μ is singular, we conclude that $\mathbb{S}_{sc}$ is μ^+-distributive. Therefore the elements of $\mathbb{T}_{sc,n}$ are members of the ground model. Let us show now that $\mathbb{S}_{sc} * \mathbb{T}_{sc,n}$ contains a dense μ_n-directed closed subset.

Let D be the set of all $\langle \langle d, s \rangle, \check{c} \rangle \in \mathbb{S}_{sc} * \mathbb{T}_{sc,n}$ such that $c \in V$, and $\max \operatorname{dom} s = \max d = \max c$. D is μ_n-directed closed, since for every sequence of pairwise compatible elements of length $\rho < \mu_n$, $\{\langle \langle s_i, d_i \rangle, \check{c}_i \rangle \mid i < \rho\}$, has a lower bound. The only thing that we need to verify is that one can add a member of $\prod_{n<\omega} \mu_n$ in the top of the scale $\bigcup_{i<\rho} s_i$ in a way that will make it a good point and this is witnessed by the club $\bigcup_{i<\rho} c_i$.

Let us show that D is dense. Let $p \in \mathbb{S}_{sc} * \mathbb{T}_{sc,n}$. By extending p, if necessary, we may assume that $p = \langle \langle s, d \rangle, \check{c} \rangle$. Using the strategy, we know that we can extend $\langle s, d \rangle$ to a condition $\langle s', d' \rangle$ such that $\max \operatorname{dom} s' = \max d$. Moreover, we may pick the last element in s' to be above all elements in $s \restriction c$ in all its coordinates, besides the first n. Thus, we can extend c to include $\max d$. □

The next two lemmas show that the threading forcing corresponding to the weak square forcing and the very good scale forcing cannot add a new branch to a $P_\kappa\lambda$-tree.

Lemma 15. *Let κ, λ be regular cardinals such that κ is not strong limit and $\kappa \leq \lambda$. Let $\nu \leq \mu$ be cardinals with $\kappa < \nu$. Let $\mathbb{S}$ be the forcing for adding a $\square_{\mu,<\nu}$ sequence. Let $\mathbb{T}$ be the threading forcing with approximations of order type less than κ. Then forcing with $\mathbb{T}$ over $V[\mathbb{S}]$ does not add any new branch to a thin $P_\kappa\lambda$ tree.*

Proof. Assume otherwise, and let $\dot{b}$ be a name for this branch. Let ρ be the least cardinal for which $2^\rho \geq \kappa$. Since κ is not strong limit, $\rho < \kappa$ and $2^{<\rho} < \kappa$. Let $s \in \mathbb{S}$ and $t \in \mathbb{T}$ be arbitrary. Since $\dot{b}$ is new, one can extend the condition $\langle s, t \rangle \in \mathbb{S} * \mathbb{T}$ to pair of conditions $\langle s', t_0 \rangle$, $\langle s', t_1 \rangle$ that force different values for $\dot{b}$ at some $x \in P_\kappa\lambda$.

Let $\{\eta_i \mid i < 2^{<\rho}\}$ be an enumeration of all elements of ${}^{<\rho}2$, such that if $\eta_i \trianglelefteq \eta_j$ then $i \leq j$. Let us define by induction a sequence of conditions s_α, $\alpha < 2^{<\rho}$ and t_η, $\eta \in {}^{<\rho}2$, such that:

(1) For every $\alpha < 2^{<\rho}$, $s_\alpha \Vdash t_{\eta_\alpha} \in \mathbb{T}$.
(2) For every $\eta \in {}^{<\rho}2$ there is $x_\eta \in P_\kappa\lambda$ such that s_α forces $t_{\eta^\frown\langle 0\rangle}, t_{\eta^\frown\langle 1\rangle}$ are stronger than t_η and force different values for $\dot{b}$ in x_η where α is such that $\eta_\alpha = \eta$.
(3) If $\eta_\alpha \triangleleft \eta$ and $\eta_\alpha \neq \eta$ then $\max t_\eta \geq \max \operatorname{dom} s_\alpha$.

For $\eta \in {}^{\rho}2$ let $t_\eta = \bigcup_{i<\rho} t_{\eta\restriction i}$. Let $\mathcal{A}$ be a set of κ many different elements in 2^ρ.

Let $s_\star$ be the condition $\bigcup_\alpha s_\alpha^\frown\langle\{t_\eta \mid \eta \in \mathcal{A}\}\rangle$. Let $x_\star = \bigcup_{\eta\in{}^{<\rho}2} x_\eta$. $s_\star \Vdash t_\eta \in \mathbb{T}$ for all $\eta \in \mathcal{A}$, and it forces that if $\eta \neq \eta'$ and $\eta, \eta' \in \mathcal{A}$ then $t_\eta, t_{\eta'}$ force different values on $\dot{b}$ in $x_\star$ - but this contradicts the assumption that the tree is thin. □

Lemma 16. *Let μ be a singular cardinal of countable cofinality, $\lambda_n \to \mu$. Force with $\mathbb{S}_{sc}$. If $\kappa < \lambda_n$ is not a strong limit, then the forcing $\mathbb{T}_{sc,n}$ cannot add a new branch to a $P_\kappa\lambda$ tree.*

The proof is essentially the same as the previous lemma. We are now ready to complete the proof of the main theorem.

Proof. Let W be the Cummings-Foreman model for the tree property at $\aleph_n$ for $n \geq 2$. Let $\mathbb{S}$ be the forcing to add either a $\square_{\aleph_\omega,<\aleph_\omega}$-sequence or a very good scale of length $\aleph_{\omega+1}$. We claim that $W[\mathbb{S}]$ is the desired model. Clearly $W[\mathbb{S}]$ has either the appropriate weak square sequence or a very good scale based on the choice of $\mathbb{S}$. So it remains to show that the super tree property holds at $\aleph_{n+2}$ for all $n < \omega$.

Let $n < \omega$ and let $\mathbb{T}_n$ be the appropriate threading forcing so that $\mathbb{S} * \mathbb{T}_n$ is $\aleph_{n+2}$-directed closed in W. By Lemma 9, the super tree property holds at $\aleph_{n+2}$ in $W[\mathbb{S} * \mathbb{T}_n]$. So by either Lemma 15 or 16 applied with $\kappa = \aleph_{n+2}$, the super tree property holds at $\aleph_{n+2}$ in $W[\mathbb{S}]$. □

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