

# Approximating $\{0, 1, 2\}$ -Survivable Networks with Minimum Number of Steiner Points

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**Abstract.** We consider low connectivity variants of the **Survivable Network with Minimum Number of Steiner Points (SN-MSP)** problem: given a finite set  $R$  of terminals in a metric space  $(M, d)$ , a subset  $B \subseteq R$  of “unstable” terminals, and connectivity requirements  $\{r_{uv} : u, v \in R\}$ , find a minimum size set  $S \subseteq M$  of additional points such that the unit-disc graph of  $R \cup S$  contains  $r_{uv}$  pairwise internally edge-disjoint and  $(B \cup S)$ -disjoint  $uv$ -paths for all  $u, v \in R$ . The case when  $r_{uv} = 1$  for all  $u, v \in R$  is the **Steiner Tree with Minimum Number of Steiner Points (ST-MSP)** problem, and the case  $r_{uv} \in \{0, 1\}$  is the **Steiner Forest with Minimum Number of Steiner Points (SF-MSP)** problem. Let  $\Delta$  be the maximum number of points in a unit ball such that the distance between any two of them is larger than 1. It is known that  $\Delta = 5$  in  $\mathbb{R}^2$ . The previous known approximation ratio for ST-MSP was  $\lfloor (\Delta + 1)/2 \rfloor + 1 + \epsilon$  in an arbitrary normed space [19], and  $2.5 + \epsilon$  in the Euclidean space  $\mathbb{R}^2$  [5]. Our approximation ratio for ST-MSP is  $1 + \ln(\Delta - 1) + \epsilon$  in an arbitrary normed space, which in  $\mathbb{R}^2$  reduces to  $1 + \ln 4 + \epsilon < 2.3863 + \epsilon$ . For SN-MSP with  $r_{uv} \in \{0, 1, 2\}$ , we give a simple  $\Delta$ -approximation algorithm. In particular, for SF-MSP, this improves the previous ratio  $2\Delta$ .

## 1 Introduction

### 1.1 Problems considered

A large research effort is focused on developing algorithms for finding a “cheap” network that satisfies a certain property. In wired networks, where connecting any two nodes incurs a cost, many problems can be cast as finding a subgraph of minimum cost that satisfies some prescribed connectivity requirements. Following previous work on min-cost connectivity problems, we use the following generic notion of connectivity.

**Definition 1.** Let  $G = (V, E)$  be a graph and let  $Q \subseteq V$ . The  $Q$ -connectivity  $\lambda_G^Q(u, v)$  of  $u, v$  in  $G$  is the maximum number of pairwise  $(E \cup Q \setminus \{u, v\})$ -disjoint  $uv$ -paths in  $G$ . Given connectivity requirements  $r = \{r_{uv} : u, v \in R \subseteq V\}$  on a subset  $R \subseteq V$  of terminals, we denote by  $D_r = \{uv : u, v \in R, r_{uv} > 0\}$  the set of “demand edges” of  $r$ . We say that  $G$  is  $(r, Q)$ -connected, or simply  $r$ -connected if  $Q$  is understood, if  $\lambda_G^Q(u, v) \geq r_{uv}$  for all  $uv \in D_r$ .

Note that edge-connectivity is the case  $Q = \emptyset$  and node-connectivity is the case  $Q = V$ . The members of  $E \cup Q$  will be called *elements*, hence  $\lambda_G^Q(u, v)$  is the maximum number of pairwise internally element-disjoint  $uv$ -paths in  $G$ . Variants of the following classic problem were extensively studied in the literature.

**Survivable Network (SN)**

*Instance:* A graph  $G = (V, E)$  with edge costs,  $Q \subseteq V$ , and connectivity requirements  $r = \{r_{uv} : uv \in R \subseteq V\}$ .

*Objective:* Find a minimum-cost  $(r, Q)$ -connected subgraph  $H$  of  $G$ .

In practical networks the connectivity requirements are rather small, usually  $r_{uv} \in \{0, 1, 2\}$  – so called  $\{0, 1, 2\}$ -SN. Particular cases in this setting are **Minimum Spanning Tree (MST)** ( $r_{uv} = 1$  for all  $u, v \in V$ ), **Steiner Tree** ( $r_{uv} = 1$  for all  $u, v \in R$ ) and **Steiner Forest** ( $r_{uv} \in \{0, 1\}$  for all  $u, v \in R$ ), and **2-Connected Subgraph** ( $r_{uv} = 2$  for all  $u, v \in V$ ).

In wireless networks, the range and the location of the transmitters determines the resulting communication network. We consider adding a minimum number of transmitters such that the resulting communication network is  $(r, Q)$ -connected. If the range of the transmitters is fixed, our goal is to add a minimum number of transmitters, and we get the following type of problems.

**Definition 2.** Let  $(M, d)$  be a metric space and let  $V \subseteq M$ . The unit-disk graph of  $V$  has node set  $V$  and edge set  $\{uv : u, v \in V, d(u, v) \leq 1\}$ .

**Survivable Network with Minimum Number of Steiner Points (SN-MSP)**

*Instance:* A finite set  $R \subseteq M$  of *terminals* in a metric space  $(M, d)$ , a set  $B \subseteq R$  of “unstable” terminals, connectivity requirements  $\{r_{uv} : uv \in R\}$ .

*Objective:* Find a minimum size set  $S \subseteq M$  such that the unit-disk graph of  $R \cup S$  is  $(r, Q)$ -connected, where  $Q = B \cup S$ .

As in previous work, we will allow to place several points at the same location, and assume that the maximum distance between terminals is polynomial in the number of terminals.

## 1.2 Previous work and our results

On previous work on high connectivity variants of SN problem we refer the reader to a survey in [17] and here only mention some work relevant to this paper. The **Steiner Tree** problem was studied extensively, c.f. [24,25,23,20,2,9], and the currently best approximation ratio for it is  $\ln 4 + \epsilon$  [2]. Let  $\tau^*$  denote the optimum value of a standard cut-LP relaxation for SN (see Section 3). In [10] is given a combinatorial primal-dual algorithm for **Steiner Forest** that computes a solution of cost at most  $2\tau^*$ . For  $\{0, 1, 2\}$ -SN a similar results is achieved by the iterative rounding method [8]; a combinatorial primal-dual algorithm that computes a solution of cost at most  $3\tau^*$  is given in [21].

We survey some relevant literature on SN-MSP problems. **ST-MSP** is NP-hard even in  $\mathbb{R}^2$ , and arises in various wireless network design problems, c.f.

[1,3,4,5,12,13,18,19] for only a sample of papers in the area, where it is studied both in  $\mathbb{R}^2$  and in general metric spaces. In the latter case, the approximation ratio is usually expressed in terms of the following parameter. Let  $\Delta$  be the maximum number of “independent” points in the unit ball, such that the distance between any two of them is larger than 1. It is known [22] that  $\Delta$  equals the maximum degree of a minimum-degree Minimum Spanning Tree in the normed space. For Euclidean distances we have  $\Delta = 5$  in  $\mathbb{R}^2$  and  $\Delta = 11$  in  $\mathbb{R}^3$ , and in  $\mathbb{R}^\ell$   $\Delta$  is at most the Hadwiger number [22]; hence  $\Delta \leq 2^{0.401\ell(1+o(1))}$ , by [11].

In finite metric spaces, ST-MSP is equivalent to the variant of the Node Weighted Steiner Tree problem when all terminals have costs 0 and the other nodes have cost 1. Klein and Ravi [16] proved that this variant is Set-Cover hard to approximate, and gave an  $O(\ln |R|)$ -approximation algorithm for general weights. Hence up to constants, even for finite metric spaces, the ratio  $O(\ln |R|)$  of [16] is the best possible unless P=NP. Note however, that this does not exclude constant ratios for metric spaces with small  $\Delta$ , e.g.,  $\Delta = 5$  in  $\mathbb{R}^2$ .

Most algorithms for SN-MSP problems applied the following reduction method, by solving the corresponding SN instance obtained as follows.

**Definition 3.** *Given a finite set  $R$  of points in a metric space  $(M, d)$  and an integer  $k \geq 1$ , the (multi)graph  $K_R$  has node set  $R$  and  $k$  parallel edges between every pair of nodes. The costs of the  $k$  edges between  $u, v$  are defined as follows. Let  $\hat{d}_{uv} = \max\{\lceil d(u, v) \rceil - 1, 0\}$ . If  $\hat{d}_{uv} > 0$ , then all the  $k$  edges have cost  $\hat{d}_{uv}$ . If  $\hat{d}_{uv} = 0$ , then one edge has cost 0 and the others have cost 1.*

Let  $\text{opt}$  denote the optimal solution value of a problem instance at hand. It is easy to see that any solution of cost  $C$  to the corresponding SN instance with  $k = \max_{uv \in D_r} r_{uv}$  defines a solution  $S$  of size  $C$  to the original SN-MSP instance, where every node in  $S$  has degree exactly 2; such a solution is called a *bead solution*. Conversely, any bead solution  $S$  can be converted into a solution to the SN instance of cost at most  $|S|$  (see [12,3]). Due to this bijective correspondence, we simply define a bead solution as a solution to the corresponding SN instance, and denote the optimal value of a bead solution to an instance  $I$  by  $\tau = \tau(I)$ . If the SN instance admits a  $\rho$ -approximation algorithm, and if for the given SN-MSP instance there exists a bead solution  $S$  of size  $\leq \alpha \text{opt}$ , then we get a  $\rho\alpha$ -approximation algorithm for the SN-MSP instance. Equivalently, for a class  $\mathcal{I}$  of SN-MSP instances, define a parameter  $\alpha$  by  $\alpha = \alpha(\mathcal{I}) = \sup_{I \in \mathcal{I}} \frac{\text{opt}(I)}{\tau(I)}$ . Then approximation ratio  $\rho$  for SN instances that correspond to the class  $\mathcal{I}$  implies approximation ratio  $\alpha\rho$  for SN-MSP instances in class  $\mathcal{I}$ .

Măndoiu and Zelikovsky [18] showed that for ST-MSP  $\alpha = \Delta - 1$ . Since the instance of SN that corresponds to ST-MSP is the MST problem that can be solved in polynomial time, this gives a  $(\Delta - 1)$ -approximation algorithm for ST-MSP. A more general method, uses a reduction to the Minimum  $k$ -Connected Spanning Subhypergraph problem, see Section 2. This method was initiated by Zelikovsky [24], improved in a long series of papers (part of them are [24,20,23]), and culminated in the paper of Byrka, Grandoni, Rothvoß, and Sanità [2]. For ST-MSP in  $\mathbb{R}^2$ , Chen and Du [5] applied this method to get the currently best

known ratio  $2.5 + \epsilon$ . In arbitrary metric spaces, the ratio  $\Delta - 1$  of [18] was improved to  $\lfloor (\Delta + 1)/2 \rfloor + 1 + \epsilon$  in [19], also using the same method. These works assume that ST-MSP instances with a constant number of terminals can be solved in polynomial time, which holds in  $\mathbb{R}^2$  if the maximum distance between terminals is polynomial in the number of terminals, see [4, Lemma 11] and the discussion there. In this paper we apply a variant due to Zelikovsky [25], and obtain the following result.

**Theorem 1.** *ST-MSP with constant  $\Delta$  admits an approximation scheme with ratio  $1 + \ln(\Delta - 1) + \epsilon$ , provided that ST-MSP instances with a constant number of terminals can be solved in polynomial time. In particular, in  $\mathbb{R}^2$  the ratio is  $1 + \ln 4 + \epsilon < 2.3863 + \epsilon$ .*

We now discuss SN-MSP problems with  $k = \max_{uv \in V} r_{uv} \geq 2$ . Bredin, Demaine, Hajiaghayi, and Rus [1] considered a related problem of adding a minimum size  $S$  such that the unit disc graph of  $R \cup S$  is  $k$ -node-connected (note that we require  $k$ -connectivity only between terminals). For this problem in  $\mathbb{R}^2$ , they gave an  $O(k^5)$ -approximation algorithm, but essentially they implicitly proved that for this class of problems  $\alpha = O(\Delta k^3)$ . Recently, it was shown in [19] that  $\alpha = \Theta(\Delta k^2)$  for node-connectivity SN-MSP instances in any normed space.

Kashyap, Khuller, and Shayman [13] considered the 2-edge/node-connectivity version of SN-MSP, where  $r_{uv} = 2$  for all  $u, v \in R$ . They used the reduction method described in Definition 3, namely, their algorithm constructs an SN instance as in Definition 3 and then converts its solution into a bead solution to the SN-MSP instance. Although they analyzed a performance of specific 2-approximation algorithms – the algorithm of Khuller and Vishkin [15] for 2-edge-connectivity and the algorithm of Khuller and Raghavachari [14] for 2-node-connectivity, they essentially proved that  $\alpha = \Delta$  in both cases. This implies ratio  $2\Delta$  in both cases. The analysis of these specific algorithms was recently improved by Calinescu [3], showing that their tight performance is  $\Delta$  for node-connectivity and  $2\Delta - 1$  for edge-connectivity. Note that the edge-connectivity version is *not* included in our model, since in our SN-MSP instances every non-terminal node is in  $Q$ , namely, the paths are required to be  $S$  disjoint.

Let  $\tau^* = \tau^*(I)$  denote the optimal value of a *fractional* bead solution of an SN-MSP instance  $I$ , namely,  $\tau^*$  is the optimum of a standard cut-LP relaxation for the corresponding SN instance (see Section 3). Here we observe, that if the algorithm we use for the corresponding SN instance computes a solution of cost at most  $\rho\tau^*$ , then the relevant parameter is the following.

**Definition 4.** *For a class  $\mathcal{I}$  of SN-MSP instances, let  $\alpha^* = \alpha^*(\mathcal{I}) = \sup_{I \in \mathcal{I}} \frac{\text{opt}(I)}{\tau^*(I)}$ .*

**Theorem 2.** *For  $Q$ -connectivity  $\{0, 1, 2\}$ -SN-MSP  $\alpha^* = \frac{\Delta}{2}$ . Thus if  $Q$ -connectivity  $\{0, 1, 2\}$ -SN admits a polynomial time algorithm that computes a solution of cost at most  $\rho\tau^*$ , then  $Q$ -connectivity  $\{0, 1, 2\}$ -SN-MSP admits approximation ratio  $\rho \cdot \frac{\Delta}{2}$ . In particular, for  $\rho = 2$  the ratio is  $\Delta$ , and thus  $\{0, 1, 2\}$ -SN-MSP admits a  $\Delta$ -approximation algorithm.*

Theorems 1 and 2 are proved in Sections 2 and 3, respectively.

## 2 Proof of Theorem 1

We consider a generic problem defined in [19], that includes both ST-MSP and the classic Steiner Tree problem.

### Generalized Steiner Tree

*Instance:* A (possibly infinite) graph  $G = (V, E)$ , a finite set  $R \subseteq V$  of terminals, and a monotone subadditive cost function  $c$  on subgraphs of  $G$ .

*Objective:* Find a minimum-cost connected finite subtree  $T$  of  $G$  containing  $R$ .

Instead of considering optimal connections only between pairs of terminals, we consider optimal connections of terminal subsets of size at most  $k$ .

**Definition 5.** For an instance of Generalized Steiner Tree and an integer  $k$ ,  $2 \leq k \leq |R|$ , the hypergraph  $\mathcal{H}_k = (R, \mathcal{E}_k)$  has node set  $R$  and hyperedge set  $\mathcal{E}_k = \{A \subseteq R : 2 \leq |A| \leq k\}$ . The cost  $c^*(A)$  of  $A \in \mathcal{E}_k$  is the cost of an optimal solution  $T_A$  to the Generalized Steiner Tree instance with terminal set  $A$ .

Given a hypergraph  $\mathcal{H}$  with hyperedge costs, the Minimum Connected Spanning Sub-hypergraph problem seeks a minimum cost subset of hyperedges that connects any two nodes. The construction in Definition 5 converts the Generalized Steiner Tree problem into the Minimum Connected Spanning Sub-hypergraph problem in a hypergraph  $\mathcal{H}_k$  of rank  $k$ . Any solution of cost  $C$  to this problem correspond to a solution of value at most  $C$  to Generalized Steiner Tree, by the subadditivity and monotonicity of the cost function in the Generalized Steiner Tree problem. The inverse is not true in general, and this reduction invokes a fee in the approximation ratio, given in the following definition.

**Definition 6.** Given an instance  $I$  of Generalized Steiner Tree let  $\tau_k(I)$  denote the minimum cost of a connected spanning sub-hypergraph of  $\mathcal{H}_k$ . The  $k$ -ratio for a class  $\mathcal{I}$  of Generalized Steiner Tree instances is defined by  $\alpha_k = \sup_{I \in \mathcal{I}} \frac{\tau_k(I)}{\text{opt}(I)}$ .

Note that for  $\mathcal{I}$  being the class of ST-MSP instances,  $\alpha_2$  is the parameter  $\alpha$  defined in the introduction, and that by [18] we have  $\alpha_2 = \alpha = \Delta - 1$ . We have  $\alpha_k = 1$  for instances with  $|R| = k$ , and in general  $\alpha_k$  is monotone decreasing and approaching 1 when  $k$  becomes larger.

In Section 2.1 we prove the following statement, which is of independent interest, and may find applications in other network design problems.

**Theorem 3.** There exists polynomial time algorithm that given a hypergraph  $\mathcal{H} = (R, \mathcal{E})$  with hyper-edge cost  $\{c(A) : A \in \mathcal{E}\}$  and a spanning tree  $T^*$  of (edges of size 2 of)  $\mathcal{H}$  computes a spanning connected sub-hypergraph  $\mathcal{T}$  of  $\mathcal{H}$  of cost at most  $\tau \left(1 + \ln \frac{c(T^*)}{\tau}\right)$ , where  $\tau$  is the minimum-cost of a connected spanning sub-hypergraph of  $\mathcal{H}$ .

**Corollary 1.** For any constant  $k$ , Generalized Steiner Tree admits an approximation ratio  $\alpha_k (1 + \ln \alpha_2)$ , provided that for any  $A \in \mathcal{E}_k$ , the instance with the terminal set  $A$  can be solved in polynomial time.

*Proof.* By the assumptions, the hypergraph  $\mathcal{H}_k$ , and the costs  $c^*(A)$  with the corresponding trees  $T_A$  for  $A \in \mathcal{E}_k$ , can be computed in polynomial time. We can also compute in polynomial time an optimal spanning tree  $T^*$  in  $\mathcal{H}_2$ ; note that  $c(T^*) \leq \alpha_2 \cdot \text{opt}$ . Then we apply the algorithm in Theorem 3 to compute a sub-hypergraph  $\mathcal{T}$  of  $\mathcal{H}_k$  of  $c^*$ -cost at most  $\tau \left(1 + \ln \frac{c(T^*)}{\tau}\right)$ , where  $\tau$  is the minimum-cost of a connected spanning sub-hypergraph of  $\mathcal{H}_k$ . Let  $\text{opt}$  denote the optimal solution value for the Generalized Steiner Tree instance. Note that  $\text{opt} \leq \tau \leq \alpha_k \text{opt}$ . Let  $T = \cup_{A \in \mathcal{T}} T_A$ . Since  $\mathcal{T}$  is a connected hypergraph,  $T$  is a feasible solution to the Generalized Steiner Tree instance. We have  $c(T) \leq \sum_{A \in \mathcal{T}} c(T_A) = c^*(\mathcal{T})$ , by the monotonicity and the subadditivity of the  $c$ -costs. Thus we have:

$$c(T) \leq c^*(\mathcal{T}) \leq \tau \left(1 + \ln \frac{c(T^*)}{\tau}\right) = \tau \left(1 + \ln \frac{c(T^*)/\text{opt}}{\tau/\text{opt}}\right) \leq \alpha_k \text{opt} (1 + \ln \alpha_2) .$$

□

Du and Zhang [7] showed that for the classic Steiner Tree problem,  $\alpha_k \leq 1 + 1/\lfloor \lg k \rfloor$ , where  $\lg k = \log_2 k$  denotes logarithm base 2. In Section 4 we prove the following.

**Theorem 4.** For ST-MSP,  $\alpha_k \leq 1 + \frac{2}{\lfloor \lg \lfloor k/(\Delta-1) \rfloor \rfloor}$  for any integer  $k \geq 2\Delta - 2$ .

Note that  $k \geq \Delta$  is necessary if we want  $\alpha_k < 2$ . Otherwise, for an instance  $I$  of  $\Delta$  points on the unit ball we have  $\frac{\tau(I)}{\text{opt}(I)} = \frac{k}{\Delta}$ , so  $\alpha_k \geq \frac{k}{\Delta}$  if  $k \leq \Delta$ .

From Corollary 1 and Theorem 4 we conclude that for any constant  $k \geq 2\Delta - 2$ , it is possible to compute in polynomial time a solution to an ST-MSP instance of size at most  $\alpha_k (1 + \ln(\Delta - 1)) \text{opt}$ , where  $\alpha_k$  is as in Theorem 4. For the metric space  $\mathbb{R}^2$ , and given a constant  $\epsilon > 0$  let  $k = 2^{O(1/\epsilon)}$  with sufficient large constant. Then by Theorem 4,  $\alpha_k \leq 1 + \epsilon/(1 + \ln 4)$ , and the approximation ratio of our algorithm is  $1 + \ln 4 + \epsilon$ . This completes the proof of Theorem 1.

## 2.1 Proof of Theorem 3

For the proof of Theorem 3 we need the following definition.

**Definition 7.** Given a tree  $T = (R, F)$  we say that  $A \subseteq R$  overlaps  $F' \subseteq F$  if the graph obtained from  $T \setminus F'$  by shrinking  $A$  into a single node is a tree. Given edge cost  $\{c(e) : e \in F\}$  let  $F(A)$  be a maximum cost edge set overlapped by  $A$ .

Note that  $F \setminus F(A)$  is an edge set of a minimum cost spanning tree in the graph obtained from  $T$  by shrinking  $A$  into a single node; hence  $F(A)$  can be computed in polynomial time. The following statement appeared in [24] (see also [2]); we provide a proof for completeness of exposition.

**Lemma 1.** Let  $T = (R, F)$  be a tree with edge costs  $\{c(e) : e \in F\}$  and let  $(R, \mathcal{E})$  be a connected hypergraph. Then  $\sum_{A \in \mathcal{E}} c(F(A)) \geq c(F)$ . Thus there exists  $A \in \mathcal{E}$  such that

$$\frac{c(F(A))}{c(A)} \geq \frac{c(F)}{c(\mathcal{E})} .$$

*Proof.* For a node  $v \in A$ , let  $C_v$  be the connected component in  $T \setminus F(A)$  that contains  $v$ . For an edge  $e \in F(A)$  that connects two components  $C_u, C_v$ , let  $y(e) = uv$  be the replacement edge of  $e$ , of cost  $c(y(e)) = c(e)$ . The graph  $T \cup \{y(e)\}$  contains a single cycle and  $y(e)$  is the heaviest edge in this cycle, since otherwise  $F(A)$  is not minimal. For a hyperedge  $A \in \mathcal{E}$  let  $y(A) = \cup_{e \in F(A)} y(e)$  be the replacement set of  $A$ , and let  $y(\mathcal{E}) = \cup_{A \in \mathcal{E}} y(A)$ . It is easy to see that  $y(A)$  span  $A$ , and  $y(\mathcal{E})$  span  $R$ . Consider a MST on  $T \cup y(\mathcal{E})$ . By the cycle property of a MST, no edge from  $y(\mathcal{E})$  would participate in that MST, so  $c(T) \leq c(y(\mathcal{E}))$ . Finally,  $c(y(\mathcal{E})) = \sum_{A \in \mathcal{E}} y(A) = \sum_{A \in \mathcal{E}} c(F(A))$ , and the lemma follows.  $\square$

### Local Replacement Algorithm

*Input:* A hypergraph  $\mathcal{H} = (R, \mathcal{E})$  with hyper-edge cost  $\{c(A) : A \in \mathcal{E}\}$ , and a spanning tree  $T^* = (R, F^*)$  of (edges of size 2 of)  $\mathcal{H}$ .

*Initialization:*  $\mathcal{J} \leftarrow \emptyset$ ,  $F \leftarrow F^*$ ,  $T \leftarrow (R, F)$ .

*While*  $c(F) > 0$  *do:*

Find  $A \in \mathcal{E}$  with  $\frac{c(F(A))}{c(A)}$  maximum.

- *If*  $c(F(A)) > c(A)$  *then do:*

- Update  $T, \mathcal{H}$ : remove  $F(A)$  and shrink  $A$  into a single node.

-  $F \leftarrow F \setminus F(A)$  and  $\mathcal{J} \leftarrow \mathcal{J} \cup \{A\}$ .

- *Else STOP* and *Return*  $\mathcal{T} = (R, F \cup \mathcal{J})$ .

*EndWhile*

*Return*  $\mathcal{T} = (R, F \cup \mathcal{J})$ .

At every iteration  $|F|$  decreases by at least 1, hence the algorithm runs in polynomial time, and clearly it computes a feasible solution. We prove the approximation ratio. Let  $F_i$  and  $\mathcal{J}_i$  be the set stored in  $F$  and  $\mathcal{J}$ , respectively, at the beginning of iteration  $i+1$ , and let  $A_i$  be the hyperedge picked at iteration  $i$ . Denote  $f_i = c(F_i)$  and  $s_i = c(A_i)$ , and recall that  $\tau$  denotes the minimum cost of a connected spanning sub-hypergraph of  $\mathcal{H}$ . At iteration  $i$  we remove  $F_{i-1}(A_i)$  from  $F_{i-1}$  after verifying that  $c(F_{i-1}(A_i)) > c(A_i) = s_i$ . Hence

$$f_i \leq f_{i-1} - \max\{c(F_{i-1}(A_i)), c(A_i)\} = f_{i-1} - s_i \cdot \max\left\{\frac{c(F_{i-1}(A_i))}{c(A_i)}, 1\right\}$$

By Lemma 1,  $\frac{c(F_{i-1}(A_i))}{c(A_i)} \geq \frac{f_{i-1}}{\tau}$ . Thus we have

$$f_i \leq f_{i-1} - s_i \cdot \max\{f_{i-1}/\tau, 1\}. \quad (1)$$

The algorithm stops if either  $c(F_q) = 0$  or  $c(F(A)) \leq c(A)$  at iteration  $q+1$ . In the latter case,  $1 \geq c(F_q)/\tau$  follows by Lemma 1. In both cases, we have that there exists an index  $q$  such that  $f_{q-1} > \tau \geq f_q$  holds. Now we use the following statement from [6].

**Lemma 2.** *Let  $\tau > 0$  and  $f_0, \dots, f_q$  and  $s_1, \dots, s_q$  be sequences of positive reals satisfying  $f_0 > \tau \geq f_q$ , such that (1) holds. Then  $f_q + \sum_{i=1}^q s_i \leq \tau(1 + \ln(f_0/\tau))$ .*

Let  $q$  be an index such that  $f_{q-1} > \tau \geq f_q$  holds. We may assume that  $f_0 = c(F^*) > \tau > 0$ . Note that  $c(\mathcal{J}_q) = \sum_{i=1}^q s_i$  and that  $c(F_i) + c(\mathcal{J}_i) \leq c(F_{i-1}) + c(\mathcal{J}_{i-1})$  for any  $i$ . Hence from Lemma 2 we conclude that

$$c(\mathcal{T}) \leq c(F_q) + c(\mathcal{J}_q) = f_q + \sum_{i=1}^q s_i \leq \tau(1 + \ln(f_0/\tau)) = \tau \left(1 + \ln \frac{c(T^*)}{\tau}\right).$$

This finishes the proof of Theorem 3.

### 3 Proof of Theorem 2

To illustrate our idea, we first prove Theorem 2 for a particular simple case – the Steiner Forest with Minimum Number of Steiner Points (SF-MSP) problem, when  $r_{uv} \in \{0, 1\}$ .

**Definition 8.** For a subset  $C$  of nodes of a graph  $G = (V, E)$  let us use the following notation:  $\Gamma_G(C)$  is the set of neighbors of  $C$  in  $G$ ;  $\delta_G(C) = \delta_E(C)$  is the set of edges in  $E$  with exactly one endnode in  $C$ ;  $E(C)$  is the set of edges in  $E$  with both endnodes in  $C$ . Given  $R \subseteq V$ , an  $R$ -component of  $G$  is a subgraph of  $G$  with node set  $C \cup \Gamma_G(C)$  and edge set  $E(C) \cup \delta_G(C)$ , where  $C$  is a connected component of  $G \setminus R$ .

The cut-LP relaxation for Steiner Forest is:

$$\begin{aligned} \tau^* = \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta_E(Y)} x_e \geq f(Y) \quad \forall \emptyset \neq Y \subset V \\ & 0 \leq x_e \leq 1 \quad \forall e \in E \end{aligned}$$

where  $f(Y) = 1$  if there are  $u, v \in V$  with  $r_{uv} = 1$  and  $|\{u, v\} \cap Y| = 1$ , and  $f(Y) = 0$  otherwise.

Robins and Salowe [22] proved that if  $V$  is a set of points in a metric space, then there exists a tree  $T = (V, E)$  of minimum total length  $\sum_{uv \in E} d(u, v)$  that has maximum degree  $\leq \Delta$ . Since any inclusion-minimal solution to a Steiner Forest instance is a forest, this implies the following.

**Lemma 3.** For any instance of SF-MSP there exists an optimal solution  $S, G$  such that  $G$  has maximum degree  $\Delta$ .  $\square$

The following statement was first observed in [13].

**Lemma 4.** Let  $R$  be a set of terminals and  $S$  a set of points in a normed space such that the unit-disc graph of  $R \cup S$  contains a tree  $T$  with leaf set  $R$ . Let  $S'$  be obtained from  $S$  by replacing each  $v \in S$  by  $\deg_T(v)$  copies of  $v$ . Then the unit disc graph of  $R \cup S'$  contains a simple cycle on  $R \cup S'$ .

*Proof.* Traverse the tree  $T$  in a DFS order; each time a node  $v \in S$  is visited, choose a different copy of  $v$ .  $\square$

Given a tree  $T$ , we will call a cycle as in the lemma above a *DFS cycle* of  $T$ .

Now we can prove Theorem 2 for the SF-MSP case. Let  $S$  be an inclusion minimal solution to an SF-MSP instance. By Lemma 3, the unit-disc graph of  $R \cup S$  contains an  $r$ -connected forest  $H$  such that  $\deg_H(v) \leq \Delta$  for every  $v \in S$ . Every  $R$ -component  $T$  of  $H$  (a.k.a. full Steiner component) is a tree with leaf set in  $R$  and all internal nodes in  $S$ . It is easy to see that by replacing every  $R$ -component  $T$  of  $H$  by a DFS cycle of capacity  $1/2$  results in a feasible solution to the cut-LP relaxation, which proves Theorem 2 for the SF-MSP case.

Now we prove Theorem 2 for  $\{0, 1, 2\}$ -SN-MSP. We start by describing the cut-LP relaxation for SN. We need some definitions.

**Definition 9.** An ordered pair  $\hat{X} = (X, X^+)$  of subsets of a groundset  $V$  is called a *bisetet* if  $X \subseteq X^+$ ;  $X$  is the inner part and  $X^+$  is the outer part of  $\hat{X}$ ,  $\Gamma(\hat{X}) = X^+ \setminus X$  is the boundary of  $\hat{X}$ , and  $X^* = V \setminus X^*$  is the complementary set of  $\hat{X}$ . An edge  $e = uv$  covers a biset  $\hat{X}$  if it has one endnode in  $X$  and the other in  $V \setminus X^+$ . For a biset  $\hat{X}$  and an edge-set/graph  $J$  let  $\delta_J(\hat{X})$  denote the set of edges in  $J$  covering  $\hat{X}$ .

By Menger's Theorem, a graph  $G = (V, E)$  is  $(r, Q)$ -connected if, and only if,  $|\delta_E(\hat{Y})| \geq f(\hat{Y})$ , where  $f$  is a biset-function defined by

$$f(\hat{Y}) = \begin{cases} \max_{uv \in \delta_{D_r}(\hat{Y})} r_{uv} - |\Gamma(\hat{Y})| & \text{if } \Gamma(\hat{Y}) \subseteq Q \\ 0 & \text{otherwise} \end{cases}$$

The cut-LP relaxation for SN is

$$\begin{aligned} \tau^* = \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta_E(\hat{Y})} x_e \geq f(\hat{Y}) \quad \forall \text{ biset } \hat{Y} \\ & 0 \leq x_e \leq 1 \quad \forall e \in E \end{aligned}$$

We will say that a graph with edge capacities  $x_e$  is *fractionally  $(r, Q)$ -connected* if  $x$  is a feasible solution to the above cut-LP relaxation.

To prove Theorem 2, we prove in the next sections the following two theorems about  $\{0, 1, 2\}$ -connected graphs, that are of independent interest, and may find further applications in low connectivity network design. An  $r$ -connected graph  $G$  is *minimally  $r$ -connected* if no proper subgraph of  $G$  is  $r$ -connected.

**Theorem 5.** Let  $G$  be a minimally  $(r, Q)$ -connected graph such that  $Q \cup R = V$  and  $r_{uv} \in \{0, 1, 2\}$  for all  $u, v \in R$ . Then every  $R$ -component is a tree. Furthermore, for any subset  $\mathcal{C}$  of connected components of  $G \setminus R$ , replacing for each  $C \in \mathcal{C}$  the corresponding tree by a DFS cycle of capacity  $1/2$  results in a fractionally  $(r, Q)$ -connected graph.

**Theorem 6.** *Let  $R$  be a set of terminals in a normed space, let  $B \subseteq R$ , and let  $r$  be a  $\{0, 1, 2\}$  requirement function on  $R$ . Let  $S$  be an inclusion minimal set of points such that the unit-disc graph of  $R \cup S$  is  $(r, B \cup S)$ -connected. Among all  $(r, B \cup S)$ -connected spanning subgraphs of the unit-disc graph of  $R \cup S$ , let  $G = (V, E)$  be one of minimum total length  $\sum_{uv \in E} d(u, v)$ . Then  $\deg_G(v) \leq \Delta$  for all  $v \in S$ .*

Particular cases of Theorem 6 were proved by Robins and Salowe [22] for  $r \equiv 1$ , and by Calinescu [3] for  $r \equiv 2$ . We prove Theorems 5 and 6 in Sections 3.1 and 5, respectively, relying on these particular cases. From Theorem 5, Theorem 6, and Lemma 4, we obtain the following corollary, that implies Theorem 2.

**Corollary 2.** *For any feasible solution  $S, G$  to an instance of  $\{0, 1, 2\}$ -SN-MSP there exists a half integral bead solution of value at most  $\Delta|S|/2$ .*

### 3.1 Proof of Theorem 5

A *block* of a graph  $G$  is an inclusion-maximal 2-connected subgraph of  $G$ , or a graph induced by a bridge of  $G$ . It is known that every edge belongs to exactly one block, hence the blocks of a graph partition its edge set. Furthermore, any two blocks have at most one node in common.

**Lemma 5.** *Let  $G = (V, E)$  be a minimally  $(r, Q)$ -connected graph such that  $r_{uv} \in \{0, 1, 2\}$  for all  $uv \in D_r$  and  $Q \cup R = V$ . Let  $G' = (V', E')$  be a 2-connected block of  $G$  and let  $R' = R \cap V'$ . Then  $|R \cap V'| \geq 2$  and no proper 2-connected subgraph of  $G'$  that contains  $R'$  exists.*

*Proof.* We may assume that  $G$  is connected, as otherwise we may consider each connected component of  $G$  separately. Any  $V'$ -component  $C$  has exactly one node in  $V'$ , which we call the *attachment node* of  $C$ . Note that if  $r_{uv} = 2$  such that  $v$  belongs to a  $V'$ -components  $C_v$  of  $G$  and  $u \notin C_v$ , then the attachment node of  $C_v$  is in  $V \setminus Q$ , and hence is in  $R$ , by the assumption  $Q \cup R = V$ .

We prove that  $|R'| \geq 2$ . Since  $G'$  is 2-connected, and  $G$  is minimally  $(r, Q)$ -connected, there exists  $uv \in D_r$  with  $r_{uv} = 2$  such that  $u \in V'$ , or  $u, v$  belong to disjoint  $V'$ -components. Suppose that  $u \in V'$ . If  $v \in V'$  then we are done. Else,  $v$  belongs to a  $V'$ -component, and the attachment node of this component is in  $R$ . If  $u, v$  belong to disjoint  $V'$ -components, then the attachment nodes of these components are distinct and belong to  $R$ . In all cases, we have  $|R'| \geq 2$ .

We prove that if  $G'' = (V'', E'')$  is a 2-connected subgraph of  $G'$  that contains  $R'$ , then  $G'' = G'$ . Suppose that  $G'' \neq G'$ . Let  $A$  be the set of attachment nodes that are in  $V' \setminus V''$ . Note that  $A \subseteq Q \setminus R$ . In  $G'$ , shrink  $V''$  into a single node  $v''$ , and take  $F$  to be the edge set of some inclusion minimal tree in  $G'$  that contains  $A \cup \{v''\}$ . Let  $I = E'' \cup F$ . If  $A = \emptyset$  then  $F = \emptyset$ , and  $I = E''$ . Otherwise, there is  $a \in A$  that has degree exactly 1 in  $(V', I)$ . In both cases,  $I$  must be a proper subset of  $E' \setminus E''$ . Let  $\hat{G}$  be obtained from  $G$  by replacing  $E'$  by  $I$ . It is not hard to verify that  $\hat{G}$  is  $(r, Q)$ -connected, since  $A \subseteq Q \setminus R$ . Furthermore,  $\hat{G}$  is a proper subgraph of  $G$ , since  $I$  is a proper subset of  $E' \setminus E''$ . This contradicts the minimality of  $G$ .  $\square$

A path  $P$  is an  $L$ -chord path of a cycle  $L$  in a graph  $G$  if the endnodes of  $P$  are in  $L$  but no internal node of  $P$  is in  $L$ . Relying on ear decomposition of 2-connected graphs, Calinescu [3] proved the following.

**Lemma 6 ([3]).** *Let  $G' = (V', E')$  be a 2-connected graph and let  $R' \subseteq V$  with  $|R'| \geq 2$ . Suppose that no proper 2-connected subgraph of  $G$  that contains  $R'$  exists. Then any cycle  $L$  in  $G'$  contains at least 2 nodes in  $R'$ , and any  $L$ -chord path contains at least one node in  $R'$  that does not belong to  $L$ .<sup>1</sup>*

We generalize this to  $\{0, 1, 2\}$ - $Q$ -connectivity, as follows.

**Lemma 7.** *Let  $G = (V, E)$  be a minimally  $(r, Q)$ -connected graph such that  $r_{uv} \in \{0, 1, 2\}$  for all  $uv \in D_r$  and  $Q \cup R = V$ . Then any cycle  $L$  in  $G$  contains at least 2 nodes in  $R$ , and any  $L$ -chord path contains at least one node in  $R$  that does not belong to  $L$ .*

*Proof.* Let  $L$  be a cycle in  $G$ . Then  $L$  is contained in some 2-connected block  $G' = (V', E')$  of  $G$ ; moreover, any  $L$ -chord path is also contained in  $G'$ . Let  $R' = R \cap V'$ . By Lemma 5,  $G', R'$  satisfy the conditions of Lemma 6; hence the statement follows from Lemma 6.  $\square$

By Lemma 7, the graph  $G \setminus R$  is a forest, and every  $v \in R$  has at most one neighbor in each connected component of  $G \setminus R$ . This implies the first part of Theorem 5. Now we prove the second part, namely, the following.

**Lemma 8.** *Let  $G$  be a minimally  $(r, Q)$ -connected graph such that  $r_{uv} \in \{0, 1, 2\}$  for all  $uv \in D_r$  and  $Q \cup R = V$ . Then for any subset  $\mathcal{C}$  of connected components of  $G \setminus R$ , replacing for each  $C \in \mathcal{C}$  the corresponding tree  $T_C$  by a DFS cycle on  $T_C(C)$  of capacity  $1/2$  results in a fractionally  $(r, Q)$ -connected graph  $H$ .*

*Proof.* Suppose to the contrary that there exists  $uv \in D_r$  such that  $u, v$  are not fractionally  $(r_{uv}, Q)$ -connected in  $H$ . This may happen only if  $r_{uv} = 2$  and there exists  $C \in \mathcal{C}$  such that  $u, v$  can be disconnected by removing two elements  $a, b$  of  $T_C$  from  $G$ . If one of  $a, b$  is an edge we can replace it by its endnode in  $T_C$ , hence we may assume that each of  $a, b$  is a node. Note that  $a \neq b$ , since otherwise  $u, v$  can be disconnected by removing the single element  $a$ , contradicting that  $\lambda_G^Q(u, v) \geq r_{uv} = 2$ . Let  $P_{ab}$  be the  $ab$ -path in  $T_C$ . Note that all the internal nodes of  $P_{ab}$  are in  $C$ , so none of them is a terminal. Consider two  $(Q \cup E)$ -disjoint  $u, v$  paths in  $G$ . One of them must contain  $a$  and the other contains  $b$ ; denote these paths by  $P_a$  and  $P_b$ , respectively. The union of the paths  $P_a$  and  $P_b$  contains a simple cycle  $L$  that contains  $a, b$ . Hence the path  $P_{ab}$  has a subpath  $P$  such that  $P$  is an  $L$ -chord path. This contradicts Lemma 7, since no internal node of  $P$  is a terminal.  $\square$

The proof of Theorem 5 is complete.

<sup>1</sup> This statement is not true for edge-connectivity; for example, if  $R = \{s, t\}$  and  $G$  consists of 2 edge-disjoint  $st$ -paths that have 2 nodes  $u, v$  in common, then the simple cycle that contains  $u, v$  contains no node from  $R$ .

## 4 Proof of Theorem 4

For a tree  $T = (V, F)$  and  $A \subseteq V$  let  $T_A = (V_A, F_A)$  be the inclusion minimal subtree of  $T$  that contains  $A$ . To prove Theorem 4 it is sufficient to prove the following.

**Lemma 9.** *Let  $T = (V, F)$  be a tree of maximum degree  $\Delta \geq 2$ , let  $R \subseteq V$ , and let  $S = V \setminus R$ . Then for any integer  $k \geq 2\Delta - 2$  there exists a connected hypergraph  $\mathcal{H} = (R, \mathcal{E})$  of rank  $\leq k$  such that  $\sum_{A \in \mathcal{E}} |V_A \cap S| \leq \left(1 + \frac{2}{\lfloor \lg \lfloor k/(\Delta-1) \rfloor \rfloor}\right) |S|$ .*

To prove Lemma 9 we prove the following.

**Lemma 10.** *Let  $T = (V, F)$  be a tree with edge costs  $\{c(e) \geq 1 : e \in F\}$  and let  $R \subseteq V$ . Then for any integer  $p \geq 2$  there exists a connected hypergraph  $\mathcal{H} = (R, \mathcal{E})$  of rank  $\leq p$  such that  $\sum_{A \in \mathcal{E}} c(F_A) + |\mathcal{E}| - 1 \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) c(T)$ .*

Lemma 10 will be proved later. Now we show that it implies Lemma 9. An  $R$ -component of  $T$  is a maximal inclusion subtree of  $T$  such that all its leaves are in  $R$  but no its internal node is in  $R$ . It is easy to see that it is sufficient to prove Lemma 9 for each  $R$ -component separately, hence we may assume that  $R$  is the set of leaves of  $T$ .

If  $T$  is a star, then since  $k \geq 2\Delta - 2 \geq \Delta$ , we let  $\mathcal{E}$  to consist of a single hyperedge  $A = R$ . Then  $|V_A \cap S| = 1 = |S|$ , and Lemma 9 holds in this case.

Henceforth assume that  $T$  is not a star. For  $v \in S$  let  $R(v)$  be the set of neighbors of  $v$  in  $R$ , and note that  $|R(v)| \leq \Delta - 1$ . Let  $T' = (V', F') = T \setminus R$  and let  $R' = \{v \in S : R(v) \neq \emptyset\}$ . Applying Lemma 10 on  $T'$  with unit edge-costs and  $R'$ , we obtain that for  $p = \lfloor k/(\Delta - 1) \rfloor$  there exists a connected hypergraph  $\mathcal{H}' = (R', \mathcal{E}')$  of rank  $\leq p$  such that  $\sum_{A' \in \mathcal{E}'} |F'_{A'}| + |\mathcal{E}'| - 1 \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) |F'|$ . Note that  $|F'| = |V'| - 1$  and that  $|V'_{A'}| = |F'_{A'}| - 1$  for every  $A' \in \mathcal{E}'$ . Hence

$$\sum_{A' \in \mathcal{E}'} |V'_{A'}| - 1 \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) (|V'| - 1) \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) |V'| - 1.$$

For  $A' \in \mathcal{E}'$  let  $A = \cup_{v \in A'} R(v)$ ; then  $|A| \leq p(\Delta - 1)$ . Let  $\mathcal{E} = \{A : A' \in \mathcal{E}'\}$ . Then  $\mathcal{H} = (R, \mathcal{E})$  is a connected hypergraph of rank  $\leq p(\Delta - 1) \leq k$ , and

$$\sum_{A \in \mathcal{E}} |V_A \cap S| = \sum_{A' \in \mathcal{E}'} |V'_{A'}| \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) |V'| = \left(1 + \frac{2}{\lfloor \lg \lfloor k/(\Delta - 1) \rfloor \rfloor}\right) |S|.$$

In the rest of this section we prove Lemma 10, by extending the proof of Du and Zhang [7] of an existence of a connected hypergraph  $\mathcal{H} = (R, \mathcal{E})$  of rank  $\leq p$  such that  $\sum_{A \in \mathcal{E}} c(F_A) \leq \left(1 + \frac{1}{\lfloor \lg p \rfloor}\right) c(T)$ . We have an extra term of  $|\mathcal{E}| - 1$ , and we show that this term can be bounded by  $\frac{c(T)}{\lfloor \lg p \rfloor}$ .

We start by transforming the tree into a (rooted) binary tree  $T$  with edge-costs, which node set is partitioned into a set  $R$  of terminals and a set  $S$  of non-terminals, such that the following properties hold:

- (A)  $R$  is the set of leaves of  $T$ .
- (B) The cost of any edge of  $T$  is either 0 or is at least 1, and among the edges that connect a node in  $S = V \setminus R$  to its children, at most one has cost 0.
- (C)  $T$  is a full binary tree, namely, every  $v \in S$  has exactly 2 children.

To obtain such a tree, root  $T$  at an arbitrary non-leaf node  $\hat{s} \in S = V \setminus R$ , and apply the following standard reductions.

1. While  $T$  has a leaf in  $S$ , remove this leaf; hence every leaf of  $T$  is in  $R$ . Then, for every  $v \in R$  that is not a leaf, add to  $T$  a new node  $v'$  and an edge  $vv'$  of cost 0, add  $v'$  to  $R$ , and move  $v$  from  $R$  to  $S$ . After this step, properties (A) and (B) hold.
2. While there is  $v \in S$  that has one child, replace the path  $P$  of length 2 that contains  $v$  by a single edge of cost  $c(P)$ , and exclude  $v$  from  $S$ . After this step, every  $v \in S$  has at least 2 children.
3. While there is  $v \in S$  that has more than 2 children, do the following. Let  $u$  be a child of  $v$  such that the cost of the edge  $vu$  is at least 1. Add a new node  $v'$  and the edge  $vv'$  of cost 0, and for every child  $u'$  of  $v$  distinct from  $u$  replace the edge  $vu'$  by the edge  $vu'$ . After this step, all the three properties (A), (B), and (C) hold.

Consequently, to prove Lemma 10, it is sufficient to prove the following.

**Lemma 11.** *Let  $T = (V, F)$  be a tree with edge costs  $\{c(e) : e \in F\}$  and leaf set  $R$ , satisfying properties (A),(B),(C). Then for any integer  $p \geq 2$  there exists a connected hypergraph  $\mathcal{H} = (R, \mathcal{E})$  of rank  $\leq p$  such that  $\sum_{A \in \mathcal{E}} c(F_A) + |\mathcal{E}| - 1 \leq \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) c(T)$ .*

Let  $T = (V, F)$  be a rooted tree with leaf set  $R$  and let  $S = V \setminus R$ . For two nodes  $u, v$  of  $T$  let  $P_T(u, v)$  denote the unique path in  $T$  between  $u$  and  $v$ .

**Definition 10.** *We say that  $T$  is proper if every node in  $S$  has at least 2 children. We say that a mapping  $f : S \rightarrow R$  is  $T$ -proper if*

- For every  $u \in S$ ,  $f(u)$  is a descendant of  $u$ .
- The paths  $\{P_T(u, f(u)) : u \in S\}$  are edge disjoint.

*Given a subtree  $T'$  of  $T$  with leaf set  $L'$  and a proper mapping  $f$ , the set of terminal connecting paths of  $T'$  is  $\{P_T(u, f(u)) : u \in L' \setminus R\}$ . Let  $\hat{T}'$  denote the tree obtained from  $T'$  by adding to  $T'$  all the terminal connecting paths.*

Du and Zhang [7] proved that any proper tree  $T$  admits a proper mapping. We prove the following.

**Lemma 12.** *Let  $T = (V, F)$  be a proper tree and let  $F_1 \subseteq F$  be such that any  $u \in S$  has a child connected to  $u$  by an edge in  $F_1$ . Then there exists a  $T$ -proper mapping  $f$  such that for every  $u \in S$ , the path  $P_T(u, f(u))$  contains at least one edge in  $F_1$ .*

*Proof.* The proof is by induction on the height of the tree. Let  $T$  be a tree as in the lemma of height  $h$ . If  $h = 1$ , then  $T$  has one internal node (the root), say  $u$ , and we set  $f(u)$  to be the node that is connected to  $u$  by an edge in  $F_1$ . Suppose that the statement is true for trees with height  $h - 1 \geq 1$ , and we prove it for trees of height  $h$ . Let  $T'$  be obtained from  $T$  by removing nodes of distance  $h$  from the root. By the induction hypothesis, for  $T'$  there exists a mapping  $f'$  as in the lemma. Let  $u$  be an internal node of  $T$ . Consider two cases.

Suppose that  $u$  is an internal node of  $T'$ . If  $f'(u)$  is a leaf of  $T$ , then define  $f(u) = f'(u)$ . If  $f'(u)$  is an internal of  $T$ , then  $f'(u)$  is a leaf of  $T'$ , and all its children in  $T$  are leaves. Then we set  $f(u)$  to be a child of  $f'(u)$  that is connected to  $f'(u)$  by an edge in  $F_1$ .

Suppose that  $u$  is a leaf of  $T'$ . Then the children of  $u$  in  $T$  are leaves, and we set  $f(u)$  to be a child of  $u$  that is connected to  $u$  by an edge in  $F_1$ .

It is easy to verify that the obtained mapping  $f$  meets the requirements.  $\square$

The following statement is implicitly proved by Du and Zhang [7].

**Lemma 13 ([7]).** *Let  $T$  be a proper binary tree with non-negative edge costs and let  $f$  be a proper mapping. Then for any integer  $p \geq 2$  there exists an edge-disjoint partition  $\mathcal{T}$  of  $T$  into subtrees such that the following holds:*

- (i) *The hypergraph with node set  $R$  and hyperedge set  $\mathcal{E} = \{\hat{T}' \cap R : T' \in \mathcal{T}\}$  is connected and has rank at most  $p$ .*
- (ii) *The total number of terminal connecting paths of all subtrees in  $\mathcal{T}$  is at least  $|\mathcal{T}| - 1$ , and their total cost is at most  $c(T)/\lfloor \lg p \rfloor$ .*

We now finish the proof of Lemma 11, and thus also of Lemma 10. Let  $F_1 = \{e \in F : c(e) \geq 1\}$  and let  $f$  be a proper mapping as in Lemma 12. Let  $\mathcal{T}$  be a partition as in Lemma 13, and let  $\mathcal{E}$  be as in Lemma 13(i), so the hypergraph  $\mathcal{H} = (R, \mathcal{E})$  is connected and has rank at most  $p$ . By Lemma 13(ii), the total number of terminal connecting paths of all subtrees is at least  $|\mathcal{T}| - 1 = |\mathcal{E}| - 1$ , while their total cost is at most  $c(T)/\lfloor \lg p \rfloor$ . Every terminal connecting path contains an edge from  $F_1$ , by Lemma 12, and thus has cost at least 1. Hence the total cost of all terminal connecting paths is at least  $|\mathcal{E}| - 1$ . Consequently

$$|\mathcal{E}| - 1 \leq \frac{c(T)}{\lfloor \lg p \rfloor}.$$

For  $A = \hat{T}' \cap R \in \mathcal{E}$  let  $P(T')$  denote the union of the edge sets of the terminal connecting paths of  $T'$ . Then  $c(F_A) \leq c(\hat{T}') = c(T) + c(P(T'))$ , hence

$$\sum_{A \in \mathcal{E}} c(F_A) \leq \sum_{T' \in \mathcal{T}} [c(T') + c(P(T'))] = \sum_{T' \in \mathcal{T}} c(T') + \sum_{T' \in \mathcal{T}} c(P(T')) \leq c(T) + \frac{c(T)}{\lfloor \lg p \rfloor}.$$

Summarizing, we have

$$\sum_{A \in \mathcal{E}} c(F_A) + |\mathcal{E}| - 1 \leq c(T) + \frac{c(T)}{\lfloor \lg p \rfloor} + \frac{c(T)}{\lfloor \lg p \rfloor} = \left(1 + \frac{2}{\lfloor \lg p \rfloor}\right) c(T).$$

The proof of Lemma 11, and thus also of Lemma 10 and Theorem 4 is now complete.

## 5 Proof of Theorem 6

To prove Theorem 6, we use the following result of Calinescu [3].

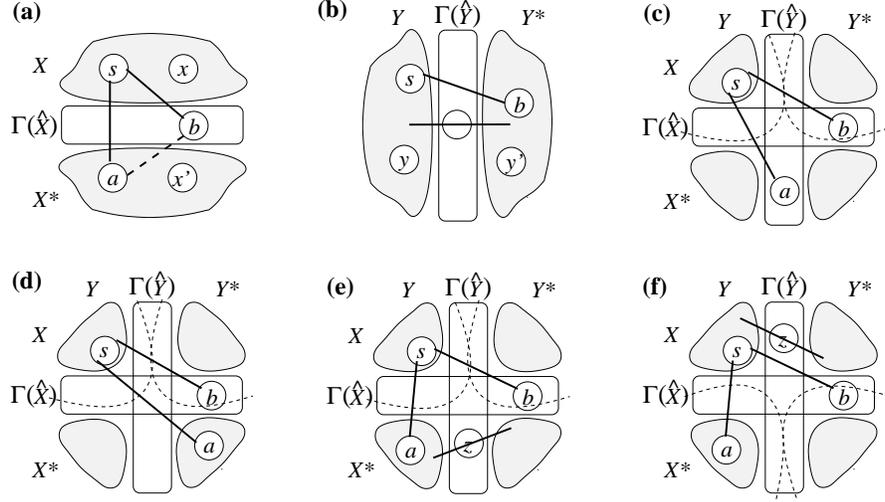
**Lemma 14 ([3]).** *Let  $R'$  be a set of terminals in a normed space and let  $S'$  be an inclusion minimal set of points such that the unit-disc graph of  $R' \cup S'$  is 2-connected. Among all 2-connected spanning subgraphs of the unit-disc graph of  $R' \cup S'$ , let  $G' = (V', E')$  be one of minimum total length  $\sum_{uv \in E'} d(u, v)$ . Then  $\deg_{G'}(v) \leq \Delta$  for all  $v \in S'$ .*

Let  $G, B, S, r$  be as in Theorem 6. As in the proof of Theorem 5, we may assume that  $G$  is connected. Consider a 2-connected block  $G' = (V, E')$  of  $G$ . Let  $R' = R \cap V'$  and  $S' = S \cap V'$ . Then by Lemma 5 no proper 2-connected subgraph of  $G'$  that contains  $R'$  exists, hence  $S'$  is an inclusion minimal set of points such that the unit-disc graph of  $R \cup S$  is 2-connected. Furthermore, since  $G$  has minimum total length, so is  $G'$ . Thus by Lemma 14,  $\deg_{G'}(v) \leq \Delta$  for all  $v \in S'$ . Consequently,  $\deg_G(v) \leq \Delta$  holds for any  $s \in S$  that belongs to exactly one block of  $G$ . A node  $s$  is a *cut-node* of a connected graph if its removal disconnects the graph. It is known that  $s$  is a cut-node of a graph if and only if  $s$  belongs to at least two blocks of the graph. Our goal now is to show that  $\deg_G(v) \leq \Delta$  holds for any cut-node  $s \in S$  of  $G$ .

Let  $s \in S$  be a cut-node of  $G$ . Suppose to the contrary that  $\deg_G(v) \geq \Delta + 1$ . Then by [22] there are neighbors  $a, b$  of  $s$  in  $G$  such that  $d(a, b) \leq d(a, s)$ . By a reduction from [22,3], we may assume that all the lengths of the edges in  $G$  are distinct, hence  $d(a, b) < d(a, s)$ . Let  $H$  be obtained from  $G$  by replacing the edge  $sa$  by the edge  $ab$ . We claim that  $H$  is  $(r, Q)$ -connected, which gives a contradiction, since  $H$  has smaller total length than  $G$ . Thus to finish the proof of Theorem 6, it is sufficient to prove the following.

**Lemma 15.** *Let  $G = (V, E)$  be an  $(r, Q)$ -connected graph with  $r_{uv} \in \{0, 1, 2\}$  for all  $uv \in D_r$ , and let  $sa, sb \in E$  be a pair of  $(r, Q)$ -connectivity critical edges with  $s \in Q \setminus R$ . Then the graph  $H$  obtained from  $G$  by replacing the edge  $sa$  by the edge  $ab$  is also  $(r, Q)$ -connected.*

*Proof.* Suppose to the contrary that there is  $xx' \in D_r$  such that  $\lambda_H^Q(xx') \leq r_{xx'} - 1$ . It is easy to see that any  $u, v$  that are connected in  $G$  also connected in  $H$ , hence we must have  $r_{xx'} = 2$ . Consider the graph  $J = G \setminus \{sa\}$ . Since  $\lambda_{J \cup \{sb\}}^Q(x, x') = 1$  and  $\lambda_{J \cup \{sa\}}^Q(x, x') = 2$ , then by Menger's Theorem, there exists a biset  $\hat{X}$  such that  $s \in X$ ,  $a \in X^*$ ,  $b \in \Gamma(\hat{X}) \subseteq Q$ ,  $\delta_G(\hat{X}) = \{sa\}$ , and one of  $x, x'$  belongs to  $X$  and the other to  $X^*$ , say  $x \in X$  and  $x' \in X^*$ ; see Figure 5(a). Similarly, since the edge  $sb$  is  $(r, Q)$ -connectivity critical, there exist  $yy' \in D_r$  with  $r_{yy'} = 2$  and a biset  $\hat{Y}$ , such that  $s \in Y$ ,  $b \in Y^*$ ,  $sb \in \delta_G(\hat{Y})$ ,  $|\delta_G(\hat{Y})| + |\Gamma(\hat{Y})| = 2$ ,  $\Gamma(\hat{Y}) \subseteq Q$ , and one of  $y, y'$  belongs to  $Y$  and the other to  $Y^*$ , say  $y \in Y$  and  $y' \in Y^*$ ; see Figure 5(b). Now we consider the three cases,  $a \in \Gamma(\hat{Y})$ ,  $a \in Y^*$ , and  $a \in Y$ , and at each of them arrive to a contradiction.



**Fig. 1.** Illustration to the proof of Lemma 15.

Suppose that  $a \in \Gamma(\hat{Y})$ ; see Figure 5(c). Then  $x \notin \Gamma(\hat{Y})$ , so  $x \in X \cap Y$  or  $x \in X \cap Y^*$ . If  $x \in X \cap Y^*$  then the biset  $\hat{Z} = \hat{X} \setminus \hat{Y}$  satisfies  $|\Gamma(\hat{Z})| + |\delta_G(\hat{Z})| = 1$  (since  $\Gamma_G(\hat{Z}) = \{b\}$  and  $\delta_G(\hat{Z}) = \emptyset$ ,  $x \in Z$ , and  $x' \in Z^*$ ; this contradicts the assumption  $\lambda_G^Q(x, x') = 2$ . In the case  $x \in X \cap Y$ , we obtain a similar contradiction for the biset  $\hat{Z} = (X \cap Y \setminus \{s\}, X \cap Y)$ .

The analysis of the case  $a \in Y^*$ , see Figure 5(d), is similar to that of the case  $a \in \Gamma(\hat{Y})$ .

Now suppose that  $a \in Y$ ; see Figure 5(e,f). Since  $|\Gamma(\hat{Y})| + |\delta_G(\hat{Y})| = 2$  and since  $sb \in \delta_G(\hat{Y})$ , there is another element  $z \in \Gamma(\hat{Y}) \cup \delta_G(\hat{Y})$ . Note that if  $z$  is a node then  $z \in X^* \cap \Gamma(\hat{Y})$  (Figure 5(e)) or  $z \in X \cap \Gamma(\hat{Y})$  (Figure 5(f)). If  $z$  is an edge then  $z$  connects  $Y \cap X^*$  and  $Y^* \setminus X$  (Figure 5(e)) or  $X \cap Y$  and  $Y^* \setminus X^*$  (Figure 5(f)). In the cases in Figure 5(e), when  $z \in X^* \cap \Gamma(\hat{Y})$  is a node, or  $z$  is an edge that connects  $Y \cap X^*$  and  $Y^* \setminus X$ , the contradiction is obtained in the same way as in the case  $a \in \Gamma(\hat{Y})$ . We therefore are left with the cases in Figure 5(f), when  $z \in X \cap \Gamma(\hat{Y})$  or  $z$  is an edge that connects  $X \cap Y$  and  $Y^* \setminus X^*$ . Then we consider the location of  $x'$ . Note that  $x' \notin \Gamma(\hat{Y})$ , hence  $x' \in Y$  or  $x' \in Y^*$ . In the case  $x' \in Y$  we obtain a contradiction by considering the biset  $\hat{Z} = \hat{Y} \setminus \hat{X}$ , and in the case  $x' \in Y^*$  we obtain a contradiction by considering the biset  $\hat{Z} = \hat{X} \cup \hat{Y}$ .  $\square$

The proof of Theorem 6 is complete.

## 6 Conclusions

In this paper we considered the Survivable Network with Minimum Number of Steiner Points problem in a normed space. The main results of this paper are a

$(1 + \ln(\Delta - 1) + \epsilon)$ -approximation scheme for ST-MSP, and a  $\Delta$ -approximation algorithm for  $\{0, 1, 2\}$ -SN-MSP. For ST-MSP in  $\mathbb{R}^2$  this improves the ratio  $2.5 + \epsilon$  of [5]. For  $\{0, 1, 2\}$ -SN-MSP, no nontrivial approximation algorithm was known before, but for the specific case of SF-MSP this improves the ratio  $2\Delta$  that can be deduced from the work of [13]. Obtaining even better approximation ratios is an important future work.

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