NOTES ON DEFINABLY COMPLETE LOCALLY O-MINIMAL EXPANSIONS OF ORDERED GROUPS

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ABSTRACT. We study definably complete locally o-minimal expansions of ordered groups in this paper. A definable continuous function defined on a closed, bounded and definable set behave like a continuous function on a compact set. We demonstrate uniform continuity of a definable continuous function on a closed, bounded and definable set and Arzela-Ascoli-type theorem. We propose a notion of special submanifolds with tubular neighborhoods and show that any definable set is decomposed into finitely many special submanifolds with tubular neighborhoods.

1. INTRODUCTION

We study definably complete locally o-minimal expansions of ordered groups in this paper. An o-minimal structure enjoys tame properties such as monotonicity and definable cell decomposition [2, 9, 12]. Toffalori and Vozoris studied locally o-minimal structures in [16]. Roughly speaking, a locally o-minimal structure is defined by simply localizing the definition of an o-minimal structure. However, their study reveals that the local version of monotonicity theorem is unavailable in a locally o-minimal structure. Local o-minimal structures are studied also in [8].

The above two papers do not assume definable completeness. Fornaisero made a comprehensive study on definably complete locally o-minimal expansions of ordered fields in [3]. This study showed that definably complete locally o-minimal structures enjoy tame topological properties when they are expansions of ordered fields. The author and his collaborators have demonstrated that definably complete locally o-minimal structures still enjoy tame topological properties without algebraic assumptions such as that they are expansions of ordered fields [4, 5, 6, 7]. These are introduced in Section 2. As a preliminary, we also demonstrate a definable choice lemma for a definably complete expansion of an ordered group and its immediate consequences in this section.

In this paper, as we said at the beginning of the paper, we employ a weak algebraic assumption; that is, we consider definably complete locally o-minimal expansions of ordered groups. We first demonstrate that a definable continuous function defined on a closed, bounded and definable set behave like a continuous function on a compact set in Section 3. More precisely, definable version of uniform continuity of continuous functions on closed, bounded and definable sets and definable Arzela-Ascoli-type theorem are demonstrated.

The notion of quasi-special submanifolds is introduced in [6]. Any set definable in a definably complete locally o-minimal structure is decomposed into finitely

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many quasi-quadratic submanifolds [7, Proposition 2.11]. This notion is obtained by relaxing the definitions of multi-cells in [3] and special submanifolds in [11], which define the same notion in our setting as we demonstrate later. Miller, Thamrongthanyalak and Fornasiero demonstrated that a definable set is decomposed into finitely many special submanifolds/multi-cells under the assumption that the structure is an expansion of an ordered field [3, 11, 15]. (Miller and Thamrongthanyalak assumed d-minimality instead of local o-minimality.) In Section 4, we extend the definition of special submanifolds to the case in which the structures are expansions of dense linear orders without endpoints. We prove that any set definable in a definably complete locally o-minimal expansion of an ordered group is decomposed into finitely many special submanifolds. In addition, we introduce the notion of special submanifolds with tubular neighborhoods and demonstrate a decomposition theorem into them.

We introduce the terms and notations used in this paper. The term 'definable' means 'definable in the given structure with parameters' in this paper. For a linearly ordered structure $\mathcal{M} = (M, <, ...)$, an open interval is a definable set of the form $\{x \in R \mid a < x < b\}$ for some $a, b \in M \cup \{\pm \infty\}$. It is denoted by (a, b) in this paper. Elements in M^2 are denoted by the same notation, but it will not confuse readers. We define a closed interval similarly. It is denoted by [a, b]. An open box in M^n is the direct product of n open intervals. We set

$$\mathcal{B}_m(x,\varepsilon) = \{ y = (y_1, \dots, y_m) \in M^m \mid |x_i - y_i| < \varepsilon \text{ for all } 1 \le i \le m \}$$

for any $x = (x_1, \ldots, x_m) \in M^m$ and $\varepsilon > 0$. The notation $M_{>r}$ denotes the set $\{x \in M \mid x > r\}$ for any $r \in M$. We set $|x| := \max_{1 \le i \le n} |x_i|$ for any vector $x = (x_1, \ldots, x_n) \in M^n$ when the addition is definable in \mathcal{M} . The function |x - y| defines a distance in M^n when \mathcal{M} is an expansion of an ordered abelian group. Let $\operatorname{int}(A)$ and $\operatorname{cl}(A)$ denote the interior and the closure of a subset A of a topological space, respectively.

2. Preliminary

2.1. Results in previous studies. We first recall basic definitions.

Definition 2.1. An expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, ...)$ is *definably complete* if every definable subset of M has both a supremum and an infimum in $M \cup \{\pm \infty\}$ [10].

An expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, ...)$ is *locally o-minimal* if, for every definable subset X of M and for every point $a \in M$, there exists an open interval I containing the point a such that $X \cap I$ is a finite union of points and open intervals [16].

A definably complete locally o-minimal structure is a model of DCTC if any definable discrete subset of M is bounded.

The definition given above is not the same as the original definition of a model of DCTC in [13]. But they are equivalent by [13, Corollary 2.8].

We also recall the definition of local monotonicity.

Definition 2.2 (Local monotonicity). A function f defined on an open interval I is *locally constant* if, for any $x \in I$, there exists an open interval J such that $x \in J \subseteq I$ and the restriction $f|_J$ of f to J is constant. A function f defined on an open interval I is *locally strictly increasing* if, for any $x \in I$, there exists an open interval

J such that $x \in J \subseteq I$ and f is strictly increasing on the interval J. We define a *locally strictly decreasing* function similarly. A *locally strictly monotone* function is a locally strictly increasing function or a locally strictly decreasing function. A *locally monotone* function is locally strictly monotone or locally constant.

The following monotonicity theorem holds true.

Theorem 2.3 (Monotonicity theorem). Let $\mathcal{M} = (M, <, ...)$ be a definably complete locally o-minimal structure. Let I be an interval and $f: I \to M$ be a definable function. There exists a mutually disjoint partition $I = X_d \cup X_c \cup X_+ \cup X_-$ of I into definable sets satisfying the following conditions:

- (1) the definable set X_d is discrete and closed;
- (2) the definable set X_c is open and f is locally constant on X_c ;
- (3) the definable set X_+ is open and f is locally strictly increasing and continuous on X_+ ;
- (4) the definable set X₋ is open and f is locally strictly decreasing and continuous on X₋.

Proof. [7, Theorem 2.3].

We also need the following lemma in [4]:

Lemma 2.4. Let $\mathcal{M} = (M, <, ...)$ be a definably complete local o-minimal structure. A locally monotone definable function defined on an open interval is monotone.

Proof. [4, Proposition 3.1]

The following proposition guarantees the existence of the limit.

Proposition 2.5. Let $\mathcal{M} = (M, <, 0, +, ...)$ be a definably complete locally ominimal expansion of an ordered group. Let s > 0 and $f : (0, s) \to M^n$ be a bounded definable map.

- (1) There exists a positive 0 < u < s such that the restriction of f to (0, u) is continuous. In addition, it is monotone when n = 1;
- (2) There exists a unique point $x \in M^n$ satisfying the following condition:

 $\forall \varepsilon > 0, \exists \delta > 0, \forall t, \ 0 < t < \delta \Rightarrow |x - f(t)| < \varepsilon.$

The notation $\lim_{t\to+0} f(t)$ denotes the point x.

Proof. We first reduce to the case in which n = 1. Assume that the proposition holds true for n = 1. Let π_i be the projection onto the *i*-th coordinate for all $1 \leq i \leq n$. Apply the proposition to the composition $\pi_i \circ f$. There exists $0 < u_i < s$ such that the restriction of $\pi_i \circ f$ to $(0, u_i)$ is continuous and monotone for each $1 \leq i \leq n$. Set $u = \min_{1 \leq i \leq n} u_i$. The restriction of f to (0, u) is continuous. Set $x_i = \lim_{t \to +0} \pi_i \circ f(t)$ for all $1 \leq i \leq n$. It is obvious that $x = (x_1, \ldots, x_n)$ is the unique point satisfying the condition in the assertion (2). We have succeeded in reducing to the case in which n = 1.

Set I = (0, s). Applying Theorem 2.3 to f, we get a partition $I = X_d \cup X_c \cup X_+ \cup X_-$ into definable sets such that X_d , X_c , X_+ and X_- satisfy the conditions in Theorem 2.3. Take a sufficiently small open interval J containing the point 0. The intersections of J with X_d , X_c , X_+ and X_- are finite unions of points and open intervals. Shrinking the interval I if necessary, we have $I = X_c$, $I = X_+$ or

 $I = X_{-}$. We have demonstrated the assertion (1) by Lemma 2.4. The remaining task is to show the assertion (2). We only consider the case in which $I = X_{-}$. We can prove the corollary similarly in the other cases.

The function f is strictly decreasing because $I = X_-$. Set $x = \inf_{0 < t < s} f(t)$, which exists because f is bounded. It is obvious the point x satisfies the required condition because f is strictly decreasing. Let x' be another point satisfying the condition. We fix an arbitrary $\varepsilon > 0$. There exists $\delta > 0$ with $|x - f(t)| < \varepsilon$ whenever $0 < t < \delta$. There exists $\delta' > 0$ with $|x' - f(t)| < \varepsilon$ whenever $0 < t < \delta'$. Set $\delta'' = \min\{\delta, \delta'\}$. We have $|x - x'| \le |x - f(t)| + |x' - f(t)| < 2\varepsilon$ whenever $0 < t < \delta''$. We get x = x' because ε is an arbitrary positive element.

Definition 2.6 (Dimension). Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, <, ...)$. Let X be a nonempty definable subset of M^n . The dimension of X is the maximal nonnegative integer d such that $\pi(X)$ has a nonempty interior for some coordinate projection $\pi : M^n \to M^d$. Here, we consider that M^0 is a singleton equipped with the trivial topology and the projection $\pi : M^n \to M^0$ is the trivial map. We set $\dim(X) = -\infty$ when X is an empty set.

We only review the assertions on dimension which are frequently used in this study.

Proposition 2.7. Consider a definably complete locally o-minimal structure $\mathcal{M} = (M, <, ...)$. The following assertions hold true.

- (1) A definable set is of dimension zero if and only if it is discrete. When it is of dimension zero, it is also closed.
- (2) Let $X \subseteq Y$ be definable sets. Then, the inequality $\dim(X) \leq \dim(Y)$ holds true.
- (3) Let σ be a permutation of the set $\{1, \ldots, n\}$. The definable map $\overline{\sigma} : M^n \to M^n$ is defined by $\overline{\sigma}(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Then, we have dim $(X) = \dim(\overline{\sigma}(X))$ for any definable subset X of M^n .
- (4) Let X and Y be definable sets. We have $\dim(X \times Y) = \dim(X) + \dim(Y)$.
- (5) Let X and Y be definable subsets of M^n . We have

$$\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}.$$

- (6) Let $f: X \to M^n$ be a definable map. We have $\dim(f(X)) \leq \dim X$.
- (7) Let $f : X \to M^n$ be a definable map. Let $\mathcal{D}(f)$ denote the set of points at which the map f is discontinuous. The inequality $\dim(\mathcal{D}(f)) < \dim X$ holds true.
- (8) Let X be a definable set. Let ∂X denote the frontier of X defined by $\partial X = cl(X) \setminus X$. We have $dim(\partial X) < dim X$.
- (9) Let $\varphi : X \to Y$ be a definable surjective map whose fibers are equi-dimensional; that is, the dimensions of the fibers $\varphi^{-1}(y)$ are constant. We have dim $X = \dim Y + \dim \varphi^{-1}(y)$ for all $y \in Y$.
- (10) Let X be a definable subset of M^n . There exists a point $x \in X$ such that we have $\dim(X \cap B) = \dim(X)$ for any open box B containing the point x.

Proof. See [7, Proposition 2.8].

Corollary 2.8. Consider a definably complete locally o-minimal structure $\mathcal{M} = (M, <, ...)$. Let X and Y be definable sets such that $X \subseteq Y$ and dim $X = \dim Y$.

Then, there exists a nonempty open box B such that $X \cap B = Y \cap B$ and dim $X \cap B = \dim X$.

Proof. Set $d = \dim X$, $Z = \operatorname{cl}(Y \setminus X)$, $V = X \cap Z$ and $W = X \setminus Z$. The set V is contained in the frontier of $Y \setminus X$, and it is of dimension smaller than d by Proposition 2.7(2),(5),(8). We have dim W = d by Proposition 2.7(5). Take a point $x \in W$ such that, for any open box B containing the point x, the equality dim $B \cap W = d$ holds true by Proposition 2.7(10). By the definitions of Z and W, if we take a sufficiently small open box B containing the point x, the intersection $B \cap W$ has an empty intersection with Z. It implies that $X \cap B = Y \cap B$.

We also need the fact that a definably complete locally o-minimal structure is definably Baire.

Definition 2.9. Consider an expansion of a linearly ordered structure $\mathcal{R} = (R, < , 0, ...)$. A parameterized family of definable sets $\{X\langle x\rangle\}_{x\in S}$ is the family of the fibers of a definable set; that is, there exists a definable set \mathcal{X} with $X\langle x\rangle = \mathcal{X}_x$ for all x in a definable set S.

A parameterized family of definable sets $\{X\langle r\rangle\}_{r>0}$ is called a *definable increasing family* if $X\langle r\rangle \subseteq X\langle r'\rangle$ whenever 0 < r < r'. A definably complete expansion of a densely linearly ordered structure is *definably Baire* if the union $\bigcup_{r>0} X\langle r\rangle$ of any definable increasing family $\{X\langle r\rangle\}_{r>0}$ with $\operatorname{int}\left(\overline{X\langle r\rangle}\right) = \emptyset$ has an empty interior.

Proposition 2.10. A definably complete locally o-minimal structure is definably Baire.

Proof. It immediately follows from [6, Proposition 2.9] and [7, Theorem 2.5]. \Box

Proposition 2.11. Consider a definably complete locally o-minimal structure. Let $\{X\langle r\}\}_{r>0}$ be a definable increasing family. We have

$$\dim\left(\bigcup_{r>0} X\langle r\rangle\right) = \sup\{\dim(X\langle r\rangle) \mid r>0\}.$$

Proof. Let $\mathcal{M} = (M, <, +, 0, ...)$ be the given definably complete locally o-minimal structure. Set $d = \sup\{\dim(X\langle r \rangle) \mid r > 0\}$ and $Y = \bigcup_{r>0} X\langle r \rangle$. We lead to a contradiction assuming that $\dim(Y) > d$.

There exist a definable open box B in M^{d+1} and a definable continuous injective map $\varphi: U \to Y$ which is homeomorphic onto its image by [7, Proposition 2.8(9)]. Set $V\langle r \rangle = \varphi^{-1}(X\langle r \rangle)$. We have $B = \bigcup_{r>0} V\langle r \rangle$ and $\dim V\langle r \rangle \leq d$ for all r > 0by Proposition 2.7(6). We also get $\dim \overline{V\langle r \rangle} \leq d$ by Proposition 2.7(5),(8). The definable sets $\overline{V\langle r \rangle}$ have empty interiors for all r > 0 by the definition of dimension. Since $B = \bigcup_{r>0} V\langle r \rangle$, the open box B has an empty interior by Proposition 2.10. Contradiction.

We next prove a definable choice lemma.

Lemma 2.12 (Definable choice lemma). Consider a definably complete expansion of an ordered group $\mathcal{M} = (M, <, 0, + ...)$. Let $\pi : M^{m+n} \to M^m$ be a coordinate projection. Let X and Y be definable subsets of M^m and M^{m+n} , respectively, satisfying the equality $\pi(Y) = X$. There exists a definable map $\varphi : X \to Y$ such that $\pi(\varphi(x)) = x$ for all $x \in X$. *Proof.* We may assume that π is the coordinate projection onto the first m coordinate without loss of generality. We fix a positive element $c \in M$. We prove the lemma by induction on m.

When m = 1, we define $\varphi : X \to Y$ as follows: Fix a point $x \in X$. Consider the fiber $Y_x = \{y \in M \mid (x,y) \in Y\}$. Set $Y_x^+ = \{y \in Y_x \mid y \ge 0\}$ and $Y_x^- = \{y \in Y_x \mid y \le 0\}$. When the definable set Y_x^+ is not an empty set, consider the element $y_1 = \inf(Y_x^+)$. If $y_1 \in Y_x$, set $\varphi(x) = (x, y_1)$. Otherwise, consider $y_2 = \sup\{y > y_1 \mid y' \in Y_x^+$ for all $y_1 < y' < y\} \in M \cup \{\infty\}$. When $y_2 = \infty$, put $\varphi(x) = (x, y_1 + c)$. Otherwise, set $\varphi(x) = (x, (y_1 + y_2)/2)$. We can define $\varphi : X \to Y$ in the same manner considering Y_x^- instead of Y_x^+ when Y_x^+ is an empty set. We next consider the case in which n > 1. Let $\pi_1 : M^{m+n} \to M^{m+n-1}$ and

We next consider the case in which n > 1. Let $\pi_1 : M^{m+n} \to M^{m+n-1}$ and $\pi_2 : M^{m+n-1} \to M^m$ be the coordinate projections forgetting the last coordinate and forgetting the last n-1 coordinates, respectively. Set $Z = \pi_1(Y)$. We have $\pi_2(Z) = X$. Applying the induction hypothesis to the pair of X and Z, we get a definable map $\varphi_2 : X \to Z$ such that the composition $\pi_2 \circ \varphi_2$ is the identity map. Applying the lemma for n = 1 to the pair of Z and Y, we obtain a definable map $\varphi_1 : Z \to Y$ with $\pi_1(\varphi_1(z)) = z$ for all $z \in Z$. The composition $\varphi = \varphi_1 \circ \varphi_2$ is the definable map we are looking for.

The following curve selection lemma is worth to be mentioned.

Corollary 2.13. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let X be a definable subset of M^n which is not closed. Take a point $a \in cl(X) \setminus X$. There exist a small positive ε and a definable continuous map $\gamma : (0, \varepsilon) \to X$ such that $\lim_{t\to+0} \gamma(t) = a$.

Proof. Let $\pi: M^{n+1} \to M$ be the projection onto the last coordinate. Set $Y = \{(x,t) \in X \times M \mid |a-x| = t\}$. Since \mathcal{M} is locally o-minimal, the intersection $(-\delta, \delta) \cap \pi(Y)$ is a finite union of points and open intervals for a sufficiently small $\delta > 0$. Since the point a belongs to the closure of X, the intersection $(-\delta, \delta) \cap \pi(Y)$ contains an open interval of the form $(0, \varepsilon)$ for some $\varepsilon > 0$. There exists a definable map $\gamma: (0, \varepsilon) \to X$ with $(\gamma(t), t) \in Y$ for all $0 < t < \varepsilon$ by Lemma 2.12. It is obvious that the map γ is bounded. Taking a smaller $\varepsilon > 0$ if necessary, we may assume that γ is continuous by Proposition 2.5(1). The equality $\lim_{t\to +0} \gamma(t) = a$ is obvious by the definition of γ .

We also use the following lemma:

Lemma 2.14. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C and P be definable subsets of M^m and M^n , respectively. Let X be a definable subset of $C \times P$. Let $\pi : M^{m+n} \to M^n$ denotes the projection onto the last n coordinates. Assume that $\dim \pi(X) = \dim P$. Then there exists a point $(c, p) \in X$ such that $\dim \pi(X \cap W) = \dim P$ for all open boxes W in M^{m+n} containing the point (c, p).

Proof. We can find a definable map $\tau : \pi(X) \to X$ such that the composition $\pi \circ \tau$ is the identity map on $\pi(X)$ by Lemma 2.12. Let D be the closure of the set of points at which τ is discontinuous. We have dim $D < \dim \pi(X) = \dim P$ by Proposition 2.7(5),(7),(8). Set $E = \pi(X) \setminus D$. We obtain dim $E = \dim P$ by Proposition 2.7(5). Therefore there exists a point $p \in E$ with dim $(E \cap U) = \dim P$ for all open box Uin M^n containing the point p by Proposition 2.7(10). Set $(c, p) = \tau(p)$. We demonstrate that the point (c, p) satisfies the condition in the lemma. Take an arbitrary sufficiently small open box W in M^{m+n} containing the point (c, p). We may assume that $D \cap \pi(W) = \emptyset$ because $p \notin D$ and D is closed. Since τ is continuous on E, the set $\tau^{-1}(W) = \pi(\tau(E) \cap W)$ is open in E. There exists an open box U in \mathbb{R}^n such that $p \in U$ and $E \cap U \subset \pi(\tau(E) \cap W)$. Shrinking U if necessary, we may assume that U is contained in $\pi(W)$. We have dim $P = \dim E \cap U$ by the definition of the point p. We then get dim $P = \dim E \cap U \leq \dim \pi(\tau(E) \cap W) \leq$ $\dim \pi(X \cap W) \leq \dim \pi(X) \leq \dim P$ by Proposition 2.7(2). We have demonstrated the lemma. \Box

Definition 2.15. Consider a definably complete expansion of an ordered group $\mathcal{M} = (M, <, 0, + ...)$. Let X and Y be definable subsets of M^n . The *distance* of X to Y is given by $\inf\{|x - y| \mid x \in X, y \in Y\}$.

Lemma 2.16. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, + ...)$. Let X and Y be mutually disjoint definable, closed and bounded subsets of M^n . Then, the distance of X to Y is positive.

Proof. We prove the contraposition of the lemma. Assume that the distance of X to Y is zero. We prove that the intersection $X \cap Y$ is not empty when X and Y are closed and bounded. By local o-minimality and the assumption, the set

$$Z = \{ |x - y| \mid x \in X, \ y \in Y \}$$

contains the origin or an open interval of the form $(0, \delta)$, where δ is a positive element. It is obvious that the intersection $X \cap Y$ is not empty when Z contains the origin. We concentrate on the case in which Z contains the open interval $(0, \delta)$ in the rest of the proof. Applying Lemma 2.12 to Z, we can choose definable maps $x, y : (0, \delta) \to M^n$ such that, for any $0 < t < \delta$, $x(t) \in X$, $y(t) \in Y$ and |x(t) - y(t)| = t. These maps are bounded because X and Y are bounded. We may assume that they are continuous by taking a smaller δ if necessary by Proposition 2.5(1). Set $a = \lim_{t\to 0} x(t)$ and $b = \lim_{t\to 0} y(t)$, which exist by Proposition 2.5(2). Since both X and Y are closed, and both the maps x and y are continuous, we have $a \in X, b \in Y$ and |a - b| = 0. It means that the point a(= b) is a common point of X and Y.

3. Functions definable in a definably complete o-minimal expansion of an ordered group

We prove several properties enjoyed by definable functions in this section. We first prove a technical lemma.

Lemma 3.1. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable, closed and bounded subset of \mathbb{R}^m . Let $\varphi, \psi : C \to M_{>0}$ be two definable functions. Assume that the following condition is satisfied:

$$\forall x \in C, \exists \delta > 0, \forall x' \in C, \ |x' - x| < \delta \Rightarrow \varphi(x') \ge \psi(x).$$

Then the inequality $\inf \varphi(C) > 0$ holds true.

Proof. Set $l = \inf \varphi(C) \ge 0$, which exists by the definable completeness of \mathcal{M} . We have only to show that l > 0. Since \mathcal{M} is locally o-minimal, we have $l \in \varphi(C)$ or there exists $u \in \mathcal{M}$ with l < u and $(l, u) \subseteq \varphi(C)$. It is obvious that l > 0 in the former case. We consider the latter case in the rest of the proof.

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Let Γ be the graph of the function φ . Let $\pi_1 : M^{m+1} \to M^m$ and $\pi_2 : M^{m+1} \to M$ be the projections onto the first m coordinates and onto the last coordinate, respectively. We can take a definable map $\eta : (l, u) \to \Gamma$ such that the composition $\pi_2 \circ \eta$ is the identity map on (l, u) by Lemma 2.12. Note that the map η is bounded because the domain of definition C of φ is bounded and the interval (l, u) is bounded. By Proposition 2.5(1), we may assume that η is continuous by taking a smaller u if necessary.

Set $z = \lim_{t \to l+} \eta(t)$, which uniquely exists by Proposition 2.5(2). We have $\pi_2(z) = l$ by the definition of η . Set $c = \pi_1(z)$. It belongs to C because C is bounded and closed. For any t > l sufficiently close to $l, \pi_1(\eta(t)) \in C$ is close to the point c. We have $\pi_2(\eta(t)) = \varphi(\pi_1(\eta(t))) \ge \psi(c)$ for such t by the assumption. We finally obtain $l = \lim_{t \to l+} \pi_2(\eta(t)) \ge \psi(c) > 0$.

We investigate the properties of functions definable in a definably complete ominimal expansion of an ordered group.

Definition 3.2. Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$. Let C and P be definable sets. Let $f : C \times P \to M$ be a definable function. The function f is *equi-continuous* with respect to P if the following condition is satisfied:

$$\forall \varepsilon > 0, \ \forall x \in C, \ \exists \delta > 0, \ \forall p \in P, \ \forall x' \in C, \ |x - x'| < \delta \Rightarrow |f(x, p) - f(x', p)| < \varepsilon.$$

The function f is *uniformly equi-continuous* with respect to P if the following condition is satisfied:

 $\forall \varepsilon > 0, \ \exists \delta > 0, \ \forall p \in P, \ \forall x, x' \in C, \ |x - x'| < \delta \Rightarrow |f(x, p) - f(x', p)| < \varepsilon.$

The function f is *pointwise bounded* with respect to P if the following condition is satisfied:

$$\forall x \in C, \ \exists N > 0, \ \forall p \in P, \ |f(x,p)| < N.$$

Proposition 3.3. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C and P be definable sets. Let $f : C \times P \to M$ be a definable function. Assume that C is closed and bounded. Then f is equi-continuous with respect to P if and only if it is uniformly equi-continuous with respect to P.

Proof. A uniformly equi-continuous definable function is always equi-continuous. We prove the opposite implication.

Take a positive $c \in M$. Consider the definable function $\varphi : C \times M_{>0} \to M_{>0}$ given by

$$\varphi(x,\varepsilon) = \sup\{0 < \delta < c \mid \forall p \in P, \ \forall x' \in C, \ |x - x'| < \delta \Rightarrow |f(x,p) - f(x',p)| < \varepsilon\}.$$

Since f is equi-continuous with respect to P, we have $\varphi(x,\varepsilon) > 0$ for all $x \in C$ and $\varepsilon > 0$. Fix arbitrary $x \in C$ and $\varepsilon > 0$. We also fix an arbitrary point $x' \in C$ with $|x' - x| < \frac{1}{2}\varphi(x, \frac{\varepsilon}{2})$. We have $|f(x', p) - f(x, p)| < \frac{\varepsilon}{2}$ for all $p \in P$ by the definition of φ .

For all $y \in C$ with $|x'-y| < \frac{1}{2}\varphi(x,\frac{\varepsilon}{2})$, we have $|x-y| \le |x-x'|+|x'-y| < \varphi(x,\frac{\varepsilon}{2})$. We get $|f(y,p) - f(x,p)| < \frac{\varepsilon}{2}$ for all $p \in P$ by the definition of φ . We finally obtain $|f(y,p) - f(x',p)| \le |f(x',p) - f(x,p)| + |f(y,p) - f(x,p)| < \varepsilon$ for all $p \in P$. It means that $\varphi(x',\varepsilon) \ge \frac{1}{2}\varphi(x,\frac{\varepsilon}{2})$ whenever $|x'-x| < \frac{1}{2}\varphi(x,\frac{\varepsilon}{2})$. Apply Lemma 3.1 to the definable functions $\varphi(\cdot,\varepsilon)$ and $\frac{1}{2}\varphi(\cdot,\frac{\varepsilon}{2})$ for a fixed $\varepsilon > 0$. We have $\inf \varphi(C,\varepsilon) > 0$. For any $\varepsilon > 0$, set $\delta = \inf \varphi(C, \varepsilon)$. For any $p \in P$ and $x, x' \in C$, we have $|f(x, p) - f(x', p)| < \varepsilon$ whenever $|x - x'| < \delta$ by the definition of φ . It means that f is uniformly equi-continuous.

It is well-known that a continuous function defined on a compact set is uniformly continuous. The following corollary claims that a similar assertion holds true for a definable function defined on a definable, closed bounded set.

Corollary 3.4. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable, closed and bounded set. A definable continuous function $f : C \to M$ is uniformly continuous.

Proof. Let P be a singleton. Apply Proposition 3.3 to the function $g: C \times P \to M$ defined by g(x, p) = f(x).

We define a definable family of functions and investigate its properties. Equicontinuity, convergence and uniform convergence are defined for sequences of functions in classical analysis. We consider similar notions for a definable family of functions.

Definition 3.5. Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, \ldots)$. Let C be a definable set and s be a positive element in M. A family $\{f_t : C \to M\}_{0 < t < s}$ of functions with the parameter variable t is a definable family of functions if there exists a definable function $F : C \times (0, s) \to M$ such that $f_t(x) = F(x, t)$ for all $x \in C$ and 0 < t < s. We call it a definable family of continuous functions if every function f_t is continuous.

Consider a definable family of functions $\{f_t : C \to M\}_{0 < t < s}$. Set I = (0, s). The map $F : C \times I \to M$ given by $F(x,t) = f_t(x)$ is a definable function by the definition. The family is a *definable family of equi-continuous functions* if Fis equi-continuous with respect to I. It is a *definable family of pointwise bounded* functions if F is pointwise bounded with respect to I.

A definable family of functions $\{f_t : C \to M\}_{0 < t < s}$ is *pointwise convergent* if for any positive $\varepsilon > 0$ and for any $x \in C$, there exists s' > 0 such that $|f_t(x) - f_{t'}(x)| < \varepsilon$ for all $t, t' \in (0, s')$.

The following lemma is proved following a typical argument in classical analysis.

Lemma 3.6. Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable set and s be a positive element in M. Consider a pointwise convergent definable family of functions $\{f_t : C \to M\}_{0 < t < s}$. For any $x \in C$, there exists s' > 0 such that the set $\{f_t(x) \mid 0 < t < s'\}$ is bounded.

Proof. Fix $x \in C$. Take a positive $\varepsilon > 0$. There exists s' > 0 such that $|f_t(x) - f_{t'}(x)| < \varepsilon$ for all $t, t' \in (0, s')$. Fix $u \in (0, s')$. For any $t \in (0, s')$, we have

$$|f_t(x)| \le |f_u(x)| + |f_u(x) - f_t(x)| < |f_u(x)| + \varepsilon.$$

It means that the set $\{f_t(x) \mid 0 < t < s'\}$ is bounded.

We also get the following converse when \mathcal{M} is a definably complete locally o-minimal expansion of a densely linearly ordered abelian group.

Lemma 3.7. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable set and s be a positive element in M. A definable family of pointwise bounded functions $\{f_t : C \to M\}_{0 < t < s}$ is pointwise convergent.

Proof. Fix $x \in C$. Set I = (0, s). Consider the definable function $g: I \to M$ given by $g(t) = f_t(x)$. It is bounded. There exists a limit $y = \lim_{t \to +0} g(t)$ by Proposition 2.5(2).

Take an arbitrary positive $\varepsilon > 0$. There exists s' > 0 such that $|y - g(t)| < \varepsilon/2$ for all $t \in (0, s')$. We have $|f_t(x) - f_{t'}(x)| \le |f_t(x) - y| + |y - f_{t'}(x)| < \varepsilon$ whenever $t, t' \in (0, s')$. It means that the family $\{f_t : C \to M\}_{0 < t < s}$ is pointwise convergent. \Box

We define the limit of a pointwise convergent definable family of functions.

Definition 3.8. Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable set and s be a positive element in M. Consider a pointwise convergent definable family of functions $\{f_t : C \to M\}_{0 < t < s}$. For any $x \in C$, consider the function $g_x : (0,s) \to M$ given by $g_x(t) = f_t(x)$. Taking a smaller s > 0 if necessary, we may assume that g_x is bounded by Lemma 3.6. There exists a unique limit $\lim_{t\to+0} g_x(t)$ exists by Proposition 2.5(2). The limit $\lim_{t\to+0} f_t : C \to M$ of the family $\{f_t : C \to M\}_{0 < t < s}$ is defined by $(\lim_{t\to+0} f_t)(x) = \lim_{t\to+0} g_x(t)$.

Definition 3.9. Consider an expansion of a densely linearly ordered abelian group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable set and s be a positive element in M. A definable family of functions $\{f_t : C \to M\}_{0 < t < s}$ is uniformly convergent if for any positive $\varepsilon > 0$, there exists s' > 0 such that $|f_t(x) - f_{t'}(x)| < \varepsilon$ for all $x \in C$ and $t, t' \in (0, s')$.

The following proposition and its proof is almost the same as the counterparts in classical analysis.

Proposition 3.10. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable set and s be a positive element in M. Consider a uniformly convergent definable family of continuous functions $\{f_t : C \to M\}_{0 < t < s}$. The limit $\lim_{t \to +0} f_t : C \to M$ is continuous.

Proof. Fix arbitrary $\varepsilon > 0$ and $x \in C$. Since the family is uniformly convergent, we may assume that $|f_t(x') - f_{t'}(x')| < \frac{\varepsilon}{5}$ for all $x' \in C$ and $t, t' \in (0, s)$ by taking a smaller s > 0 if necessary. Fix t_0 with $0 < t_0 < s$. There exists $\delta > 0$ such that $|f_{t_0}(x') - f_{t_0}(x)| < \frac{\varepsilon}{5}$ whenever $|x - x'| < \delta$ because f_{t_0} is continuous. Fix an arbitrary point $x' \in C$ with $|x - x'| < \delta$. We can take $t_1, t_2 \in (0, s)$ with $|(\lim_{t \to +0} f_t)(x) - f_{t_1}(x)| < \frac{\varepsilon}{5}$ and $|(\lim_{t \to +0} f_t)(x') - f_{t_2}(x')| < \frac{\varepsilon}{5}$ by the definition of the limit $\lim_{t \to +0} f_t$. We finally have

$$\begin{aligned} &|(\lim_{t \to +0} f_t)(x') - (\lim_{t \to +0} f_t)(x)| \\ &\leq |(\lim_{t \to +0} f_t)(x') - f_{t_2}(x')| + |f_{t_2}(x') - f_{t_0}(x')| + |f_{t_0}(x') - f_{t_0}(x)| \\ &+ |f_{t_0}(x) - f_{t_1}(x)| + |f_{t_1}(x) - (\lim_{t \to +0} f_t)(x)| \\ &\leq \varepsilon. \end{aligned}$$

We have proven that $\lim_{t\to+0} f_t$ is continuous.

The following Arzela-Ascoli-type theorem is a main theorem of this paper.

Theorem 3.11. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable, closed and bounded set. A pointwise convergent definable family of equi-continuous functions $\{f_t : C \to M\}_{0 \le t \le s}$ is uniformly convergent.

Proof. Set I = (0, s). Consider the map $F : C \times I \to M$ given by $F(x, t) = f_t(x)$. It is an equi-continuous definable function with respect to I by the definition. Set $g = \lim_{t \to +0} f_t$. It is well-defined by Definition 3.8 because the family is pointwise convergent.

Take c > 0. Consider the definable function $\varphi: C \times M_{>0} \to M_{>0}$ given by

$$\varphi(x,\varepsilon) = \sup\{0 < \delta < c \mid \forall t, t' \in (0,\delta), |F(x,t) - F(x,t')| < \varepsilon\}.$$

We first show that it is well-defined. Fix $x \in C$ and $\varepsilon > 0$. There exists $\delta > 0$ such that $|F(x, u) - g(x)| < \frac{\varepsilon}{2}$ for all $u \in (0, \delta)$ by the definition of g. For any $t, t' \in (0, \delta)$, we have $|F(x, t) - F(x, t')| \leq |F(x, t) - g(x)| + |g(x) - F(x, t')| < \varepsilon$. The definable set $\{0 < \delta < c \mid \forall t, t' \in (0, \delta), |F(x, t) - F(x, t')| < \varepsilon\}$ is not empty and the function φ is well-defined.

We fix $x \in C$ and $\varepsilon > 0$ again. Since F is equi-continuous with respect to I, there exists $\delta' > 0$ such that

$$\forall t \in (0,s), \ \forall x' \in C, \ |x - x'| < \delta' \Rightarrow |F(x,t) - F(x',t)| < \frac{\varepsilon}{3}.$$

Fix an arbitrary $x' \in C$ with $|x - x'| < \delta'$. For any $t, t' \in (0, \varphi(x, \frac{\varepsilon}{3}))$, we have $|F(x,t) - F(x,t')| < \frac{\varepsilon}{3}$ by the definition of φ . We finally get

$$|F(x',t) - F(x',t')| \le |F(x',t) - F(x,t)| + |F(x,t) - F(x,t')| + |F(x,t') - F(x',t')| < \varepsilon$$

whenever $t, t' \in (0, \varphi(x, \frac{\varepsilon}{3}))$. It means that $\varphi(x', \varepsilon) \ge \varphi(x, \frac{\varepsilon}{3})$. Apply Lemma 3.1 to the definable functions $\varphi(\cdot, \varepsilon)$ and $\varphi(\cdot, \frac{\varepsilon}{3})$ for a fixed $\varepsilon > 0$. We have $\inf \varphi(C, \varepsilon) > 0$ for all $\varepsilon > 0$.

Fix $\varepsilon > 0$. Set $\delta = \inf \varphi(C, \varepsilon) > 0$. We have $|f_t(x) - f_{t'}(x)| = |F(t, x) - F(t', x)| < \varepsilon$ for all $x \in C$ and $t, t' \in (0, \delta)$. It means that the family $\{f_t : C \to M\}_{0 < t < s}$ is uniformly convergent.

The above theorem together with the curve selection lemma yields the following corollary:

Corollary 3.12. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C and P be definable sets. Assume that C is closed and bounded. Let $f : C \times P \to M$ be a definable function which is equi-continuous and pointwise bounded with respect to P. Take $p \in cl(P)$. There exists a definable continuous curve $\gamma : (0, \varepsilon) \to P$ such that $\lim_{t\to+0} \gamma(t) = p$ and the definable family of functions $\{g_t : C \to M\}_{0 < t < \varepsilon}$ defined by $g_t(x) = f(x, \gamma(t))$ is uniformly convergent.

Proof. The corollary follows from Corollary 2.13, Lemma 3.7 and Theorem 3.11. \Box

Consider a parameterized function $f: C \times P \to M$ which is equi-continuous with respect to P. We show that the projection image of the set at which f is discontinuous onto the parameter space P is of dimension smaller than dim P when C is closed.

Theorem 3.13. Consider a definably complete expansion of an ordered group $\mathcal{M} = (M, <, +, 0, ...)$. Let C be a definable closed set and P be a definable set. Let $\pi : C \times P \to P$ be the projection. Consider a definable function $f : C \times P \to M$ which is equi-continuous with respect to P. Set $D = \{(x,q) \in C \times P \mid f \text{ is discontinuous at } (x,q)\}$. We have $\dim \pi(D) < \dim P$.

Proof. Let C and P be definable subsets of M^m and M^n , respectively.

We first consider the case in which C is bounded. Consider the set

 $S = \{(x, p) \in D \mid \exists U \subset M^m : \text{open box with } x \in U \text{ and } C \cap U = D_p \cap U\},\$

where the notation D_p denotes the fiber $\{x \in C \mid (x, p) \in D\}$. We demonstrate that $\dim \pi(S) < \dim P$.

Assume for contradiction that $\dim \pi(S) = \dim P$. We can take an open box Bin M^m such that $\dim \pi(S) \cap B = \dim P$ and $\pi(S) \cap B = P \cap B$ by Corollary 2.8. Using Lemma 2.12, we can construct a definable map $\tau : P \cap B \to S$ such that the composition $\pi \circ \tau$ is the identity map on $P \cap B$. The set of points at which τ is discontinuous is of dimension smaller than $\dim P$ by Proposition 2.7(7). By shrinking B if necessary, we may assume that τ is continuous by Proposition 2.7(5) and Corollary 2.8. Fix a positive K > 0. Consider the definable map $\varphi : P \cap B \to M$ given by

$$\varphi(x) = \sup\{0 < \lambda < K \mid C \cap \mathcal{B}_m(\tau(p), \lambda) = D_p \cap \mathcal{B}_m(\tau(p), \lambda)\}.$$

We may assume that φ is continuous in the same manner as above. Set

$$W = \bigcup_{p \in P \cap B} (\mathcal{B}_m(\tau(p), \varphi(p)) \cap C) \times \{p\}.$$

It is obvious that W is an open subset of $C \times P$ and $W \subseteq D$. Since W is open in $C \times P$ and $W \subseteq D$, the restriction of f to W is discontinuous everywhere. It contradicts Proposition 2.7(7). We have demonstrated the inequality dim $\pi(S) < \dim P$.

We next demonstrate that $\dim \pi(D) < \dim P$. We lead to a contradiction assuming the contrary. Set $T = D \setminus \pi^{-1}(\pi(S))$. We have $\dim \pi(T) = \dim P$ by Proposition 2.7(5) because $\dim \pi(S) < \dim P$. There exists a point $(c, p) \in T$ such that $\dim \pi(T \cap W) = \dim P$ for all open box W in M^{m+n} containing the point (c, p)by Lemma 2.14. Fix an arbitrary $\varepsilon > 0$. Since f is uniformly equi-continuous with respect to P by the assumption and Proposition 3.3, there exists $\delta > 0$ satisfying the following condition:

(1)
$$\forall q \in P, \ \forall x, x' \in C, \ |x - x'| < \delta \Rightarrow |f(x, q) - f(x', q)| < \varepsilon/3.$$

On the other hand, we have $D_p \cap U \subsetneq C \cap U$ for any open box U containing the point c by the definition of the set S because $(c, p) \notin S$. In particular, there exists $c_0 \in C$ such that $|c - c_0| < \delta/2$ and $(c_0, p) \notin D$. It implies that the function f is continuous at the point (c_0, p) . There exists $\delta' > 0$ such that

(2)
$$\forall q \in P, \ |q-p| < \delta' \Rightarrow |f(c_0,q) - f(c_0,p)| < \varepsilon/3.$$

Consider an arbitrary point $(c', p') \in C \times P$ with $|c - c'| < \delta/2$ and $|p - p'| < \delta'$. We have $|f(c_0, p) - f(c, p)| < \varepsilon/3$ by the inequality (1) because $|c - c_0| < \delta/2$. We also have $|f(c', p') - f(c_0, p')| < \varepsilon/3$ by (1) because $|c' - c_0| \le |c' - c| + |c - c_0| < \delta$. We get

$$|f(c',p') - f(c,p)| \le |f(c',p') - f(c_0,p')| + |f(c_0,p') - f(c_0,p)| + |f(c_0,p) - f(c,p)| < \varepsilon$$

by the above inequalities together with the inequality (2). We have demonstrated that f is continuous at (c, p). It is a contradiction to the condition that $(c, p) \in T \subset D$. We have demonstrated the theorem when C is bounded.

We next treat the general case. The definable closed set C is not necessarily bounded. For any r > 0, we set $B\langle r \rangle = [-r, r]^n$ and $C\langle r \rangle = B\langle r \rangle \cap C$. We also

set $D\langle r \rangle = \{(x,q) \in C\langle r \rangle \times P \mid f|_{C\langle r \rangle \times P}$ is discontinuous at $(x,q)\}$ for all r > 0. Here, the notation $f|_{C\langle r \rangle \times P}$ denote the restriction of f to $C\langle r \rangle \times P$. We obviously have $D = \bigcup_{r>0} D\langle r \rangle$ and $\pi(D) = \bigcup_{r>0} \pi(D\langle r \rangle)$. The family $\{\pi(D\langle r \rangle)\}_{r>0}$ is a definable increasing family. Since the theorem holds true when C is bounded, we have $\dim \pi(D\langle r \rangle) < \dim P$ for all r > 0. We get $\dim \pi(D) < \dim P$ by Proposition 2.11.

4. Decomposition into special submanifolds

4.1. **Definition of quasi-special/special submanifolds.** We first introduce our definition of special manifolds.

Definition 4.1. Consider an expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, ...)$. Let $\pi : M^n \to M^d$ be a coordinate projection, where n is a positive integer and d is a non-negative integer with $d \leq n$. We consider that M^0 is a singleton equipped with the trivial topology and the projection $\pi : M^n \to M^0$ is the trivial map when d = 0. Let τ be the unique permutation of $\{1, ..., n\}$ such that

- (a) $\tau(i) < \tau(j)$ for $1 \le i < j \le n$ when $\tau(i) > d$ and $\tau(j) > d$.
- (b) The composition $\pi \circ \overline{\tau}$ is the projection onto the first *d* coordinates, where $\overline{\tau}: M^n \to M^n$ is the map defined by $\overline{\tau}(x_1, \ldots, x_n) = (x_{\tau(1)}, \ldots, x_{\tau(n)})$.

Set

$$\operatorname{fib}(X,\pi,x) = \{ y \in M^{n-d} \mid (x,y) \in \overline{\tau}^{-1}(X) \}$$

for $x \in \pi(X)$. Note that fib $(X, \pi, x) = \{y \in M^{n-d} \mid (x, y) \in X\}$ when π is the projection onto the first d coordinates.

When π is the coordinate projection onto the first d coordinate, a definable subset X of M^n is a π -special submanifold if, for any $x \in M^d$, there exist an open box U in M^d containing the point x and a family $\{V_y\}_{y \in \operatorname{fib}(X,\pi,x)}$ of mutually disjoint open boxes in M^n indexed by the set $\operatorname{fib}(X,\pi,x)$ such that

- (1) $\pi(V_y) = U$ for all $y \in \operatorname{fib}(X, \pi, x)$;
- (2) $X \cap \pi^{-1}(U)$ is contained in $\bigcup_{y \in \operatorname{fib}(X,\pi,x)} V_y$, and
- (3) $V_y \cap X$ is the graph of a continuous map defined on U for each $y \in fib(X, \pi, x)$.

We do not require that the union $\bigcup_{y \in \operatorname{fib}(X,\pi,x)} V_y$ is definable.

When π is not the coordinate projection onto the first d coordinate, we say that a definable subset X of M^n is π -special submanifold if $\overline{\tau}^{-1}(X)$ is $\pi \circ \overline{\tau}$ -special submanifold. We omit the prefix π when it is clear from the context.

Note that a discrete, closed definable subset of M^n is always a π -special submanifold, where $\pi: M^n \to M^0$ is the trivial map.

We also need the following definition:

Definition 4.2. Consider an expansion of a dense linear order without endpoints $\mathcal{M} = (M, <, ...)$. Let $\pi : M^n \to M^d$ be a coordinate projection. Let X be a definable subset of M^n . Let τ be the permutation of $\{1, ..., n\}$ satisfying the conditions in Definition 4.1.

A point $(a, b) \in M^n$ is (X, π) -normal if there exist a definable neighborhood A of a in M^d and a definable neighborhood B of b in M^{n-d} such that either $A \times B$ is disjoint from $\overline{\tau}^{-1}(X)$ or $(A \times B) \cap \overline{\tau}^{-1}(X)$ is the graph of a definable continuous map $f: A \to B$.

We get the following:

Lemma 4.3. Let $\mathcal{M} = (M, <, 0, +, ...)$ be a definably complete locally o-minimal expansion of an ordered group. Let $\pi : M^n \to M^d$ denote the projection onto the first d coordinates. Let X be a definable subset of M^n . Assume that any point in X is (X, π) -normal. Then, for any $x \in \pi(X)$ and sufficiently small positive $\varepsilon \in M$, there exists an open box U in M^d containing the point x such that the intersection $X \cap (U \times \mathcal{B}_{n-d}(y, \varepsilon))$ is the graph of a continuous map for each $y \in \operatorname{fib}(X, \pi, x)$.

In particular, when X is a π -special submanifold, for any $x \in \pi(X)$ and sufficiently small positive $\varepsilon \in M$, there exists an open box U in M^d containing the point x such that the pair $(U, \{U \times \mathcal{B}_{n-d}(y, \varepsilon)\}_{y \in \operatorname{fib}(X,\pi,x)})$ satisfies the conditions (1) through (3) in Definition 4.1.

Proof. We fix $x \in \pi(X)$. Set $D = \operatorname{fib}(X, \pi, x)$ for simplicity. Fix a positive element $c \in M$. We temporarily fix $y \in D$. Because the point (x, y) is (X, π) -normal, the set $(\mathcal{B}_d(x, \delta) \times \mathcal{B}_{n-d}(y, \varepsilon')) \cap X$ is the graph of a definable continuous map defined on $\mathcal{B}_d(x, \delta)$ when δ and ε' are sufficiently small positive elements. Consider the function $\sigma : D \to M$ given by

$$\sigma(y) = \sup\{0 < \varepsilon' < c \mid \exists \delta > 0 \quad (\mathcal{B}_d(x, \delta) \times \mathcal{B}_{n-d}(y, \varepsilon')) \cap X \text{ is}$$
the graph of a continuous map}.

It is definable and the image $\sigma(D)$ is discrete and closed by Proposition 2.7(1),(6). Set $\varepsilon_0 = \inf \sigma(D)$.

Fix a sufficiently small $\varepsilon > 0$ so that $\varepsilon < \varepsilon_0$. Consider the function $\tau : D \to M$ given by

$$\tau(y) = \sup\{0 < \delta < c \mid (\mathcal{B}_d(x, \delta) \times \mathcal{B}_{n-d}(y, \varepsilon)) \cap X \text{ is}$$

the graph of a continuous map}.

The image $\tau(D)$ is discrete and closed for the same reason. Set $\tilde{\delta} = \inf \tau(D)$. The open box $U = \mathcal{B}_d(x, \tilde{\delta})$ satisfies the requirement of the lemma.

Let us consider the case in which X is a π -special manifold. Let $(U', \{V_y\}_{y \in D})$ be a pair satisfying the conditions (1) through (3) in Definition 4.1. It is easy to check that the pair $(U \cap U', \{(U \cap U') \times \mathcal{B}_{n-d}(y, \varepsilon)\}_{y \in D})$ satisfies the conditions (1) through (3) in Definition 4.1. We omit the details.

We recall the definition of a quasi-special submanifold.

Definition 4.4. Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, <, ...)$. Let $\pi : M^n \to M^d$ be a coordinate projection. A definable subset is a π -quasi-special submanifold or simply a quasi-special submanifold if, for every point $x \in \pi(X)$, we can take an open box U in M^d containing the point xand a family $\{V_y\}_{y \in \operatorname{fib}(X,\pi,x)}$ of mutually disjoint open boxes in M^n indexed by the set $\operatorname{fib}(X,\pi,x)$ satisfying the conditions (1) and (3) in Definition 4.1.

It is obvious that a special submanifold is always a quasi-special submanifold. The following example illustrates that the converse is false in general.

Example 4.5. Consider the ordered field of reals $(\mathbb{R}, <, 0, 1, +, \cdot)$. The set

$$\{(x,0) \mid x \in \mathbb{R}\} \cup \{(x,1/x) \mid x > 0\}$$

is definable and a quasi-special submanifold, but it is not a special submanifold. We can not take an open box U and a family of open boxes $\{V_y\}_{y \in \operatorname{fib}(X,\pi,x)}$ satisfying the condition (2) in Definition 4.1 at x = 0.

We use the following lemma:

Lemma 4.6. Consider a definably complete locally o-minimal structure $\mathcal{M} = (M, <, ...)$. Let X be a definable subset of M^n and $\pi : M^n \to M^d$ be a coordinate projection. Assume that all the points $x \in X$ are (X, π) -normal. Then, X is a π -quasi-special submanifold.

Proof. It immediately follows from [6, Lemma 4.2] and [7, Theorem 2.5]. \Box

We also need the following technical definition:

Definition 4.7. Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, <, ...)$. Let X be a definable subset of M^n and $\pi : M^n \to M^d$ be a coordinate projection. Let x be a point in $\pi(X)$. We say that (X, π) is *locally bounded at* x if there exist a bounded open box U in M^d containing the point x such that $X \cap \pi^{-1}(U)$ is bounded.

We give a sufficient condition for a quasi-special submanifold to be a special submanifold.

Lemma 4.8. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, ...)$. Let $\pi : M^n \to M^d$ be a coordinate projection. A π -quasi-special submanifold X in M^n is a π -special submanifold if it is closed in $\pi^{-1}(\pi(X))$ and (X, π) is locally bounded at every point in $\pi(X)$.

Proof. We may assume that π is the coordinate projection onto the first d coordinates by permuting coordinates if necessary. Fix an arbitrary point $x \in \pi(X)$. Set $Y = \operatorname{fib}(X, \pi, x) = \{y \in M^{n-d} \mid (x, y) \in X\}$. Note that every point in X is (X, π) -normal by the definition of quasi-special submanifolds. Fix a sufficiently small positive $\varepsilon > 0$. We can take an open box B containing the point x such that $X \cap (B \times \mathcal{B}_{n-d}(y, \varepsilon))$ is the graph of a continuous map defined on B for each $y \in Y$ by Lemma 4.3. We may assume that B is bounded and $X \cap (\operatorname{cl}(B) \times \mathcal{B}_{n-d}(y, \varepsilon))$ is the graph of a continuous map defined on E for each $y \in Y$ by Lemma 4.3. We may assume that B is bounded and $X \cap (\operatorname{cl}(B) \times \mathcal{B}_{n-d}(y, \varepsilon))$ is the graph of a continuous map defined on cl(B) by shrinking B if necessary.

Since (X, π) is locally bounded at x, by shrinking B again if necessary, $\pi^{-1}(B) \cap X$ is bounded. Take a bounded open box W so that $\pi^{-1}(B) \cap X$ is contained in $B \times W$. Put $Z = \operatorname{cl}(W) \setminus \bigcup_{y \in Y} \mathcal{B}_{n-d}(y, \varepsilon)$. It is easy to show that the definable sets

$$C_1 = \{x\} \times Y$$
 and
 $C_2 = X \cap (\operatorname{cl}(B) \times Z)$

are closed and bounded, and their intersection is empty. The proof is left to readers. Let \mathfrak{d} be the distance of C_1 to C_2 . It is positive by Lemma 2.16. Choose $\overline{\delta} > 0$ so that $\overline{\delta} < \mathfrak{d}$ and $\mathcal{B}_d(x,\overline{\delta}) \subseteq B$. The pair $(\mathcal{B}_d(x,\overline{\delta}), \{\mathcal{B}_d(x,\overline{\delta}) \times \mathcal{B}_{n-d}(y,\varepsilon)\}_{y \in Y})$ satisfies the conditions (1) through (3) in Definition 4.1. We have proven that X is a π -special submanifold.

4.2. Comparison with other definitions. We demonstrate that a special manifold defined in [11] and a multi-cell in [3] coincide with a special manifold in our sense when the structure is definably complete locally o-minimal. We first recall Miller's definition of special manifolds. **Definition 4.9** ([11]). We only consider an expansion of the ordered set of reals $(\mathbb{R}, <)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be a coordinate projection. A *d*-dimensional submanifold X of \mathbb{R}^n is π -special if, for each $x \in \pi(X)$, there exists an open box U in \mathbb{R}^d containing the point x such that each connected component C of $X \cap \pi^{-1}(U)$ projects homeomorphically onto U.

Note that there are no connected definable sets other than singletons in some ordered structure whose universe is not \mathbb{R} . For instance, let \mathbb{R}_{alg} denote the set of algebraic real numbers. The structure ($\mathbb{R}_{alg}, 0, 1, +, \cdot$) is o-minimal because sets definable in this structures are semialgebraic by Tarski-Seidenberg principle [1, Theorem 2.2.1] and any nonempty open interval is not connected by [1, Example 2.4.1]. We can easily derive that a connected definable set is a singleton in this case. This example illustrates that Miller's definition of special submanifold cannot be extended literally to the non-real cases.

We show that Miller's definition coincides with ours when the structure is a locally o-minimal expansion of the ordered set of reals $(\mathbb{R}, <)$.

Proposition 4.10. Consider an expansion of the ordered set of reals $(\mathbb{R}, <)$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ be a coordinate projection. A definable subset of \mathbb{R}^n is a π -special submanifold in the sense of Definition 4.9 if it is a π -special submanifold in the sense of Definition 4.1. The opposite implication holds true when the structure is locally o-minimal.

Proof. We may assume that π is the projection onto the first d coordinates without loss of generality. We fix a definable subset X of \mathbb{R}^n .

Assume first that X is π -special submanifold in the sense of Definition 4.1. Fix an arbitrary point $x \in \pi(X)$. Take an open box U in \mathbb{R}^d containing the point x and a family $\{V_y\}_{y \in \operatorname{fib}(X,\pi,x)}$ of open boxes in \mathbb{R}^n satisfying the conditions in Definition 4.1. Since $\{V_y\}_{y \in \operatorname{fib}(X,\pi,x)}$ is a family of mutually disjoint open boxes, $X \cap V_y$ are connected components of $X \cap \pi^{-1}(U)$. They are graphs of continuous maps, and they project homeomorphically onto U. It means that X is a π -special submanifold in the sense of Definition 4.9.

We next show the opposite implication. Assume that X is a π -special submanifold in the sense of Definition 4.9. It is obvious that all the points $x \in X$ are (X, π) -normal. Therefore, X is a π -quasi-special submanifold by Lemma 4.6. Fix an arbitrary point $x \in \pi(X)$. Since X is a quasi-special submanifold, we can take an open box U containing the point x and, for each $y \in X$ with $\pi(y) = x$, there exists an open box V_y such that $y \in V_y$, $\pi(V_y) = U$ and $V_y \cap X$ is the graph of a continuous function defined on U. Shrinking U if necessary, we may assume that any connected component of $X \cap \pi^{-1}(U)$ projects homeomorphically onto U by the assumption. Consequently, each connected component of $X \cap \pi^{-1}(U)$ is the graph of a continuous map defined on U and contained in some V_y . It means that $X \cap \pi^{-1}(U) \subseteq \bigcup_{y \in \operatorname{fib}(X,\pi,x)} V_y$. Therefore, X is a π -special submanifold in the sense of Definition 4.1.

We next recall the definition of Fornasiero's multi-cell.

Definition 4.11. Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, ...)$ be an expansion of an ordered commutative field. Let X be a definable subset of F^n of dimension d and $\pi : F^n \to F^d$ be a coordinate projection. Take the permutation τ of $\{1, \ldots, n\}$ satisfying the conditions (a) and (b) in Definition 4.1. The notation $\overline{\tau}$ denotes the map defined

in Definition 4.1. We first consider the case in which $\overline{\tau}^{-1}(X) \subseteq F^d \times (0,1)^{n-d}$. A point $a \in F^d$ is (X,π) -bad if it is the projection of a non- (X,π) -normal point; otherwise, the point a is called (X,π) -good.

Consider the case in which X does not satisfy the previous condition. Let ϕ : $F \to (0, 1)$ be a definable homeomorphism. Consider the map

$$\psi: \mathrm{id}^d \times \phi^{n-d}: F^d \times F^{n-d} \to F^d \times (0,1)^{n-d}.$$

We say that a is (X, π) -good if it is $(\psi(\overline{\tau}^{-1}(X)), \pi \circ \overline{\tau})$ -good. We define (X, π) -bad points etc. similarly.

The definable set X is a π -multi-cell if every point of $\pi(X)$ is (X, π) -good.

Fornasiero concentrates on the case in which the structure is an expansion of an ordered commutative field. The following fact is well-known. We give a proof for the readers' convenience.

Lemma 4.12. A discrete set definable in a definably complete locally o-minimal expansion of an ordered field is bounded.

Proof. Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, \ldots)$ be a definably complete locally o-minimal expansion of an ordered field. Let X be a definable discrete subset of F^n . We first demonstrate X is bounded when n = 1. Consider the set $Y_+ = \{1/x \mid x > 0 \text{ and } x \in X\}$. By local o-minimality, there exists r > 0 such that the intersection $Z_+ = (-r, r) \cap Y_+$ is a finite union of points and open intervals. If it contains an open interval, X also contains an open interval. It is a contradiction. If Z_+ is empty, $X \subseteq (-\infty, 1/r)$. Otherwise, there exists the smallest element $s \in Z_+$. We have $X \subseteq (-\infty, 1/s)$ in this case. We have demonstrated that X is bounded above. We can prove that X is bounded below in the same manner. We omit the details. We have demonstrated that X is bounded when n = 1.

We consider the case in which n > 1. Let $\pi_i : F^n \to F$ denote the projection onto the *i*-th coordinate for $1 \le i \le n$. The projection image $\pi_i(X)$ is discrete by Proposition 2.7(1),(6), and it is bounded by this lemma for n = 1. There exists a bounded open interval I_i with $\pi_i(X) \subseteq I_i$ for each *i*. It is obvious that $X \subseteq \prod_{i=1}^n I_i$. It implies that X is bounded.

Fornasiero's multi-cell is equivalent to our special submanifold when the structure is a definably complete locally o-minimal expansion of an ordered field.

Proposition 4.13. Let $\mathcal{F} = (F, <, +, 0, \cdot, 1, ...)$ be a definably complete locally ominimal expansion of an ordered field. Let $\pi : F^n \to F^d$ be a coordinate projection. A definable set is a π -special submanifold in the sense of Definition 4.1 if and only if it is a π -multi-cell.

Proof. Let X be a definable subset of F^n . We may assume that π is the projection onto the first d coordinates as usual. We first demonstrate that, if X is a π -special submanifold, it is a π -multi-cell. Let $\psi : F^d \times F^{n-d} \to F^d \times (0,1)^{n-d}$ be the definable homeomorphism given in Definition 4.11. By the definition of π -special manifold, it is obvious that any point in $\pi(X) \times F^{n-d}$ is $(\psi(X), \pi)$ -normal out of the set $\pi(X) \times \partial((0,1)^{n-d})$, where $\partial((0,1)^{n-d})$ denotes the frontier of $(0,1)^{n-d}$. Fix an arbitrary point $x \in \pi(X)$ and arbitrary $t \in \partial((0,1)^{n-d})$. We can choose a bounded closed box B in F^{n-d} such that the set $\{y \in F^{n-d} \mid (x,y) \in X\}$ is contained in B by Lemma 4.12. Thanks to Lemma 4.3, expanding B a little bit, we can take an open box U containing the point x such that $X \cap \pi^{-1}(U)$ is contained in $U \times B$. The image $\psi(B)$ is contained in $(0,1)^{n-d}$ and closed in F^{n-d} . We can take an open box V in F^{n-d} containing the point t and having an empty intersection with $\psi(B)$. The intersection $(U \times V) \cap \psi(X)$ is empty. It means that (x,t) is $(\psi(X), \pi)$ -normal. We have demonstrated that X is π -multi-cell.

We next show the opposite implication. Assume that X is a π -multi-cell. As an initial step, we show that (X, π) is locally bounded at every point in $\pi(X)$. Fix a point $x \in \pi(X)$. Assume for contradiction that $(\mathcal{B}_d(x, t) \times F^{n-d}) \cap X$ is not bounded for any t > 0. It implies that the set

$$S(t) := (\mathcal{B}_d(x, t) \times (F^{n-d} \setminus \mathcal{B}_{n-d}(O, 1/t))) \cap X$$

is not empty for any t > 0, where O denotes the origin of F^d . By Lemma 2.12, we can find a definable function $f: (0, \infty) \to F^n$ such that f(t) is an element of S(t) for t > 0. There exists a unique limit $z = \lim_{t\to 0} \psi(f(t)) \in \partial((0, 1)^{n-d})$ by Proposition 2.5(2). The point $(x, z) \in F^d \times F^{n-d}$ is in the frontier of $\psi(X)$. In particular, it is not contained in $\psi(X)$, and any open box containing it has a nonempty intersection with $\psi(X)$. It implies that the point (x, z) is not $(\psi(X), \pi)$ -normal. It contradicts the assumption that X is a π -multi-cell. We have demonstrated that (X, π) is locally bounded at every point in $\pi(X)$.

It is obvious that any point in X is (X, π) -normal. The π -multi-cell X is a π -quasi-submanifold by Lemma 4.6. It is obvious that X is closed in $\pi^{-1}(\pi(X))$ from the definition of multi-cells. The set X is a π -special submanifold by Lemma 4.8.

4.3. **Decomposition into special submanifolds.** We finally demonstrate the main theorem which asserts that any definable set is decomposed into finitely many special submanifolds. We consider the following cases separately.

- (A) There exists a definable, discrete, closed and unbounded subset of M;
- (B) The structure is a model of DCTC.

We first consider the case (A).

Lemma 4.14. Let $\mathcal{M} = (M, <, ...)$ be a definably complete structure. Let X be a definable discrete closed subset of M such that $\sup(X) = \infty$. Then, there exists a definable map succ : $X \to X$ such that $x < \operatorname{succ}(x)$ and there are no elements in X between x and $\operatorname{succ}(x)$.

Proof. Define $\operatorname{succ}(x) = \inf\{y \in X \mid y > x\}$. We have $\operatorname{succ}(x) \in X$ because X is closed. We get $x < \operatorname{succ}(x)$ because X is discrete. It is obvious that there are no elements in X between x and $\operatorname{succ}(x)$.

Lemma 4.15. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, ...)$. Assume that there exists a definable, discrete, closed and unbounded subset D of M. Let $\pi : M^n \to M^d$ be a coordinate projection. A π -quasi-special submanifold X of M^n which is closed in $\pi^{-1}(\pi(X))$ is a π -special submanifold.

Proof. We may assume that π is the projection onto the first d coordinate without loss of generality. We may further assume that $\sup(D) = \infty$ and $\inf(D) = -\infty$ by replacing D with $D \cup \{-x \mid x \in D\}$. Set e = n - d. Let succ : $D \to D$ denote the successor map given in Lemma 4.14. For any $x = (x_1, \ldots, x_e) \in D^e$, we set

 $\mathcal{C}(x) = \{ (y_1, \dots, y_e) \in M^e \mid x_i \le y_i \le \operatorname{succ}(x_i) \text{ for all } 1 \le i \le e \}.$

The set $\mathcal{C}(x)$ is a bounded closed box.

We fix an arbitrary point a in $\pi(X)$. Take a bounded open box B such that B contains the point a and the closure cl(B) is contained in $\pi(X)$. It is possible because $\pi(X)$ is open. Set $E = fib(x, \pi, a) = \{y \in M^e \mid (a, y) \in X\}$. Fix a sufficiently small positive element $\varepsilon \in M$. By Lemma 4.6, we may assume that $X \cap (B \times \mathcal{B}_e(y, \varepsilon))$ is the graph of a continuous function defined on B for each $y \in E$ by shrinking B if necessary. Set

$$Y = ((\mathrm{cl}(B) \times M^e) \cap X) \setminus \bigcup_{y \in E} B \times \mathcal{B}_e(y, \varepsilon).$$

It is closed because X is closed in $\pi(X) \times M^e = \pi^{-1}(\pi(X))$ and $\operatorname{cl}(B) \subseteq \pi(X)$. If Y is an empty set, the pair $(B, \{B \times \mathcal{B}_e(y, \varepsilon)\}_{y \in E})$ satisfies the conditions (1) through (3) in Definition 4.1.

We next consider the case in which $Y \neq \emptyset$. Set $F = \{x \in D^e \mid (\operatorname{cl}(B) \times \mathcal{C}(x)) \cap Y \neq \emptyset\}$. It is not empty. In addition, we have $\bigcup_{x \in F} (\operatorname{cl}(B) \times \mathcal{C}(x)) \cap Y = Y$ by the definition of D. Define the function $\rho : F \to M$ so that $\rho(x)$ is the distance of the singleton $\{a\}$ to the definable set $\pi((\operatorname{cl}(B) \times \mathcal{C}(x)) \cap Y)$. The set $\pi((\operatorname{cl}(B) \times \mathcal{C}(x)) \cap Y)$ does not contain the point a by the definition of Y. It is closed and bounded by [10, Lemma 1.7] because $(\operatorname{cl}(B) \times \mathcal{C}(x)) \cap Y$ is bounded and closed. They imply that $\rho(x) > 0$ for all $x \in F$ by Lemma 2.16. The definable set F is of dimension zero by Proposition 2.7(1),(2),(4). The image $\rho(F)$ is of dimension zero and it is discrete and closed by Proposition 2.7(1),(6). In particular, we have $\inf \rho(F) \in \rho(F)$ and $\inf \rho(F) > 0$. Take $\delta > 0$ so that $\delta < \inf \rho(F)$ and the box $\mathcal{B}_d(a, \delta)$ is contained in B. It is obvious that $\pi^{-1}(\mathcal{B}_d(a, \delta)) \cap X \subseteq \bigcup_{y \in E} \mathcal{B}_d(a, \delta) \times \mathcal{B}_e(y, \varepsilon)$. It implies that the pair $(\mathcal{B}_d(a, \delta), \{\mathcal{B}_d(a, \delta)) \times \mathcal{B}_e(y, \varepsilon)\}_{y \in E})$ satisfies the conditions (1) through (3) in Definition 4.1.

We next treat the case (B).

Lemma 4.16. Consider a model of DCTC $\mathcal{M} = (M, <, 0, +, ...)$. Let $\pi : M^n \to M^d$ be a coordinate projection and X be a π -quasi-special submanifold X of M^n which is closed in $\pi^{-1}(\pi(X))$. The definable set NLB (X, π) given by

$$NLB(X,\pi) = \{x \in \pi(X) \mid (X,\pi) \text{ is not locally bounded at } x\}$$

has an empty interior.

Proof. It is obvious that $NLB(X, \pi)$ is definable. We omit the details. The remaining task is to show that it has an empty interior.

Assume for contradiction that $\operatorname{NLB}(X, \pi)$ contains a nonempty open box B. As usual, we may assume that π is the projection onto the first d coordinates. Set e = n - d. Let $\rho_i : M^e \to M$ be the projection to the *i*-th coordinate for $1 \leq i \leq e$. Set $Y_{i,x} = \rho_i(\operatorname{fib}(X, \pi, x)) = \rho_i(\{y \in M^e \mid (x, y) \in X\})$ for $x \in \pi(X)$ and $1 \leq i \leq e$. By the definition of quasi-special submanifolds, the set $\operatorname{fib}(X, \pi, x)$ is discrete. The sets $Y_{i,x}$ are discrete and closed by Proposition 2.7(1),(6). They are bounded by the definition of a model of DCTC. Consider the functions $u_i, l_i : B \to M$ given by $l_i(x) = \inf Y_{i,x}$ and $u_i(x) = \sup Y_{i,x}$ for $1 \leq i \leq e$. Shrinking B if necessary, we may assume that u_i and l_i are continuous by Proposition 2.7(7).

Fix a point $a \in B$. Take a closed box C such that $a \in int(C)$ and $C \subseteq B$. There are L_i and U_i in M such that $L_i < l_i(x) \le u_i(x) < U_i$ for all $x \in C$ and $1 \le i \le e$ by [10, Proposition 1.10]. By the definitions of L_i and U_i , we have

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 $\pi^{-1}(\operatorname{int}(C)) \cap X \subseteq \operatorname{int}(C) \times (\prod_{i=1}^{e} (L_i, U_i))$. It means that (X, π) is locally bounded at the point *a*. It is a contradiction because $a \in \operatorname{NLB}(X, \pi)$. We have demonstrated that $\operatorname{NLB}(X, \pi)$ has an empty interior. \Box

The following is the main part of the proof:

Lemma 4.17. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, ...)$. Let X be a definable subset of M^n . There exists a family $\{C_i\}_{i=1}^N$ of mutually disjoint special submanifolds with $X = \bigcup_{i=1}^N C_i$. Furthermore, the number N of special submanifolds is not greater than the number uniquely determined only by n.

Proof. By [6, Lemma 4.3], the definable set X is decomposed into finitely many mutually disjoint quasi-special submanifolds and the number of the quasi-special submanifolds is not greater than the number uniquely determined only by n. Therefore, we may assume that X is a π -quasi-special submanifold of dimension d, where $\pi: M^n \to M^d$ is a coordinate projection.

We prove the lemma by induction on d. The definable set X is obviously a special submanifold when d = 0. We consider the case in which d > 0. The frontier ∂X of X is of dimension < d by Proposition 2.7(8). When X is closed, set $Z_1 = \emptyset$ and $X_1 = X$. It is obvious that X_1 is closed in $\pi^{-1}(\pi(X_1))$ in this case. We next consider the case X is not closed. The closure of the image $Y_1 := cl(\pi(\partial X))$ is of dimension < d by Proposition 2.7(5),(6),(8). The intersection $\pi^{-1}(x) \cap X$ is discrete by the definition of quasi-special submanifolds for each $x \in Y_1$. It is of dimension zero by Proposition 2.7(1). Set $Z_1 = \pi^{-1}(Y_1) \cap X$ and $X_1 = X \setminus Z_1$. We have dim $Z_1 = \dim Y_1 + \dim(\pi^{-1}(x) \cap X) = \dim Y_1 < d$ for any $x \in Y_1$ by Proposition 2.7(9). Since each point in X_1 is (X_1, π) -normal, X_1 is a quasi-special submanifold by Lemma 4.6. It is obvious that X_1 is closed in $\pi^{-1}(\pi(X_1))$.

We treat two separate cases. We first consider the case in which there exists an unbounded definable discrete subset of M. The quasi-special submanifold X_1 is a special submanifold by Lemma 4.15. By the induction hypothesis, Z_1 is decomposed into mutually disjoint special submanifolds C_1, \ldots, C_N . The decomposition $X = X_1 \cup \bigcup_{i=1}^N C_i$ is the desired decomposition.

The latter case is the case in which all definable discrete subsets of M are bounded. The structure \mathcal{M} is a model of DCTC by Definition 2.1. The definable set NLB (X_1, π) is of dimension $\langle d \rangle$ by Lemma 4.16. Set $Y_2 = \operatorname{cl}(\operatorname{NLB}(X_1, \pi))$. We have dim $Y_2 \langle d \rangle$ by Proposition 2.7(5),(8). Set $Z_2 = \pi^{-1}(Y_2) \cap X_1$ and $X_2 = X_1 \setminus Z_2$. The pair (X_2, π) is obviously locally bounded at every point in $\pi(X_2)$. Apply the same argument for X_1 and Z_1 to X_2 and Z_2 . The definable set X_2 is a quasi-special submanifold which is closed in $\pi^{-1}(\pi(X_2))$ and dim $Z_2 \langle d$. We obtain dim $Z_1 \cup Z_2 \langle d$ by Proposition 2.7(5). The definable set X_2 is a special submanifold by Lemma 4.8. By the induction hypothesis, $Z_1 \cup Z_2$ is decomposed into mutually disjoint special submanifolds C_1, \ldots, C_N . The decomposition $X = X_2 \cup \bigcup_{i=1}^N C_i$ is the desired decomposition. \Box

Definition 4.18. Consider an expansion of a densely linearly order without endpoints $\mathcal{M} = (M, <, ...)$. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n . A decomposition of M^n into special submanifolds partitioning $\{X_i\}_{i=1}^m$ is a finite family of special submanifolds $\{C_i\}_{i=1}^N$ such that

•
$$\bigcup_{i=1}^{N} C_i = M^n$$
,

- $C_i \cap C_j = \emptyset$ when $i \neq j$ and
- either C_i has an empty intersection with X_j or it is contained in X_j

for any $1 \leq i \leq m$ and $1 \leq j \leq N$. A decomposition $\{C_i\}_{i=1}^N$ of M^n into special submanifolds *satisfies the frontier condition* if the closure of any special manifold $cl(C_i)$ is the union of a subfamily of the decomposition.

Theorem 4.19. Consider a definably complete locally o-minimal expansion of an ordered group $\mathcal{M} = (M, <, 0, +, ...)$. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n . There exists a decomposition $\{C_i\}_{i=1}^N$ of M^n into special submanifolds partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. Furthermore, the number N of special submanifolds is not greater than the number uniquely determined only by m and n.

Proof. The proof is literally same as the proof of [6, Theorem 4.4, Theorem 4.5] except that we use Lemma 4.17 instead of [6, Lemma 4.3]. We omit the details. \Box

Remark 4.20. Lemma 4.3 implies that there exists a family $\{U_y\}_{y \in \text{fib}(X,\pi,x)}$ of open boxes U_y parameterized by the definable set $\text{fib}(X,\pi,x)$ such that

- (a) the union $\bigcup_{y \in \operatorname{fib}(X,\pi,x)} \{y\} \times U_y$ is definable and
- (b) $X \cap U_y$ is the graph of a definable map for each $y \in \operatorname{fib}(X, \pi, x)$.

This fact is essentially used in our proof of the decomposition into special submanifolds.

Consider a definably complete locally o-minimal structure $\mathcal{M} = (M, <, ...)$ which is not necessarily an expansion of an ordered group. A definable, discrete closed subset D in M is always a special submanifold. In this case, a family satisfying the conditions (a) and (b) is a family $\{I_x\}_{x\in D}$ of mutually disjoint open intervals I_x containing the point x such that $\bigcup_{x\in D} \{x\} \times I_x$ is definable. The author does not know whether such a family $\{I_x\}_{x\in D}$ exists when a definable choice lemma such as Lemma 2.12 is unavailable. He does not know whether Theorem 4.19 still holds true when we drop the assumption that the structure is an expansion of an ordered group, neither.

4.4. Decomposition into special submanifolds with tubular neighborhoods. A tubular neighborhood of a submanifold in a Euclidean space is a convenient tool for geometric studies of semialgebraic sets [1] and others [14]. We define a special

submanifolds with a tubular neighborhood and give a decomposition theorem into special submanifolds with tubular neighborhoods for future use.

We define a special submanifold with a tubular neighborhood.

Definition 4.21. Let $\mathcal{M} = (M, <, +, 0, ...)$ be an expansion of an ordered abelian group. Let X be a π -special submanifold in M^n , where $\pi : M^n \to M^d$ is a coordinate projection. Let τ be the unique permutation of $\{1, \ldots, n\}$ satisfying the conditions (a) and (b) in Definition 4.1. Set $U = \pi(X)$.

When dim X < n, the tuple (X, π, T, η, ρ) is a special submanifold with a tubular neighborhood if

- (a) T is a definable open subset of $\pi^{-1}(U)$;
- (b) $\eta: U \to F$ is a positive bounded definable continuous function such that, for all $u \in U$, we have

$$\operatorname{fib}(T, \pi, u) = \bigcup_{x \in \operatorname{fib}(X, \pi, u)} \mathcal{B}_{n-d}(x, \eta(u))$$

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and

$$\mathcal{B}_{n-d}(x_1,\eta(u))\cap\mathcal{B}_{n-d}(x_2,\eta(u))=\emptyset$$

- for all $x_1, x_2 \in \operatorname{fib}(X, \pi, u)$ with $x_1 \neq x_2$;
- (c) $\rho: T \to X$ is a definable continuous retraction such that, for any $u \in U$, we have $\rho(\pi^{-1}(u) \cap T) \subseteq \pi^{-1}(u) \cap X$ and $\rho(\overline{\tau}(u, y)) = \overline{\tau}(u, x)$ for all $x \in \operatorname{fib}(X, \pi, u)$ and $y \in \mathcal{B}_{n-d}(x, \eta(u))$.

When dim X = n, the tuple (X, π, T, η, ρ) is a special submanifold with a tubular neighborhood if X is open, T = X, $\eta \equiv 0$, and ρ is the identity map on X.

A decomposition of M^n into special submanifolds with tubular neighborhoods is a finite family of special submanifolds with tubular neighborhoods $\{(X_i, \pi_i, T_i, \eta_i, \rho_i)\}_{i=1}^N$ such that $\{(X_i, \pi_i)\}_{i=1}^N$ is a decomposition of M^n into special submanifolds. We say that a decomposition $\{(X_i, \pi_i, T_i, \eta_i, \rho_i)\}_{i=1}^N$ of M^n into special submanifolds with tubular neighborhoods partitions a given finite family of definable sets and satisfies the frontier condition if so does the decomposition into special submanifolds $\{(X_i, \pi_i)\}_{i=1}^N$.

This definition seems to be technical, but a decomposition into special submanifolds with tubular neighborhoods is useful. The following theorem guarantees the existence of the decomposition.

Theorem 4.22. Let $\mathcal{M} = (M, <, +, 0, ...)$ be a definably complete locally o-minimal expansion of an ordered group. Let $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of M^n . There exists a decomposition of M^n into special submanifolds with tubular neighborhoods partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. In addition, the number of special submanifolds with tubular neighborhoods is bounded by a function of m and n.

Proof. We first demonstrate the following claim:

Claim 1. Let $\pi : M^n \to M^d$ be a coordinate projection and $C \subseteq M^n$ be a π -special submanifold. There exists a special submanifolds with tubular neighborhoods (X, π, T, η, ρ) such that $X \subseteq C$ and dim $C \setminus X < d$.

The claim is obvious when d = n. We have only to set X = T = C, $\eta = 0$ and $\rho = \text{id}$. We next consider the case in which d < n. For simplicity of notations, we assume that π is the coordinate projection onto the first d coordinates. Note that fib (C, π, u) are discrete and closed for all $u \in U = \pi(C)$ by the definition of special submanifolds and Proposition 2.7(1). We also note that fib $(C, \pi, u) \setminus \{x\}$ is also closed and discrete for any $x \in \text{fib}(C, \pi, u)$. Take a positive element $c \in M$. Consider the map $\eta' : U \to F$ defined by

$$\eta'(u) = \frac{1}{3} \inf \left(\{c\} \cup \bigcup_{x \in \operatorname{fib}(C,\pi,u)} \{|y-x| \mid y \in \operatorname{fib}(C,\pi,u) \setminus \{x\}\} \right)$$

The map η' is obviously definable. For a fixed $u \in U$, the set in the round brackets is a discrete and closed set contained in the open interval $(0, \infty)$ by Proposition 2.7(1),(6). In particular, we have $\eta'(u) > 0$ for all $u \in U$.

Let D be the closure of the set of points at which η' is discontinuous. We have dim D < d by Proposition 2.7(5),(7),(8). Set $V = U \setminus D$, which is a definable open subset of M^d . We define X, T, η and ρ as follows:

•
$$X = C \cap \pi^{-1}(V);$$

- T = ⋃_{u∈V} ⋃_{x∈fib(C,π,u)} {u} × B_{n-d}(x, η'(u));
 η is the restriction of η' to V;
- $\rho: T \to X$ is the map such that $\rho(x)$ is the unique $y \in X$ with u = $\pi(x) = \pi(y)$ and $\Pi(x) \in B_{n-d}(\Pi(y), \eta'(u))$, where Π is the projection of M^n forgetting the first *d*-coordinates.

We show that (X, π, T, η, ρ) is a special submanifold with a tubular neighborhood. They obviously satisfy the conditions (a) and (b) in Definition 4.21. It is also obvious that ρ is definable and satisfies the inclusion and the equality given in Definition 4.21(c). The remaining task is to demonstrate that ρ is continuous.

Take a point $x \in T$. Set $u = \pi(x)$. There exists a unique point $y \in X$ such that $\pi(y) = u$ and $\Pi(x) \in \mathcal{B}_{n-d}(\Pi(y), \eta'(u))$. There exist an open box W containing the point y and a definable continuous map ζ defined on $\pi(W)$ such that $W \cap C$ is the graph of ζ . We may assume that $\pi(W)$ is contained in V, shrinking W if necessary. Set $W' = \bigcup_{u \in \pi(W)} \{u\} \times \mathcal{B}_{n-d}(\zeta(u), \eta'(u))$. The set W' is an open subset of T containing the point x. For $(u',t) \in W'$, we get $\rho(u',t) = (u',\zeta(u'))$. This equality implies that ρ is continuous on W'. Therefore, the map ρ is continuous on its domain of definition.

The set $C \setminus X$ is given by $C \cap \pi^{-1}(D)$. Since dim D < d and dim $\pi^{-1}(u) \cap C = 0$ for all $u \in D$, we get dim $C \setminus X < d$ by Proposition 2.7(9). We have now demonstrated the claim.

For any definable subset X of M^n , we define a decomposition of X into special submanifolds in the same manner as the case in which $X = M^n$. We next demonstrate the following claim. The theorem is a direct corollary of this claim.

Claim 2. Let X be a definable subset of M^n and $\{X_i\}_{i=1}^m$ be a finite family of definable subsets of X. There exists a decomposition of X into special submanifolds with tubular neighborhoods partitioning $\{X_i\}_{i=1}^m$ and satisfying the frontier condition. In addition, the number of special submanifolds with tubular neighborhoods is bounded by a function of m and n.

We prove the claim by induction on $d = \dim X$. We first consider the case in which d = 0. Apply Theorem 4.19 and find a decomposition of M^n into special submanifolds partitioning $\{X\} \cup \{X_i\}_{i=1}^m$. A subfamily of this partition gives a decomposition of X into special submanifolds $\{C_i\}_{i=1}^N$ partitioning $\{X_i\}_{i=1}^m$. Since dim $C_i = 0$, the special submanifolds C_i are closed for all $1 \le i \le N$ by Proposition 2.7(1). Apply Claim 1 to the special submanifold C_i for each $1 \leq i \leq N$, and we get a special submanifold with tubular neighborhoods $(C_i, \pi_i, T_i, \eta_i, \rho_i)$. The family $\{(C_i, \pi_i, T_i, \eta_i, \rho_i)\}_{i=1}^N$ is a desired decomposition.

We next consider the case in which d > 0. Apply Theorem 4.19. We get a decomposition of X into special submanifolds $\{C_i\}_{i=1}^N$ partitioning $\{X_i\}_{i=1}^m$. We may assume that dim $C_i = d$ for all $1 \le i \le L$ and dim $C_i < d$ for all i > L without loss of generality. Apply Claim 1 to the special submanifold C_i for each $1 \leq i \leq L$, we get a definable subset C'_i of C_i with dim $C_i \setminus C'_i < d$ and a special submanifold with a tubular neighborhood $(C'_i, \pi_i, T_i, \eta_i, \rho_i)$. Set $D_i = C_i \setminus C'_i$ for $1 \le i \le L$ and $D_i = C_i$ for $L < i \le N$. We put $X' = \bigcup_{i=1}^N D_i$. We obtain dim X' < dby Proposition 2.7(5). Apply the induction hypothesis to X', then there exists a decomposition $\{(E_i, \pi'_i, T'_i, \eta'_i, \rho'_i)\}_{i=1}^{N'}$ of X' into special submanifolds with tubular neighborhoods partitioning $\{D_i\}_{i=1}^{N} \cup \{D_i \cap \partial C'_j\}_{1 \leq i \leq N, \ 1 \leq j \leq L}$ and satisfying the frontier condition. The family $\{(C'_i, \pi_i, T_i, \eta_i, \rho_i)\}_{i=1}^L \cup \{(E_i, \pi'_i, T'_i, \eta'_i, \rho'_i)\}_{i=1}^{N'}$ is a desired decomposition.

The 'in addition' part is clear from the proof.

Remark 4.23. Let r be a positive integer. When the structure is a definably complete locally o-minimal expansion of an ordered field, Theorem 4.22 can be easily extended to a decomposition into special C^r -submanifolds with tubular neighborhoods by minor modifications of the definition of the neighborhood $\mathcal{B}_m(x,r)$ and proofs.

Several assertions demonstrated in the o-minimal setting using the definable cell decomposition theorem and the stratification theorem also hold true in definably complete locally o-minimal expansions of ordered fields. We use decomposition into special C^r -submanifolds with tubular neighborhoods instead of the definable cell decomposition theorem and the stratification theorem. They will be proved in the forthcoming paper.

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