

PATHOLOGICAL EXAMPLES OF STRUCTURES WITH O-MINIMAL OPEN CORE

ALEXI BLOCK GORMAN, ERIN CAULFIELD, AND PHILIPP HIERONYMI

ABSTRACT. This paper answers several open questions around structures with o-minimal open core. We construct an expansion of an o-minimal structure \mathcal{R} by a unary predicate such that its open core is a proper o-minimal expansion of \mathcal{R} . We give an example of a structure that has an o-minimal open core and the exchange property, yet defines a function whose graph is dense. Finally, we produce an example of a structure that has an o-minimal open core and definable Skolem functions, but is not o-minimal.

1. INTRODUCTION

Introduced by Miller and Speissegger [5] the notion of an open core has become a mainstay of the model-theoretic study of ordered structures. However, there are still many rather basic questions, in particular about structures with o-minimal open cores, that have remained unanswered. In this paper, we are able to settle some of the questions raised by Dolich, Miller and Steinhorn [2, 3].

Throughout this paper, \mathcal{R} denotes a fixed, but arbitrary expansion of a dense linear order $(R, <)$ without endpoints. We now recall several definitions from the aforementioned papers. We denote by \mathcal{R}° the structure $(R, (U))$, where U ranges over the open sets of all arities definable in \mathcal{R} , and call this structure **the open core of \mathcal{R}** .

Given two structures \mathcal{S}_1 and \mathcal{S}_2 with the same universe S , we say \mathcal{S}_1 and \mathcal{S}_2 are **interdefinable** (short: $\mathcal{S}_1 =_{\text{df}} \mathcal{S}_2$) if \mathcal{S}_1 and \mathcal{S}_2 define the same sets. For a given theory T extending the theory of dense linear orders, we say that a theory T' is an **open core of T** if for every $\mathcal{M} \models T$ there exists $\mathcal{M}' \models T'$ such that $\mathcal{M}^\circ =_{\text{df}} \mathcal{M}'$.

Question 1 ([2, p. 1408]). *If $S \subseteq R$ and $(\mathcal{R}, S)^\circ$ is o-minimal, is $(\mathcal{R}, S)^\circ =_{\text{df}} \mathcal{R}^\circ$?*

We give a negative answer to this question by constructing an expansion of the real field by a single unary predicate whose open core is o-minimal, but defines an irrational power function. It is clear from the construction in Section 2 that there are similar examples of expansions of the real ordered additive group that do not define an ordered field.

We say \mathcal{R} is **definably complete** (short: $\mathcal{R} \models \text{DC}$) if every definable unary set has both a supremum and an infimum in $R \cup \{\pm\infty\}$. We denote by $\text{dcl}_{\mathcal{R}}$ the definable closure operator in \mathcal{R} , and often drop the subscript \mathcal{R} . We say that \mathcal{R} has the

Date: January 8, 2021.

This is a preprint version. Later versions might contain significant changes.

exchange property (short: $\mathcal{R} \models \text{EP}$) if $b \in \text{dcl}(S \cup \{a\})$ for all $S \subseteq R$ and $a, b \in R$ such that $a \in \text{dcl}(S \cup \{b\}) \setminus \text{dcl}(S)$. For a theory T , we say T has the exchange property (short: $T \models \text{EP}$) if every model of T has the exchange property. We write $\mathcal{R} \models \text{NIP}$ if its theory does not have the independence property as introduced by Shelah [6]. We refer the reader to Simon [8] for a modern treatment of NIP and related model-theoretic tameness notions.

Question 2 ([2, p. 1409]). *If $\mathcal{R} \models \text{DC} + \text{EP} + \text{NIP}$ and expands an ordered group, is \mathcal{R} o-minimal?*

By [2, p. 1374] we know that \mathcal{R} has o-minimal open core if $\mathcal{R} \models \text{DC} + \text{EP}$ and expands an ordered group. Thus Question 2 asks whether there is a combinatorial model-theoretic tameness condition that can be added to force o-minimality. Again, we give a negative answer to this question. We construct a counterexample as follows: Let $\mathbb{Q}(t)$ be the field of rational functions in a single variable t . We consider an expansion \mathcal{R}_t of the ordered real additive group $(\mathbb{R}, <, +, 0, 1)$ into a $\mathbb{Q}(t)$ -vector space such that for all $c \in \mathbb{Q}(t) \setminus \mathbb{Q}$, the graph of the function $x \mapsto cx$ is dense in \mathbb{R}^2 . We show that $\mathcal{R}_t \models \text{DC} + \text{EP} + \text{NIP}$, but is not o-minimal.

The structure \mathcal{R}_t has infinite dp-rank. By Simon [7], if $\mathcal{R} \models \text{DC}$, expands an ordered group and has dp-rank 1, then \mathcal{R} is o-minimal. However, we do not know whether \mathcal{R} is o-minimal if $\mathcal{R} \models \text{DC} + \text{EC}$, expands an ordered group and has finite dp-rank.

In addition to showing that $\mathcal{R}_t \models \text{DC} + \text{EP} + \text{NIP}$, we prove that its open core is interdefinable with its o-minimal reduct $(\mathbb{R}, <, +, 0, 1)$. Since the graph of $x \mapsto cx$ is dense for $c \in \mathbb{Q}(t) \setminus \mathbb{Q}$, the theory of \mathcal{R}_t provides a negative answer to the following question.

Question 3 ([3, p. 705]). *Let T be a complete o-minimal extension of the theory of densely ordered groups. If \tilde{T} is any theory (in any language) having T as an open core, and some model of \tilde{T} defines a somewhere dense graph, must EP fail for \tilde{T} ?*

Our counterexample \mathcal{R}_t does not expand a field and we don't know whether Question 2 (or Question 3) has a positive answer if we require \mathcal{R} (or T) to expand an ordered field.

We say that \mathcal{R} has **definable Skolem functions** if for every definable set $A \subseteq R^m \times R^n$ there is a definable function $f : R^m \rightarrow R^n$ such that $(a, f(a)) \in A$ whenever $a \in A$ and there exists $b \in R^n$ with $(a, b) \in A$. Every o-minimal expansion of an ordered group with a distinguished positive element has definable Skolem functions, but all documented examples of non-o-minimal structures with o-minimal core do not.

Question 4 ([2, p. 1409]). *If \mathcal{R} has definable Skolem functions and \mathcal{R}° is o-minimal, is \mathcal{R} o-minimal?*

The answer is again negative. We say \mathcal{R} satisfies **uniform finiteness** (short: $\mathcal{R} \models \text{UF}$) if for every $m, n \in \mathbb{N}$ and every $A \subseteq R^{m+n}$ definable in \mathcal{R} there exists $N \in \mathbb{N}$ such that for every $a \in R^m$ the set $\{b \in R^n : (a, b) \in A\}$ is either infinite or contains at most N elements. By [2, Theorem A], if $\mathcal{R} \models \text{DC} + \text{UF}$ and expands an ordered group, then \mathcal{R}° is o-minimal. Using a construction due to Winkler [9] and following

a strategy of Kruckman and Ramsey [4], we establish that if $\mathcal{R} \models \text{DC} + \text{UF}$, then \mathcal{R} has an expansion \mathcal{S} such that \mathcal{S} has definable Skolem functions and satisfies $\mathcal{S}^\circ =_{\text{df}} \mathcal{R}^\circ$. Thus if \mathcal{R} also expands an ordered group, then \mathcal{R}° is o-minimal and so is \mathcal{S}° .

Acknowledgements. A.B.G. was supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE – 1746047. P.H. was partially supported by NSF grant DMS-1654725. We thank Pantelis Eleftheriou for helpful conversations on the topic of this paper, in particular for bringing Question 1 to the attention of P.H. at a conference at the Bedlewo Conference Center. We thank Chris Miller for helpful comments on a draft of this paper.

Notation. We will use m, n for natural numbers and κ for a cardinal. Let \mathcal{L} be a language and T an \mathcal{L} -theory. Let $\mathcal{M} \models T$. We follow the usual convention to denote the universe of \mathcal{M} by M . In this situation, \mathcal{L} -definable means \mathcal{L} -definable with parameters. Let b be a tuple of elements in M , and let $A \subseteq M$. We write $\text{tp}_{\mathcal{L}}(b|A)$ for the \mathcal{L} -type of b over A . If \mathcal{N} is another model of T and h is an embedding of a substructure of \mathcal{M} containing A into \mathcal{N} , then $h \text{tp}(b|A)$ is the type containing all formulas of the form $\varphi(x, h(a))$ where $\varphi(x, a) \in \text{tp}(b|A)$.

2. QUESTION 1

Let $\overline{\mathbb{R}}$ be the real ordered field $(\mathbb{R}, <, +, \cdot)$ and let \mathbb{R}_{exp} be the expansion of the real field by the exponential function exp . Let $I \subseteq \mathbb{R}$ be a dense $\text{dcl}_{\mathbb{R}_{\text{exp}}}$ -independent set. Let $\tau \in I$ be such that $\tau > 1$. Set

$$J := \bigcup_{a \in I \setminus \{\tau\}} \{|a|, |a|^\tau, |a| + |a|^\tau\}.$$

By [3, 2.25] the open core of $(\mathbb{R}_{\text{exp}}, I)^\circ$ is interdefinable with \mathbb{R}_{exp} . Since $(\overline{\mathbb{R}}, J)$ is a reduct of $(\mathbb{R}_{\text{exp}}, I)$, we have that $(\overline{\mathbb{R}}, J)^\circ$ is a reduct of $(\mathbb{R}_{\text{exp}}, I)^\circ$. As the latter structure is o-minimal, we have that $(\overline{\mathbb{R}}, J)^\circ$ is o-minimal as well. In order to show that Question 1 has a negative answer, it is left to show that $(\overline{\mathbb{R}}, J)^\circ$ defines a set not definable in $\overline{\mathbb{R}}$. Since $\overline{\mathbb{R}}$ only defines raising to rational powers, it suffices to prove the definability of $x \mapsto x^\tau$ on an unbounded interval.

Lemma 2.1. *Let $u_1, u_2, u_3 \in J$ such that $1 < u_1 < u_2$ and $u_1 + u_2 = u_3$. Then there is a $a \in I \setminus \{\tau\}$ such that $u_1 = |a|$ and $u_2 = |a|^\tau$.*

Proof. For $a \in I \setminus \{\tau\}$ observe that $|a|, |a|^\tau$ and $|a| + |a|^\tau$ are interdefinable in \mathbb{R}_{exp} over τ . Since $u_1 + u_2 = u_3$, we have u_1, u_2, u_3 are $\text{dcl}_{\mathbb{R}_{\text{exp}}}$ -dependent. Because I is $\text{dcl}_{\mathbb{R}_{\text{exp}}}$ -independent, there are $a \in I \setminus \{\tau\}$ and $i, j \in \{1, 2, 3\}$ such that

$$u_i, u_j \in \{|a|, |a|^\tau, |a| + |a|^\tau\}.$$

Let $\ell \in \{1, 2, 3\}$ such that $\ell \neq i$ and $\ell \neq j$. Note u_ℓ is $\text{dcl}_{\mathbb{R}_{\text{exp}}}$ -dependent over u_i and u_j . Thus $u_\ell \in \{|a|, |a|^\tau, |a| + |a|^\tau\}$. Since $|a| > 0$, we obtain from $u_1 + u_2 = u_3$ that

$$u_1, u_2 \in \{|a|, |a|^\tau\}.$$

Since $1 < u_1 < u_2$ and $\tau > 0$, we have that $u_1 = |a|$ and $u_2 = |a|^\tau$. □

Proposition 2.2. *The graph of $x \mapsto x^\tau$ on $\mathbb{R}_{\geq 1}$ is definable in $(\overline{\mathbb{R}}, J)^\circ$.*

Proof. By Lemma 2.1 the structure $(\overline{\mathbb{R}}, J)$ defines

$$\{(|a|, |a|^\tau) : |a| > 1, a \in I \setminus \{\tau\}\}.$$

Since I is dense in \mathbb{R} , the closure of this set is the graph of $x \mapsto x^\tau$ on $\mathbb{R}_{\geq 1}$, and hence definable in $(\overline{\mathbb{R}}, J)^\circ$. \square

We conclude that $(\overline{\mathbb{R}}, J)^\circ$ is a proper expansion of $\overline{\mathbb{R}}$.

3. QUESTIONS 2 AND 3

In this section we give negative answers to Questions 2 and 3. Let $\mathbb{Q}(t)$ be the field of rational functions in the variable t . We expand $(\mathbb{R}, <, +, 0, 1)$ to a $\mathbb{Q}(t)$ -vector space such that for each non-constant $q(t) \in \mathbb{Q}(t)$ the graph of multiplication by $q(t)$ is dense.

We now construct such a $\mathbb{Q}(t)$ -vector space structure on \mathbb{R} . Let $\mathbf{1}$ be the multiplicative identity of $\mathbb{Q}(t)$. We fix a dense basis \mathcal{B} of \mathbb{R} as a \mathbb{Q} -vector space, and a basis I of $\mathbb{Q}(t)$ as a \mathbb{Q} -vector space such that $\mathbf{1} \in I$. We choose a sequence of functions $\{\widetilde{f}_\gamma : I \rightarrow \mathcal{B}\}_{\gamma \in 2^{\aleph_0}}$ such that

$$\mathcal{B} = \bigcup_{\gamma \in 2^{\aleph_0}} \widetilde{f}_\gamma(I)$$

and for all $\gamma \in 2^{\aleph_0}$:

- \widetilde{f}_γ is injective,
- for all $\eta \in 2^{\aleph_0}$ with $\eta \neq \gamma$, $\widetilde{f}_\eta(I) \cap \widetilde{f}_\gamma(I) = \emptyset$,
- for all open intervals $J_1, \dots, J_m \subseteq \mathbb{R}$ open intervals and all pairwise distinct $p_1(t), \dots, p_m(t) \in I$ there exists $\gamma \in 2^{\aleph_0}$ such that

$$\widetilde{f}_\gamma(p_1(t)) \in J_1, \dots, \widetilde{f}_\gamma(p_m(t)) \in J_m.$$

Since the order topology on the real line has a countable base, it is easy to see that such a sequence of functions exists. For each $\gamma \in 2^{\aleph_0}$, \widetilde{f}_γ is defined on the basis I of $\mathbb{Q}(t)$. Therefore, we can extend each $\widetilde{f}_\gamma : I \rightarrow \mathcal{B}$ to a \mathbb{Q} -linear map $f_\gamma : \mathbb{Q}(t) \rightarrow \mathbb{R}$.

Lemma 3.1. *Let $a \in \mathbb{R}$. Then there are unique $\gamma_1, \dots, \gamma_n \in 2^{\aleph_0}$ and $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ such that*

$$a = f_{\gamma_1}(p_1(t)) + \dots + f_{\gamma_n}(p_n(t)).$$

Proof. Since \mathcal{B} is a basis of \mathbb{R} as a \mathbb{Q} -vector space, there are unique $b_1, \dots, b_n \in \mathcal{B}$ and $u_1, \dots, u_n \in \mathbb{Q}$ such that $a = u_1 b_1 + \dots + u_n b_n$. By the above construction, there are unique $\gamma_1, \dots, \gamma_n \in 2^{\aleph_0}$ and $q_1(t), \dots, q_n(t) \in I$ such that $b_i = f_{\gamma_i}(q_i(t))$ for $i = 1, \dots, n$. Then by \mathbb{Q} -linearity of the f_{γ_i} 's

$$\begin{aligned} a &= u_1 b_1 + \dots + u_n b_n \\ &= u_1 f_{\gamma_1}(q_1(t)) + \dots + u_n f_{\gamma_n}(q_n(t)) \\ &= f_{\gamma_1}(u_1 q_1(t)) + \dots + f_{\gamma_n}(u_n q_n(t)). \end{aligned}$$

Set $p_i := u_i q_i(t)$ for $i = 1, \dots, n$. \square

We now introduce a \mathbb{Q} -linear map $\lambda : \mathbb{Q}(t) \times \mathbb{R} \rightarrow \mathbb{R}$. Let $q(t) \in \mathbb{Q}(t)$ and $a \in \mathbb{R}$. By Lemma 3.1 there are unique $\gamma_1, \dots, \gamma_n \in 2^{\aleph_0}$ and $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ such that

$$a = f_{\gamma_1}(p_1(t)) + \dots + f_{\gamma_n}(p_n(t)).$$

We define

$$\lambda(q(t), a) := f_{\gamma_1}(q(t) \cdot p_1(t)) + \cdots + f_{\gamma_n}(q(t) \cdot p_n(t)).$$

By Lemma 3.1, the function λ is well-defined. For $q(t) \in \mathbb{Q}(t)$, we write $\lambda_{q(t)}$ for the map taking $a \in \mathbb{R}$ to $\lambda(q(t), a)$.

Proposition 3.2. *The additive group $(\mathbb{R}, +)$ with λ as scalar multiplication is an $\mathbb{Q}(t)$ -vector space.*

Proof. We only verify the following vector spaces axioms: for all $a \in \mathbb{R}$ and for all $q_1(t), q_2(t) \in \mathbb{Q}(t)$.

$$\lambda_{q_1(t) \cdot q_2(t)}(a) = \lambda_{q_1(t)}(\lambda_{q_2(t)}(a)).$$

The other axioms can be checked using similar arguments and we leave this to the reader.

Let $a \in \mathbb{R}$ and let $q_1(t), q_2(t) \in \mathbb{Q}(t)$. By Lemma 3.1 there are unique $\gamma_1, \dots, \gamma_n \in 2^{\aleph_0}$ and $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ such that

$$a = f_{\gamma_1}(p_1(t)) + \cdots + f_{\gamma_n}(p_n(t)).$$

We obtain

$$\begin{aligned} \lambda_{q_1(t)}(\lambda_{q_2(t)}(a)) &= \lambda_{q_1(t)}(\lambda_{q_2(t)}(\sum_{i=1}^n f_{\gamma_i}(p_i(t)))) \\ &= \lambda_{q_1(t)}(\sum_{i=1}^n f_{\gamma_i}(q_2(t) \cdot p_i(t))) \\ &= \sum_{i=1}^n f_{\gamma_i}(q_1(t) \cdot (q_2(t) \cdot p_i(t))) \\ &= \lambda_{q_1(t) \cdot q_2(t)}(\sum_{i=1}^n f_{\gamma_i}(p_i(t))) \\ &= \lambda_{q_1(t) \cdot q_2(t)}(a). \end{aligned}$$

□

Let \mathcal{L} be the language of $(\mathbb{R}, <, +, 0, 1)$, and let T be its theory; that is the theory of ordered divisible abelian groups with a distinguished positive element. It is well-known that T has quantifier-elimination and is o-minimal. We will use various consequences of this fact throughout this section. Most noteworthy: when $\mathcal{M} \models T$, $X \subseteq M^n$ is \mathcal{L} -definable over A and there is $b = (b_1, \dots, b_n) \in X$ such that b_1, \dots, b_n are \mathbb{Q} -linearly independent over A , then X has interior.

Let $\mathcal{R}_t = (\mathbb{R}, <, +, 0, 1, (\lambda_{q(t)})_{q(t) \in \mathbb{Q}(t)})$ be the expansion of $(\mathbb{R}, <, +, 0, 1)$ by function symbols for $\lambda_{q(t)}$ where $q(t) \in \mathbb{Q}(t)$. We denote the language of \mathcal{R}_t by \mathcal{L}_t .

3.1. Density. Let $p = (p_1(t), \dots, p_n(t)) \in \mathbb{Q}(t)^n$. Let $\lambda_p : \mathbb{R} \rightarrow \mathbb{R}^n$ denote the function from \mathbb{R} that maps a to $(\lambda_{p_1(t)}(a), \dots, \lambda_{p_n(t)}(a))$. The main goal of this subsection is to show the density of the image of λ_p when the coordinates of p are \mathbb{Q} -linearly independent.

Lemma 3.3. *Let $p = (p_1(t), \dots, p_n(t)) \in I$ be such that $p_i(t) \neq p_j(t)$ for $i \neq j$. Then the image of λ_p is dense in \mathbb{R}^n .*

Proof. Let J_1, \dots, J_n be open intervals in \mathbb{R} . Since $p_1(t), \dots, p_n(t)$ are distinct elements of I , there exists $\gamma \in 2^{\mathbb{N}_0}$ such that

$$f_\gamma(p_1(t)) \in J_1, \dots, f_\gamma(p_n(t)) \in J_n.$$

For each $i \in \{1, \dots, n\}$, we have

$$\lambda_{p_i(t)}(f_\gamma(\mathbf{1})) = f_\gamma(p_i(t))$$

by definition of $\lambda_{p_i(t)}$. Therefore,

$$(\lambda_{p_1(t)}(f_\gamma(\mathbf{1})), \dots, \lambda_{p_n(t)}(f_\gamma(\mathbf{1}))) \in J_1 \times J_2 \times \dots \times J_n.$$

□

Proposition 3.4. *Let $q = (q_1(t), \dots, q_m(t)) \in \mathbb{Q}(t)^m$ be such that $q_1(t), \dots, q_m(t)$ are \mathbb{Q} -linearly independent. Then the image of λ_q is dense in \mathbb{R}^n .*

Proof. Let $n \in \mathbb{N}$, let $p_1(t), \dots, p_n(t) \in I$ be distinct non-constant, and let $A = (u_{i,j})_{i=1, \dots, m, j=0, \dots, n}$ be an $m \times (n+1)$ matrix with rational entries such that

$$q_1(t) = u_{1,0}\mathbf{1} + u_{1,1}p_1(t) + \dots + u_{1,n}p_n(t)$$

$$q_2(t) = u_{2,0}\mathbf{1} + u_{2,1}p_1(t) + \dots + u_{2,n}p_n(t)$$

$$\vdots$$

$$q_m(t) = u_{m,0}\mathbf{1} + u_{m,1}p_1(t) + \dots + u_{m,n}p_n(t).$$

By definition of $\lambda_{\mathbf{1}}, \lambda_{p_1(t)}, \dots, \lambda_{p_n(t)}$, we have for each $i \in \{1, \dots, m\}$

$$\lambda_{q_i(t)}(x) = u_{i,0}\lambda_{\mathbf{1}}(x) + u_{i,1}\lambda_{p_1(t)}(x) + \dots + u_{i,n}\lambda_{p_n(t)}(x).$$

Therefore,

$$A\lambda_{(\mathbf{1}, p_1(t), \dots, p_n(t))} = \lambda_{(q_1(t), \dots, q_m(t))}.$$

Since $q_1(t), \dots, q_m(t)$ are \mathbb{Q} -linearly independent, the matrix A has rank m . Thus multiplication by A is a surjective map from \mathbb{R}^n to \mathbb{R}^m . Since matrix multiplication is continuous and continuous surjections preserve density, the image of $A\lambda_{(\mathbf{1}, p_1(t), \dots, p_n(t))}$ is dense in \mathbb{R}^m by Lemma 3.3. □

3.2. Axiomatization and QE. In this subsection, we will find an axiomatization of \mathcal{R}_t and show that this theory has quantifier elimination. Indeed, we will prove that the following subtheory of the \mathcal{L}_t -theory of \mathcal{R}_t already has quantifier-elimination.

Definition 3.5. *Let T_t be the \mathcal{L}_t -theory extending T by axiom schemata stating that for every model $\mathcal{M} = (M, <, +, 0, 1, (\lambda_{q(t)})_{q(t) \in \mathbb{Q}(t)}) \models T_t$*

(T1) *$(M, +, 0, (\lambda_{q(t)})_{q(t) \in \mathbb{Q}(t)})$ is a $\mathbb{Q}(t)$ -vector space.*

(T2) *If $q_1(t), \dots, q_m(t) \in \mathbb{Q}(t)$ are \mathbb{Q} -linearly independent, then the image of the $\lambda_{(q_1(t), \dots, q_m(t))}$ is dense in M^m .*

By Proposition 3.2 and Proposition 3.4 we know that $\mathcal{R}_t \models T_t$. Let $\mathcal{M} \models T_t$. We observe that by (T1) the \mathcal{L}_t -substructures of \mathcal{M} are precisely the $\mathbb{Q}(t)$ -linear subspaces of \mathcal{M} containing 1.

Lemma 3.6. *Let $\mathcal{M} \models T_t$ and let A be an \mathcal{L}_t -substructure of \mathcal{M} . Let $b \in M \setminus A$ and let $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ be \mathbb{Q} -linearly independent. Then $\lambda_{p_1(t)}(b), \dots, \lambda_{p_n(t)}(b)$ are \mathbb{Q} -linearly independent over A .*

Proof. Since \mathcal{A} is a $\mathbb{Q}(t)$ -linear subspace of \mathcal{M} , we know that $\lambda_{q(t)}(b) \notin A$ for all non-zero $q(t) \in \mathbb{Q}(t)$. Let $u_1, \dots, u_n \in \mathbb{Q}$. Since \mathcal{M} is a $\mathbb{Q}(t)$ -vector space,

$$u_1 \lambda_{p_1(t)}(b) + \dots + u_n \lambda_{p_n(t)}(b) = \lambda_{u_1 p_1(t) + \dots + u_n p_n(t)}(b).$$

Because $p_1(t), \dots, p_n(t) \in \mathbb{Q}(t)$ are \mathbb{Q} -linearly independent and $b \notin A$, we get that

$$u_1 \lambda_{p_1(t)}(b) + \dots + u_n \lambda_{p_n(t)}(b) \in A \Rightarrow u_1 = \dots = u_n = 0.$$

Thus $\lambda_{p_1(t)}(b), \dots, \lambda_{p_n(t)}(b)$ are \mathbb{Q} -linearly independent of A . \square

Proposition 3.7. *The theory T_t has quantifier elimination.*

Proof. Let $\mathcal{M}, \mathcal{N} \models T_t$ be such that \mathcal{N} is $|\mathcal{M}|^+$ -saturated. Let $\mathcal{A} \subseteq \mathcal{M}$ be a substructure that embeds into \mathcal{N} via the embedding $h : \mathcal{A} \hookrightarrow \mathcal{N}$. Let $b \in M \setminus A$. To prove quantifier elimination of T_t , it is enough to show that the embedding h extends to an embedding of the \mathcal{L}_t -substructure generated by \mathcal{A} and b .

Consider the tuple $(\lambda_{p(t)}(b))_{p(t) \in I}$. We first find $c \in N$ such that

$$(1) \quad h \operatorname{tp}_{\mathcal{L}}((\lambda_{p(t)}(b))_{i \in I} | A) = \operatorname{tp}_{\mathcal{L}}((\lambda_{p(t)}(c))_{i \in I} | h(A))$$

By saturation of \mathcal{N} it is enough to show that for every

- pairwise distinct $p_1(t), \dots, p_n(t) \in I$,
- \mathcal{L} -formula $\psi(x, y)$ and $a \in A^{|y|}$ such that

$$\mathcal{M} \models \psi(\lambda_{p_1(t)}(b), \dots, \lambda_{p_n(t)}(b), a),$$

there is $c \in N \setminus h(A)$ such that

$$\mathcal{N} \models \psi(\lambda_{p_1(t)}(c), \dots, \lambda_{p_n(t)}(c), h(a)).$$

Because I is a \mathbb{Q} -linear basis of $\mathbb{Q}(t)$, the sequence $(\lambda_{p(t)}(b))_{i \in I}$ is \mathbb{Q} -linear independent over A by Lemma 3.6. Thus the set

$$\{d \in N^n : \mathcal{N} \models \psi(d, h(a))\}$$

has interior. The existence of c now follows from (T2) and saturation of \mathcal{N} .

Let $c \in N$ be such that (1) holds. Let \mathcal{X} be the \mathbb{Q} -linear subspace of \mathcal{M} generated by $(\lambda_{p(t)}(b))_{i \in I}$ and \mathcal{A} . Let \mathcal{Y} be the \mathbb{Q} -linear subspace of \mathcal{N} generated by $(\lambda_{p(t)}(c))_{i \in I}$ and $h(\mathcal{A})$. Observe that \mathcal{X} is the $\mathbb{Q}(t)$ -subspace of \mathcal{M} generated by b and \mathcal{A} , and \mathcal{Y} is the $\mathbb{Q}(t)$ -subspace of \mathcal{N} generated by c and $h(\mathcal{A})$. Hence \mathcal{X} and \mathcal{Y} are \mathcal{L}_t -structures of \mathcal{M} and \mathcal{N} respectively. Since c satisfies (1), there is an \mathcal{L} -isomorphism $h' : \mathcal{X} \rightarrow \mathcal{Y}$ extending h and mapping $\lambda_{p(t)}(b)$ to $\lambda_{p(t)}(c)$ for each $p(t) \in I$. It follows easily that this h' is $\mathbb{Q}(t)$ -linear and hence an \mathcal{L}_t -isomorphism extending h . \square

Corollary 3.8. *Let $\mathcal{M}, \mathcal{N} \models T_t$, let $\mathcal{A} \subseteq \mathcal{M}$ be an \mathcal{L}_t -substructure such that $h : \mathcal{A} \hookrightarrow \mathcal{N}$ is an \mathcal{L}_t -embedding. Let $b \in M \setminus A$ and $c \in N \setminus h(A)$ such that*

$$h \operatorname{tp}_{\mathcal{L}}((\lambda_{p(t)}(b))_{p(t) \in I} | A) = \operatorname{tp}_{\mathcal{L}}((\lambda_{p(t)}(c))_{p(t) \in I} | h(A)).$$

Then $\operatorname{tp}_{\mathcal{L}_t}(b | A) = \operatorname{tp}_{\mathcal{L}_t}(c | h(A))$.

Proof. Let \mathcal{X} be the \mathbb{Q} -linear subspace of \mathcal{M} generated by $(\lambda_{p(t)}(b))_{i \in I}$ and \mathcal{A} , and let \mathcal{Y} be the \mathbb{Q} -linear subspace of \mathcal{N} generated by $(\lambda_{p(t)}(c))_{i \in I}$ and $h(\mathcal{A})$. It is easy to check that \mathcal{X} and \mathcal{Y} are \mathcal{L}_t -substructures of \mathcal{M} and \mathcal{N} respectively. By our assumption on b and c , the embedding h extends on an \mathcal{L} -isomorphism h' between \mathcal{X} and \mathcal{Y} mapping $\lambda_{p(t)}(b)$ to $\lambda_{p(t)}(c)$ for each $p(t) \in I$. Since h is an \mathcal{L}_t -embedding, it

follows easily that h' is an \mathcal{L}_t -isomorphism between \mathcal{X} to \mathcal{Y} . Since T_t has quantifier elimination, we get that $\text{tp}_{\mathcal{L}_t}(b|A) = \text{tp}_{\mathcal{L}_t}(c|h(A))$. \square

Proposition 3.9. *The theory of \mathcal{R}_t is axiomatized by the theory T_t in conjunction with the axiom scheme that specifies $\text{tp}_{\mathcal{L}}((\lambda_{p(t)}(1))_{p(t) \in I})$.*

Proof. Let T_t^* be the theory described in the statement. Since $\mathcal{R}_t \models T_t$, we immediately get that $\mathcal{R}_t \models T_t^*$. It is left to show that T_t^* is complete. Let \mathcal{M} and \mathcal{N} be model of T_t^* of size $\kappa > \aleph_0$. By Corollary 3.8, $1_{\mathcal{M}}$ and $1_{\mathcal{N}}$ satisfy the same \mathcal{L}_t -type. Thus there is an \mathcal{L}_t -isomorphism h mapping the \mathcal{L}_t -substructure of \mathcal{M} generated by $1_{\mathcal{M}}$ to the \mathcal{L}_t -substructure of \mathcal{N} generated by $1_{\mathcal{N}}$. By the proof of Proposition 3.7 this isomorphism h extends to an \mathcal{L}_t -isomorphism between \mathcal{M} and \mathcal{N} . \square

3.3. Exchange property. In this subsection we establish that every model of T_t has the exchange property. We will do so by showing that the definable closure in such a model is equal to the $\mathbb{Q}(t)$ -linear span.

Lemma 3.10. *Let $\mathcal{M} \models T_t$ and let $\mathcal{A} \subseteq \mathcal{M}$ be an \mathcal{L}_t -substructure. Then \mathcal{A} is definably closed.*

Proof. Without loss of generality, we can assume that \mathcal{M} is $|A|^+$ -saturated. Let $b \in M \setminus A$. It is enough to show that there exists $c \in M$ such that $b \neq c$ and $\text{tp}_{\mathcal{L}_t}(b|A) = \text{tp}_{\mathcal{L}_t}(c|A)$. By Corollary 3.8 it is sufficient to find $c \in M$ such that $b \neq c$ and

$$\text{tp}_{\mathcal{L}}((\lambda_{p(t)}(b))_{p(t) \in I} | A) = \text{tp}_{\mathcal{L}}((\lambda_{p(t)}(c))_{p(t) \in I} | A).$$

Let $\varphi(x, y)$ be an \mathcal{L} -formula, $p_1(t), \dots, p_m(t) \in I$ and $a \in A^n$ such that

$$\mathcal{M} \models \varphi(\lambda_{p_1(t)}(b), \dots, \lambda_{p_m(t)}(b), a).$$

By saturation of \mathcal{M} , we only need to find $c \in M$ such that $c \neq b$ and $\mathcal{M} \models \varphi(\lambda_{p_1(t)}(c), \dots, \lambda_{p_m(t)}(c), a)$. By Lemma 3.6, $(\lambda_{p(t)}(b))_{p(t) \in I}$ is \mathbb{Q} -linear independent over A . Thus the set

$$X := \{d \in M^m : \mathcal{M} \models \varphi(d, a)\}$$

has interior. By axiom (T2) the intersection

$$\{(\lambda_{p_1(t)}(c), \dots, \lambda_{p_m(t)}(c)) : c \in M\} \cap X$$

is dense in X . \square

Corollary 3.11. *Let $\mathcal{M} \models T_t$ and let $Z \subseteq M$. Then the \mathcal{L}_t -definable closure of Z is the $\mathbb{Q}(t)$ -subspace of \mathcal{M} generated by Z .*

Proof. By Lemma 3.10 the definable closure of Z is the \mathcal{L}_t -substructure generated by Z . However, the latter is just the $\mathbb{Q}(t)$ -subspace of \mathcal{M} generated by Z . \square

The exchange property for T_t follows immediately from Corollary 3.11 and the classical Steinitz exchange lemma for vector spaces.

Proposition 3.12. *The theory T_t has the exchange property.*

3.4. Open core. Let $\mathcal{M} \models T_t$. Then by Axiom (T2) it defines functions from M to M whose graph is dense. We already know that \mathcal{M} has EP by Proposition 3.12. To give a negative answer to Question 3, it is left to show that every open subset of M^n definable in \mathcal{M} is already definable in the reduct $(M, <, +, 0, 1)$.

Theorem 3.13. *The theory T is an open core of T_t .*

Proof. Let $\mathcal{M} \models T_t$. We prove that every open set is \mathcal{L} -definable. Without loss of generality, we can assume that \mathcal{M} is \aleph_0 -saturated. Let C be a finite subset. Using Boxall and Hieronymi [1, Corollary 3.1], we will show that for every $n \in \mathbb{N}$ and every subset of M^n that is \mathcal{L}_t -definable over C , is also \mathcal{L} -definable over C . Let $n \in \mathbb{N}$ and $p_1(t), \dots, p_{n-1}(t) \in I$ be distinct and non-constant. We define

$$D := \{(a, \lambda_{p_1(t)}(a), \dots, \lambda_{p_{n-1}(t)}(a)) : a \notin \text{dcl}_{\mathcal{L}_t}(C)\}.$$

From (T2) and saturation of \mathcal{M} , it follows easily that D is dense in M^n . Thus Condition (1) of [1, Corollary 3.1] is satisfied. Condition (3) of [1, Corollary 3.1] holds by Corollary 3.8. It is only left to establish Condition (2).

Let $b \in D$ and $a \notin \text{dcl}_{\mathcal{L}_t}(C)$ be such that $b = (a, \lambda_{p_1(t)}(a), \dots, \lambda_{p_{n-1}(t)}(a))$. Let $U \subseteq M^n$ be open and suppose that $\text{tp}_{\mathcal{L}}(b|C)$ is realized in U . We need to show that $\text{tp}_{\mathcal{L}}(b|C)$ is realized in $U \cap D$. By Lemma 3.6 we know that the coordinates of b are \mathbb{Q} -linearly independent over $\text{dcl}_{\mathcal{L}_t}(C)$. Thus the set of realizations of $\text{tp}_{\mathcal{L}}(b|C)$ is open, and so is its intersection with the open set U . Denote this intersection by V . By (T2) and \aleph_0 -saturation of \mathcal{M} , we find $a' \notin \text{dcl}_{\mathcal{L}_t}(C)$ such that $b' = (a', \lambda_{p_1(t)}(a'), \dots, \lambda_{p_{n-1}(t)}(a'))$. Now b' is the desired realization of $\text{tp}_{\mathcal{L}}(b|C)$ in $U \cap D$. \square

By Theorem 3.13 every model of T_t has o-minimal open core and thus is definably complete.

3.5. Neostability results. We will now show that T_t is NIP, but not strong. We use an equivalent definition of the independence property in the theorem below, namely that in a monster model \mathcal{M} of T_t there is no formula $\varphi(x, y)$ and no element $a \in M$ such that for some indiscernible sequence $(b_i)_{i < \omega}$ of tuples in $M^{|y|}$ we have

$$\mathcal{M} \models \varphi(a, b_i) \text{ if and only if } i < \omega \text{ is even.}$$

For a proof that this is equivalent to the classic definition of NIP, see [8].

Theorem 3.14. *Every completion of the theory T_t has NIP.*

Proof. We let $\mathcal{M} \models T_t$ be a monster model of T_t . We suppose for a contradiction that there is an \mathcal{L}_t -formula $\varphi(x, y)$ along with an element $a \in M$ and indiscernible sequence $(b_i)_{i < \omega}$ of elements in $M^{|y|}$ that witnesses IP, i.e. $\mathcal{M} \models \varphi(a, b_i)$ precisely if i is even. Let $|y| = n$ and for each $i < \omega$ we denote the j -coordinate of b_i by $b_{i,j}$. By quantifier elimination in the language \mathcal{L}_t , we can assume that the formula $\varphi(a, y)$ is equal to a boolean combination of formulas of the form

$$\begin{aligned} (1) \quad & a - \sum_{j=1}^n \lambda_{q_j(t)}(y_j) = 0 \\ (2) \quad & a - \sum_{j=1}^n \lambda_{q_j(t)}(y_j) > 0 \end{aligned}$$

where $q_1, \dots, q_n(t) \in \mathbb{Q}(t)$. Since NIP is preserved under boolean combinations, we can assume that φ is of the form (1) or (2). For ease of notation, let $f(b_i) = \sum_{j=1}^n \lambda_{q_j(t)}(b_{i,j})$ for each $i < \omega$.

Suppose that φ is of form (1). We suppose without loss of generality that $a - f(b_i) = 0$ holds if and only if $i < \omega$ is odd. Then we have that $f(b_1) = f(b_3)$, but $f(b_2) \neq f(b_1)$. Thus we conclude $\text{tp}_{\mathcal{L}_t}(b_1b_2) \neq \text{tp}_{\mathcal{L}_t}(b_2b_3)$, contradicting indiscernability.

Now assume that φ is of the form (2). Without loss of generality assume that $a - f(b_i) > 0$ holds if and only if $i < \omega$ is odd. Then for all $i < \omega$, we have $a < f(b_{2i})$ and $a > f(b_{2i+1})$. However, this means that $f(b_1) < f(b_2)$ and $f(b_2) > f(b_3)$. So we again obtain $\text{tp}_{\mathcal{L}_t}(b_1b_2) \neq \text{tp}_{\mathcal{L}_t}(b_2b_3)$, contradicting indiscernability. \square

Thus $\mathcal{R}_t \models \text{DC} + \text{EP} + \text{NIP}$, but \mathcal{R}_t is not o-minimal. This gives a negative answer to Question 2.

Proposition 3.15. *No completion of the theory T_t is strong.*

Proof. Fix a family $(q_j(t))_{j \in \mathbb{N}}$ of distinct elements of I . Consider the family of \mathcal{L}_t -formulas given by $(\lambda_{q_j(t)}(x) \in (a_k, b_k))_{j,k \in \mathbb{N}}$ such that

- $(a_k, b_k) \cap (a_\ell, b_\ell) = \emptyset$, for all $\ell \neq k \in \mathbb{N}$, and
- the tuples $(a_k, b_k)_{k \in \mathbb{N}}$ form an indiscernible sequence.

In the array that corresponds to varying $j \in \mathbb{N}$ along the rows and the $k \in \mathbb{N}$ along the columns, it is easy to see that formulas in the same row are pairwise inconsistent. However, for every path $(\lambda_{q_{\gamma(k)}(t)}(x) \in (a_{\gamma(k)}, b_{\gamma(k)}))_{k \in \mathbb{N}}$, every finite subset of these formulas are consistent by our axiom scheme (T2). So by compactness, every path through the entire array is consistent. \square

4. QUESTION 4

Let T be a theory extending the theory of dense linear orders without endpoints in an language \mathcal{L} . We write $T \models \text{UF}$ if every model of T satisfies UF. The main goal of this section is to establish the following theorem.

Theorem 4.1. *Suppose that $T \models \text{DC} + \text{UF}$. Let T' be an open core of T . There is a theory T_{Sk}^∞ extending T such that T_{Sk}^∞ has definable Skolem functions and T' is an open core of T_{Sk}^∞ .*

This immediately gives a negative answer to Question 4, as there are many documented examples of a theory T with $T \models \text{DC} + \text{UF}$ and o-minimal open core that is not o-minimal itself. To prove Theorem 4.1 we follow a strategy of Kruckman and Ramsey [4] and rely on a construction due to Winkler [9] allowing us to successively add definable Skolem functions to the language \mathcal{L} of a given theory T while preserving uniform finiteness. As explained below, this construction preserves the open definable sets by a result from [1]. We begin by recalling notations and results from [9].

4.1. Skolem expansions. Let \mathcal{L} be a language and $\Theta = \{\theta_t(x, y) : t < |\mathcal{L}|\}$ be an enumeration of all \mathcal{L} -formulas $\varphi(x, y)$ where the variable y has length 1. Define \mathcal{L}_{Sk} to be $\mathcal{L} \cup \{f_t : t < |\mathcal{L}|\}$, where the arity of f_t is the length of the tuple x appearing in $\theta_t(x, y)$.

The **Skolem expansion T_+ of T** is the \mathcal{L}_{Sk} -theory

$$T_+ = T \cup \{\forall x \exists y (\theta_t(x, y) \rightarrow \theta_t(x, f_t(x))) : t < |\mathcal{L}|\}.$$

We refer to the f_t 's as Skolem functions.

From here on we assume that T has quantifier elimination in the language \mathcal{L} and assume that for each \mathcal{L} -definable function f there is an \mathcal{L} -term t such that $T \models \forall x f(x) = t(x)$. Let $\mathcal{M}_+ \models T_+$, and denote its reduct to \mathcal{L} by \mathcal{M} . For $A \subseteq M$ we denote by $\langle A \rangle_{\text{Sk}}$ the \mathcal{L}_{Sk} -substructure generated by A .

Following [9], we say an \mathcal{L}_{Sk} -formula $\chi(x_1, \dots, x_n)$ is a **uniform configuration** if it is a conjunction of equalities of the form $f_t(x_{i_1}, \dots, x_{i_m}) = x_{i_0}$ involving Skolem functions. We need the following result about uniform configurations from [9].

Fact 4.2 ([9, p. 448]). *Let $\chi(x)$ be a uniform configuration. Then there exists an \mathcal{L} -formula $\chi'(x)$ such that for all $\mathcal{A} \models T_+$ and $a \in A^{|x|}$ the following are equivalent:*

- $\mathcal{A} \models \chi'(a)$.
- *The result of altering the Skolem structure of \mathcal{A} precisely so that $\mathcal{A} \models \chi(a)$ is again a model of T_+ .*

In the case of Fact 4.2, we say that $\chi'(x)$ **codes the eligibility of the configuration** $\chi(x)$.

Lemma 4.3. *Let $t_1(x), \dots, t_n(x)$ be \mathcal{L}_{Sk} -terms such that for every $i \leq n$ there is an \mathcal{L}_{Sk} -function symbol f_i with*

$$t_i(x) = f_i(x, t_1(x), \dots, t_{i-1}(x)).$$

Then there is an \mathcal{L} -formula $\varphi(x, y)$ and a uniform configuration $\chi(x, y)$ such that

$$T_+ \models \forall x \forall y \left((\varphi(x, y) \wedge \chi(x, y)) \leftrightarrow \left(\bigwedge_{i=1}^n y_i = f_i(x, y_1, \dots, y_{i-1}) \right) \right).$$

Proof. Let $J \subseteq \{1, \dots, n\}$ be the set of all i such that $f_i \in \mathcal{L}_{\text{Sk}} \setminus \mathcal{L}$. Let $\chi(x, y)$, where $y = (y_1, \dots, y_n)$, be the uniform configuration given by

$$\bigwedge_{i \in J} f_i(x, y_1, \dots, y_{i-1}) = y_i$$

and let $\varphi(x, y)$ be the \mathcal{L} -formula given by

$$\bigwedge_{i \in \{1, \dots, n\} \setminus J} f_i(x, y_1, \dots, y_{i-1}) = y_i.$$

It is easy check this pair of formulas has the desired property. \square

One of the main results in [9] is that if $T \models \text{UF}$, then the Skolem expansion has a model companion. Indeed, more is true:

Fact 4.4 ([9, Theorem 2, Corollary 3]). *Let $T \models \text{UF}$. Then the Skolem expansion T_+ has a model companion T_{Sk} that satisfies UF.*

From here on we assume that $T \models \text{UF}$. We have the following axiomatization of the model companion of the Skolem expansion.

Fact 4.5 ([9, p. 447]). *The theory T_{Sk} is axiomatized as the expansion of T_+ by the set Φ of all sentences of the form $\forall x_1 \dots \forall x_k \psi(x)$, where $x = (x_1, \dots, x_n)$ and*

- (i) $\psi(x) = \exists^\infty x_{k+1} \dots x_n \varphi(x) \wedge \chi'(x) \rightarrow \exists x_{k+1}, \dots, x_n \varphi(x) \wedge \chi(x)$,
- (ii) $\varphi(x)$ is a quantifier free \mathcal{L} -formula,

- (iii) $\chi(x)$ is a uniform configuration,
- (iv) $\chi'(x)$ codes the eligibility of the configuration $\chi(x)$.

Let \mathcal{M}_{Sk} be an $|\mathcal{L}|^+$ -saturated model of T_{Sk} with underlying set M , and denote its reduct to \mathcal{L} by \mathcal{M} . We need following easy corollary of the axiomatization of T_{Sk} .

Fact 4.6. *Let \mathcal{I} be the set of partial \mathcal{L} -elementary maps $\iota : X \rightarrow Y$ between \mathcal{M}_{Sk} and itself such that*

- ι is a partial \mathcal{L}_{Sk} -isomorphisms and
- $X = \langle X \rangle_{\text{Sk}}$ and $Y = \langle Y \rangle_{\text{Sk}}$.

Then \mathcal{I} is a back-and-forth system.

Proof. Let $\iota : X \rightarrow Y$ in \mathcal{I} . Let $a \in M \setminus X$. By symmetry, it is enough to find $a' \in M$ such that there exists $\iota' \in \mathcal{I}$ extending ι such that $\iota(a) = a'$. By saturation of \mathcal{M}_{Sk} , we just need to find $a' \in M$ such that for all $\mathcal{L}_{\text{Sk}}(X)$ -terms $t(x) = (t_1(x), \dots, t_n(x))$

$$\text{tp}_{\mathcal{L}}(a', t(a')|Y) = \iota \text{tp}_{\mathcal{L}}(a, t(a)|X).$$

Without loss of generality, we can assume that there is $c \in X^m$ such that for every $i \in \{1, \dots, n\}$ there is a function symbol $f_i \in \mathcal{L}_{\text{Sk}}$ with

$$t_i(x) = f_i(x, t_1(x), \dots, t_{i-1}(x), c).$$

Let $\varphi(x, y_1, \dots, y_n, z)$ be the \mathcal{L} -formula and $\chi(x, y_1, \dots, y_n, z)$ be the uniform configuration given by Lemma 4.3. Let the \mathcal{L} -formula $\chi'(x, y_1, \dots, y_n, z)$ code the eligibility of $\chi(x, y_1, \dots, y_n, z)$. For ease of notation, set $y := (y_1, \dots, y_n)$.

Consider an \mathcal{L} -formula $\psi(x, y, z)$ and $c' \in X^{|c|}$ such that $\psi(x, y, c') \in \text{tp}_{\mathcal{L}}(a, t(a)|X)$. Extending c , we can assume that $c = c'$. By saturation of \mathcal{M}_{Sk} it suffices to find $a' \in M$ such that $\mathcal{M}_{\text{Sk}} \models \psi(a', t(a'), \iota(c))$. Since $\mathcal{M} \models \psi(a, t(a), c) \wedge \varphi(a, t(a), c) \wedge \chi'(a, t(a), c)$ and $a \notin X$, we have that

$$\mathcal{M} \models \exists^\infty xy \psi(x, y, c) \wedge \varphi(x, y, c) \wedge \chi'(x, y, c).$$

Since ι is \mathcal{L} -elementary,

$$\mathcal{M} \models \exists^\infty xy \psi(x, y, \iota(c)) \wedge \varphi(x, y, \iota(c)) \wedge \chi'(x, y, \iota(c)).$$

Thus from the axiomatization of T_{Sk} we know that there is $(a', a'_1, \dots, a'_n) \in M^{1+n}$ such that

$$\mathcal{M} \models \psi(a', a'_1, \dots, a'_n, \iota(c)) \wedge \varphi(a', a'_1, \dots, a'_n, \iota(c)) \wedge \chi(a', a'_1, \dots, a'_n, \iota(c)).$$

By our choice of φ and χ , we have that $a'_i = t_i(a')$ for each i . Thus $\mathcal{M} \models \psi(a', t(a'), c)$. \square

We now collect the following easy corollary of Fact 4.6.

Fact 4.7. *Let $a, a' \in M^n$ and let $\sigma : M \rightarrow M$ be an \mathcal{L}_{Sk} -automorphism fixing C such that $\sigma(a) = a'$ and for all \mathcal{L}_{Sk} -terms $t(x) = (t_1(x), \dots, t_n(x))$*

$$\text{tp}_{\mathcal{L}}(t(a)|C) = \text{tp}_{\mathcal{L}}(t(a')|C).$$

Then $\text{tp}_{\mathcal{L}_{\text{Sk}}}(a|C) = \text{tp}_{\mathcal{L}_{\text{Sk}}}(a'|C)$.

4.2. No new definable open sets in T_{Sk} . Let \mathcal{M}_{Sk} be an $|\mathcal{L}_{\text{Sk}}|^+$ -saturated model of T_{Sk} with underlying set M , and denote its reduct to \mathcal{L} by \mathcal{M} . Fix a subset $C \subseteq M$ of cardinality at most $|\mathcal{L}_{\text{Sk}}|$.

Theorem 4.8. *Let $C = \langle C \rangle_{\text{Sk}}$. Then every open set definable over C in \mathcal{M}_{Sk} is definable in \mathcal{M} .*

Proof. By [1, Theorem 2.2] it is enough to show that for every $a \in M^n$ for which the set of realisations of $\text{tp}_{\mathcal{L}}(a|C)$ is dense in an open set, the set of realisations of $\text{tp}_{\mathcal{L}_{\text{Sk}}}(a|C)$ is dense in the set of realizations of $\text{tp}_{\mathcal{L}}(a|C)$. Let $U \subseteq M^n$ be an open definable set such that the set of realizations of $\text{tp}_{\mathcal{L}}(a|C)$ intersected with U is dense in U . It is left to show that there is $a' \in U$ such that $a' \models \text{tp}_{\mathcal{L}_{\text{Sk}}}(a|C)$. By Fact 4.7 and saturation of \mathcal{M}_{Sk} , it is enough to find for

- every tuple $t = (t_1, \dots, t_m) : M^n \rightarrow M^m$ of $\mathcal{L}_{\text{Sk}}(C)$ -terms and
- every $\mathcal{L}(C)$ -definable set $X \subseteq M^{n+m}$ with $(a, t(a)) \in X$

an $a' \in U$ such that $(a', t(a')) \in X$. Fix t and X . After increasing m , we can assume that there is $c \in C^\ell$ such that X is $\mathcal{L}(c)$ -definable and for every $i \leq m$

$$t_i(x) = f_i(x, t_1(x), \dots, t_{i-1}(x), c)$$

where f_i is a function symbol in \mathcal{L}_{Sk} . Let $\varphi(x, y_1, \dots, y_n, c)$ be the \mathcal{L} -formula and $\chi(x, y_1, \dots, y_n, c)$ be the uniform configuration given by Lemma 4.3. Set $y = (y_1, \dots, y_n)$. Let the \mathcal{L} -formula $\chi'(x, y, c)$ code the eligibility of $\chi(x, y, c)$.

We now prove the existence of a' . Let d_0 be a realization of $\text{tp}_{\mathcal{L}}(a|C)$ in U . Let $d_1, \dots, d_m \in M^m$ be such that $(d_0, d_1, \dots, d_m) \in X$ and

$$\mathcal{M} \models (\varphi \wedge \chi')(d_0, d_1, \dots, d_m, c).$$

Since there are infinitely many realizations of $\text{tp}_{\mathcal{L}}(a|C)$ in U , there are infinitely many $e \in M^{n+m}$ such that $e \in X \cap U$ and $\mathcal{M} \models (\varphi \wedge \chi')(e, c)$. Thus by Fact 4.5, there is $e = (e_0, e_1, \dots, e_m) \in M^{n+m}$ such that

$$(e_0, e_1, \dots, e_m) \in X \cap U \text{ and } \mathcal{M}_{\text{Sk}} \models \chi(e_0, e_1, \dots, e_m, c).$$

Thus $(e_1, \dots, e_m) = t(e_0)$ and we can set $a' = e_0$. \square

Corollary 4.9. *Let T' be an open core of T . Then T' is an open core of T_{Sk} .*

Proof. Let \mathcal{L}' be the language of T' . Without loss of generality, we can assume that $\mathcal{L}' \cap \mathcal{L}_{\text{Sk}} = \emptyset$. Let \mathcal{L}^* be the union of \mathcal{L}' and \mathcal{L}_{Sk} . Let $\mathcal{M}_{\text{Sk}} \models T_{\text{Sk}}$. Since T' is an open core of T , we can expand \mathcal{M}_{Sk} to a model \mathcal{M}^* of the \mathcal{L}^* -theory $T' \cup T_{\text{Sk}}$. Let $X \subseteq M^n$ be an open set given by

$$X := \{a \in M^n : \mathcal{M}_{\text{Sk}} \models \varphi(a, c)\},$$

where φ is an \mathcal{L}_{Sk} -formula with parameters $c \in M^m$. Let \mathcal{N} be an elementary extension of \mathcal{M}^* that is $|\mathcal{L}|^+$ -saturated. Set $Y := \{a \in N^n : \mathcal{N} \models \varphi(a, c)\}$. Since X is open, so is Y . By Theorem 4.8 there is an \mathcal{L}' -formula $\psi(x, y)$ such that there is $d \in M^\ell$ with $Y = \{a \in N^n : \mathcal{N} \models \psi(a, d)\}$. Since $\mathcal{M}^* \preceq \mathcal{N}$, there is $d' \in M^\ell$ such that $X = \{a \in M^n : \mathcal{M}^* \models \psi(a, d')\}$. Thus X is \mathcal{L}' -definable. \square

Proof of Theorem 4.1. We are now able to complete the proof of Theorem 4.1 using the same argument as in [4, Corollary 4.9]. Suppose $T \models \text{DC} + \text{UF}$ and let T' be an open core of T . Set T_0 be the Morleyization of T in a language \mathcal{L}_0 . For every $n > 0$, we will now construct a language \mathcal{L}_n and an \mathcal{L}_n -theory T_n such that

- (1) T_n has quantifier-elimination,
- (2) $T_n \models \text{UF}$, and
- (3) T' is an open core of T_n .

Let $n \geq 0$, and suppose we already constructed a language \mathcal{L}_n and an \mathcal{L}_n -theory T_n with the properties (1)-(3). Let Φ be the set of \mathcal{L}_n -formulas $\varphi(x, y)$ such that $|y| = 1$ and

$$T_n \models \forall x \exists! y \varphi(x, y),$$

For each $\varphi(x, y) \in \Phi$ we introduce a new function symbol f_φ of arity $|x|$. Let \tilde{T} be the union of the \mathcal{L}_n and $\{f_\varphi : \varphi \in \Phi\}$. Let \tilde{T} be the union of T_n with the set of all $\tilde{\mathcal{L}}$ -sentence of the form

$$\forall x \forall y (f_\varphi(x) = y) \leftrightarrow \varphi(x, y),$$

where $\varphi \in \Phi$. Since \tilde{T} is an expansion of T_n by definitions, it is easy to check that \tilde{T} satisfies (1)-(3). Now consider the model companion $(\tilde{T})_{\text{Sk}}$ of the Skolem expansion $(\tilde{T})_+$. Let T_{n+1} be the Morleyization of $(\tilde{T})_{\text{Sk}}$ in an expanded language \mathcal{L}_{n+1} . We know $T_{n+1} \models \text{UF}$ by Fact 4.4. By Corollary 4.9, the theory T' is an open core of T_{n+1} .

Now set $T_{\text{Sk}}^\infty := \bigcup_{i \in \mathbb{N}} T_n$. From the construction, it follows immediately that T' is an open core of T_{Sk}^∞ and that T_{Sk}^∞ has definable Skolem functions. \square

REFERENCES

- [1] Gareth Boxall and Philipp Hieronymi, Expansions which introduce no new open sets. *J. Symb. Log.*, 77(1):111-121, (2012).
- [2] Alfred Dolich, Chris Miller, and Charles Steinhorn, Structures having o-minimal open core. *Trans. Am. Math. Soc.*, 362(3):1371-1411, (2010).
- [3] Alfred Dolich, Chris Miller, and Charles Steinhorn, Expansions of o-minimal structures by dense independent sets. *Ann. Pure Appl. Logic*, 167(8):684-706, (2016).
- [4] Alex Kruckman and Nicholas Ramsey, Generic expansion and Skolemization in NSOP1 theories *Ann. Pure Appl. Logic.*, 169(8): 755-774, (2018).
- [5] Chris Miller and Patrick Speissegger, Expansions of the real line by open sets: o-minimality and open cores. *Fund. Math.*, 162(3):193-208 (1999).
- [6] Saharon Shelah. Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory, *Ann. Pure Appl. Logic*, 3 (3): 271-362 (1971)
- [7] Pierre Simon, On dp-minimal structures *J. Symb. Logic*, 76(2): 448-460 (2011)
- [8] Pierre Simon, A guide to NIP theories. *Lect. Notes Log.*, vol. 44, Cambridge University Press, 2015.
- [9] Peter Winkler, Model completeness and skolem expansions. in *Model Theory and Algebra*, Springer (pp. 408-463), 1975.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, 1409 WEST GREEN STREET, URBANA, IL 61801

Email address: atb2@illinois.edu

DEPARTMENT OF MATHEMATICS & STATISTICS, MCMASTER UNIVERSITY, HAMILTON HALL, 1280 MAIN STREET WEST, HAMILTON, ONTARIO, CANADA, L8S 4K1

Email address: caulfiee@mcmaster.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, 1409 WEST GREEN STREET, URBANA, IL 61801

Email address: phierony@illinois.edu

URL: <http://faculty.math.illinois.edu/~phierony>