Constructing Quasiminimal Structures

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Abstract

Quasiminimal structures play an important role in non-elementary categoricity. In this paper we explore possibilities of constructing quasiminimal models of a given first-order theory. We present several constructions with increasing control of the properties of the outcome using increasingly stronger assumptions on the theory. We also establish an upper bound on the Hanf number of the existence of arbitrarily large quasiminimal models.

1 Introduction

An uncountable structure is called *quasiminimal* if every (first order) definable subset is either countable or cocountable. Quasiminimal structures carrying a homogeneous pregeometry play an important role in non-elementary categoricity (see Zilber [2005b]). Several key analytic structures such as \mathbb{C}_{exp} are conjectured to be quasiminimal (see Zilber [2005a]).

The study of quasiminimal structures from the first order perspective is pioneered in Pillay and Tanović [2011]. They isolate a key notion of a strongly regular type for an arbitrary theory (see Definition 2.1). An important result is that assuming the generic type (i.e. the type containing formulas defining cocountable sets) of a quasiminimal structure is definable, its unique global heir is strongly regular. In this paper we establish the converse of this: if there is a definable strongly regular type, then we can construct a quasiminimal model. The definability condition holds in particular for groups. Thus given a regular group, there is a quasiminimal group elementarily equivalent to it. (The converse stating that the monster model of a quasiminimal group is regular is due to Pillay and Tanović [2011].) This reduces the existence of non-commutative quasiminimal groups and quasiminimal fields that are not algebraically closed to respective problems for regular groups and fields. See the next section for a detailed discussion of this.

The techniques of Pillay and Tanović [2011] provide more. If the cardinality of a quasiminimal structure is strictly greater than \aleph_1 , then the global heir of the generic type is symmetric (i.e. Morley sequence are totally indiscernible). Thus to construct quasiminimal models of arbitrarily large cardinalities we need additional assumptions. Here we present two constructions of arbitrarily large quasiminimal models: one assuming the theory has definable Skolem functions and the other assuming the theory is stable. We also use a variant of Skolem functions to prove an upper bound on the Hanf number of having arbitrarily large quasiminimal models.

If we strengthen the stability assumption to ω -stability, we can use the existence of prime models. We can then easily get quasiminimal models by taking prime models over Morley sequences in the strongly regular type. Here we show that a stronger conclusion holds: the class of all such models is quasiminimal excellent in the sense of Zilber [2005b]. Since this class is clearly uncountably categorical, its excellence was expected. However Shelah's results on deducing excellence from categoricity require additional set theoretic assumptions and only apply to $L_{\omega_1,\omega}$ sentences. So they can't be used to deduce excellence in this generality.

2 Preliminaries

The model theoretic notation is standard. The notions left undefined are either standard or can be found in Pillay and Tanović [2011] (or both) on which this paper is based on.

We work in a monster model \mathfrak{C} of a first order theory T in a countable language. Types over \mathfrak{C} are called global types. A global type $\mathfrak{p}(x) \in S_1(\mathfrak{C})$ is an ultrafilter on the boolean algebra of definable subsets of \mathfrak{C} . If we think of sets in \mathfrak{p} as "large" and sets outside \mathfrak{p} as "small", we can, in analogy with the algebraic closure, define the closure of A to be the union of all small sets definable over A. More formally define

$$cl_{\mathfrak{p}}(A) = \{ b \in \mathfrak{C} : b \not\models \mathfrak{p}|A \}$$

A natural question is when is cl_p a closure operator (monotone, idempotent, finitary operator) or better a pregeometry (closure operator with exchange). To answer that question we need the notion of a strongly regular type.

Definition 2.1. Let A be a subset, $\mathfrak{p}(\bar{x})$ be a global A-invariant type and $\phi(\bar{x}) \in \mathfrak{p}$ be a formula over A. The pair (\mathfrak{p}, ϕ) is called *strongly regular* if for all $B \supseteq A$ and \bar{a} satisfying ϕ either $\bar{a} \models \mathfrak{p}|_B$ or $\mathfrak{p}|_B \vdash \mathfrak{p}|_{B\bar{a}}$.

Note that the following statements are all equivalent to $\mathfrak{p}|_B \vdash \mathfrak{p}|_{B\bar{a}}$.

- The type $\mathfrak{p}|_B$ has a unique extension to a type over $B\bar{a}$.
- If $\bar{b} \models \mathfrak{p}|_B$, then $\operatorname{tp}_{\bar{x}}(\bar{b}/B) \cup \operatorname{tp}_{\bar{y}}(\bar{a}/B)$ determines a complete type over B in $\bar{x}\bar{y}$.
- If $\bar{b} \models \mathfrak{p}|_B$, then $\operatorname{tp}(a/B) \vdash \operatorname{tp}(a/Bb)$.

In our case \mathfrak{p} will be a type in one variable, $\phi(x)$ will always be the formula x = x (so we surpass ϕ from the notation) and we will assume $A = \emptyset$ (this can be achieved by extending the language, if A is countable).

Fact 2.2 (Pillay and Tanović [2011]). Let $\mathfrak{p} \in S_1(\mathfrak{C})$ be an \emptyset -invariant type. Then $cl_{\mathfrak{p}}$ is a closure operator if and only if \mathfrak{p} is strongly regular. Moreover, $cl_{\mathfrak{p}}$ is a pregeometry if in addition all (equivalently some) Morley sequences in \mathfrak{p} are totally indiscernible.

Here by a Morley sequence in \mathfrak{p} we mean a sequence of $(a_i : i < \alpha)$ satisfying $a_i \models \mathfrak{p}|_{\{a_j:j < i\}}$. Since \mathfrak{p} is \emptyset -invariant, a Morley sequence is always indiscernible. Also it is easy to see that for the sequence to be totally indiscernible it is necessary and sufficient that $a_1a_0 \equiv a_0a_1$. In this case \mathfrak{p} is called *symmetric* and it is called *asymmetric* otherwise.

We can also start from the other end. Let M be a structure and cl an infinite dimensional pregeometry on it. Assume that the pregeometry is related to the language in the following way: for every finite $B \subset M$ the set of elements of M outside cl(B) is the set of realisations of a complete type $p_B(x)$ over B. Such a pregeometry is called *homogeneous* in Pillay and Tanović [2011]. In the context of quasiminimal excellent classes this property is referred as the *uniqueness of the generic type*. Now consider $p_{cl}(x) = \bigcup_{B \subset fin} p_B(x)$. It can be seen that $p_{cl}(x) \in S_1(M)$ is a complete type over M. Results of Pillay and Tanović [2011] show that this type extends to a global strongly regular type.

Fact 2.3 (Pillay and Tanović [2011]). Let (M, cl) be a homogeneous pregeometry. Then p_{cl} (defined above) is definable and its unique global heir \mathfrak{p} is \emptyset -invariant, generically stable, symmetric strongly regular type. In particular $cl = cl_{\mathfrak{p}}|_{M}$.

Thus homogeneous pregeometries (such as those in quasiminimal excellent classes) come from symmetric regular types and conversely symmetric regular types induces homogeneous pregeometries. On the other hand we can also find strongly regular types if we start from quasiminimal structures. Let M be a quasiminimal structure. Then

 $\{\phi(x) \in L(M) : \phi \text{ defines a cocountable set}\}$

is a complete type over M. We call this type the *generic type* of M. Under suitable conditions we can also extend this type to a strongly regular global type.

Fact 2.4 (Pillay and Tanović [2011]). Assume that the generic type p of a quasiminimal structure M is \emptyset -definable, then its unique global heir \mathfrak{p} is strongly regular. Moreover if $|M| > \aleph_1$, then the definability condition holds (after adding parameters) and \mathfrak{p} is symmetric and generically stable.

However there are no guarantees that the strongly regular type \mathfrak{p} is symmetric (unless $|M| > \aleph_1$). For example the structure ($\aleph_1 \times \mathbb{Q}, <$) in the lexicographic order is quasiminimal. Indeed it has quantifier elimination (as a dense linear order) and every formula of the form x < a defines a countable set, while every formula of the form x > a defines a countable set. Now the generic type is the completion of $\{x > a : a \in \aleph_1 \times \mathbb{Q}\}$, so is \emptyset -definable. But its unique global heir is clearly asymmetric (no indiscernible sequence is totally indiscernible because of the order).

The definability condition holds in particular if (M, \cdot) is a quasiminimal group. The generic type is definable using the fact that a definable set $X \subseteq M$ is cocountable if and only if $X \cdot X = M$. Indeed, if X is countable, then so is $X \cdot X$ and hence it can't be the whole of M. Conversely if X is cocountable, then so is mX^{-1} for a given $m \in M$. Hence $mX^{-1} \cap X$ is nonempty and therefore $m \in X \cdot X$. Thus using the Fact 2.4, we see that the monster model of M has a strongly regular type. Such groups are called *regular* in Pillay and Tanović [2011].

One of the questions Pillay and Tanović [2011] asks is whether every regular group is commutative. A similar question of whether every quasiminimal group is commutative has been around for some time. By the above, given a non-commutative quasiminimal group, its monster model is a regular noncommutative group. Our results imply the converse of this: if there is a regular non-commutative group, there is a quasiminimal non-commutative group. Pillay and Tanović [2011] eludes to a possible construction of a regular non commutative group, but to the best of our knowledge the problem is still open.

A related question is whether every regular (or quasiminimal) field is algebraically closed. It is known for regular fields where the strongly regular type is symmetric and hence for quasiminimal fields of cardinality $> \aleph_1$ (see Hyttinen et al. [2005] and Gogacz and Krupiński [2014]). However the case of an asymmetric regular field and a quasiminimal field of cardinality \aleph_1 remain open. Our results together with Fact 2.4 reduce these two cases to each other.

In the rest of the paper we are mainly concerned with constructing quasiminimal models. From the above we know that in well behaved quasiminimal structures (certainly for large quasiminimal structures) the generic type is strongly regular and induces a closure operator. So a natural question is given a theory with a strongly regular type, can we construct a quasiminimal model. Note also that a model will be quasiminimal if and only if it satisfies the countable closure property (i.e. closure of a countable set is countable). We present several constructions with increasing control of properties of the outcome using increasingly stronger assumptions on the theory.

3 Arbitrary Theories

In this section we describe a method for constructing a quasiminimal structures which we then apply to construct a quasiminimal model of a theory with a strongly regular type. Our method is based on the original method of Morley and Vaught [1962] of constructing an (\aleph_1, \aleph_0) model from a Vaughtian pair. The method is widely known among model theorists (although not used often outside of its original context) and expositions are available in many model theory texts.

Definition 3.1. A special pair for a theory T is a pair of models $M \prec N$ where N is a proper elementary extension and there is an \emptyset -definable type $p \in S_1(M)$ such that all $a \in N \setminus M$ realise p.

Let $M \prec N$ be a special pair of models of T. Let $p \in S_1(M)$ be the type of elements in $N \setminus M$ and d_p be the schema defining p (note that all schemas defining p are equivalent). Add a new predicate symbol R to the language and consider the theory \hat{T} of N where R is interpreted as M. The theory \hat{T} in particular encodes the following

- The universe is a model of T;
- *R* defines a proper elementary substructure;
- $\{\phi(x,\bar{a}): \bar{a} \in R \land d_p x \phi(x,\bar{a})\}$ is a complete type over R;
- $\forall x \notin R \; \forall \bar{y} \in R(\phi(x, \bar{y}) \leftrightarrow d_p x \phi(x, \bar{y}))$ (every element outside R realises the above type).

Thus any model of \hat{T} provides a special pair for T. So if we have a special pair for T, we can construct a countable one. Therefore without lose of generality assume that N is countable. Further by iteratively realising types in M and N we may assume that both M and N are homogeneous and realise the same types over \emptyset . Indeed we can construct a sequence

$$(N_0, M_0) \preccurlyeq (N_1, M_1) \preccurlyeq (N_2, M_2) \preccurlyeq \dots$$

of countable models of \hat{T} such that

- if $p \in S_n(T)$ is realised in N_{3i} , then it is realised in M_{3i+1} ;
- if $\bar{a}, \bar{b}, c \in M_{3i+1}$ and $\operatorname{tp}^{M_{3i+1}}(\bar{a}) = \operatorname{tp}^{M_{3i+1}}(\bar{b})$, then there is $d \in M_{3i+2}$ such that $\operatorname{tp}^{M_{3i+2}}(\bar{a}, c) = \operatorname{tp}^{M_{3i+2}}(\bar{b}, d)$;
- if $\bar{a}, \bar{b}, c \in N_{3i+2}$ and $\operatorname{tp}^{N_{3i+2}}(\bar{a}) = \operatorname{tp}^{N_{3i+2}}(\bar{b})$, then there is $d \in N_{3i+3}$ such that $\operatorname{tp}^{N_{3i+3}}(\bar{a}, c) = \operatorname{tp}^{N_{3i+3}}(\bar{b}, d)$;

Then take $(N, M) = \bigcup_{i < \omega} (N_i, M_i)$. Now M and N would be homogeneous by the second and third close and realise the same types over \emptyset by the first close. In particular M and N would be isomorphic. This is a standard step in Vaught's two cardinal theorem and more details can be found in standard textbooks such as Marker [2002].

Proposition 3.2. If T has a special pair, then it has a quasiminimal model of cardinality \aleph_1 .

Proof. Let $M \prec N$ be a special pair, p the type of elements in $N \setminus M$ and d_p the defining scheme. By the above discussion we may assume that M and N are countable, homogeneous and isomorphic. We construct an elementary sequence $(A_{\alpha} : \alpha < \aleph_1)$ of structures, each isomorphic to M (and hence N) such that for each $\beta < \alpha$ every element of $A_{\alpha} \setminus A_{\beta}$ realises the unique heir of p over A_{β} . The construction is by transfinite recursion.

- Take $A_0 = M$.
- If δ is a limit ordinal, then take $A_{\delta} = \bigcup_{\alpha < \delta} A_{\alpha}$. Note that A_{δ} is homogeneous, countable and realises the same types (over \emptyset) as M. Hence $A_{\delta} \cong M$. For $\beta < \delta$ every element in $A_{\delta} \setminus A_{\beta}$ is in some A_{α} with $\beta < \alpha < \delta$. Therefore it realises the unique heir of p over A_{β} .
- Assume A_{α} is constructed. Let $f: M \to A_{\alpha}$ be an isomorphism. Then f extends to an isomorphism $\hat{f}: N \to B$ for some structure B. Take $A_{\alpha+1} = B$. Then $A_{\alpha} \prec A_{\alpha+1}$ and $A_{\alpha+1} \cong N \cong M$. It remains to show

that every $c \in A_{\alpha+1} \setminus A_{\alpha}$ realises the unique heir of p over A_{α} . Let $\bar{a} \in A_{\alpha}$ and $A_{\alpha+1} \models d_p x \phi(x, \bar{a})$. Let $c' = \hat{f}^{-1}(c), \bar{a}' = \hat{f}^{-1}(\bar{a})$. Note that $c' \in N \setminus M$. Since \hat{f} is an isomorphism, we have $N \models d_p x \phi(x, \bar{a}')$. But \hat{f} extends f and since $\bar{a} \in A_{\alpha}$, we have $\bar{a}' = f^{-1}(\bar{a}) \in M$. Therefor $N \models \phi(c', \bar{a}')$ and so $A_{\alpha+1} \models \phi(c, \bar{a})$.

Now take $A = \bigcup_{\alpha < \aleph_1} A_{\alpha}$. If $\bar{a} \in A$, then $\bar{a} \in A_{\alpha}$ for some countable α . Now if $d_p x \phi(x, \bar{a})$ is satisfied, then every element in $A \setminus A_{\alpha}$ satisfies $\phi(x, \bar{a})$ and hence it defines a cocountable set. Otherwise $\phi(x, \bar{a})$ defines a countable set. Thus A is quasiminimal. Also observe that countability is weakly definable in A in the sense of Zilber [2003].

Now we can apply this method to construct a quasiminimal model from a strongly regular type.

Theorem 3.3. Let T be a theory with $\mathfrak{p} \neq \emptyset$ -definable strongly regular type. Then there is a quasiminimal model of T.

Proof. Let $M \prec \mathfrak{C}$ be a small model. Consider $N = \mathrm{cl}_{\mathfrak{p}}(M)$. Since $\mathfrak{p}|_M$ must have a realisation, we have $N \subsetneq \mathfrak{C}$. We claim that N is an elementary substructure. We use the Tarski-Vaught test. Assume that $\overline{a} \in N$ and $\phi(b, \overline{a})$ holds for some $b \in \mathfrak{C}$. If $b \in N$, then we are done. So assume that $b \notin N$. Then b realises $\mathfrak{p}|_M$ and hence $\mathfrak{p}|_N$. Now since $\overline{a} \in N$, we have that $\mathfrak{p}|_M \vdash \mathfrak{p}|_N$ and therefore some L(M)-formula $\psi(x)$ implies $\phi(x, \overline{a})$. Since M is a model, ψ must be realised in M and hence in N. Thus ϕ is realised in N and $N \prec \mathfrak{C}$. Now every element of $\mathfrak{C} \setminus N$ realises $\mathfrak{p}|_N$ and hence N and \mathfrak{C} are a special pair.

We can apply this to regular groups, since by Pillay and Tanović [2011] the strongly regular type in a group is \emptyset -definable.

Corollary 3.4. If G is a regular group, then there is a quasiminimal group H elementarily equivalent to G.

Without additional assumptions we cannot construct quasiminimal models of cardinalities larger than \aleph_1 . Indeed by Fact 2.4 the generic type of such a structure is symmetric. So we at least need to assume that **p** is symmetric.

4 Theories with definable Skolem functions

We say that a theory T has definable Skolem functions if for every L-formula $\phi(y, \bar{x})$ there is a \emptyset -definable function $f_{\phi}(\bar{x})$ such that $T \models \forall \bar{x} (\exists y \phi(y, \bar{x}) \rightarrow \forall \bar{x}) (\exists y \phi(y, \bar{x}))$

 $\phi(f_{\phi}(\bar{x}), \bar{x}))$. In this section we study symmetric strongly regular types in theories with definable Skolem functions. So let T be such a theory and let \mathfrak{p} be a symmetric strongly regular type. Note that by the Fact 2.3 the type \mathfrak{p} is \emptyset -definable.

We will show that T has a quasiminimal model of arbitrary uncountable cardinality κ . Since there are definable Skolem functions, for every subset A there is the least model containing it, namely the submodel with universe dcl(A). Thus if there is a quasiminimal model containing a Morley sequence $A = (a_{\alpha} : \alpha < \kappa)$, then dcl(A) would be one. Thus we need to show that dcl(A) is quasiminimal. Further since dcl(A) embeds in every model containing A, all we need is to construct a model M containing A such that for a fixed countable $A_0 \subseteq A$ we have that $cl_p(A_0) \cap M$ is countable. For that we will use the technique of self-extending models originally due to Vaught [1965].

The type \mathfrak{p} is an ultrafilter on definable subsets. We can use this ultrafilter for a definable analogue of the ultrapower construction. More specifically given a model M of T we can associate a canonical extension M^* defined as follows. The elements of M^* are all definable (with parameters) functions $f: M \to M$ modulo agreeing on a large set (i.e. a member of \mathfrak{p}). That is f, g: $M \to M$ are considered equal if $f(x) = g(x) \in \mathfrak{p}$. The nonlogical symbols of L are interpreted as follows. Given an n-ary relation R and elements f_1, \ldots, f_n of M^* , the formula $R(f_1, \ldots, f_n)$ holds in M^* if and only if $R(f_1(x), \ldots, f_n(x)) \in$ \mathfrak{p} . Just as in the regular ultrapower construction, one sees that this does not depend on the representatives f_1, \ldots, f_n . Similar definitions are applied to the interpretation of constant and functional symbols. We have the analogue of Loś's Theorem.

Lemma 4.1. Given an *L*-formula $\phi(y_1, ..., y_n)$ and elements $f_1, ..., f_n \in M^*$ we have $M^* \models \phi(f_1, ..., f_n)$ if and only if $\phi(f_1(x), ..., f_n(x)) \in \mathfrak{p}$.

Proof. By induction of ϕ . The only nontrivial case is when ϕ is of the form $\exists y_0 \psi(y_0, y_1, ..., y_n)$. Assuming $M^* \models \exists y_0 \psi(y_0, f_1, ..., f_n)$, there exists $f_0 \in M^*$ such that $M^* \models \psi(f_0, f_1, ..., f_n)$. Then by the induction hypothesis $\psi(f_0(x), f_1(x), ..., f_n(x)) \in \mathfrak{p}$. And since it implies $\exists y_0 \psi(y_0, f_1(x), ..., f_n(x)))$, the latter is also in \mathfrak{p} . Conversely assume that $\exists y_0 \psi(y_0, f_1(x), ..., f_n(x)) \in \mathfrak{p}$. Since T has definable Skolem functions, there is a definable function f_{ψ} : $M \to M$ such that $\forall x (\exists y_0 \psi(y_0, f_1(x), ..., f_n(x)) \to \psi(f_{\psi}(x), f_1(x), ..., f_n(x)))$. Therefore $\psi(f_{\psi}(x), f_1(x), ..., f_n(x)) \in \mathfrak{p}$ and by the induction hypothesis $M^* \models \exists y_0 \psi(y_0, f_1, ..., f_n)$.

Corollary 4.2. If we identify each element of M with the constant function in M^* , then M^* is an elementary extension of M.

Using the fact that \mathfrak{p} is a strongly regular type, the result of the previous Lemma can also be expressed as follows. Assume that a is generic over parameters defining $f_1, ..., f_n$ in M, then $M^* \models \phi(f_1, ..., f_n)$ if and only if $M \models \phi(f_1(a), ..., f_n(a)).$

We call M^* the canonical extension of M. (This explains the terminology self-extending.) The following property of the canonical extension is crucial for our purposes.

Proposition 4.3. If M is an infinite dimensional model, then its canonical extension M^* is a proper extension where every new element realises $\mathfrak{p}|_M$.

Proof. First note that since $\mathfrak{p}|_M$ is not isolated, the identity function in M^* is different from all constant functions modulo \mathfrak{p} . Hence M^* is a proper extension of M.

On the other hand, let $f \in M^*$ be an element that does not realise $\mathfrak{p}|_M$. Then there is an L(M)-formula $\psi(\bar{a}, y) \notin \mathfrak{p}$ such that $M^* \models \psi(\bar{a}, f)$. Let $\phi(\bar{b}, x, y)$ define f in M. Pick an element c generic over $\bar{a}\bar{b}$. By the assumption d = f(c) satisfies $\psi(\bar{a}, y)$ in M. But then $d \in \mathrm{cl}_{\mathfrak{p}}(\bar{a})$ and since c is generic over $\bar{a}\bar{b}$ we see that f(x) = d for every generic x. Thus f is a constant function (modulo \mathfrak{p}) and so is already in M.

Theorem 4.4. Let \mathfrak{p} be a symmetric strongly regular type in a theory T with definable Skolem functions. Then T has quasiminimal models of arbitrarily large cardinalities.

Proof. We follow the approach outlined in the beginning of the section. Let κ be an uncountable cardinal and let $A = (a_{\alpha} : \alpha < \kappa)$ be a Morley sequence in \mathfrak{p} . We show that dcl(A) is quasiminimal. Let $A_0 \subseteq A$ be countably infinite. Let M be a countable model containing A_0 . Iterating the previous Proposition we obtain an extension N such that $\dim_{\mathfrak{p}}(N) = \kappa$ and $\operatorname{cl}_{\mathfrak{p}}(A_0) \cap N = \operatorname{cl}_{\mathfrak{p}}(A_0) \cap M$ is countable. By conjugating N with an automorphism if necessary we may assume that $A \subseteq N$. But then dcl(A) $\subseteq N$ and hence $\operatorname{cl}_{\mathfrak{p}} \cap \operatorname{dcl}(A)$ is also countable. This shows that dcl(A) is quasiminimal. \Box

In the rest of this section we do not assume that T has definable Skolem function. Instead we add functional symbols to the language that will resemble Skolem functions. We use this to prove that if T has a quasiminimal model of cardinality \beth_{ω_1} , then it has quasiminimal models of all uncountable cardinalities. The existence of such a cardinal follows from a very general argument due to Hanf, and so the least such cardinal is often called the Hanf number. In this terminology we show that the Hanf number of the existence of quasiminimal models is at most \beth_{ω_1} . **Theorem 4.5.** If T has a quasiminimal model of cardinality \beth_{ω_1} , then it has quasiminimal models of arbitrarily large cardinalities.

Proof. Let $M \models T$ with $|M| = \beth_{\omega_1}$. Then by the Fact 2.4 (after adding parameters) there is a global symmetric regular type \mathfrak{p} that extends the type of cocountable subsets of |M|. Since M is quasiminimal, its dimension (with respect to $cl_{\mathfrak{p}}$) is \beth_{ω_1} and so there is a Morley sequence $(a_{\alpha} : \alpha < \beth_{\omega_1})$ in \mathfrak{p} .

Expand the language by *n*-arey functional symbols f_n^i for each $i, n < \omega$ (nullary functional symbols being constant symbols). Interpret f_n^i in M such that for every $\alpha_1 < \ldots < \alpha_n < \kappa$ the set $\{f_n^i(a_{\alpha_1}, \ldots, a_{\alpha_n}) : i < \omega\}$ enumerates the closure $cl_p(a_{\alpha_1}, \ldots, a_{\alpha_n})$ in M. Denote the resulting language, theory and structure by L', T' and M' respectively.

Given a cardinal κ , there is an indiscernible sequence $B = (b_{\beta} : \beta < \kappa)$ in the monster model \mathfrak{C}' of T' such that for every m there are $\alpha_1 < \ldots < \alpha_m$ satisfying

$$\operatorname{tp}^{\mathfrak{C}'}(b_0, ..., b_{m-1}) = \operatorname{tp}^{M'}(a_{\alpha_1}, ..., a_{\alpha_m}).$$

(This is commonly known as "Morley's method". A proof can be found for example in Tent and Ziegler [2012] or Casanovas [2011] where the length of the sequence is assumed to be $\beth_{(2^{|T|})^+}$. The better bound of \beth_{ω_1} for countable theories is from Grossberg et al. [2002].)

Now let $N = \{f_n^i(b_{\beta_1}, ..., b_{\beta_n}) : i, n < \omega, \beta_1 < ... < \beta_n < \kappa\} \subseteq \mathfrak{C}'$ be the closure of this indiscernible sequence under all f_n^i -s. Note that $B \subset N$ is a Morley sequence in \mathfrak{p} . Also each $f_n^i(b_{\beta_1}, ..., b_{\beta_n}) \in \mathrm{cl}_{\mathfrak{p}}(b_{\beta_1}, ..., b_{\beta_n})$ (since this is encoded in the type of the corresponding sequence in M'). Hence $N \subseteq \mathrm{cl}_{\mathfrak{p}}(B)$ has dimension κ .

We claim that N is a quasiminimal model of T. This essentially follows from the fact that for every $\beta_1 < ... < \beta_n < \kappa$ there are $\alpha_1 < ... < \alpha_n < \beth_{\omega_1}$ such that the mapping sending $f_n^i(b_{\beta_1}, ..., b_{\beta_n}) \mapsto f_n^i(a_{\alpha_1}, ..., a_{\alpha_n})$ is elementary in L. Now by embedding a suitable fragment of N inside M in this way we see that the elements of $N \setminus \{f_n^i(b_{\beta_1}, ..., b_{\beta_n}) : i < \omega\}$ are generic over $b_{\beta_1}, ..., b_{\beta_n}$. This shows that N is quasiminimal. Similarly each satisfiable L-formula over $f_n^{i_1}(b_{\beta_1}, ..., b_{\beta_n}), ..., f_n^{i_m}(b_{\beta_1}, ..., b_{\beta_n})$ is either in \mathfrak{p} and hence realised in B (since B is infinite dimensional), or not in \mathfrak{p} and realised in $\{f_n^i(b_{\beta_1}, ..., b_{\beta_n}) : i < \omega\}$. This shows that N is an L-elementary substructure of \mathfrak{C}' and hence a model of T.

5 Stable Theories

In this section we assume that apart from having a strongly regular type \mathfrak{p} , the theory T is stable. This automatically makes \mathfrak{p} symmetric (all indis-

cernible sequence are totally indiscernible in stable theories). And since all symmetric global regular types are \emptyset -definable, the results of Section 3 apply here.

Some standard background on stable theories would be assumed. To construct a quasiminimal model we use the technology of local isolation and local atomicity. It was first introduced by Lachlan [1972] in order to prove a two cardinal theorem for stable theories. Here we present the background on local isolation in order to make our account more complete. More details can be found in advanced books on stability theory such as Shelah [1990], Baldwin [1988]. Our notion of local isolation coincides with $\mathbf{F}_{\aleph_0}^l$ -isolation of Shelah [1990].

Definition 5.1. A type $p(\bar{x}) \in S(A)$ is called *locally isolated* if for every *L*-formula $\phi(\bar{x}, \bar{y})$ there is a formula $\psi_{\phi}(\bar{x}) \in p$ such that $\psi_{\phi}(\bar{x}) \vdash p(\bar{x})|_{\phi}$. (Here $p(\bar{x})|_{\phi} = \{\phi(\bar{x}, \bar{a}) : \phi(\bar{x}, \bar{a}) \in p\} \cup \{\neg \phi(\bar{x}, \bar{a}) : \neg \phi(\bar{x}, \bar{a}) \in p\}$ is the ϕ -part of p.) A set B is called *locally atomic* (over A) if every type (over A) realised in B is locally isolated.

The notion of local isolation can be seen through the topology of ϕ -types as follows. Let $S_{\phi}(A)$ be the space of all complete ϕ -types over A. Topologise $S_{\phi}(A)$ by taking the basic clopen sets to be of the form $[\psi] = \{p \in S_{\phi}(A) : p \vdash \psi\}$ where ψ is any boolean combination of ϕ -formulas over A. Then $S_{\phi}(A)$ is a boolean topological space (compact, Hausdorff and totally disconnected) and the canonical restriction $\sigma_{\phi} : S(A) \to S_{\phi}(A)$ is continuous (and hence also closed). Now a type $p \in S(A)$ will be locally isolated if and only if $\sigma_{\phi}^{-1}(\sigma_{\phi}(p))$ is a neighbourhood of p for every ϕ (i.e. contains an open set containing p).

Recall that in stable theories $S_{\phi}(\mathfrak{C})$ is a scattered topological space of finite Cantor-Bendixson rank (see Casanovas [2011]). For a partial type p we denote by $CB_{\phi}(p)$ and $Mlt_{\phi}(p)$ the Cantor-Bendixson rank and degree of the set $\sigma_{\phi}([p]) = \{q \in S_{\phi}(\mathfrak{C}) : p \cup q \text{ is consistent}\}$ in $S_{\phi}(\mathfrak{C})$.

We would like to show that for every subset A, there is a model $M \supseteq A$ that is locally atomic over A. For that we prove

Lemma 5.2. For any set A, locally isolated types are dense in S(A).

Proof. Given a formula $\chi(\bar{x})$ over A, we need to find a locally isolated type that contains χ . Enumerate $(\phi_n(\bar{x}, \bar{y}_n) : n < \omega)$ all formulas in L. Put $p_0 = {\chi(\bar{x})}$ and $p_{n+1} = p_n \cup {\psi_n(\bar{x}, \bar{a}_n)}$ where ψ_n is chosen such that $CB_{\phi_n}(p_{n+1})$ is the least possible and among those $Mlt_{\phi_n}(p_{n+1})$ is the least possible. We claim that $p = \bigcup_{n < \omega} p_n$ has a unique completion which is locally isolated. Indeed given $\phi_n(\bar{x}, \bar{y}_n)$ and $\bar{b}_n \in A$, we can't have both $\phi_n(\bar{x}, \bar{b}_n)$ and $\neg \phi_n(\bar{x}, b_n)$ consistent with p_{n+1} as it contradicts the choice of ψ_n . Hence p_{n+1} , which is finite isolates the ϕ_n -part of the completion of p.

This allows us to iteratively realise formulas by locally isolated types similar to the constructible models in ω -stable theories. To make the whole construction locally atomic over A we need

Lemma 5.3. If A is locally atomic over BC and B is locally atomic over C, then AB is locally atomic over C.

Proof. Let $\bar{a} \in A$ and $b \in B$ and $\phi(\bar{x}, \bar{y}, \bar{z})$ be a formula. It is enough to find a formula $\psi(\bar{x}, \bar{y}) \in \operatorname{tp}(\bar{a}\bar{b}/C)$ such that for every $\bar{c} \in C$ for which $\phi(\bar{a}, \bar{b}, \bar{c})$ holds we have $\psi(\bar{x}, \bar{y}) \to \phi(\bar{x}, \bar{y}, \bar{c})$. Since $\operatorname{tp}(\bar{a}/BC)$ is locally isolated, we have a formula $\chi(\bar{x}, \bar{b}')$ over C in $\operatorname{tp}(\bar{a}/BC)$ such that $\chi(\bar{x}, \bar{b}') \to \phi(\bar{x}, \bar{b}, \bar{c})$ whenever $\phi(\bar{a}, \bar{b}, \bar{c})$ holds. Now we have $\forall \bar{x}(\chi(\bar{x}, \bar{z}) \to \phi(\bar{x}, \bar{y}, \bar{c})) \in \operatorname{tp}(\bar{b}\bar{b}'/C)$. By the local isolation of the latter there is a formula $\sigma(\bar{y}, \bar{z}) \in \operatorname{tp}(\bar{b}'\bar{b}/C)$ such that

$$\sigma(\bar{y}, \bar{z}) \to \forall \bar{x}(\chi(\bar{x}, \bar{z}) \to \phi(\bar{x}, \bar{y}, \bar{c}))$$

whenever $\forall x(\chi(\bar{x}, \bar{b}') \to \phi(\bar{x}, \bar{b}, \bar{c}))$ holds. Or equivalently

$$(\exists \bar{z}(\sigma(\bar{y},\bar{z}) \land \chi(\bar{x},\bar{z}))) \to \phi(\bar{x},\bar{y},\bar{c}).$$

Thus $\exists \bar{z}(\sigma(\bar{y}, \bar{z}) \land \chi(\bar{x}, \bar{z}))$ is the required formula $\psi(\bar{x}, \bar{y})$.

Now the two combine to give

Proposition 5.4. For every A in a stable theory there is a model $M \supseteq A$ that is locally atomic over A and $|M| = |A| + \aleph_0$.

Proof. We imitate the construction of prime models in ω -stable theories. Given B construct B' as follows: enumerate $(\phi_{\alpha}(\bar{x}_{\alpha}) : \alpha < \kappa)$ all consistent L(B) formulas that are not realised in B. Then add realisations b_{α} to B' such that $\operatorname{tp}(b_{\alpha}/B\{b_{\beta}:\beta<\alpha\})$ is locally isolated. By Lemma 5.3, B' is locally atomic over B. Now take $A_0 = A$ and $A_{n+1} = A'_n$. Finally $M = \bigcup_{n < \omega} A_n$ is the required model.

The final tool that we need for working with local isolation is the analogue of the Open Mapping Theorem.

Proposition 5.5. (Local Open Mapping Theorem) If $p \in S(A)$ does not fork over $B \subseteq A$ and is locally isolated, then so is $p|_B$.

Proof. Let N(A/B) be the set of types in S(A) that don't fork over B. By the usual Open Mapping Theorem the restriction map $\pi : N(A/B) \to S(B)$ is open. Now fix a formula $\phi(\bar{x}, \bar{y})$. By the local isolation of p we have that $\sigma_{\phi}^{-1}(\sigma_{\phi}(p)) \cap N(A/B)$ is a neighbourhood of p in N(A/B). Hence its image under π , which is contained in $\sigma_{\phi}^{-1}(\sigma_{\phi}(p|_B))$ is a neighbourhood of $p|_B$. \Box

We can use locally atomic models to construct a proper extension of a given model M where all the new elements realise the type $\mathfrak{p}|_M$. (In the previous section we used the canonical extension of a model for theories with definable Skolem functions for this purpose.) Indeed let a realise $\mathfrak{p}|_M$ and let N be locally atomic over Na. Now let $b \in N \setminus M$. Then $\operatorname{tp}(b/Ma)$ is locally isolated and $\operatorname{tp}(b/M)$ is not (otherwise it would be realised). Hence by the Local Open Mapping Theorem $b \not\perp_M a$. But then by symmetry we have $a \not\perp_M b$, which means that $a \not\models \mathfrak{p}|_{Mb}$, i.e. $a \in \operatorname{cl}_\mathfrak{p}(Mb)$. Hence $b \notin \operatorname{cl}_\mathfrak{p}(M)$, as otherwise $a \in \operatorname{cl}_\mathfrak{p}(M)$.

However the situation is a bit more delicate here. The problem is that there is no equivalent notion of local primness. That is we cannot in general embed a locally atomic or a locally constructible model over A in an arbitrary model containing A. So we cannot use the same idea as in the previous section to construct arbitrarily large quasiminimal models.

As an alternative we could try to construct a locally atomic model over a Morley sequence A in the following way. Enumerate $A = (a_{\alpha} : \alpha < \kappa)$. Then build a sequence $(M_{\alpha} : \alpha < \kappa)$ such that $M_{\alpha+1}$ is locally atomic over $M_{\alpha}a_{\alpha}$. Then all the elements in $M_{\alpha+1} \setminus M_{\alpha}$ will be generic over M_{α} , so we do not extend the closure. The problem with this, however, is that for $\alpha \geq \aleph_1$ the models M_{α} will be uncountable. And so in general $M_{\alpha+1}$ adds uncountably many elements.

To remedy this we employ a construction along the lines of Shelah's Excellence/NOTOP. The technique was adapted for local atomicity in Bays et al. [2014b]. Some of our arguments are adapted from that paper. First let us introduce some notation.

Definition 5.6. Let I be a downward-closed set of subsets (i.e. $s \subseteq t \in I$ implies $s \in I$). An *I*-system is a collection $(M_s : s \in I)$ of models such that $M_s \preceq M_t$ whenever $s \subseteq t$. If $J \subseteq I$ we denote $M_J = \bigcup_{s \in J} M_s$.

For $s \in I$ denote $\langle s := \mathcal{P}(s) \setminus \{s\}$ and $\not\geq s := \{t \in I : t \not\supseteq s\}.$

The system is *independent* if $M_s \, \bigcup_{M_{\leq s}} M_{\geq s}$.

An enumeration of I is an ordering $(s_{\alpha} : \alpha \in \kappa)$ of I such that $\alpha \leq \beta$ whenever $s_{\alpha} \subseteq s_{\beta}$. If an enumeration is fixed we write $< \alpha$ for $\{s_{\beta} : \beta < \alpha\}$. I.e. $M_{<\alpha} = \bigcup_{\beta < \alpha} M_{s_{\beta}}$. **Definition 5.7.** Given two subsets $A \subseteq B$ we say that A is Tarski-Vaught in B (in symbols $A \subseteq_{TV} B$) if every formula over A realised in B is already realised in A.

We use the Tarski-Vaught condition to lift local isolation to larger sets.

Proposition 5.8. If $A \subseteq_{TV} B$ and \bar{c} is locally isolated over A, then it is locally isolated over B.

Proof. Let $\phi(\bar{x}, \bar{y})$ be a formula without parameters and assume that $\psi(\bar{x})$ isolates the ϕ -part of $\operatorname{tp}(\bar{c}/A)$. Then $\psi(\bar{x})$ isolates a ϕ -type over B. Indeed if for some $\bar{b} \in B$ we have $\exists \bar{x}(\phi(\bar{x}) \land \phi(\bar{x}, \bar{b})) \land \exists \bar{x}(\phi(\bar{x}) \land \neg \phi(\bar{x}, \bar{b}))$, then the same should be true for some $\bar{a} \in A$. Thus $\psi(\bar{x})$ isolates the ϕ -part of $\operatorname{tp}(\bar{c}/B)$.

Finally we need the following Lemma from Shelah [1990].

Lemma 5.9 (TV Lemma, [Shelah, 1990, XII.2.3(2)]). Let $(M_s : s \in I)$ be an independent system in a stable theory. Assume that $J \subseteq I$ is such that for all $s \in I$ if $s \subseteq \bigcup J$, then $s \in J$. Then $M_J \subseteq_{TV} M_I$.

Theorem 5.10. Let \mathfrak{p} be a strongly regular type in a stable theory T. Then T has quasiminimal models of arbitrarily large cardinalities.

Proof. Let I be the set of all finite subsets of some uncountable cardinal κ . We inductively build an independent I-system of countable models as follows. Enumerate $I = \{s_{\alpha} : \alpha < \kappa\}$ so that $s_{\alpha} \subseteq s_{\beta}$ implies $\alpha \leq \beta$. Note that this implies $s_0 = \emptyset$. For M_{\emptyset} pick a countable infinite dimensional model. Given an ordinal α , assume that $M_{s_{\beta}}$ has been constructed for $\beta < \alpha$. If $s_{\alpha} = \{\delta\}$ is a singleton, pick a_{δ} a realisation of $\mathfrak{p}|_{M_{<\alpha}}$ and let $M_{\{\delta\}}$ be a countable locally atomic model over $M_{\emptyset}a_{\delta}$ that is independent from $M_{<\alpha}$ over M_{\emptyset} . Otherwise let M_s be a countable locally atomic model over $M_{<s}$. Then $(M_s : s \in I)$ is an independent system (this is [Shelah, 1990, Lemma XII.2.3(1)], whose proof is "an exercise in non-forking").

Now M_I is a model (by Tarski-Vaught test) and $|M_I| = \kappa$. We claim that M_I is quasiminimal. Fix $s = \{\alpha_1, ..., \alpha_n\} \subset \kappa$. We claim that all elements of $M_I \setminus M_s$ realise $\mathfrak{p}|_{M_s}$ over M_s . Since the later is countable, this implies that every subset of M_I definable over M_s is either countable or cocountable. Since s was arbitrary we conclude that M_I is quasiminimal.

Let $A = \{a_{\alpha} : \alpha < \kappa\}$. We show that M_I is locally atomic over M_sA . With the above enumeration of I we show by induction on α that $M_{<\alpha}$ is locally atomic over M_sA . This is clear if $\alpha = 0$ or α is a limit ordinal. So assume that it holds for α and let us prove it for $\alpha + 1$. Consider several cases.

- If $s_{\alpha} \subseteq s$, then clearly $M_{s_{\alpha}}$ is locally isolated over $M_{<\alpha}M_sA$.
- If $|s_{\alpha}| > 1$, then $M_{s_{\alpha}}$ is locally atomic over $M_{\langle s_{\alpha}}$. By the first clause we may assume that $s_{\alpha} \not\subseteq s$. But by TV Lemma (with $I = \{s_{\beta} : \beta < \alpha\} \cup \mathcal{P}(s)$ and $J = \mathcal{P}(s_{\alpha}) \setminus \{s_{\alpha}\}$) we have $M_{\langle s_{\alpha}} \subseteq_{TV} M_{\langle \alpha} M_s$. Also $M_{\langle \alpha} M_s \subseteq_{TV} M_{\langle \alpha} M_s A$ since M_{\emptyset} is infinite dimensional. Hence $M_{s_{\alpha}}$ is locally atomic over $M_{\langle \alpha} M_s A$.
- If $s_{\alpha} = \{\delta\}$ is a singleton, then $M_{s_{\alpha}}$ is locally atomic over $M_{\emptyset}a_{\delta}$. By the first clause we may assume that $\delta \notin s$ and hence a_{δ} realises \mathfrak{p} over $M_{<\alpha}M_s$. Now since M_{\emptyset} is a model, we have $M_{\emptyset} \subseteq_{TV} M_{<\alpha}M_s$. Using the definability of \mathfrak{p} we can show that $M_{\emptyset}a_{\delta} \subseteq_{TV} M_{<\alpha}M_sa_{\delta}$ and as before $M_{<\alpha}M_sa_{\delta} \subseteq_{TV} M_{<\alpha}M_sA$. Thus we conclude that $M_{s_{\alpha}}$ is locally atomic over $M_{<\alpha}M_sA$.

Thus in all cases we have $M_{s_{\alpha}}$ is locally atomic over $M_{<\alpha}M_sA$. But by the induction hypothesis $M_{<\alpha}$ is locally atomic over M_sA . Hence $M_{s_{\alpha}}M_{<\alpha} = M_{<\alpha+1}$ is locally atomic over M_sA . This completes the induction.

Now given $b \in M \setminus M_s$, it is locally atomic over M_sA . Since b is not locally atomic over M_s (otherwise its type would be realised in M_s), we conclude by the Local Open Mapping Theorem that $b \not\perp_{M_s} A$. Pick a finite subset $A' \subset A$ such that $b \not\perp_{M_s} A'$. We can also assume that A' is disjoint from M_s . Now by symmetry we have $A' \not\perp_{M_s} b$. Since A' is a Morley sequence in \mathfrak{p} over M_s , we conclude that $\dim(A'/M_s b) < \dim(A'/M_s)$ (where dim is in the sense of pregeometry $cl_{\mathfrak{p}}$). Hence $b \notin cl_{\mathfrak{p}}(M_s)$. This finishes the proof. \Box

6 ω -stable Theories

In this section we assume that the theory T is ω -stable. This allows us to construct prime models over every subset. It is easy to see that a prime model over an uncountable Morley sequence in \mathfrak{p} must be quasiminimal. Here we show more: the class \mathcal{C} of prime models over Morley sequences in \mathfrak{p} is a quasiminimal excellent class (see Definition 6.4). Since \mathcal{C} is clearly uncountably categorical, its excellence was expected and possibly known to experts. But we are not aware of a published proof.

Quasiminimal excellent classes play an important role in nonelementary categoricity. They were originally introduced in Zilber [2005b] where it is proven that a quasiminimal excellent class is uncountably categorical. The original formulation contains a technical axiom called *excellence*. It was thought to be the key to categoricity, until Bays et al. [2014a] showed that excellence follows from the rest of the axioms (see also a direct proof of categoricity in Haykazyan [2014]). Bays et al. [2014a] called an infinite dimensional structure in such a class a *quasiminimal pregeometry structure* and following this we called the entire class *quasiminimal pregeometry class* in Haykazyan [2014]. However in this paper we are dealing with structure that are quasiminimal and have a pregeometry but are not quasiminimal pregeometry structures in the sense of Bays et al. [2014a]. So we have reverted the terminology back to quasiminimal excellent.

The following simple observation will be used repeatedly, so it is worth stating explicitly. We have already used a variant of it for locally atomic model in the previous section.

Proposition 6.1. Let M be a model $a \notin cl_{\mathfrak{p}}(M)$ and N prime over Ma. Then for every $b \in N \setminus M$ we have $a \in cl_{\mathfrak{p}}(Mb)$. Hence by the exchange property also $b \in cl_{\mathfrak{p}}(Ma) \setminus cl_{\mathfrak{p}}(M)$.

Proof. Indeed since tp(b/Ma) is isolated, whereas tp(b/Ma) is not, by the Open Mapping Theorem we have $b \not\perp_M a$. Hence $a \not\perp_M b$. So that $a \in cl_{\mathfrak{p}}(Mb)$.

Let us first show that each uncountable model in C is quasiminimal. One consequence of quasiminimality is that there are no uncountable indiscernible sequences except in p. This will help us establish primeness of models in some cases.

Lemma 6.2. If A is an uncountable Morley sequence in \mathfrak{p} , then the prime model over A is quasiminimal.

Proof. This follows the standard pattern we have used already in previous sections. For a fixed countable $A_0 \subseteq A$ we can construct a model N containing A such that $\operatorname{cl}_{\mathfrak{p}}(A_0) \cap N$ is countable. To do so enumerate $A \setminus A_0 = (a_\alpha : \alpha < \kappa)$ and take $N_{\alpha+1}$ to be prime over $N_\alpha a_\alpha$. Let N_0 be an arbitrary countable model containing A_0 and $N_\delta = \bigcup_{\alpha < \delta} N_\delta$ for a limit ordinal δ . Then take $N = N_\kappa$. Finally M embeds into N over A showing that $\operatorname{cl}_{\mathfrak{p}}(A_0) \cap M$ is also countable. \Box

In order to satisfy some technical conditions of Definition 6.4, we make two further assumptions, which can be achieved by expanding the language. Firstly we assume that the theory T has quantifier elimination. Otherwise we can consider its Morleysation, i.e. its expansion with predicate symbols for all \emptyset -definable relations. Secondly we assume that $\mathfrak{p}|_A$ is not isolated for all finite A. (Note that this implies that $\mathfrak{p}|_A$ is not isolated for all A.) Otherwise we can add a countably infinite Morley sequence in \mathfrak{p} to the language. The result of the second assumption is **Lemma 6.3.** 1. If M is a model and $A \subseteq M$, then $cl_{\mathfrak{p}}(A) \cap M$ is a model.

- If M is prime over a Morley sequence A in p, then A is a basis for M (i.e. M ⊆ cl_p(A)).
- *Proof.* 1. Let $N = cl_{\mathfrak{p}}(A) \cap M$. We use the Tarski-Vaught test. Let $\phi(x, b)$ be a formula over N. Since $\mathfrak{p}|_{\bar{b}}$ is not isolated, there is a consistent formula $\psi(x, \bar{b})$ that implies ϕ and is not in \mathfrak{p} . Now every element in $M \setminus N$ realises $\mathfrak{p}|_{\bar{b}}$. Hence a realisation of ψ must be in N.
 - 2. Given $b \in M$, we have that $\operatorname{tp}(b/A)$ is isolated. Hence $b \not\models \mathfrak{p}|_A$ and so $b \in \operatorname{cl}_\mathfrak{p}(A)$.

Note that both assertions can fail without assuming that $\mathfrak{p}|_A$ is isolated for all finite A. For example both assertions fail in the theory of an infinite set in the empty language, where \mathfrak{p} is the unique non-algebraic type (so the theory is strongly minimal).

Now get prepared for a large definition. In the following a partial embedding is a partial map that preserves quantifier free formulas.

Definition 6.4. A quasiminimal excellent class is a collection C of pairs (H, cl_H) where H is a structure and cl_H is a pregeometry on H satisfying the following conditions.

- 1. Closure under isomorphisms If $(H, cl_H) \in \mathcal{C}$ and $f: H \to H'$ is an isomorphism, then $(H', cl_{H'}) \in \mathcal{C}$, where $cl_{H'}$ is defined as $cl_{H'}(X') = f(cl_H(f^{-1}(X')))$ for $X' \subseteq H'$.
- 2. Quantifier free theory If $(H, cl_H), (H', cl'_H) \in C$, then H and H' satisfy the same quantifier free sentences.
- 3. Pregeometry
 - For each $(H, cl_H) \in \mathcal{C}$ the closure of any finite set is countable.
 - If $(H, cl_H) \in \mathcal{C}$ and $X \subseteq H$, then $cl_H(X)$ is a substructure of H and together with the restriction of cl_H it is in \mathcal{C} .
 - If $(H, cl_H), (H', cl_{H'}) \in C, X \subseteq H, y \in H$ and $f : H \to H'$ is a partial embedding defined on $X \cup \{y\}$, then $y \in cl_H(X)$ if and only if $f(y) \in cl_{H'}(f(X))$.

- 4. Uniqueness of the generic type over countable closed models Let $(H, cl_H), (H', cl_{H'}) \in \mathcal{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty and $g: G \to G'$ be an isomorphism. If $x \in H, x' \in H'$ are independent from G and G' respectively, then $g \cup \{(x, x')\}$ is a partial embedding.
- 5. \aleph_0 -homogeneity over countable closed models Let $(H, \operatorname{cl}_H), (H', \operatorname{cl}_{H'}) \in \mathcal{C}$, subsets $G \subseteq H, G' \subseteq H'$ be countable closed or empty and $g: G \to G'$ be an isomorphism. If $g \cup f: H \to H'$ is a partial embedding, $X = \operatorname{dom}(f)$ is finite and $y \in \operatorname{cl}_H(X \cup G)$, then there is $y' \in H'$ such that $g \cup f \cup \{(y, y')\}$ is a partial embedding.

We don't elaborate on the definition any further. The interested reader can consult Zilber [2005b], Bays et al. [2014a] or Haykazyan [2014]. Note however that given a strongly regular type \mathfrak{p} in an arbitrary theory, the class of elementary submodels of the monster model \mathfrak{C} (closed under isomorphisms) together with the restriction of $cl_{\mathfrak{p}}$ satisfies axioms 1 and 2. If further the theory has quantifier elimination and $\mathfrak{p}|_A$ is not isolated for all finite A, then the class satisfies axioms 3 and 4, except the countable closure property. To satisfy the countable closure property we need to take the class of elementary submodels of a large quasiminimal structure (if such a structure exists). Finally satisfying axiom 5 is the real challenge. For that we need to show that in a prime model over a Morley sequence, if we pick a different Morley sequence, then the respective closure is prime over it. The following lemma is extracted from Makkai [1984].

Lemma 6.5. Assume M is a model $a \models \mathfrak{p}|_M$ and N is a prime model over Ma. Let $b \in N \setminus M$. Then N is prime over Mb. Consequently there is an automorphism in $\operatorname{Aut}(N/M)$ taking a to b.

Proof. By Proposition 6.1 we have $a \not\perp_M b$. Let $\theta(x,b) \in \operatorname{tp}(a/Mb)$ be a formula over Mb such that any type containing it forks over M. Let $\phi(a, y)$ be a formula over Ma that isolates $\operatorname{tp}(b/Ma)$. We claim that $\theta(x,b) \wedge \phi(x,b)$ isolates $\operatorname{tp}(a/Mb)$. It is enough to show that for every $a' \in N$ such that $\theta(a',b) \wedge \phi(a',b)$ we have $a \equiv_{Mb} a'$. Given such an a' we have $a' \not\perp_M b$. Then $a' \in N \setminus M$. Hence by Proposition 6.1 again $a' \models \mathfrak{p}_M$. Now we can get the desired conclusion from $\phi(a',b)$. Indeed let $f \in \operatorname{Aut}(\mathfrak{C}/M)$ be an automorphism that maps a' to a. Then $\phi(a, f(b))$ holds. Since $\phi(a, y)$ isolates $\operatorname{tp}(b/Ma)$ we have $ba \equiv_M f(b)a \equiv_M ba'$. Hence $a \equiv_{Mb} a'$.

Now since N is constructible over Ma and tp(a/Mb) is isolated, we conclude that N is constructible and hence prime and atomic over Mb.

Proposition 6.6. Assume that M is prime over a Morley sequence $A = (a_{\alpha} : \alpha < \kappa)$ in \mathfrak{p} . Let $B = (b_{\alpha} : \alpha < \mu) \subseteq M$ is another Morley sequence in \mathfrak{p} . Then $\mathrm{cl}_{\mathfrak{p}}(B) \cap M$ is prime over B.

Proof. By Lemma 6.3, the closure $cl_{\mathfrak{p}}(B) \cap M$ is a model. Also by the countable closure property it does not contain an uncountable indiscernible sequence over B. Thus it remains to show that $cl_{\mathfrak{p}}(B) \cap M$ is atomic over B.

We can assume that B is finite. Indeed every $a \in cl_p(B) \cap M$ is in the closure of a finite subset $B_0 \subseteq B$. Assume we prove that a is atomic over B_0 . Then given $b \in B \setminus B_0$, we have that $tp(a/B_0) \vdash tp(a/B_0b)$ (see the discussion after Definition 2.1). Hence $tp(a/B_0b)$ is also isolated. Iterating this we obtain that tp(a/B) is isolated. So assume that $B = \{b_0, ..., b_{n-1}\}$ is finite.

Further we may assume that B is a subset of A. Indeed by the exchange property there is $a \in A$ such that for $A' = A \setminus \{a\}$ we have that $A' \cup \{b_0\}$ is a basis of M. Now $N = cl_p(A') \cap M$ is a model by Lemma 6.3. Further M is prime over Na (it is atomic over Na as Na is normal over A) and $b_0 \in M \setminus Na$. Thus by Lemma 6.5 there is an automorphism of M fixing A' and taking a to b_0 . Thus we can assume that $b_0 \in A$. Iterating this construction we can assume that $B \subseteq A$.

But now the claim follows from the Open Mapping Theorem. Indeed if $c \in cl_{\mathfrak{p}}(B) \cap M$, then $A \, {\textstyle igstyle _B} c$ and hence $c \, {\textstyle igstyle _B} A$. Since tp(c/A) is isolated so is tp(c/B).

Now we are ready to prove the main result of this section.

Theorem 6.7. Let C be the class of prime models over Morley sequences in \mathfrak{p} . Then C together with restrictions of $cl_{\mathfrak{p}}$ is a quasiminimal excellent class.

Proof. The only axiom not covered so far is the \aleph_0 -homogeneity over countable closed models. Let $H, H' \in \mathcal{C}$ be prime over Morley sequences A and A' respectively. Let $G \subseteq H, G' \subseteq H'$ be countable closed or empty. Let $g: G \to G'$ be an isomorphism. If G (and hence G') is nonempty, we may assume by Proposition 6.6 that $G = \operatorname{cl}_{\mathfrak{p}}(A_0) \cap H, G' = \operatorname{cl}_{\mathfrak{p}}(A'_0) \cap H'$ where $A_0 \subseteq A, A'_0 \subseteq A$ and g maps A_0 to A'_0 .

Let $f : H \to H'$ be a partial embedding with a finite domain. Assume that dom $(f) = \bar{a}\bar{b}$ where \bar{a} is independent over G and $\bar{b} \in cl_{\mathfrak{p}}(G\bar{a})$. Then $cl_{\mathfrak{p}}(G\bar{a}) \cap H$ is prime and constructible over $G\bar{a}$. (In case if G is nonempty, $cl_{\mathfrak{p}}(G\bar{a}) \cap H$ is prime over $A_0\bar{a}$ by Proposition 6.6 and $G\bar{a}$ is normal over $A_0\bar{a}$.) Since $tp(\bar{b}/G\bar{a})$ is isolated, $cl_{\mathfrak{p}}(G\bar{a}) \cap H$ is also constructible and so prime over $G\bar{a}\bar{b}$. Thus the elementary mapping $g \cup f$ extends to any element of $cl_{\mathfrak{p}}(G\bar{a}\bar{b}) \cap H$.

7 Conclusion

We have shown that \beth_{ω_1} is an upper bound for the Hanf number of the existence of quasiminimal models. We don't have an example to show that it is sharp, we don't even know if the Hanf number is \aleph_2 . One way of proving it would be to show that the existence of a symmetric strongly regular type is enough to construct arbitrarily large quasiminimal models. We have shown so using additional assumptions: either stability or presence of definable Skolem functions. One could try to expand the language with Skolem functions, but doing so naively will destroy the regular type. Detailed examination of the proof however reveals that not all Skolem functions are necessary, which gives this approach some hope.

On the other direction one could ask whether the stability assumptions we use for our constructions are sharp. For Theorem 5.10 one can ask whether there is a simple theory with a strongly regular type and no arbitrarily large quasiminimal model. A candidate for such a theory could be ACFA where the type of a transformally transcendental element is strongly regular. By Theorem 3.3 there is a quasiminimal model of ACFA of cardinality \aleph_1 , but we don't know what happens in other cardinalities.

One can also try to construct a quasiminimal excellent class from a weaker assumption than ω -stability. A reasonable assumption could be a strongly regular type in a superstable theory with NOTOP. There is a deep analogy between NOTOP and excellence and one could try to exploit it. The methods of Bays et al. [2014a] allow one to construct a quasiminimal excellent class if the theory is superstable and all types are definable over finite subsets. This is however not very far from an ω -stable theory. Every such theory which is in addition small (i.e. has countably many pure types) is ω -stable.

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