UNIFORMLY LOCALLY O-MINIMAL OPEN CORE

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ABSTRACT. This paper discusses sufficient conditions for a definably complete densely linearly ordered expansion of an abelian group having the uniformly locally o-minimal open cores of the first/second kind and strongly locally ominimal open core, respectively.

1. INTRODUCTION

The open core of a structure is its reduct generated by its definable open sets. Dolich et al. first introduced the notion of open core and gave a sufficient condition for the structure having an o-minimal open core in [2]. Fornasiero also investigated necessary and sufficient conditions for a definably complete expansion of an ordered field having a locally o-minimal open core in [3].

A uniformly locally o-minimal structure was first introduced in [7] and a systematic study was made in [4]. The purpose of this paper is to give sufficient conditions for structures having uniformly locally o-minimal open cores of the first/second kind and having strongly locally o-minimal open core, respectively. The following theorem is our main theorem. The notations in the theorem are defined in Section 2.

Theorem 1.1. Consider an expansion of a densely linearly ordered abelian group $\mathcal{R} = (R, <, +, 0, ...)$ with $\mathcal{R} \models DC$.

- If $\mathcal{R} \models SLUF$, the structure \mathcal{R} has a strongly locally o-minimal open core.
- If *R* ⊨ LUF₁, the structure *R* has a uniformly locally o-minimal open core of the first kind.
- If $\mathcal{R} \models LUF_2$, the structure \mathcal{R} has a uniformly locally o-minimal open core of the second kind.

We can prove the theorem basically following the same strategy as the proof of [2, Theorem A]. We first review the notion of D_{Σ} -families and D_{Σ} -sets used in the proof of [2, Theorem A] in Section 3. The key lemma is that, for a D_{Σ} -family $\{X_{r,s}\}_{r>0,s>0}$, the set $X_{r,s}$ has a nonempty interior for some r > 0 and s > 0when the D_{Σ} -set $X = \bigcup_{r>0,s>0} X_{r,s}$ has a non empty interior. This lemma holds true both for the o-minimal open core case [2, 3.1] and our case. However, a new investigation is necessary to demonstrate the lemma in our case. Basic lemmas including the above lemma are proved in Section 4. We cannot use the same definition of dimension of D_{Σ} -sets as [2]. We give a new definition of dimension in Section 3. We finally prove the theorem in Section 5 using this concept.

We introduce the terms and notations used in this paper. The term 'definable' means 'definable in the given structure with parameters' in this paper. A CBD

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set is a closed, bounded and definable set. A *CDD* set is a closed, discrete and definable set. For any set $X \subset \mathbb{R}^{m+n}$ definable in a structure $\mathcal{R} = (R, \ldots)$ and for any $x \in \mathbb{R}^m$, the notation X_x denotes the fiber defined as $\{y \in \mathbb{R}^n \mid (x, y) \in X\}$. For a linearly ordered structure $\mathcal{R} = (R, <, \ldots)$, an open interval is a definable set of the form $\{x \in \mathbb{R} \mid a < x < b\}$ for some $a, b \in \mathbb{R}$. It is denoted by (a, b) in this paper. We define a closed interval in the same manner and it is denoted by [a, b]. An open box in \mathbb{R}^n is the direct product of n open intervals. A closed box is defined similarly. Let A be a subset of a topological space. The notations $\operatorname{int}(A)$ and \overline{A} denote the interior and the closure of the set A, respectively. The notation |S| denotes the cardinality of a set S.

2. Definitions

We review the definitions and the assertions introduced in the previous studies. A *constructible* set is a finite boolean combination of open sets. Note that every constructible definable set is a finite boolean combination of open definable sets by [1]. The definition of a definably complete structure is found in [8]. The notation $\mathcal{R} \models DC$ means that the structure \mathcal{R} is definably complete. A locally o-minimal structure is defined and investigated in [9]. Readers can find the definitions of uniformly locally o-minimal structures of the first/second kind in [4]. The open core of a structure is defined in [2].

We give the definitions used in Theorem 1.1.

Definition 2.1 (Local uniform finiteness). A densely linearly ordered structure $\mathcal{R} = (R, <, ...)$ satisfies *locally uniform finiteness of the second kind* – for short, $\mathcal{R} \models \text{LUF}_2$ if, for any definable subset $X \subset \mathbb{R}^{m+1}$, $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$, there exist

- a positive integer N,
- a closed interval I with $a \in int(I)$ and
- an open box $U \subset \mathbb{R}^m$ with $b \in U$

such that $|X_x \cap J| = \infty$ or $|X_x \cap J| \leq N$ for all $x \in U$ and all closed intervals J with $a \in int(J) \subset I$.

An easy induction shows that $\mathcal{R} \models \text{LUF}_2$ if and only if, for any definable subset $X \subset \mathbb{R}^{m+n}$ and $(b, a) \in \mathbb{R}^{m+n}$, there exist a positive integer N, a closed box B with $a \in \text{int}(B)$ and an open box U with $b \in U$ such that $|X_x \cap B'| = \infty$ or $|X_x \cap B'| \leq N$ for all $x \in U$ and all closed boxes B' with $a \in \text{int}(B') \subset B$.

The structure $\mathcal{R} = (R, <, ...)$ satisfies locally uniform finiteness of the first kind – for short, $\mathcal{R} \models \text{LUF}_1$ if $\mathcal{R} \models \text{LUF}_2$ and we can take $U = \mathbb{R}^m$ in the definition of LUF₂. The structure $\mathcal{R} = (R, <, ...)$ satisfies strongly locally uniform finiteness – for short, $\mathcal{R} \models \text{SLUF}$ if $\mathcal{R} \models \text{LUF}_1$ and we can take I independently of the definable set $X \subset \mathbb{R}^{m+1}$.

Remark 2.2. A definably complete uniformly locally o-minimal structure of the second kind satisfies locally uniform finiteness of the second kind by [4, Theorem 4.2].

Remark 2.3. Consider a locally o-minimal expansion of the group of reals $\mathbb{R} = (\mathbb{R}, <, 0, +, ...)$. The following assertion is [5, Theorem 4.3].

For any definable subset X of \mathbb{R}^{n+1} , there exist a positive element $r \in \mathbb{R}$ and a positive integer K such that, for any $a \in \mathbb{R}^n$, the definable set $X \cap (\{a\} \times (-r, r))$ has at most K connected components.

By reviewing its proof, it is easy to check that the above r can be taken independently of X because \mathbb{R} is strongly locally o-minimal by [9, Corollary 3.4]. Therefore, we have $\widetilde{\mathbb{R}} \models \text{SLUF}$.

Example 2.4. We give an example of a structure which has a strongly locally ominimal open core, but is not a locally o-minimal.

We first consider the language $L_1 = \{0, 1, +, -, <\}$. We define L_1 -structures $\widetilde{\mathbb{Q}} = (\mathbb{Q}, 0^{\widetilde{\mathbb{Q}}}, 1^{\widetilde{\mathbb{Q}}}, -^{\widetilde{\mathbb{Q}}}, -^{\widetilde{\mathbb{Q}}}, <^{\widetilde{\mathbb{Q}}})$ and $\widetilde{\mathbb{R}} = (\mathbb{R}, 0^{\widetilde{\mathbb{R}}}, 1^{\widetilde{\mathbb{R}}}, -^{\widetilde{\mathbb{R}}}, <^{\widetilde{\mathbb{R}}})$ naturally. It is easy to demonstrate that both $\widetilde{\mathbb{Q}}$ and $\widetilde{\mathbb{R}}$ have quantifier elimination. We can also easy to demonstrate that $\widetilde{\mathbb{Q}}$ is an elementary substructure of $\widetilde{\mathbb{R}}$ using the Tarski-Vaught test. They are both o-minimal structures.

The structures $[0,1)_{\mathbb{Q}}$ and $[0,1)_{\mathbb{R}}$ are the restrictions of $\widetilde{\mathbb{Q}}$ and $\widetilde{\mathbb{R}}$ to the sets $[0,1)_{\mathbb{Q}} = \{x \in \mathbb{Q} \mid 0 \leq x < 1\}$ and $[0,1)_{\mathbb{R}} = \{x \in \mathbb{R} \mid 0 \leq x < 1\}$ defined in [7, Definition 2], respectively. The structure $[0, 1]_{\mathbb{Q}}$ is again an elementary substructure of $[0,1]_{\mathbb{R}}$. The notation $\mathcal{M} = ([0,1]_{\mathbb{R}}, [0,1]_{\mathbb{Q}})$ denotes their dense pair. The definition of dense pairs is found in [10]. The dense pair \mathcal{M} satisfies uniform finiteness by [10, Corollary 4.5]. The notion of uniform finiteness is introduced in [2].

We next consider the language $L = \{0, 1, +, -, <, P_{\mathbb{Z}}, P_{\mathbb{Q}}\}$, where $P_{\mathbb{Z}}$ and $P_{\mathbb{Q}}$ are unary predicates. We define an L-structure $\mathcal{R} = (\mathbb{R}, 0^{\mathcal{R}}, 1^{\mathcal{R}}, +^{\mathcal{R}}, -^{\mathcal{R}}, <^{\mathcal{R}}, P_{\mathbb{Z}}^{\mathcal{R}}, P_{\mathbb{Q}}^{\mathcal{R}})$ as follows:

- R ⊨ P_Z^R(x) if and only if x ∈ Z;
 R ⊨ P_Q^R(x) if and only if x ∈ Q.

The following claim is proved by the induction on the complexity of the formula defining the definable set X. We omit the proof.

Claim. Let X be a subset of \mathbb{R}^n definable in \mathcal{R} . There exist finite subsets $X_1, \ldots X_k$ of $[0,1]_{\mathbb{R}}$ definable in the dense pair \mathcal{M} and a map $\iota : \mathbb{Z}^n \to \{1, \ldots, k\}$ such that, for any $z = (z_1, \ldots, z_n) \in \mathbb{Z}^n$, we have

$$X \cap \left(\prod_{i=1}^{n} [z_i, z_i + 1)\right) = z + X_{\iota(z)},$$

where $[c,d) = \{x \in R \mid c \le x < d\}$ and $z + X_{\iota(z)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (x_1 - d)\}$ $z_1,\ldots,x_n-z_n)\in X_{\iota(z)}\}.$

The structure \mathcal{R} is not locally o-minimal because the set \mathbb{Q} is definable in \mathcal{R} . We have $\mathcal{R} \models$ SLUF by the above claim because \mathcal{M} satisfies uniform finiteness. We also have $\mathcal{R} \models DC$ because the universe \mathbb{R} is complete. The structure \mathcal{R} has a strongly locally o-minimal open core by Theorem 1.1.

3. D_{Σ} -sets and its dimension

We can apply the same strategy as [2] to our problem. Dolich et al. used the notion of D_{Σ} -sets in [2]. They play an important role also in this paper.

Definition 3.1 (D_{Σ} -sets). Consider an expansion of a densely linearly ordered abelian group $\mathcal{R} = (R, <, +, 0...)$. A parameterized family of definable sets are the family of the fibers of a definable set. A parameterized family $\{X_{r,s}\}_{r>0,s>0}$ of CBD subsets of \mathbb{R}^n is called a \mathbb{D}_{Σ} -family if $X_{r,s} \subset X_{r',s}$ and $X_{r,s'} \subset X_{r,s}$ whenever $r \leq r'$ and $s \leq s'$. A definable subset X of \mathbb{R}^n is a \mathbb{D}_{Σ} -set if $X = \bigcup_{r > 0, s > 0} X_{r,s}$ for

some D_{Σ} -family $\{X_{r,s}\}_{r>0,s>0}$.

The following two lemmas are found in [2, 8].

Lemma 3.2. Consider an expansion of a densely linearly ordered abelian group \mathcal{R} with $\mathcal{R} \models DC$. The following assertions are true:

- (1) The projection image of a D_{Σ} -set is D_{Σ} .
- (2) Fibers, finite unions and finite intersections of D_{Σ} -sets are D_{Σ} .
- (3) Every constructible definable set is D_{Σ} .

Proof. (1) Immediate from [8, Lemma 1.7]. (2) [2, 1.9(1)]. (3) [2, 1.10(1)].

Lemma 3.3. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC$. A CBD set $X \subset \mathbb{R}^{n+1}$ has a nonempty interior if the CBD set

 $\{x \in \mathbb{R}^n \mid X_x \text{ contains a closed interval of length } s\}$

has a nonempty interior for some s > 0.

Proof. [2, 2.8(2)]

The notion of dimension used for o-minimal open cores in [2] is not appropriate for our setting. We give a new definition of dimension of a D_{Σ} -set. The dimension of a set definable in a locally o-minimal structure admitting local definable cell decomposition is defined in [4, Section 5]. In a definably complete uniformly locally o-minimal structure of the second kind, the dimension defined below coincides with the dimension defined in [4, Section 5] by [4, Corollary 5.3].

Definition 3.4 (Dimension). Let $\mathcal{R} = (R, <, ...)$ be an expansion of a densely linearly ordered structure. Consider a D_{Σ} -subset X of \mathbb{R}^n and a point $x \in \mathbb{R}^n$. The *local dimension* dim_x X of X at x is defined as follows:

- $\dim_x X = -\infty$ if there exists an open box B with $x \in B$ and $B \cap X = \emptyset$.
- Otherwise, $\dim_x X$ is the supremum of nonnegative integers d such that, for any open box B with $x \in B$, the image $\pi(B \cap X)$ has a nonempty interior for some coordinate projection $\pi: \mathbb{R}^n \to \mathbb{R}^d$.

The dimension of X is defined by dim $X = \sup\{\dim_x X \mid x \in \mathbb{R}^n\}$. The projective dimension proj. dim X of X is defined as follows:

- proj. dim $X = -\infty$ if X is an empty set.
- Otherwise, proj. dim X is the supremum of nonnegative integers d such that the image $\pi(X)$ has a nonempty interior for some coordinate projection $\pi: \mathbb{R}^n \to \mathbb{R}^d$.

The following lemma illustrates that the dimension and the projective dimension coincides in some open box.

Lemma 3.5. Let $\mathcal{R} = (R, <, ...)$ be an expansion of a densely linearly ordered structure. Consider a D_{Σ} -subset X of \mathbb{R}^n of dimension d. Take a point $x \in \mathbb{R}^n$ with $\dim_x X = d$. We have $\dim(X \cap B) = \operatorname{proj.dim}(X \cap B) = d$ for any sufficiently small open box B in \mathbb{R}^n with $x \in B$.

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Proof. Since $\dim_x X = d$, the projection image $\pi(B \cap X)$ has an empty interior for any coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^{d+1}$ when B is a sufficiently small open box with $x \in B$. In particular, proj. $\dim B \cap X \leq d$. It is obvious that $d = \dim_x X \leq \dim B \cap X \leq \dim X = d$. We have shown that $\dim B \cap X = d$ and proj. $\dim B \cap X \leq \dim B \cap X$. The opposite inequality $\dim B \cap X \leq \operatorname{proj.} \dim B \cap X$ is obvious from the definition. \Box

4. Basic Lemmas

We introduce basic lemmas in this section. We first prove two lemmas. We can prove the lemmas by localizing the arguments in [2, 2.4].

Lemma 4.1. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$.

For any definable set $X \subset \mathbb{R}^{m+n}$ and a point $(b, a) \in \mathbb{R}^m \times \mathbb{R}^n$, there exist a positive integer N, a closed box B with $a \in int(B)$ and an open box U containing the point b such that, if $X_x \cap B$ is discrete, we have $|X_x \cap B| \leq N$ for any $x \in U$.

In addition, we can take $U = R^m$ if $\mathcal{R} \models LUF_1$. We can take B independently of $X \subset R^{m+n}$ if $\mathcal{R} \models SLUF$.

Proof. We first demonstrate the following claim:

Claim. Let X be a definable subset of \mathbb{R}^{m+n} . Assume that the fiber X_x is CDD for any $x \in \mathbb{R}^m$. For any $(b, a) \in \mathbb{R}^m \times \mathbb{R}^n$, there exists a closed box B with $a \in int(B)$ and an open box U containing the point b such that $X_x \cap B$ is a finite set for any $x \in U$. In addition, we can take $U = \mathbb{R}^m$ if $\mathcal{R} \models LUF_1$. We can take B independently of $X \subset \mathbb{R}^{m+n}$ if $\mathcal{R} \models SLUF$.

We prove the claim. Assume the contrary. We can find a point $(b, a) \in \mathbb{R}^m \times \mathbb{R}^n$ such that, for any closed box B with $a \in int(B)$ and any open box U with $b \in U$, $X_x \cap B$ is infinite for some $x \in U$. Consider the set

$$Y = \{(x, y_1, y_2) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mid (x, y_1) \in X, \ (x, y_2) \in X, \\ y_1 \ge y_2 \text{ in the lexicographic order} \}.$$

Since $\mathcal{R} \models \text{LUF}_2$, there exist a positive integer N, a closed box B with $a \in \text{int}(B)$ and an open box V with $(b, a) \in V = U_1 \times U_2 \subset \mathbb{R}^m \times \mathbb{R}^n$ such that, for any $(x, y_1) \in V$, we have $|Y_{(x,y_1)} \cap B| = \infty$ or $|Y_{(x,y_1)} \cap B| \leq N$. Shrinking B if necessary, we may assume that B is contained in U_2 . Fix such a closed box B. Take a point $x \in U_1$ such that $X_x \cap B$ is infinite. Such a point x exists by the assumption. Note that $X_x \cap B$ is CBD. We construct a point $z_k \in B$ with $|Y_{(x,z_k)} \cap B| = k$ inductively. Take $z_1 = \text{lexmin}(X_x \cap B)$, then $Y_{(x,z_1)} \cap B = \{z_1\}$. The notation lexmin denotes the lexicographic minimum defined in [8]. Take $z_2 = \text{lexmin}(X_x \cap B \setminus \{z_1\})$, then $Y_{(x,z_2)} \cap B = \{z_1, z_2\}$. Take z_3, z_4, \ldots in this manner. We have $|Y_{(x,z_{N+1})} \cap B| =$ N + 1, which is a contradiction.

It is obvious that $U = R^m$ if $\mathcal{R} \models LUF_1$ and B is common to all X if $\mathcal{R} \models SLUF$. We have finished the proof of the claim.

We return to the proof of the lemma. Take a bounded closed box C with $b \in int(C)$. Set $D = \{(s, x, y) \in R \times C \times R^n \mid \prod_{i=1}^n [y_i - s, y_i + s]^n \cap X_x = \{y\}\}$, where $y = (y_1, \ldots, y_n)$. The following assertions are trivial.

- The fiber $D_{(s,x)}$ is CDD.
- $\bigcup_{s>0} D_s = \{(x, y) \mid y \text{ is a discrete point in } X_x\}.$

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Apply the claim to the set D. We can take an open box U with $b \in U$ and a closed box B with $a \in int(B)$ such that $D_{(s,x)} \cap B$ are finite sets for all $x \in U$ and all sufficiently small s > 0. Since $\mathcal{R} \models LUF_2$, shrinking B and U if necessary, we have $|D_{(s,x)} \cap B| \leq N$ for some positive integer N, any $x \in U$ and any sufficiently small s > 0. Since $D_{(s,x)} \subset D_{(s',x)}$ for all s > s' > 0, we have $|\bigcup_{s>0} D_{(s,x)} \cap B| \leq N$. The 'in addition' parts of the lemma are obvious.

Lemma 4.2. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$.

For any definable set $X \subset \mathbb{R}^{m+1}$ and a point $(b, a) \in \mathbb{R}^m \times \mathbb{R}$, there exist positive integers N_1 , N_2 , an open interval I with $a \in I$ and an open box U containing the point b such that, for any $x \in U$,

- the open set $int(X_x) \cap I$ is the union of at most N_1 open intervals in I;
- the closed set $\overline{X_x} \cap \overline{I}$ is the union of at most N_1 points and N_2 closed intervals in \overline{I} .

In addition, we can take $U = R^m$ if $\mathcal{R} \models LUF_1$. We can take I independently of $X \subset R^{m+1}$ if $\mathcal{R} \models SLUF$.

Proof. The assertion on the closure follows the assertion on the interior by considering $\mathbb{R}^{m+1} \setminus X$ in place of X. We only prove the latter. We may assume that X_x is open for any $x \in \mathbb{R}^m$ without loss of generality. Consider the set

$$C = \{ (r, x, y) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \mid r > 0, \exists \varepsilon > 0, \ (y - \varepsilon, y + \varepsilon) \subset X_x \cap (a - r, a + r) \\ y - \varepsilon, y + \varepsilon \in \partial(X_x \cap (a - r, a + r)) \},$$

where $\partial(X_x \cap (a-r, a+r))$ denotes the boundary of the set $X_x \cap (a-r, a+r)$ in R. The fiber $C_{(r,x)}$ is discrete. By Lemma 4.1, there exist a positive integer N, a positive element s > 0, a closed interval I' with $a \in int(I')$ and an open box Ucontaining the point b such that $|C_{(r,x)} \cap I'| \leq N$ for all $x \in U$ and 0 < r < s. Take a sufficiently small r > 0 with $[a-r, a+r] \subset I'$ and set I = (a-r, a+r). The definable set $X_x \cap I$ consists of at most N open intervals for any $x \in U$.

Lemma 4.3 is the key lemma introduced in Section 1. We prove the lemma combining the arguments of [2] and [6]. Lemma 4.4 is another key lemma corresponding to [4, Theorem 3.3]. They are proved simultaneously.

Lemma 4.3. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Let $\{X_{r,s} \subset \mathbb{R}^n\}_{r>0,s>0}$ be a D_{Σ} -family. Set $X = \bigcup_{r,s} X_{r,s}$. One of the following conditions is satisfied:

- The D_{Σ} -set X has an empty interior and it is locally finite when n = 1.
- The CBD set $X_{r,s}$ has a nonempty interior for some r > 0 and s > 0.

Lemma 4.4. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Consider a D_{Σ} -subset X of \mathbb{R}^n with a nonempty interior. Let $X = X_1 \cup X_2$ be a partition into two D_{Σ} -sets. At least one of X_1 and X_2 has a nonempty interior.

Proof. We prove the lemmas by the induction on n. We first consider the case in which n = 1. We first demonstrate Lemma 4.3. Assume that $\operatorname{int}(X_{r,s}) = \emptyset$ for all r > 0 and s > 0. We have only to show that X is locally finite. Fix an arbitrary point $a \in R$ and r > 0. Set $X_r = \bigcup_{s>0} X_{r,s}$. There exist a closed interval I with $a \in \operatorname{int}(I)$, a positive element $t \in R$, positive integers N_1 and N_2 such that the

intersection $I \cap X_{r,s}$ is the union of at most N_1 points and N_2 closed intervals for any 0 < s < t by Lemma 4.2. The set $I \cap X_{r,s}$ consists of at most N_1 points because $\operatorname{int}(X_{r,s}) = \emptyset$. We have $|X_r \cap I| \leq N_1$ because $\{X_{r,s}\}_{s>0}$ is a decreasing sequence. The set X_r is a CDD set.

We show that $X = \bigcup_{r>0} X_r$ is a CDD set. Assume the contrary. There exists a point $a \in R$ such that, for any open interval I containing a point a, the definable set $I \cap X$ is an infinite set. We may assume that $\{x > a\} \cap I \cap X$ is an infinite set for any open interval I containing the point a without loss of generality.

Consider the definable function $f : \{r \in R \mid r > 0\} \rightarrow \{x \in R \mid x > a\}$ defined by $f(r) = \inf\{x > a \mid x \in X_r\}$. As in the proof of [6, Theorem 4.3], we can prove the following assertions:

- The definable function f is a decreasing function and $\lim_{r\to\infty} f(r) = a$.
- Consider the image $\operatorname{Im}(f)$ of the function f. For any $b \in \operatorname{Im}(f)$, there exists a point $b_1 \in \operatorname{Im}(f)$ such that $b < b_1$ and the open interval (b, b_1) has an empty intersection with $\operatorname{Im}(f)$.

We may assume that the intersection $\overline{\text{Im}(f)} \cap \overline{I}$ of the closure of the image with \overline{I} consists of finite points and finite closed intervals by Lemma 4.2 shrinking the open interval I if necessary. We lead to a contradiction assuming that it contains a closed interval J. Take an arbitrary point $b \in \text{Im}(f)$ in the interior of the closed interval J. The open interval (b, b_1) has an empty intersection with Im(f) for some $b_1 \in \mathbb{R}$. It is a contradiction. We have shown that $I \cap \text{Im}(f)$ is a finite set. It is a contradiction to the fact that $\lim_{r\to\infty} f(r) = a$. We have shown that $X = \bigcup_{r>0} X_r$ is a CDD set. It is obviously locally finite. We have demonstrated Lemma 4.3 when n = 1.

Lemma 4.4 is immediate from Lemma 4.3 when n = 1.

We next consider the case in which n > 1. We first demonstrate Lemma 4.3. We can prove that $X_r = \bigcup_{s>0} X_{r,s}$ has an empty interior if $X_{r,s}$ have empty interiors for all s > 0 in the same way as [6, Lemma 3.3]. Now we can get Lemma 4.3 in the same way as the proof of [6, Lemma 4.1, Lemma 4.2] using Lemma 4.4 for n - 1 instead of [4, Theorem 3.3].

The remaining task is to prove Lemma 4.4 when n > 1. Take D_{Σ} -families $\{X_{r,s}^i\}_{r,s}$ with $X_i = \bigcup_{r,s} X_{r,s}^i$ for i = 1, 2. Set $X_{r,s} = X_{r,s}^1 \cup X_{r,s}^2$. It is a CBD set. We have $\operatorname{int}(X_{r,s}) \neq \emptyset$ for some r > 0 and s > 0 by Lemma 4.3 for n because $X = \bigcup_{r,s} X_{r,s}$. If at least one of $X_{r,s}^1$ and $X_{r,s}^2$ has a nonempty interior, at least one of X_1 and X_2 has a nonempty interior. Therefore, we may assume that X_1 and X_2 are CBD sets. Let B be a closed box contained in X. We have $B = (X_1 \cap B) \cup (X_2 \cap B)$. If the lemma is true for B, the lemma is also true for the original X. Hence, we may assume that X is a closed box.

Let π be the coordinate projection forgetting the last coordinate. For i = 1, 2and s > 0, we set

 $S_s^i = \{x \in \pi(X_i) \mid \text{ the fiber } (X_i)_x \text{ contains a closed interval of length } s\}$

They are CBD sets. Set $T_i = \bigcup_{s>0} S_s^i$, which is D_{Σ} . It is obvious that $T_i = \{x \in \pi(X_i) \mid (X_i)_x \text{ contains an open interval}\}$. Since X is a closed box, we have $\pi(X) = T_1 \cup T_2$ by Lemma 4.4 for n = 1. At least one of T_1 and T_2 has a nonempty interior by the induction hypothesis. We may assume that $\operatorname{int}(T_1) \neq \emptyset$ without loss of generality. We have $\operatorname{int}(S_s^1)$ for some s > 0 by Lemma 4.3 for n - 1. The CBD set

 X_1 has a non-empty interior by Lemma 3.3. We have finished the proof of Lemma 4.4.

We finally demonstrate that a definable map whose graph is D_{Σ} is continuous on a dense set.

Lemma 4.5. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Consider a definable map $f: U \to \mathbb{R}^n$ defined on an open set U whose graph is a D_{Σ} -set. There exists a nonempty open box V contained in U such that the restriction of f to V is continuous.

Proof. We can prove the lemma in the same way as [6, Lemma 5.1]. We use Lemma 4.3 instead of [6, Lemma 3.4]. We omit the proof. \Box

5. Uniformly locally o-minimal open core

We demonstrate Theorem 1.1 in this section. The following lemma claims that all D_{Σ} -subsets of R are constructible.

Lemma 5.1. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Consider a definable subset X of R. The followings are equivalent:

- (1) The set X is a D_{Σ} -set.
- (2) For any $x \in R$, there exists an open interval I such that $x \in I$ and $X \cap I$ is a finite union of points and open intervals.
- (3) The set X is constructible.

Proof. (1) \Rightarrow (2): The difference $X \setminus \operatorname{int}(X)$ is D_{Σ} by Lemma 3.2. It is locally finite by Lemma 4.3. For any $x \in R$, the intersection $\operatorname{int}(X) \cap I$ is a finite union of open intervals for some open interval I with $x \in I$ by Lemma 4.2. We get the assertion (2).

 $(2) \Rightarrow (3)$: The difference $X \setminus int(X)$ is locally finite by the assertion (2). It means that X is constructible.

 $(3) \Rightarrow (1)$: Immediate from Lemma 3.2 (3).

We next demonstrate that all D_{Σ} -sets satisfy a condition satisfied by the sets definable in a uniformly locally o-minimal structure of the second kind when $\mathcal{R} \models DC$, LUF₂.

Lemma 5.2. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Consider a D_{Σ} -subset X of \mathbb{R}^{m+1} . For any $a \in \mathbb{R}$ and $b \in \mathbb{R}^m$, there exist positive integers N_1 , N_2 , an open interval I with $a \in I$ and an open box B with $b \in B$ such that $X_x \cap I$ is the union of at most N_1 points and N_2 open intervals for any $x \in B$.

Furthermore, we can take $B = R^m$ if $\mathcal{R} \models LUF_1$. We can take I independently of $X \subset R^{m+1}$ if $\mathcal{R} \models SLUF$.

Proof. The difference $X_x \setminus \operatorname{int}(X_x)$ is discrete and closed by Lemma 5.1 for all $x \in \mathbb{R}^m$. There exist a positive integer N_1 , an open box B and a closed interval I_1 with $a \in \operatorname{int}(I_1)$ and $|(X_x \setminus \operatorname{int}(X_x)) \cap I_1| \leq N_1$ for all $x \in B$ by Lemma 4.2. There exist a positive integer N_2 and an open interval I_2 with $a \in I_2$ such that $\operatorname{int}(X_x) \cap I_2$ consist of at most N_2 open intervals for all $x \in B$ also by Lemma 4.2. Set $I = \operatorname{int}(I_1) \cap I_2$, then I satisfies the conditions in the lemma. The 'furthermore' part is obvious by Lemma 4.2.

We investigate D_{Σ} -sets of dimension zero.

Lemma 5.3. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. A D_{Σ} -set of dimension zero is discrete and closed. In particular, it is constructible.

Proof. Consider a D_{Σ} subset X of \mathbb{R}^n of dimension zero. Let π_i be the projections onto the *i*-th coordinate for all $1 \leq i \leq n$. Let $x \in \mathbb{R}^n$ be an arbitrary point. Take a sufficiently small open box B with $x \in B$. The projection images $\pi_i(X \cap B)$ have empty interiors for all *i* because dim_x $X \leq 0$. By Lemma 5.1, we may assume that $\pi_i(X \cap B)$ is empty or a singleton by shrinking B if necessary. It means that X is discrete and closed.

The following three lemmas are essential parts of the proof of our main theorem.

Lemma 5.4. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Consider a D_{Σ} -subset X of \mathbb{R}^{n+1} . Set

 $\mathcal{I}(X) = \{ x \in \mathbb{R}^n \mid \text{the fiber } X_x \text{ contains an open interval} \}.$

It is a D_{Σ} -set and we have proj. dim $(\mathcal{I}(X)) < \text{proj. dim}(X)$.

Proof. Let $\{X_{r,s}\}_{r>0,s>0}$ be a D_{Σ} -family with $X = \bigcup_{r>0,s>0} X_{r,s}$. Set $Y_{r,s} = \{x \in \mathbb{R}^n \mid \exists t \in \mathbb{R}, [t-s,t+s] \subset (X_{r,s})_x\}$. The set $Y_{r,s}$ is CBD. We show that $\mathcal{I}(X) = \bigcup_{r,s} Y_{r,s}$. It is obvious that $\bigcup_{r,s} Y_{r,s} \subset \mathcal{I}(X)$. We demonstrate the opposite inclusion. Take an arbitrary point $x \in \mathcal{I}(X)$. We have $\operatorname{int}(X_x) \neq \emptyset$. We get $\operatorname{int}((X_{r,s})_x) \neq \emptyset$ for some r > 0 and s > 0 by Lemma 4.3. There exist $t \in \mathbb{R}$ and s > 0 with $[t-s,t+s] \subset (X_{r,s})_x$. It means that $x \in Y_{r,s}$. We have demonstrated that $\mathcal{I}(X) = \bigcup_{r,s} Y_{r,s}$. In particular, $\mathcal{I}(X)$ is a D_{Σ} -set.

We next demonstrate that proj. $\dim(\mathcal{I}(X)) < \operatorname{proj.} \dim(X)$. It is obvious when $\operatorname{int}(X) \neq \emptyset$ because proj. $\operatorname{dim}(\mathcal{I}(X)) \leq n$ and proj. $\operatorname{dim}(X) = n+1$ by the definition. We consider the case in which $int(X) = \emptyset$. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the coordinate projection forgetting the last coordinate. We have proj. dim $\mathcal{I}(X) \leq$ proj. dim $\pi(X) \leq \text{proj. dim } X$ because $\mathcal{I}(X) \subset \pi(X)$. We lead to a contradiction assuming that proj. dim $\mathcal{I}(X) = \text{proj. dim } X$. Set $d = \text{proj. dim } X = \text{proj. dim } \mathcal{I}(X)$. Take a coordinate projection $\pi_1 : \mathbb{R}^n \to \mathbb{R}^d$ with $\operatorname{int}(\pi_1(\mathcal{I}(X))) \neq \emptyset$. The co-ordinate projection $\pi_2 : \mathbb{R}^{n+1} \to \mathbb{R}^d$ is the composition of π_1 with π . We have $\operatorname{int}(\pi_2(X)) \neq \emptyset$ because $\pi_1(\mathcal{I}(X)) \subset \pi_2(X)$. The notation $\pi_3 : \mathbb{R}^{n+1} \to \mathbb{R}$ denotes the coordinate projection onto the last coordinate. The coordinate projection $\Pi = (\pi_2, \pi_3) : \mathbb{R}^{n+1} \to \mathbb{R}^{d+1}$ is given by $\Pi(x) = (\pi_2(x), \pi_3(x))$. Consider the set $T = \{x \in \mathbb{R}^d \mid \Pi(X)_x \text{ contains an open interval}\}$. We have $\pi_1(\mathcal{I}(X)) \subset T$. In fact, take $x \in \mathcal{I}(X)$ and open interval $J \subset X_x$. The set $\pi_1(x) \times J$ is contained in $\Pi(X)_x$. It means that $\pi_1(x) \in T$. We get $int(T) \neq \emptyset$ because $\pi_1(\mathcal{I}(X))$ has a nonempty interior. Set $T_{r,s} = \{x \in \mathbb{R}^d \mid \exists t \in \mathbb{R}, [t-s,t+s] \subset (\Pi(X_{r,s}))_x\}$. The set T is D_{Σ} and $T = \bigcup_{r,s} T_{r,s}$ as demonstrated previously. We have $int(T_{r,s}) \neq \emptyset$ for some r > 0and s > 0 by Lemma 4.3. We get $int(\Pi(X_{r,s})) \neq \emptyset$ by Lemma 3.3 and we obtain $\operatorname{int}(\Pi(X)) \neq \emptyset$. It is a contradiction to the assumption that proj. dim X = d. \square

Lemma 5.5. Let $\mathcal{R} = (R, <, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC$, LUF_2 . Let X be a D_{Σ} -subset of \mathbb{R}^n of proj. dim(X) =d. Take a coordinate projection $\pi : X \to \mathbb{R}^d$ such that $\pi(X)$ has a nonempty interior. Then, there exists a D_{Σ} -subset Z of \mathbb{R}^d such that Z has an empty interior and the fiber $X \cap \pi^{-1}(x)$ is locally finite for any $x \in \mathbb{R}^d \setminus Z$. Proof. For all $1 \leq i \leq n-d$, we can take coordinate projections $\pi_i : \mathbb{R}^{n-i+1} \to \mathbb{R}^{n-i}$ with $\pi = \pi_{n-d} \circ \cdots \circ \pi_1$. We may assume that π_i are the coordinate projections forgetting the last coordinate without loss of generality. Set $\Pi_i = \pi_i \circ \cdots \circ \pi_1$ and $\Phi_i = \pi_{n-d} \circ \cdots \circ \pi_{i+1}$. Consider the sets $T_i = \{x \in \mathbb{R}^{n-i} \mid \pi_i^{-1}(x) \cap \Pi_{i-1}(X) \text{ contains an open interval}\}$. The sets T_i are D_{Σ} and we have proj. dim $(T_i) <$ proj. dim $\Pi_{i-1}(X) =$ proj. dim X = d by Lemma 5.4. Set $U_i = \Phi_i(T_i) \subset \mathbb{R}^d$ for all $1 \leq i \leq n-d$. The projection images U_i are D_{Σ} -sets by Lemma 3.2(1). We get $\operatorname{int}(U_i) = \emptyset$ because proj. dim $(T_i) < d$. Set $Z = \bigcup_{i=1}^{n-d} U_i$. It also has an empty interior by Lemma 4.4.

The fiber $X \cap \pi^{-1}(x)$ is locally finite for any $x \in \mathbb{R}^d \setminus Z$. In fact, let $y \in \mathbb{R}^n$ be an arbitrary point with $x = \pi(y)$. Set $y_0 = y$ and $y_i = \prod_i(y)$ for $1 \leq i \leq n - d$. We have $y_{n-d} = x$ by the definition. We construct an open box B_i in \mathbb{R}^{n-d-i} for $0 \leq i \leq n-d$ such that $y_i \in B_i$ and $(\{x\} \times B_i) \cap \prod_i(X)$ consists of at most one point in decreasing order. When i = n - d, the open box $B_{n-d} = \mathbb{R}^0$. When $(\{x\} \times B_i) \cap \prod_i(X) = \emptyset$, set $B_{i-1} = B_i \times \mathbb{R}$. We have $(\{x\} \times B_{i-1}) \cap \prod_{i-1}(X) = \emptyset$. When $(\{x\} \times B_i) \cap \prod_i(X) \neq \emptyset$, the fiber $\prod_{i=1}(X) \cap \pi_i^{-1}(y_i)$ is locally finite by Lemma 5.1. Therefore, there exists an open box B_{i-1} in $\mathbb{R}^{n-d+1-i}$ such that $\pi_i(B_{i-1}) = B_i$, $y_{i-1} \in B_{i-1}$ and $(\{x\} \times B_{i-1}) \cap \prod_{i=1}(X)$ consists of at most one point. We have constructed the open boxes B_i in \mathbb{R}^{n-d-i} for all $0 \leq i \leq n-d$. The existence of B_0 implies that $X \cap \pi^{-1}(x)$ is locally finite. \square

Lemma 5.6. Let $\mathcal{R} = (R, <, ...)$ be a densely linearly ordered structure. Let X be a definable subset of \mathbb{R}^n of dimension d < n. Consider the set

 $\mathcal{G}(X) = \{x \in X \mid \text{there exist a coordinate projection } \pi : R^n \to R^d$

and an open box B with $x \in B$ such that $X \cap B$ is the graph

of a continuous map defined on $\pi(B)$.

It is definable and constructible. Furthermore, we have $\dim(X \setminus \mathcal{G}(X)) < d$ if the following conditions are all satisfied:

- $\mathcal{R} = (R, <, +, 0, ...)$ is an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$.;
- X is a D_{Σ} set;
- Any D_{Σ} set of dimension smaller than d is constructible.

Proof. It is obvious that the set $\mathcal{G}(X)$ is definable. We show that $\mathcal{G}(X)$ is constructible. Fix an arbitrary coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$. Consider the set $\mathcal{G}(X)_{\pi}$ of points $x \in \mathbb{R}^n$ such that $X \cap B$ is the graph of a continuous map defined on $\pi(B)$ for some open box B containing the point x. The set $\mathcal{G}(X)_{\pi}$ is locally closed because $\mathcal{G}(X)_{\pi}$ is locally the graph of a continuous map. Therefore, it is constructible. The set $\mathcal{G}(X)$ is also constructible because we have $\mathcal{G}(X) = \bigcup_{\pi} \mathcal{G}(X)_{\pi}$.

The difference $X \setminus \mathcal{G}(X)$ is D_{Σ} by Lemma 3.2. We next demonstrate that dim $(X \setminus \mathcal{G}(X)) < d$ under the given conditions. When d = 0, $X = \mathcal{G}(X)$ by Lemma 5.3. The lemma is obvious.

Consider the case in which d > 0. Note that we always have $\mathcal{G}(X) \cap U = \mathcal{G}(X \cap U)$ for any definable open subset U of \mathbb{R}^n by the definition of $\mathcal{G}(X)$. We also have $U \cap (X \setminus \mathcal{G}(X)) = (X \cap U) \setminus \mathcal{G}(X \cap U)$. We use this fact without mentioning. Set $Y = X \setminus \mathcal{G}(X)$. We lead to a contradiction assuming that $\dim(Y) = d$. Take a point $y \in \mathbb{R}^n$ with $\dim_y Y = d$. We can take a coordinate projection $\pi : \mathbb{R}^n \to \mathbb{R}^d$ such that we have $\pi(Y \cap U)$ has a nonempty interior for any open box U containing y. We fix a sufficiently small open box B in \mathbb{R}^n with $y \in B$. Note that $\dim_y X = d$ because $d = \dim_y Y \leq \dim_y X \leq \dim X = d$. We can prove by an easy induction that there exists a positive integer N_1 with $|X \cap B \cap \pi^{-1}(x)| \leq N_1$ or $|X \cap B \cap \pi^{-1}(x)| = \infty$ for all $x \in \pi(X \cap B)$ using Lemma 5.2. We omit the proof. We also have $\dim(X \cap B) = \dim(Y \cap B) = \operatorname{proj.dim}(X \cap B) = \operatorname{proj.dim}(X \cap B) = d$ by Lemma 3.5. We may assume that

- $|X \cap \pi^{-1}(x)| \leq N_1$ or $|X \cap \pi^{-1}(x)| = \infty$ for all $x \in \pi(X)$ and
- $d = \dim(X) = \dim(Y) = \operatorname{proj.dim}(X) = \operatorname{proj.dim}(Y)$

considering $X \cap B$ instead of X. We can take a D_{Σ} -subset Z of \mathbb{R}^d such that Z has an empty interior and $\pi^{-1}(x) \cap X$ is locally finite for any $x \in \mathbb{R}^d \setminus Z$ by Lemma 5.5. Since dim(Z) < d, Z is a constructible set by the assumption. Therefore, the sets $\pi(Y) \setminus Z$ is a D_{Σ} -set by Lemma 3.2(3). The set $\pi(Y) \setminus Z$ has a nonempty interior by Lemma 4.4. We may further assume that

- $\pi(Y)$ has a nonempty interior and
- $|X \cap \pi^{-1}(x)| \le N_1$ for all $x \in \pi(X)$

considering $X \cap \pi^{-1}(B')$ instead of X, where B' is an open box contained in $\pi(Y) \setminus Z$. We can reduce to the case in which there exists a positive integer N with $|Y \cap \pi^{-1}(x)| = N$ for all $x \in \pi(Y)$. We need the following claim:

Claim. There exists a positive integer N with $N \leq N_1$ such that the set $E = \{x \in \mathbb{R}^d \mid |\pi^{-1}(x) \cap Y| = N\}$ is D_{Σ} and $int(E) \neq \emptyset$.

We begin to prove the claim. For all $1 \leq i \leq N_1$, consider the sets $C_i = \{x \in \mathbb{R}^d \mid |\pi^{-1}(x) \cap Y| = i\}$ and $D_i = \{x \in \mathbb{R}^d \mid |\pi^{-1}(x) \cap Y| \geq i\}$. The set D_i is D_{Σ} . In fact, D_i is the projection image of the D_{Σ} -set $\{(x_1, \ldots, x_i) \in (\mathbb{R}^n)^i \mid \pi(x_1) = \cdots = \pi(x_i), x_j \neq x_k$ for all $j \neq k, x_j \in Y$ for all j. We demonstrate the claim by the induction on N_1 . When $N_1 = 1$, we have nothing to prove. We have N = 1 and $E = \pi(Y)$. When $N_1 > 1$, the set $C_{N_1} = D_{N_1}$ is D_{Σ} . If $\operatorname{int}(C_{N_1}) \neq \emptyset$, set $N = N_1$ and $E = C_{N_1}$. Otherwise, we have dim $C_{N_1} < d$. The definable set C_{N_1} is constructible by the assumption. By the induction hypothesis, we can get $N < N_1$ such that $E = \{x \in \mathbb{R}^d \mid |\pi^{-1}(x) \cap (Y \setminus (C_{N_1} \times \mathbb{R}^{n-d}))| = N\}$ is D_{Σ} and $\operatorname{int}(E) \neq \emptyset$. It is obvious that $E = \{x \in \mathbb{R}^d \mid |\pi^{-1}(x) \cap Y| = N\}$. We have proven the claim.

Let E be the D_{Σ} -set in the claim and take an open box B'' contained in E. Set $X' = X \cap \pi^{-1}(B'')$. We may assume that $|Y \cap \pi^{-1}(x)| = N$ for all $x \in \pi(X)$ and $\pi(X)$ is an open box by considering X' in place of X. Applying the same argument to the new X, we can reduce to the following case:

- We have $\pi(Y) = \pi(X) = V$ for some open box V in \mathbb{R}^d ;
- There exist positive integers N and N' such that $|Y \cap \pi^{-1}(x)| = N$ and $|X \cap \pi^{-1}(x)| = N'$ for any $x \in V$.

We demonstrate that the closure of $\mathcal{G}(X)$ has an empty intersection with Y. Let y be a point in the intersection. Let $\{y_1, \ldots, y_M\}$ be the fiber $\mathcal{G}(X) \cap \pi^{-1}(\pi(y))$, where M = N' - N. We have $y \neq y_i$ for all $1 \leq i \leq M$ and $y_i \neq y_j$ for $i \neq j$. Since $\mathcal{G}(X)$ is locally the graph of a continuous function, there are $y'_1, \ldots, y'_M \in \mathcal{G}(X)$ such that $\pi(y'_1) = \cdots = \pi(y'_M)$ and y'_i are sufficiently closed to y_i for $1 \leq i \leq M$. Since y is a point of the closure, there exists $y' \in \mathcal{G}(X)$ sufficiently close to y with $\pi(y') = \pi(y'_1)$. The fiber $\mathcal{G}(X) \cap \pi^{-1}(\pi(y'))$ contains the (M+1) points y'_1, \ldots, y_M and y'. Contradiction to the fact that $|\mathcal{G}(X) \cap \pi^{-1}(\pi(y'))| = M$. $Y_i = \{x \in \mathbb{R}^n \mid x \text{ is the } i\text{-th minimum in } Y \cap \pi^{-1}(\pi(x)) \text{ in the lexicographic order} \}$ for all $1 \leq i \leq N$. Consider the D_{Σ} -set $Z = \{(x_1, \ldots, x_N) \in (\mathbb{R}^n)^N \mid \pi(x_1) = \cdots = \pi(x_N), x_1 < x_2 < \cdots < x_N \text{ in the lexicographic order, } x_i \in X \text{ for all } 1 \leq i \leq N \}.$ The definable set Y_i is the projection image of the D_{Σ} -set Z, and it is also D_{Σ} by Lemma 3.2(1). The projection image $\pi(Y_i)$ is an open box because $\pi(Y_i) = \pi(Y) = V$. Consequently, Y_i is simultaneously a D_{Σ} -set and the graph of a definable map defined on an open box for any $1 \leq i \leq N$. Applying Lemma 4.5 iteratively to Y_i , we can find a nonempty open box W such that $Y_i \cap \pi^{-1}(W)$ are the graphs of definable continuous maps defined on W for all $1 \leq i \leq N$. Since the closure of $\mathcal{G}(X)$ has an empty intersection with Y, we have $\widetilde{B} \cap Y = \widetilde{B} \cap X$ for any point $y \in Y$ and any sufficiently small open box \widetilde{B} containing the point y. We have $Y \cap \pi^{-1}(W) \subset \mathcal{G}(X)$. Contradiction to the definition of Y.

We finally get the following theorem.

Theorem 5.7. Let $\mathcal{R} = (R, <, +, 0, ...)$ be an expansion of a densely linearly ordered abelian group with $\mathcal{R} \models DC, LUF_2$. Any D_{Σ} -set is constructible. In particular, any set definable in the open core of \mathcal{R} is constructible.

Proof. Let X be a D_{Σ} -subset of \mathbb{R}^n of dimension d. We show that X is constructible by the induction on d. When d = 0, it is clear from Lemma 5.3. When d > 0, consider the constructible set $\mathcal{G}(X)$ defined in Lemma 5.6. The difference $X \setminus \mathcal{G}(X)$ is a D_{Σ} -set of dimension smaller than d by Lemma 5.6. It is constructible by the induction hypothesis. Consequently, X is also constructible.

It is obvious that any set definable in the open core of \mathcal{R} is constructible because the projection image of a constructible set is again constructible by the assertion we have just proven and Lemma 3.2(1).

Proof of Theorem 1.1. Theorem 1.1 is now obvious by Lemma 3.2(3), Lemma 5.2 and Theorem 5.7. \Box

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