# Network Coding Algorithms for Multi-Layered Video Broadcast 

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#### Abstract

In this paper we give network coding algorithms for multi-layered video streaming. The problem is motivated by video broadcasting in a communication network to users with varying demands. We give a polynomial time algorithm for deciding feasibility for the case of two layers, and show that the problem becomes NP-hard if the task is to maximize the number of satisfied demands. For the case of three layers we also show NP-hardness of the problem. Finally, we propose a heuristic for three layers and give experimental comparison with previous approaches.


Keywords network coding • multi-layered video streaming

## 1 Introduction

The appearance of new devices (smartphones, tablets, etc.) has highly increased user diversity in communication networks. As a consequence, when watching a video stream, users may have very different quality demands depending on the resolution capability of their devices.

Multi-resolution code (MRC) is one successful way to handle this diversity, encoding data into a base layer and one or more refinement layers [8, 1]. Receivers can request cumulative layers, and the decoding of a higher layer always requires the correct reception of all lower layers (including the base layer). The multi-layer multicast problem is to multicast as many valuable layers to as many receivers as possible.

In a multi-layered streaming setup, network coding was shown to be a successful tool for increasing throughput compared to simple routing [4]. In their simple heuristic, Kim et al. [4] give a network coding scheme based on restricting the set of layers that may be encoded at certain nodes.

This paper proposes algorithms for the multi-layered video streaming problem. We give an optimal polynomial time algorithm for two layers when the goal is to send the base layer to every user, and within this constraint to maximize the number of users receiving two layers. For the case of three or more layers we show NP-hardness of the problem. Also, we show NP-hardness for the case of two layers when the goal is to maximize the total number of transmitted useful layers. We also propose a heuristic for three

[^0]layers and give experimental comparison between the best known heuristic due to Kim et al. 4 and our approach.

The rest of the paper is organized as follows: Section 2 contains the problem formulation and some definitions. In Section 3 we prove NP-hardness for some special cases of the problem. In Section 4, the notion of feasible height function is introduced, and a sufficient condition for simultaneously satisfiable receiver demands is given. In Section 5 we give an optimal algorithm for two layers. In Section 6 we present a heuristic for three layers, and give some numerical results of experimental comparison.

## 2 Problem Formulation

A video stream is divided into $k$ layers of equal size. At each time slot $t$, a set of messages $\mathbf{M}(t)=$ $\left\{M_{1}(t), M_{2}(t), \ldots M_{k}(t)\right\}$ is generated, message $M_{i}(t)$ belonging to layer $i$ and represented by an element of a finite field $\mathbb{F}_{q}$ of size $q$. The idea of network coding is to transmit linear combinations of messages on the unit rate arcs. We will construct such linear network coding schemes, where these linear combinations only combine messages from the same time slot, and the same coding scheme is applied for each message set $\mathbf{M}(t)$, so the notation ' $t$ ' will be omitted in the paper.

Let $D=(V, A)$ be a directed acyclic graph with a single source node $s$ and with unit capacity arcs. We will consider this graph fixed, except in Section 3. For a node $v \in V-s$, let $\lambda(s, v)$ denote the maximal number of arc-disjoint paths from $s$ to $v$. For a pair of nodes $u, v \in V$, a set $X \subset V$ is a $\bar{u} v$-set, if $u \notin X$ and $v \in X$. For a set $X$ of nodes let $\varrho(X)$ denote the number of entering $\operatorname{arcs}$ of $X$. Note that $\lambda(s, v) \geq k$ if and only if $\varrho(X) \geq k$ for every $\bar{s} v$-set $X$. We assume that $\lambda(s, v) \geq 1$ for every node in the graph. A set $X$ not containing $s$, and having $\varrho(X)=i$ is called an $i$-set.

The task is to multicast $\mathbf{M}=\left(M_{1}, M_{2}, \ldots M_{k}\right)$ from $s$. A network code can be represented by the vector of the coefficients $\mathbf{c}=\left(c_{1}, \ldots, c_{k}\right)$ on each arc (where $c_{i} \in \mathbb{F}_{q}$ ). Let $\mathbb{F}_{q}^{k}$ denote the $k$-dimensional vector space over $\mathbb{F}_{q}$, and let $\mathbf{e}_{i}$ denote the $i$ th unit vector. For a set $S \subseteq \mathbb{F}_{q}^{k}$ of vectors, let $\langle S\rangle$ denote the linear subspace spanned by $S$.
Definition 1 A linear network code of $k$ messages over a finite field $\mathbb{F}_{q}$ is a mapping $\mathbf{c}: A \rightarrow \mathbb{F}_{q}^{k}$ which fulfills the linear combination property: $\mathbf{c}(u v) \in\langle\{\mathbf{c}(w u) \mid w u \in A\}\rangle$ for all $u \neq s$. We will use the notation $\langle\mathbf{c}, u\rangle=\langle\{\mathbf{c}(w u) \mid w u \in A\}\rangle$. The function $\mathbf{c}$ gives the coefficients of the messages on an arc, that is, on arc a the message sent is the scalar product $\mathbf{c}(a) \cdot \mathbf{M}$. We say that a message $M_{i}$ (or layer i) has non-zero coefficient on an arc a, if $\mathbf{c}(a) \cdot \mathbf{e}_{i} \neq 0$. A node $v$ can decode message $M_{i}$ (or layer $i$ ), if $\mathbf{e}_{i} \in\langle\mathbf{c}, v\rangle$. Hence, with abuse of notation, $\mathbf{e}_{i}$ will be identified with message $M_{i}$ and layer $i$ throughout the paper.

We remark here that if a node $v$ can decode layer $i$ by the above definition, then it really can decode message $M_{i}$, as it gets all scalar products $\mathbf{c}(u v) \cdot \mathbf{M}$ and $\mathbf{e}_{i} \in\langle\mathbf{c}, v\rangle$, so it can easily calculate $M_{i}=\mathbf{e}_{i} \cdot \mathbf{M}$. Note also that simple routing can be regarded as a special case, where for each arc $u v, \mathbf{c}(u v)=\mathbf{e}_{i}$ for some $1 \leq i \leq k$.

Definition 2 In multi-resolution coding, for $i>j$ we say that layer $i$ is higher than layer $j$, and layer $j$ is lower than layer $i$. The height of a network code on an arc uv is the highest layer with non-zero coefficient on that arc. For example, the first unit vector has height one and so on, $\mathbf{e}_{i}$ has height $i$, and vector $(1,0,1,0)$ has height 3. The height of $\mathbf{c}$ is denoted by $h_{\mathbf{c}}: A \rightarrow \mathbb{N}$.

A layer $i$ is valuable for a node only if all lower layers can also be decoded at that node, i.e., for every $j \leq i$ message $M_{j}$ is decodable. The performance of a network code at a node $v$ is the index of the highest valuable layer for $v$. The performance function of $\mathbf{c}$ is denoted by $p_{\mathbf{c}}: V \rightarrow\{0,1, \ldots, k\}$, where $p(v)=0$ denotes that layer 1 is not decodable at $v$.

A demand is a sequence of mutually disjoint subsets of $V-s$ denoted by $\tau=\left(T_{1}, T_{2}, \ldots, T_{k}\right)$. The set of receiver nodes is the union of these request sets, denoted by $T=T_{1} \cup T_{2} \cup \ldots \cup T_{k}$. The nodes in $T_{i}$
request the first $i$ layers. Given a demand $\tau$, we can define a demand function $d_{\tau}: V \rightarrow\{0,1, \ldots, k\}$ on the nodes in a straightforward way by setting $d_{\tau}(v)=i$ if $v \in T_{i}$, and $d_{\tau}(v)=0$ if $v \in V \backslash T$.

A network code is feasible for demand $\tau$, if $p_{\mathbf{c}}(v) \geq d_{\tau}(v)$ for all $v \in V$, that is, for all $i$ and $j \leq i$, every receiver node $t \in T_{i}$ can decode $M_{j}$. If there exists a feasible network code for a demand, the demand is called satisfiable.

## 3 Complexity Results

In this section we prove NP-hardness of some special cases of the multi-layered network coding problem. Lehman and Lehman [6] showed NP-hardness for a more general network coding problem, where receivers may demand any subset of the messages, and there can be multiple sources, accessing disjoint subsets of the information demanded by the receivers. Here we prove NP-hardness for a special case of this problem, when there is only one source in the graph, and demands of the receivers have a layered structure as defined in the previous section 2 .

Theorem 3 Given a directed acyclic graph $D$ and a demand with three layers $\tau=\left(T_{1}, \emptyset, T_{3}\right)$, it is NP-hard to decide, whether there exists a feasible network code for $\tau$.


Fig. 1: Reduction of 3 -SAT to demand $\tau=\left(T_{1}, \emptyset, T_{3}\right)$.

Proof We reduce the well-known NP-complete 3-SAT problem [2] to our problem. Let $S=(X, C L)$ be a 3-SAT instance, where $X=\left\{x_{1}, \ldots, x_{n}\right\}$ and $C L=\left\{C_{1}, \ldots, C_{m}\right\}$ denote the set of variables and clauses, respectively. We define a network coding problem on a digraph $D$ corresponding to this instance. First we create special nodes $s, t$ with an arc $s t$, and put $t$ into $T_{1}$. For each variable $x_{i}$ we add six nodes with eleven arcs (see Figure 11, so that $a_{i}, b_{i}, c_{i} \in T_{1}$ and $d_{i} \in T_{3}$. Nodes $x_{i}$ and $\bar{x}_{i}$ correspond to literals. For each clause $C_{j}$ we add a node $C_{j}$, arcs $s C_{j}$ and $t C_{j}$ and arcs from every node corresponding to literals of $C_{j}$. Each $C_{j}$ is put into $T_{3}$. We prove that this network coding problem has a feasible solution over some finite field if and only if $S$ can be satisfied. Suppose the above defined network coding problem has a feasible network code $\mathbf{c}$. Since $t \in T_{1}, h_{\mathbf{c}}(s t)=1$ and for all $C_{j} h_{\mathbf{c}}\left(t C_{j}\right)=1$. Moreover, the arc $s C_{j}$ can transmit any message from $s$, hence $C_{j}$ can decode all three layers if and only if at least one additional arc entering $C_{j}$ has height greater than one. Note that such an arc can come only from a node corresponding to a literal in $C_{j}$.

Claim 4 If the network coding problem has a feasible network code $\mathbf{c}$, then for every variable $x_{i}$, the code $\mathbf{c}$ has height one on at least one of the arcs $s x_{i}$ and $s \bar{x}_{i}$.

Proof Let us assume indirectly that neither $\mathbf{c}\left(s x_{i}\right)$ nor $\mathbf{c}\left(s \bar{x}_{i}\right)$ have height one. Since $a_{i}, b_{i}, c_{i}$ must be able to decode the first layer, $\mathbf{c}\left(s a_{i}\right) \in\left\langle\mathbf{e}_{1}, \mathbf{c}\left(s x_{i}\right)\right\rangle$, and $\mathbf{c}\left(s c_{i}\right) \in\left\langle\mathbf{e}_{1}, \mathbf{c}\left(s \bar{x}_{i}\right)\right\rangle$, and $\mathbf{e}_{1} \in\left\langle\mathbf{c}\left(s x_{i}\right), \mathbf{c}\left(s \bar{x}_{i}\right)\right\rangle$. Hence we have

$$
\operatorname{dim}\left\langle\mathbf{c}\left(s a_{i}\right), \mathbf{c}\left(s c_{i}\right), \mathbf{c}\left(s x_{i}\right), \mathbf{c}\left(s \bar{x}_{i}\right)\right\rangle=\operatorname{dim}\left\langle\mathbf{c}\left(s x_{i}\right), \mathbf{c}\left(s \bar{x}_{i}\right)\right\rangle \leq 2,
$$

that is, these four vectors cannot span a 3 -dimensional space to transmit three layers to $d_{i}$.
By the claim we can transform a solution of the network coding problem into an assignment of $S$ by assigning value 'true' to a literal $l$ if the height of $\mathbf{c}(s l)$ is at least two. Note that if for a variable $x_{i}$ both $s x_{i}$ and $s \bar{x}_{i}$ have height one, we can choose the value of $x_{i}$ arbitrarily to get a satisfying assignment.

Similarly we can get a feasible network code $\mathbf{c}$ for the network coding problem from a truth assignment of $S$ over any field. The corresponding vectors $\mathbf{c}(e)$ are the following. Let $\mathbf{c}(s t)=(1,0,0), \mathbf{c}\left(s C_{j}\right)=$ $(1,1,1)$, and for any node $u$ with only one incoming arc $w u$, all outgoing arcs carry $\mathbf{c}(w u)$. If $x_{i}$ is true, then $\mathbf{c}\left(s a_{i}\right)=(0,1,0), \mathbf{c}\left(s x_{i}\right)=(1,1,0), \mathbf{c}\left(s \bar{x}_{i}\right)=(1,0,0), \mathbf{c}\left(s c_{i}\right)=(1,1,1), \mathbf{c}\left(a_{i} d_{i}\right)=(0,1,0), \mathbf{c}\left(b_{i} d_{i}\right)=$ $(1,0,0), \mathbf{c}\left(c_{i} d_{i}\right)=(1,1,1)$, and the code can be constructed symmetrically if $x_{i}$ is false. It is easy to check that this $\mathbf{c}$ is indeed a feasible network code.

For the general case with $k \geq 3$ layers we easily get the similar result by adding $k-3$ new $s C_{i}$ and $s d_{j}$ arcs for each $i, j$.
Corollary 5 For $k \geq 3$ layers and demand $\tau=\left(T_{1}, \emptyset, \ldots, \emptyset, T_{k}\right)$, it is NP-hard to decide whether there exists a feasible network code for $\tau$.

Theorem 6 Given a directed acyclic graph $D$ and a demand $\tau=\left(T_{1}, T_{2}\right)$ it is NP-hard to find a maximal cardinality subset $T_{1}^{\prime}$ of $T_{1}$, so that for $\tau^{\prime}=\left(T_{1}^{\prime}, T_{2}\right)$ there exists a feasible network code.

Proof We prove the theorem by reducing to this problem the NP-hard Vertex Cover problem [2], which is the following: given a graph $G=(W, E)$, find a subset of the nodes $X \subseteq W$ of minimum size such that $X \cap\{u, v\} \neq \emptyset$ for every $u v \in E$. Given an instance $G=(W, E)$ of the vertex cover problem, we construct an acyclic graph $D$ and demand $\left(T_{1}, T_{2}\right)$. First let $D$ be a single node $s$. Then for every vertex $w \in W$ we add a receiver $t_{w} \in T_{1}$ with an arc $s t_{w}$, while for every edge $u v \in E$ we add a receiver $t_{u v} \in T_{2}$ with arcs $t_{u} t_{u v}$ and $t_{v} t_{u v}$. For a given network code $\mathbf{c}$, a receiver node $t_{w} \in T_{1}$ can decode the first layer if and only if the height of the code on $s t_{w}$ is one. A receiver node $t_{u v} \in T_{2}$ can decode both layers, only if on at least one entering arc the code has height two. Let $t_{v} t_{u v}$ be such an arc. Since the tail node $t_{v}$ has exactly one entering arc $s t_{v}$, this arc must have height two also, hence $t_{v}$ does not receive the first layer in $D$. Let $T_{1}^{\prime} \subseteq T_{1}$ denote the set of nodes $t_{w} \in T_{1}$ for which the arc $s t_{w}$ has height one, and let $X \subseteq W$ be the set of nodes $w \in W$ for which $s t_{w}$ has height two. It is easy to conclude that if the code is feasible for demand $\left(T_{1}^{\prime}, T_{2}\right)$ then $X$ is a vertex cover. Conversely, from a vertex cover $X \subseteq W$ let $T_{0}=\left\{t_{w} \in T_{1} \mid w \in X\right\}$ and let $T_{1}^{\prime}=T_{1} \backslash T_{0}$. Then we get a feasible network code for demand $\tau^{\prime}=\left(T_{1}^{\prime}, T_{2}\right)$ with the following properties: the network code has height two on arcs incident to nodes in $T_{0}$, while on all other arcs it has height one. For a fieldsize large enough $(q \geq|W|)$ we can choose a network code which is pairwise linearly independent on arcs of type $s t_{w}$ for $t_{w} \in T_{0}$. Such a network code is feasible for $\tau^{\prime}$ because on one hand, a receiver $t_{w} \in T_{1}^{\prime}$ has entering arc $s t_{w}$ of height one, on the other hand, a receiver $t_{u v} \in T_{2}$ has either two entering arcs of height two with linearly independent codes or it has one entering arc of height two and one of height one, which always transmit together two valuable layers to $t_{u v}$. Hence $X$ is a minimal vertex cover if and only if demand $\left(T_{1}^{\prime}, T_{2}\right)$ is satisfiable and $T_{1}^{\prime}$ is maximal.

As a minimal mixed (vertices and edges) cover of the edges can be assumed to contain only vertices, we also get the following.

Corollary 7 Given a network $D$, a demand $\tau=\left(T_{1}, T_{2}\right)$ and a number $K$, it is NP-hard to decide whether there exists a network code satisfying at least $K$ requests.

## 4 Tools for feasible network code construction

In 4 Kim et al. gave a simple randomized network coding algorithm for the multi-layered video streaming problem. In their approach a function $h: V \rightarrow\{0,1, \ldots, k\}$ is determined, and then a randomized linear network code $\mathbf{c}$ is sent in the network such that for each arc $u v \in A$, the highest layer with non-zero coefficient in $\mathbf{c}(u v)$ is at most $h(v)$. Their algorithm ensures that the first layer can be decoded at each receiver with high probability, and some receivers may be able to decode more layers.

In this paper we give some (non-randomized) algorithms that are also based on restricting the highest layer with non-zero coefficient, but in our approach restrictions may differ for arcs entering the same node. In order to describe our algorithms, some further layer-related notions are needed.
Definition 8 A function $f: A \rightarrow\{0,1, \ldots, k\}$ is a height function if there exists a finite field $\mathbb{F}_{q}$ and a linear network code $\mathbf{c}$ over $\mathbb{F}_{q}$ with $h_{\mathbf{c}}=f$. Similarly we can define when a function $g: V \rightarrow\{0,1, \ldots, k\}$ is a performance function, i.e., if there exists a linear network code $\mathbf{c}$ over $\mathbb{F}_{q}$ with $p_{\mathbf{c}}=g$. We say that functions $f: A \rightarrow\{0,1, \ldots, k\}$ and $g: V \rightarrow\{0,1, \ldots, k\}$ form a height-performance-pair if there exists a network code $c$ with $h_{\mathbf{c}}=f$ and $p_{\mathbf{c}}=g$. Given a function $f: A \rightarrow\{0,1, \ldots, k\}$, a function $g: V \rightarrow\{0,1, \ldots, k\}$ is called a realizable extension of $f$, if they form a height-performance-pair. A height function $f$ is feasible for a demand $\tau$ if it has a realizable extension $g$ such that $g \geq d_{\tau}$.

### 4.1 Sufficient condition for feasible height functions

Our algorithms for feasible network code construction for a demand $\tau$ will always first find a function $f: A \rightarrow\{0,1, \ldots, k\}$ and then a realizable extension $g$ such that $g(v) \geq d_{\tau}$.

In this subsection we give a sufficient condition for a function $f$ to be a height function (see Corollary 14. As we will see, this condition is also necessary for two layers, leading to a characterization for that case. We use this characterization to get a new heuristic for three layers, with better performance than earlier approaches.

In this section we assume that the reader is familiar with the classical algorithm of Jaggi et al. 3]. In their algorithm, they construct a feasible network code for a demand $\tau=\left(\emptyset, \ldots, \emptyset, T_{k}\right)$ by fixing $k$ arc-disjoint paths to every receiver and constructing the network code on the arcs one by one, in the topological order of their tails. We say that an arc $a$ is processed during the algorithm, if the network code $\mathbf{c}(a)$ is defined. Jaggi et al. maintain that for every receiver, the span of the codes on the last processed arcs on the fixed $k$ paths remain the whole $k$-dimensional vector space. Their algorithm can be easily generalized for multi-layer demands.
Definition 9 For a function $f: A \rightarrow\{0,1, \ldots, k\}$, a path $P$ with arcs $a_{1}, a_{2}, \ldots, a_{r}$ is called monotone, if $f\left(a_{1}\right) \leq f\left(a_{2}\right) \leq \ldots \leq f\left(a_{r}\right)$. We define for such a monotone path $\min (P)=f\left(a_{1}\right)$ and $\max (P)$ $=f\left(a_{r}\right)$. The former $\min (P)$ is also called the value of the path.

Definition 10 Let a node $v \in V-s$, a function $f: A \rightarrow\{0,1, \ldots, k\}$ and a function $g: V \rightarrow\{0,1, \ldots, k\}$ be given. An $i$-fan of $v$ consists of $i$ pairwise arc-disjoint non-trivial (i.e., containing at least one arc) monotone paths $P_{1}, \ldots, P_{i}$ ending at $v$, where for all $j \leq i$ we have $j \leq \min \left(P_{j}\right) \leq \max \left(P_{j}\right) \leq i$, and $P_{j}$ begins at a node $v_{j}$ with $g\left(v_{j}\right) \geq \min \left(P_{j}\right)$.

Definition 11 If a function $f: A \rightarrow\{0,1, \ldots, k\}$ and a function $g: V \rightarrow\{0,1, \ldots, k\}$ is given in such a way that
$i$, for every node $v$ with $g(v)>0$ there exists a $g(v)$-fan of $v$,
ii, for every arc $v w$, either $f(v w) \leq g(v)$, or there exists an incoming arc uv with $f(u v)=f(v w)$,
then $g$ is called a fan-extension of $f$. Let us call an arc uv free if $f(u v) \leq g(u)$. Note that every starting arc of a path in a fan is free.

Theorem 12 A fan-extension $g$ of a function $f$ is always a realizable extension of $f$.
Proof If a node can decode the first $i$ layers then it can also send any linear combination of these layers.
Claim If $v$ has an $i$-fan then it also has an $i$-fan with exactly one free arc on each path.
Proof Let $a^{\prime}$ be a free arc on a path $P_{j}$ of a fan such that it is not the first arc $a$. Since $P$ is monotone, $j \leq f(a) \leq f\left(a^{\prime}\right)$, hence the fan resulting from replacing $P_{j}$ by the subpath $P_{j}^{\prime}$ starting from $a^{\prime}$ to $v$ is also an $i$-fan of $v$.

Let us fix such a fan for every node $v$ with $g(v)>0$. First we define the network code $\mathbf{c}$ on arcs covered by at least one fan. Let $L$ denote the maximum number of fans an arc is covered by. Our algorithm constructs a network code over any finite field $F_{q}$ with $q>L$. Note that since $|V|>L, q>|V|$ is always sufficient. We modify the algorithm of Jaggi et al. [3] the following way: on free arcs of a fan we construct the network code in increasing order of the $f$ values on the arcs. Since the paths in a fan satisfy that $\min \left(P_{j}\right) \geq j$ and $q>L$, we can define the network code $\mathbf{c}$ so that for every fan, $\operatorname{dim}\left\langle\mathbf{c}\left(a_{1}\right), \ldots, \mathbf{c}\left(a_{j}\right)\right\rangle=j$ for all $1 \leq j \leq i$, where $a_{j}$ is the first arc on path $P_{j}$. On non-free arcs we define the network code in a topological order of their tails. When constructing the network code on a non-free arc $u v, u \neq s$, we maintain that for every $i$-fan which contains $u v$, the span of codes on the last processed arcs on the $i$ paths remain the $i$-dimensional subspace of the first $i$ layers. We use the following lemma to prove that this is possible.

Lemma 13 [3] Let $n \leq q$. Consider pairs $\left(\mathbf{x}_{i}, \mathbf{y}_{i}\right) \in \mathbb{F}_{q}^{k} \times \mathbb{F}_{q}^{k}$ with $\mathbf{x}_{i} \cdot \mathbf{y}_{i} \neq 0$ for $1 \leq i \leq n$. There exists a linear combination $\mathbf{b}$ of vectors $\mathbf{x}_{1}, \ldots, \mathbf{x}_{n}$ such that $\mathbf{b} \cdot \mathbf{y}_{i} \neq 0$ for $1 \leq i \leq n$.
If vectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{n}$ span the subspace of the first $n$ layers, then for every $\mathbf{v}_{i}, 1 \leq i \leq n$ there is a vector $\mathbf{y}_{i}$ in this subspace with $\mathbf{v}_{j} \cdot \mathbf{y}_{i}=0, i \neq j$ and $\mathbf{v}_{j} \cdot \mathbf{y}_{j} \neq 0$. We call $\mathbf{y}_{j}$ a control vector of $\mathbf{v}_{j}$. Let $F_{1}, \ldots, F_{\ell}$ denote the set of fans containing $u v$. Consider first $F_{1}$, and suppose it is an $i$-fan. Let $P_{1}, \ldots P_{i}$ denote the paths of fan $F_{1}$. For $j=1, \ldots, i$ let $a_{j}$ denote the last processed arc of $P_{j}$, and let $\mathbf{v}_{j}=\mathbf{c}\left(a_{j}\right)$. After determining control vectors $\mathbf{z}_{1}, \ldots, \mathbf{z}_{i}$, let $\mathbf{x}_{1}=\mathbf{v}_{p}$ and $\mathbf{y}_{1}=\mathbf{z}_{p}$, where path $P_{p}$ is the one that uses arc $u v$. Clearly arc $a_{p}$ enters $u$, and $\mathbf{x}_{1}=\mathbf{c}\left(a_{p}\right)$. Let the tail of arc $a_{p}$ be denoted by $w_{1}$. For the other fans we similarly define $\mathbf{x}_{2}, \mathbf{y}_{2}, \ldots, \mathbf{x}_{\ell}, \mathbf{y}_{\ell}$ and nodes $w_{2}, \ldots w_{\ell}$. Let $w u$ be an entering arc with $f(w u)=f(u v)$. Since $u v$ is a non-free arc, such an arc exists. Define $\mathbf{x}_{\ell+1}=\mathbf{c}(w u)$ and $\mathbf{y}_{\ell+1}=\mathbf{e}_{f(u, v)}$. Now let be be linear combination provided by Lemma 13 (for $\mathbf{x}_{1}, \mathbf{y}_{1}, \ldots, \mathbf{x}_{l+1}, \mathbf{y}_{l+1}$ ) and define $\mathbf{c}(u v)=\mathbf{b}$. The height of $\mathbf{c}(u v)$ will be at most the height of arcs $w_{i} u$, hence it remains under $f(u v)$, because all $P_{j}$ 's are monotone. As $\mathbf{b} \cdot \mathbf{y}_{\ell+1}=\mathbf{b} \cdot \mathbf{e}_{f(u v)} \neq 0$, the height of $u v$ is exactly $f(u v)$. Finally, for arcs not covered by any fan we can choose carbitrarily within the height constraint. Because of property ii, in Definition 11, this can also be done in the topological order of the tails of these arcs.

Corollary 14 If a function $f: A \rightarrow\{0,1, \ldots, k\}$ has a fan-extension then $f$ is a height function.

### 4.2 Maximal fan-extensions

In this subsection we prove a key property of fan-extensions.
Theorem 15 If a function $f$ has a fan-extension, then it has a unique maximal fan-extension $g^{*}$, that is $g^{*}(v) \geq g(v)$ for every fan-extension $g$ of $f$ and every node $v$.

First we start with a very important, though straightforward observation.
Proposition 16 Given a fan-extension $g$ of a function $f$ such that there exists an $i$-fan to a node $v$ with $i>g(v)$, setting $g(v)$ to $i$ is also a fan-extension of $f$.


Fig. 2: Auxiliary graph $D_{v, 3}$.

## Proof (Proof of Theorem 15 )

Let $g^{+}$be a fan-extension for which $\sum_{v \in V} g^{+}(v)$ is maximum and assume indirectly that there exists another fan-extension $g^{\prime}$ and a node $v$ for which $g^{\prime}(v)>g^{+}(v)$. We can assume that $v$ is the first such node in a topological order. From Proposition 16 increasing $g^{+}$on $v$ to $i=g^{\prime}(v)$ would also give a fan-extension, because the $i$-fan of $v$ is also an $i$-fan for $g^{+}$.

Theorem 17 The maximal fan-extension of a function $f$ can be determined algorithmically.
Proof By Proposition 16, it is enough to prove that we can calculate the maximal fan-extension in a topological order of the nodes. Assume that $g$ is defined for any node before a node $v \in V-s$ in that order. In order to find monotone paths, we build auxiliary graphs (see Figure 2 2).

For $0 \leq i \leq k$, let $D_{v, i}=\left(V^{\prime}, A^{\prime}\right)$ denote the following auxiliary graph of $D$ : we delete all arcs with $f$ value greater than $i$. We add $i$ extra nodes to the digraph: $t_{1}, \ldots, t_{i}$. For every node $u$ before $v$ in the topological order we change the tail of every outgoing arc $u w$ from $u$ to $t_{f(u w)}$ if $g(u) \geq f(u w)$ and we define $f\left(t_{f(u w)} w\right):=f(u w)$. We replace every arc $u x \in A^{\prime}$ by two new nodes $z_{u x}^{u}$ and $z_{u x}^{x}$, and $\operatorname{arcs} z_{u x}^{u} z_{u x}^{x}$ and $z_{u x}^{x} x$, and for each $w u \in A^{\prime}$, if $f(w u) \leq f(u x)$, then add arc $z_{w u}^{u} z_{u x}^{u}$. For every vertex of the form $z_{t_{i} x}^{t_{i}}$ we also add arc $t_{i} z_{t_{i} x}^{t_{i}}$. Finally we add extra $\operatorname{arcs}: s t_{j}$ for $1 \leq j \leq i$ and $i-1$ parallel copies of $t_{j} t_{j+1}$ for $1 \leq j \leq i-1$.

Lemma 18 There exists an $i$-fan to $v \in V$ if and only if $\lambda_{D_{v, i}}(s, v)=i$.
Proof Note that a monotone path $P$ to $v$ in $D$ which has exactly one free arc, corresponds to a path in $D_{v, i}$ starting from $t_{\min (P)}$ and vice versa. Hence an $i$-fan corresponds to $i$ paths in $D_{v, i}$, each starting from a node $t_{j}$ for some $j$. Suppose indirectly that $\lambda_{D_{v, i}}(s, v)<i$, that is, there exists an $\bar{s} v$ set $X \subseteq V^{\prime}$ with $\varrho(X)<i$. Since $\lambda_{D_{v, i}}\left(s, t_{i}\right)=i, t_{i} \notin X$. Let $j$ denote the greatest integer for which $t_{j} \in X$. Since for an $i$-fan at least $i-j$ paths in the fan have value at least $j+1$, paths in $D_{v, i}$ corresponding to paths
of the fan enter $X$ on at least $i-j$ arcs. Also, there are $j$ paths to $t_{j}$ in $D_{v, i}$ using arcs between $s$ and $t_{1}, \ldots, t_{j}$ only, which are disjoint from the arcs of the fan. Hence there are at least $i$ arcs entering $X$, contradicting the assumption.

To prove the other direction, let $P_{1}, P_{2}, \ldots, P_{i}$ be $i$ arc-disjoint $s v$ paths in $D_{v, i}$. Note that $\left\{t_{1}, \ldots, t_{i}\right\}$ is a cut set in $D_{v, i}$ hence every path $P_{j}$ must go through at least one of them. Since $\varrho\left(\left\{t_{1}, \ldots, t_{j}\right\}\right)=j$, at least $i-j+1$ paths go through the set $\left\{t_{j+1}, \ldots, t_{i}\right\}$, which correspond to paths in $D$ with value at least $j+1$.

The maximal possible value of $g(v)$ is the maximal $i$ for which there exists an $i$-fan of $v$. Once $g$ is determined for every node, we can easily check property ii, in Definition 11 for $f$ and $g$.

Lemma 18 shows that the existence of a fan is equivalent with a connectivity requirement in an auxiliary graph.

Corollary 19 Given a function $f: A \rightarrow\{0,1, \ldots, k\}$ and a demand $\tau$, we can check algorithmically whether $f$ has a fan-extension $g$ such that $g \geq d_{\tau}$ by calculating the maximal fan-extension $g^{*}$ and comparing it to $d_{\tau}$.

## 5 Characterizing feasible height functions for two layers

In this section we will prove that for two layers $(k=2)$, the feasible height functions can be characterized. A demand is proper, if $\lambda\left(s, t_{i}\right) \geq i$ for all $i$ and all $t_{i} \in T_{i}$. Being a proper demand is a natural necessary condition for a demand to have a feasible network code, however, not always sufficient.

Theorem 20 function $f: A \rightarrow\{1,2\}$ is a height function, feasible for a proper demand $\tau=\left(T_{1}, T_{2}\right)$, if and only if for all arcs $u v \in A$, with $u \neq s$

1. if $f(u v)=2$, then $\exists w u \in A: f(w u)=2$,
2. if $f(u v)=1$, then either $\exists w u \in A: f(w u)=1$, or $\lambda(s, u) \geq 2$,
3. for any receiver $t \in T_{1}$ with $\lambda(s, t)=1$, there is a 1-valued arc entering $t$, and
4. for any $t \in T_{2}$ there is a 2-valued arc entering $t$.

Proof It is enough to prove sufficiency, necessity is straightforward. Let $U \subseteq V-s$ denote the set of special non-receiver nodes, where a node $u$ is special, if all entering arcs are 2 -valued, but it has a 1 valued outgoing arc (by Property 2, we know that $\lambda(s, u) \geq 2$ ). The set of receiver nodes $t \in T$ for which $\lambda(s, t)=1$ is denoted by $T_{1}^{\prime}$. As $\tau$ is proper, for each node in $T_{2}^{\prime}=U \cup T \backslash T_{1}^{\prime}$ there exist two arc disjoint paths from $s$, hence, for $T_{2}^{\prime}$ there exists a network code $\mathbf{c}$ feasible for demand $\tau_{2}=\left(\emptyset, T_{2}^{\prime}\right)$. If the field size $q$ is greater than $\left|T_{2}^{\prime}\right|$, the code can be chosen to have height two on every arc, that is, the coefficient of $\mathbf{e}_{2}$ is nonzero [3]. In order to be feasible for the original demand $\tau=\left(T_{1}^{\prime}, T_{2}^{\prime}\right)$, we modify $\mathbf{c}$ the following way: for every arc $u v$ with $f(u v)=1$ we set $\mathbf{c}(u v)=(1,0)$.

We are left to prove that $\mathbf{c}$ remains a network code, and becomes feasible for demand $\tau$.
The span of the incoming vectors can only change at nodes which have only 1 -valued incoming arcs, but in this case it has also only 1-valued outgoing arcs, so the network code has the linear combination property (note that in special nodes the span of the incoming vectors remains two-dimensional). Using Properties 3 and 4, the code clearly becomes feasible for demand $\tau$.

Corollary 21 For two layers, a function $f: A \rightarrow\{0,1, \ldots, k\}$ is a height function feasible for a demand $\tau$ if and only if it has a fan-extension $g$ with $g(v) \geq d_{\tau}$ for all $v$.

Proof We use the notations of the previous proof and define the extension $g$ to be 2 on $T_{2}^{\prime}$ and 1 on $T_{1}^{\prime}$ and zero everywhere else. It is easy to see that for a receiver node $t$, if it is in $T_{1}^{\prime}$, there is a path from
another terminal node containing 1 -valued arcs only, that is, there exists a 1 -fan to that node. If $t$ is in $T_{2}^{\prime}$, either there are two edge disjoint paths of 2-valued arcs starting from receiver nodes both in $T_{2}^{\prime}$ or there is a path of 1 -valued arcs from a node in $T_{1}^{\prime}$ and a path of 2 -valued arcs from a node in $T_{2}^{\prime}$. Both cases give a 2 -fan for $t$.

### 5.1 Optimal algorithm for two layers

In this subsection we show that given the condition that all receiver nodes have to be able to decode the first layer, there is a unique maximal set of nodes $X$ in the graph such that demand $\tau^{\prime}=(T \backslash X, X)$ is satisfiable. We will give an algorithm for finding this maximal set, as well as constructing a feasible network code.

Note that by Menger's theorem, $\lambda(s, v)$ equals the minimum of $\varrho(X)$, where $X$ is an $\bar{s} v$ set.
Proposition 22 Let $v \in V-s, \lambda(s, v)=i$, and $X, Y$ two $i$-sets with $v \in X \cap Y$. Then $X \cup Y$ is also an $i$-set.

Proof As $\varrho(X \cup Y)+\varrho(X \cap Y) \leq \varrho(X)+\varrho(Y)$, and $\varrho(X \cup Y), \varrho(X \cap Y) \geq i$, the claim follows.
From the claim we get that for every vertex $v \in V-s$ there is a unique maximal $\lambda(s, v)$-set containing $v$.

Given a proper demand $\tau=\left(T_{1}, T_{2}\right)$, the following algorithm gives a feasible height function for $\tau^{\prime}=\left(T_{1}, T_{2}^{\prime}\right)$ where $T_{2}^{\prime}$ is the unique maximal subset of $T_{2}$, such that a feasible network code for $\tau^{\prime}$ exists. As a by-product, it also decides whether demand $\tau$ is satisfiable or not. Having this height function, one can easily get a feasible network code for $\tau^{\prime}$ along the lines of the previous subsection. We remark that this code will also be feasible for $\tau^{\prime \prime}=\left(T_{1} \cup\left(T_{2} \backslash T_{2}^{\prime}\right), T_{2}^{\prime}\right)$, in other words every receiver will get at least the base layer. We will also prove, that any fieldsize $q>\left|T_{1}\right|+\left|T_{2}\right|$ will be enough for this network code.

Let $\mathcal{Z}$ be the set of maximal 1-sets which contain at least one node from $T$. For a set $Z_{i} \in \mathcal{Z}$, let $I\left(Z_{i}\right)$ denote the set of arcs with head or tail in $Z_{i}$.

Claim 23 The sets $Z_{i}$ are pairwise disjoint and so are the sets $I\left(Z_{i}\right)$.
Let $Z$ denote the set of nodes not reachable from $s$ in $D^{\prime}=\left(V, A \backslash \bigcup_{i} I\left(Z_{i}\right)\right)$. It is obvious that if every receiver in $T$ can decode the first layer, then no receiver in $Z$ can decode two layers. Let $T_{2}^{\prime}=T_{2} \backslash Z$. For an arc $u v \in A$, let $f(u v)$ be the following. If $u v \in I(Z)$, then $f(u v)=1$, otherwise let $f(u v)=2$.

Theorem 24 Function $f$ is realizable for $\tau^{\prime \prime}=\left(T_{1} \cup\left(T_{2} \backslash T_{2}^{\prime}\right), T_{2}^{\prime}\right)$. In addition, any finite field of size $q>|T|$ can be chosen for the network code (where $T=T_{1} \cup T_{2}$ ).

Proof By the definition of $Z$, it is clear that Constraint 1 of Theorem 20 is fulfilled. Suppose that $f(u v)=1$ for an arc with $u \neq s$ and there are no 1 -valued arcs entering $u$. We need to prove that $\lambda(s, u) \geq 2$.

Suppose that this is not the case, thus there is an $\bar{s} u$ set $X \subset V$ with $\varrho(X)=1$. Since $u v \in I(Z)$ but none of the arcs entering $u$ is in $I(Z)$, it follows that $v \in Z$ and $u \notin Z$. Hence $v \in Z_{i}$ for some $i$, but then $X \cup Z_{i}$ would be a subset with in-degree one, contradicting the maximality of $Z_{i}$.

For the second statement, using the proof of Theorem 20, it is enough to show that the size of the set $U$ of special non-receiver nodes defined there is not greater than the number of terminals that have demand one in $\tau^{\prime \prime}$. We claim moreover that $|U| \leq|T \cap Z|$. Every $u \in U$ is a tail of an arc entering some $Z_{i}$, and for every $Z_{i}$ there is only one entering arc. Since each of the pairwise disjoint sets $Z_{i}$ contains at least one terminal from $T \cap Z$, we are done.

We note that this algorithm has a more-or-less obvious implementation in time $O(|A|)$ using BFS. We do not detail it here, because a more general algorithm given in the next section will also do the job.

## 6 Three layers

### 6.1 Heuristics for 3 layers

In this subsection we give a new network coding algorithm for three layers. We prove that given a receiver set $T$, the algorithm sends the first layer to every receiver and within this constraint, the unique maximal set of receivers gets at least two layers, while some receivers may get three layers. Because of its properties we call our heuristic 2-Max.

Step 1 Let $W_{1}$ denote the union of maximal 1-sets which contain at least one node from $T$. In Section 4 it was proved that if all receivers get the first layer, a receiver $v$ cannot get more than one layer if and only if it is cut by $W_{1}$ from $s$, that is, if there is no directed path from $s$ to $v$ in $V \backslash W_{1}$. Let $\bar{W}_{1} \supseteq W_{1}$ denote the set of nodes cut from $s$ by $W_{1}$. We set $T_{1}=T \cap \bar{W}_{1}$. We define a set of pseudo receivers $U$ which contains nodes not in $\bar{W}_{1}$ but having an outgoing arc entering $\bar{W}_{1}$.

Step 2 Similarly to the first case, let $W_{2}$ denote the maximal 2 -sets which contain a receiver or a pseudo receiver. Let $\bar{W}_{2} \subseteq V \backslash \bar{W}_{1}$ denote the set of nodes only reachable from $s$ through $\bar{W}_{1} \cup W_{2}$. We set $T_{2}=(T \cup U) \cap \bar{W}_{2}$.

Note that for determining sets $\bar{W}_{1}$ and $\bar{W}_{2}$ we can use the distributed algorithm presented in Subsection 6.2,

Step 3 We define a function $f: A \rightarrow\{1,2,3\}$ on $D$ which is 1 on $I\left(\bar{W}_{1}\right), 2$ on $I\left(\bar{W}_{2}\right) \backslash I\left(\bar{W}_{1}\right)$ and 3 otherwise. Let $T^{*}=U \cup T$. We proceed on the nodes of $T^{*} \backslash T_{1}$ in a fixed a topological order and decrease $f$ on some arcs from 3 to 2 . Let $v$ denote the next node to be processed. We take a cost function $c: A \rightarrow\{0,1\}$ which is 1 on 3 -valued arcs and 0 everywhere else. Then we take the set of nodes $X \subseteq T$ reachable from $s$ on 3 -valued arcs and increase $c$ to $|A|$ on an $s$-arborescence (a directed tree in which every node except $s$ has in-degree 1) of 3 -valued arcs spanning $X$. Since $v \notin \bar{W}_{1}$, there are two arc-disjoint paths $P_{1}$ and $P_{2}$ from $T^{*} \cup\{s\}$ to $v$ so that $P_{2}$ does not start in $T_{1}$. Moreover, it can be assumed that the inner nodes of these paths do not intersect $T^{*}$.

Case I There are two edge-disjoint paths from $T^{*} \cup\{s\}$ to $v$, such that both avoid $T_{1}$. Let us take a minimum cost pair of paths $P_{1} \cup P_{2}$ described above according to the cost function $c$. Then we decrease $f$ on the 3 -valued arcs of $P_{1}$ and $P_{2}$.

Case II No such pair exists. We take a minimum cost $P_{1} \cup P_{2}$ from $T^{*} \cup\{s\}$ to $v$ according to the cost function $c$. Then we decrease $f$ on the 3 -valued arcs of $P_{1} \cup P_{2}$.

Step 4 Finally, we check in the topological order of the nodes, whether every 3 -valued outgoing arc has a 3 -valued predecessor, and if not, we decrease its value to 2 .

Theorem 25 Let $T_{1}$ be the set of receivers that can get at most 1 layer, and let $T_{2}$ be the set of receivers and pseudo receivers that can get at most 2 layers, as described by the algorithm above. The function $f$ constructed has a realizable extension for demand $\tau=\left(T_{1}, T_{2}^{\prime}, T_{3}^{\prime}\right)$ for which $T_{2} \subseteq T_{2}^{\prime}$ and $\left(T_{2}^{\prime} \cup T_{3}^{\prime}\right) \supseteq T \backslash T_{1}$. Heuristic 2-Max sends at least one layer to each receiver and within this constraint it sends at least two layers to the maximum number of receivers.

The running time of the algorithm is $O(|V|(|A|+|V| \log |V|))$, because steps 1,2 and 4 require time $O(|A|)$, and the processing of a node in step 3 requires time $|A|+|V| \log |V|$ applying Suurballe's algorithm as a subroutine for minimum cost arc-disjoint path pairs [9.

We remark that for more than 3 layers a network code can be determined by a straightforward generalization of the heuristic for which the number of valuable layers is 1,2 or $k$ at each receiver. A more refined network code, including intermediate performance values too, is the scope of future work.
6.2 A connectivity algorithm for determining maximal 1-sets and 2-sets

Goals: we are going to give a distributed, linear time algorithm for the following problems:

- Determine $\lambda(s, v)$ for all $v$, but if it is $\geq 3$ then only this fact should be detected.
- For each $v$ with $\lambda(s, v)=1$ determine the incoming arc of the unique maximal 1-set containing $v$.
- For each $v$ with $\lambda(s, v)=2$ determine the incoming arcs of the unique maximal 2 -set containing $v$.

We assume that $*$ is a special symbol which differs from all arcs.
During the algorithm each node $v$ (except $s$ ) waits until it hears messages along all incoming arcs, then it calculates $\lambda(s, v)$, and the 3 messages $m_{1}(v), m_{2}(v), m_{3}(v)$ it will send along all outgoing arcs.

The algorithm starts with $s$ sending $m_{1}(s):=m_{2}(s):=m_{3}(s):=*$ along all outgoing arcs.
We need to describe the algorithm for an arbitrary node $v \in V-s$. First $v$ waits until hearing the messages on the set of incoming arcs denoted by $I N(v)=\left\{a_{1}, \ldots, a_{r}\right\}$. When on an arc $a_{i}$ it hears a $*$, it replaces it by $a_{i}$. Let the messages arrived (after these replacements) on arc $a_{i}$ be $m_{1}^{i}, m_{2}^{i}, m_{3}^{i}$. Then $v$ examines the set $M_{1}(v)=\left\{m_{1}^{i}\right\}_{i=1}^{r}$. If $\left|M_{1}(v)\right|=1$ then $v$ sets $\lambda(s, v):=1$ and $m_{1}(v):=m_{2}(v):=$ $m_{3}(v):=m_{1}^{1}$, otherwise it sets $m_{1}(v):=*$.

Next $v$ examines the set $M_{2}(v)=\bigcup_{i=1}^{r}\left\{m_{2}^{i}, m_{3}^{i}\right\}$. If $\left|M_{2}(v)\right|=2$ then it sets $\left\{m_{2}(v), m_{3}(v)\right\}=M_{2}(v)$, and if $\lambda(s, v)$ was not set to 1 before, it sets it to 2 .

Let us call an entering arc $a_{j}$ important for $v$, if $m_{1}^{j} \notin \bigcup_{1 \leq i \leq r, i \neq j}\left\{m_{2}^{i}, m_{3}^{i}\right\}$, and let $I_{v}$ denote the set of important arcs for $v$. If $\left|M_{2}(v)\right|>2$, then $v$ next examines the set $M_{2}^{\prime}(v)=\bigcup_{i \in I_{v}}\left\{m_{2}^{i}, m_{3}^{i}\right\} \cup \bigcup \bigcup ~<i \not I_{v}\left\{m_{1}^{i}\right\}$, and if $\left|M_{2}^{\prime}(v)\right|=2$, then it makes the same steps with $M_{2}^{\prime}(v)$ as described before with $M_{2}(v)$.

Finally, if both $\left|M_{2}^{\prime}(v)\right|$ and $\left|M_{2}(v)\right|$ are greater than 2 and $\lambda(s, v)$ was not set to $1, v$ examines $M_{1}(v)$ again, and if $\left|M_{1}(v)\right| \leq 2$, then it sets $\left\{m_{2}(v), m_{3}(v)\right\}=M_{1}(v)$, and it sets $\lambda(s, v)$ to 2 .

If they were not set before, let $m_{2}(v):=m_{3}(v):=*$ and $\lambda(s, v)=3$.
An $s v$ cut is a set of arcs, which intersects every $s v$ path.
Claim 26 Let $v \in V-s$. Each of the sets $M_{1}(v), M_{2}(v)$ and $M_{2}^{\prime}(v)$, whenever defined, contains an sv cut.

Proof For an arc $a$, let us call an arc set an $a$-arc-cut if it intersects every directed path from $s$ ending with $a$. Note that the arc $a$ itself is an $a$-arc-cut, and the union of arc-cuts for all the entering arcs of a node $v$ form an $s v$ cut. Also, for an arc $u v$, an $s u$ cut forms a $u v$-arc-cut. To prove the claim, inductively we can assume, that on an arc $u v$ either $m_{1}(u v)=*$ or $m_{1}(u v)$ is an $s u$ cut, and also either set $\left\{m_{2}(u v), m_{3}(u v)\right\}=\{*\}$ or is an $s u$ cut. In all cases, after the replacement, node $v$ hears along arc $u v$ an $m_{1}$ that forms a $u v$-arc-cut and $m_{2}, m_{3}$ that form also a $u v$-arc-cut, proving the claim.

Theorem 27 For every node $v \in V-s$, the algorithm correctly calculates $\lambda(s, v)$. If $\lambda(s, v)=1$ then $m_{1}(v)$ is the incoming arc of the unique maximal $\bar{s} v$ set with $\varrho(X)=1$. If for the arc uw entering this set $X$ we have $\lambda(s, u)=2$, then $\left\{m_{2}(v), m_{3}(v)\right\}=\left\{m_{2}(u), m_{3}(u)\right\}$. If $\lambda(s, v)=2$ then $m_{2}(v), m_{3}(v)$ is the pair of incoming arcs of the unique maximal $\bar{s} v$ set with $\varrho(X)=2$.

Proof First suppose that $\lambda(s, v) \geq 3$. By Claim 26. $\left|M_{1}(v)\right| \geq 3,\left|M_{2}(v)\right| \geq 3$. and $\left|M_{2}^{\prime}(v)\right| \geq 3$. Consequently in this case node $v$ correctly concludes $\lambda(s, v) \geq 3$ and it will send $*$ s as messages.

Now suppose $\lambda(s, v)=1$, and let $X$ denote the unique maximal set with $s \notin X, v \in X, \varrho(X)=1$, and let $u w$ be the unique arc entering $X$. In this case clearly $m_{1}(w)=u w$ (otherwise $m_{1}(w)$ would be an arc $e$ entering another set $Y$ with $u \in Y$ and $\varrho(Y)=1$, but then $X \cup Y$ would be a bigger set with one incoming arc). It is easy to see that now along every arc inside $X$ the first message is also $u w$, so only this message arrives at $v$ as first message and then $v$ correctly sets $\lambda(s, v)=1$. Also, if $\lambda(s, u)=2$, then inductively we may assume that $\left|\left\{m_{2}(u), m_{3}(u)\right\}\right|=2$ hence $M_{2}(z)$ remains this set for every node only reachable from $u$, including $v$.

Finally suppose $\lambda(s, v)=2$, and let $X$ denote the unique maximal $\bar{s} v$ set with $\varrho(X)=2$, and let $u w$ and $u^{\prime} w^{\prime}$ be the two arcs entering $X$. Note that $\lambda(s, u), \lambda\left(s, u^{\prime}\right)>1$, otherwise $X$ would not be maximal. That is, $m_{1}(u)=m_{1}\left(u^{\prime}\right)=*$. By Claim 26, $\left|M_{1}(v)\right| \geq 2$, so $v$ does not set $\lambda(s, v)$ to one.

As $D$ is acyclic with a unique source $s$, and every node is reachable from $s$, the subgraph of $D$ spanned by $X$ either contains one source, say $w$, or contains two sources: $w$ and $w^{\prime}$ (a source must be the head of an entering arc).

Case $I w=w^{\prime}$. As $w$ is the source of $G[X]$, we have $\varrho(w)=2$, so $\left|M_{1}(w)\right|=2$ and $\left\{m_{2}(w), m_{3}(w)\right\}=$ $\left\{u w, u^{\prime} w\right\}$. Therefore every node $x$ inside $X$ has $M_{2}(x)=\left\{u w, u^{\prime} w\right\}$. As $\left|M_{2}(v)\right|=2$, $v$ sets $\lambda(s, v)=2$.

Case II $w \neq w^{\prime}$. Let $X_{1}$ denote the set of vertices $x \in X$ only reachable from one of $w$ and $w^{\prime}$. It follows that $\lambda(s, x)=1$ for all $x \in X_{1}$ hence every $x \in X_{1}$ has $M_{1}(x)=\{u w\}$ or $\left\{u^{\prime} w^{\prime}\right\}$. If node $v$ is a source of $G\left[X \backslash X_{1}\right]$, then $M_{1}(v)=u w, u^{\prime} w^{\prime}$. For a node $v \in X \backslash X_{1}$ with entering arcs from $X_{1}$ and also from $X \backslash X_{1}$, it holds that $\left\{u w, u^{\prime} w^{\prime}\right\} \subseteq M_{2}(v)$, since an entering arc a not coming from $X_{1}$ is important for $v$ and it carries $\left\{u w, u^{\prime} w^{\prime}\right\}$ in $\left\{m_{2}(a), m_{3}(a)\right\}$. An arc $b$ coming from $X_{1}$ carries $u w$ or $u^{\prime} w^{\prime}$ in $m_{1}(b)$, so $b$ is not important. Hence $M_{2}^{\prime}=\left\{u w, u^{\prime} w^{\prime}\right\}$. Finally for a node $v \in X \backslash X_{1}$ with all entering arcs from $X \backslash X_{1}$, clearly $M_{2}=\left\{u w, u^{\prime} w^{\prime}\right\}$.

The running time of the algorithm is $O(|A|)$.

### 6.3 Experimental results

We compared our heuristic 2-Max for three layers with the heuristic of Kim et al. which they called minCut [4].

We generated random acyclic networks with given number of nodes and given arc densities. Then we chose some nodes as receivers with a given probability. Finally for every receiver $t$ we calculated $i=\min (3, \lambda(s, t))$ and put $t$ randomly into one of the sets $T_{1}, \ldots, T_{i}$.

The comparison is not easy, because there is no obvious objective function that measures the quality of the solutions. Generally we can say that none of the algorithms outperformed the other. To illustrate this we show an example, which was run on random networks with 551 nodes and 2204 arcs and with probability 0.1 for selecting receivers. We describe only the number of nodes in $T_{3}$ receiving 1,2 , or 3 layers (see Figure 3).

For making more precise comparison, we had to define a realistic objective function. As both heuristics carry the base layer to every receiver, we did not give a score for these. The objective function we chose is $2 \cdot r_{2}^{2}+1.8 \cdot r_{3}^{2}+2.7 \cdot r_{3}^{3}$, where $r_{2}^{2}$ is the number of receivers in $T_{2}$ that received two layers, $r_{3}^{2}$ is the number of receivers in $T_{3}$ that received two layers, and $r_{3}^{3}$ is the number of receivers in $T_{3}$ that received three layers. The ideology behind this is the following. A receiver with demand two is absolutely satisfied if it receives two layers. A receiver with demand three is a little bit less satisfied if it receives two layers, but much more happy than one receiving only one layer. And a receiver receiving three layers is 1.5 times satisfied than one receiving only two.

We made series of random inputs with varying number of nodes. For each node number we generated 10 inputs, calculated the scores defined above, and averaged, this score makes one point in the graphs shown. Implementations were carried out with LEMON C++ library [7].

## 7 Conclusion

In this paper we investigated the multi-layered multicasting problem proposed by Kim et al. 4. We proved NP-hardness for some very special cases of the problem, including demand $\tau=\left(T_{1}, T_{2}\right)$, if we want to maximize the number of satisfied receivers. For two layers we gave a network coding algorithm which is optimal if the task is to send at least one layer to every receiver and two layers to as many receivers as


Fig. 3: Comparison on one specific example for users with demand 3.


Fig. 4: Comparison of weighted performances with varying number of nodes.
possible. For three or more layers we gave a sufficient condition for a function $f: A \rightarrow\{0,1, \ldots, k\}$ to be a feasible height function for a demand, and showed that this condition can be checked algorithmically, and is sharp for the case of two layers. Also, we presented a heuristic for three layers called 2-Max, which not only ensures that all terminals can decode the base layers, but also carries the second layer to the maximum number of receivers. The comparison of the heuristics analyzed shows that on some average of the inputs our new heuristic outperforms the other with a peremptorily chosen objective. But on a given input it is hard to predict which heuristic gives the best output, so we propose to run both, and choose the better (regarding to the objective in question).

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