Minimal diameter double-loop networks: Dense optimal families

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This article deals with the problem of minimizing the transmission delay in Illiac-type interconnection networks for parallel or distributed architectures or in local area networks. A double-loop network (also known as circulant) $G(n,h)$, consists of a loop of n vertices where each vertex i is also joined by chords to the vertices $i \pm h \bmod n$. An integer n, a hop h, and a network $G(n,h)$ are called optimal if the diameter of $G(n,h)$ is equal to the lower bound k when $n \in R[k] = \{2k^2 - 2k + 2, \ldots, 2k^2 + 2k + 1\}$. We determine new dense families of values of n that are optimal and such that the computation of the optimal hop is easy. These families cover almost all the elements of $R[k]$ if k or $k + 1$ is prime and cover 92% of all values of n up to 10^6.

1. INTRODUCTION

This article deals with a combinatorial problem arising from studies in distributed and parallel architectures and in local area networks. Several topologies have been proposed for interconnection networks [2,6]. One that is widely used because of its simplicity is the loop topology [8] that can be modeled by a cycle. This topology has some disadvantages, in particular, its high vulnerability to node or link failure and its large diameter (corresponding to a large transmission delay). One way to improve the performance of a network is to increase its connectivity and decrease its diameter. This can be done by adding links, but usually it is desirable to add as few links as possible, preferably in a homogeneous way, to allow for uniform hardware components and simpler data routing algorithms.

A simple way to achieve this improvement is by adding chord-links to a cycle

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in a regular manner, by joining vertex i to the vertices $i \pm h \bmod n$, for some given h. Although the graph obtained in this way is regular of degree 4 (if $1 < h < n/2$), 4-connected [5], and is the union of two Hamiltonian cycles [3], the computation of its minimal diameter over all possible h is not immediate. Here we exhibit dense infinite families of networks for which the diameter is the smallest possible and for which the hop h can be easily computed. these networks can be viewed as generalized Illiac-type parallel computers.

Let us first introduce some definitions and notations and some relevant results. For a survey of various types of loop-networks, we refer the reader to [1]. Let s_1, s_2, n be positive integers such that $s_1 < s_2 < n/2$. Let $\mathbf{G(n;\pm s_1,\pm s_2)}$ be the undirected graph with vertex set $\{0,1,\ldots,n-1\}$, corresponding to the integers modulo n, where vertex i is joined to the four vertices $i \pm s_1, i \pm s_2$. Let **diam G** denote the diameter of $G(n;\pm s_1,\pm s_2)$, namely, the maximum distance between any pair of vertices in the graph. Following the notation used in Ref. 9, let us define

$$\mathbf{n_k} = 2k^2 + 2k + 1; \qquad \mathbf{R[k]} = \{n_{k-1} + 1, \ldots, n_k\}.$$

Thus, $|R[k]| = 4k$. Let $\mathbf{D_n} = \min\{\text{diam } G(n;\pm s_1,\pm s_2) \,|\, 0 < s_1 < s_2 < n/2\}$.

Theorem 1 provides a lower bound on D_n (see [4,5,11,12]).

Theorem 1. For all $n \in R[k]$, $D_n \geq k$.

The next theorem asserts that the lower bound on D_n can always be achieved (see [4,5,12]).

Theorem 2.

(i) diam $G(n;\pm k,\pm(k+1)) = k$ for all $n \in R[k]$,

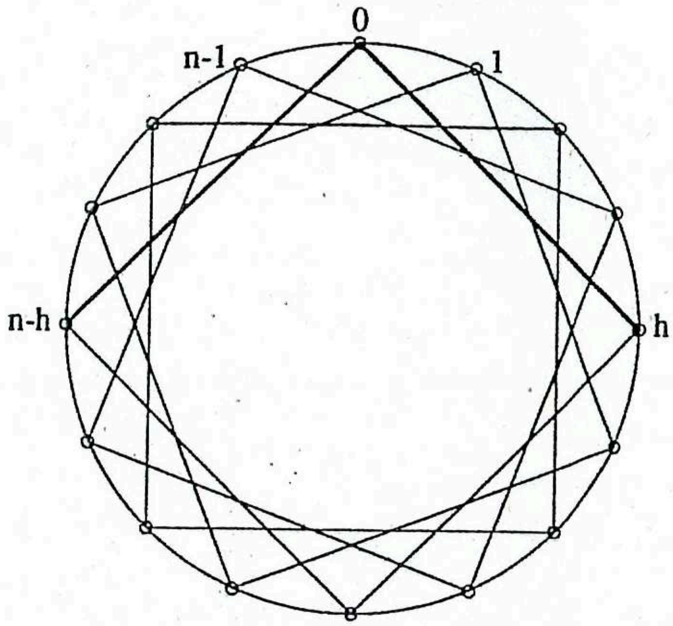
(ii) diam $G(n;\pm(k-1),\pm k) = k$ for all $n \in R[k], n \leq 2k^2 + 1$.

The networks in which we are interested in this work correspond to the special case when $s_1 = 1$ and can be represented by a cycle with chords. In what follows we shall denote these networks by $\mathbf{G(n,h)}$; Figure 1 corresponds to $n = 16$, $h = 4$. We shall refer to h as the hop size, or just the **hop**, and use the notation

$$\mathbf{D_n^*} = \min\left\{\text{diam } G(n,h) \,|\, 1 < h < \frac{n}{2}\right\}.$$

The lower bound on D_n^* equals the one mentioned in Theorem 1 for D_n, but unlike the case of $G(n;\pm s_1,\pm s_2)$, the lower bound for D_n^* may never be achieved for some values of n. In particular, we have [see [4,12] for (a) and [7,9] for (b)]:

Theorem 3. Let $k \geq 2$, then



G(n,h)

FIG. 1.

(a) For $n = n_k$, $D^*_n = \text{diam}\, G(n_k, 2k + 1) = k$.
(b) For $n = n_k - 1$, $D^*_n = k + 1$.

Let $n \in R[k]$. If a hop h^*_n exists such that $G(n,h^*_n) = k$, then we shall refer to n, h^*_n, and $G(n,h^*_n)$ as **optimal.** A subset Θ of the natural numbers is an **optimal family** if all of its members n are optimal.

Du et al. [7] identified such an infinite optimal family. That family intersects with each $R[k]$, $k \geq 3$, at 10 values, independently of k. (The cases $k = 0,1,2$ are trivial.) Note that in [7] the term "optimal" is used in a somewhat different sense than here.

In [9], Tzvieli has identified several optimal families of networks, each of which intersects each $R[k]$ in a set of cardinality $O(\sqrt{k})$. It can be shown that the families in [9] include those mentioned in [7]. Results in [9] also include bounds on optimal hops and an algorithm that computes those hops whenever they exist; sets of optimal hops are identified for some special values of n in each $R[k]$.

Here we identify "dense" optimal families of values of n, along with the corresponding optimal hops h that render optimal networks $G(n,h)$. Although by no means exhaustive, those families cover 92% of all values of n up to 1,000,000. Also, when k or k + 1 is a prime, we settle almost completely the question of determining optimality in $R[k]$. (In this case, all but one or two values in $R[k]$ are optimal.)

Additional optimal, suboptimal, and other families are identified in chapters 5 and 6 in [10].

2. DENSE INFINITE OPTIMAL FAMILIES

In the following, $gcd(a,b)$ will denote the greatest common divisor of the integers a and b.

Lemma 1. Let $k \geq 1$, $n \in R[k]$, and let q be relatively prime to n. Then

(i) diam $G(n;\pm qk,\pm q(k+1)) = k$,
(ii) diam $G(n;\pm q(k-1),\pm qk) = k$, if $n \leq 2k^2+1$;

where qk, $q(k+1)$, $q(k-1)$ are all computed modulo n.

Proof. (i) Consider the mapping f: $\{0,\ldots,n-1\} \rightarrow \{0,\ldots,n-1\}$ where for each i, $0 \leq i \leq n-1$, $f(i) \equiv qi \bmod n$. Since $gcd(q,n)=1$, f is bijective and therefore f is a graph isomorphism mapping of $G(n;\pm k,\pm(k+1))$ onto $G(n;\pm qk,\pm q(k+1))$. As the diameter of a graph is preserved under isomorphism, the result follows from (i) in Theorem 2. The proof of (ii) is similar. ■

Lemma 2. Let $n \in R[k]$, then n is optimal in each of the cases:

(a) $gcd(n,k)=1$,
(b) $gcd(n,k+1)=1$,
(c) $gcd(n,k-1)=1$ and $n \leq 2k^2+1$,

and in each case the associated optimal hop is easily determined.

Proof. Two integers m and n are relatively prime if and only if there exist integers q and r such that $qm - rn = 1$, or equivalently if and only if there exists a q such that $qm \equiv 1 \bmod n$. Note that q is necessarily relatively prime to n, as $gcd(q,n)$ must divide 1. Thus, parts (a), (b), and (c) of Lemma 2 follow from Lemma 1 by choosing $m=k$, $k+1$, and $k-1$, respectively. Furthermore, if q satisfies $qm \equiv 1 \bmod n$, then $q+\alpha n$ satisfies the same congruence and $n-q$ satisfies $(n-q)m \equiv -1 \bmod n$, so we can always choose q such that $0<q<n/2$. ■

Example 1.

(a) Let $n = n_k = 2k^2+2k+1$. We can easily prove that n is optimal [Theorem 3(a)]. Indeed, $gcd(n,k) = gcd(n,k+1) = 1$. The choices $q = 2k+2$ and $q=2k$ correspond to cases (a) and (b) of Lemma 2, respectively, with $h = 2k+1$ in both cases.
(b) Similarly, $n = n_k - 2 = 2k^2+2k-1$ is optimal. We can either use (a) of Lemma 2 with $q = 2k+2$ and $h = 2k+3$ or (b) with $q=2k$ and $h = 2k-1$.

(c) $n = 2k^2 - 3$ is optimal, using case (c) of the lemma, with $q = 2k + 2$ and $h = 2k + 3$.

Let $n \in R[k]$. To establish n's optimality according to Lemma 2, the coprimality of n and k (or $k + 1$, or $k - 1$) needs to be checked. This would be simplified if the quadratic term k^2 could be purged from n. In fact, as already hinted in Theorem 3(b), in each $R[k]$, the element $n_k - 1 = 2k^2 + 2k$ plays a special role, and we will show that it is sufficient to consider n in terms of its displacement **p** from $n_k - 1$. In the following theorem, we reformulate Lemma 2 by identifying three optimal families of values of n. (Recall that a family is said to be an optimal family if all of its members are optimal.)

Theorem 4. Let $n_k = 2k^2 + 2k + 1$. The families $\Psi_i = \bigcup_{k \geq 1} \Psi_i[k]$, $1 \leq i \leq 3$, are optimal, where

(a) $\Psi_1[k] = \{n = n_k - 1 - p \,|\, 1 \leq p < 4k - 1, gcd(k,p) = 1\} \cup \{n_k\}$,
(b) $\Psi_2[k] = \{n = n_k - 1 - p \,|\, 1 \leq p < 4k - 1, gcd(k + 1,p) = 1\} \cup \{n_k\}$,
(c) $\Psi_3[k] = \{n = n_k - 1 - p \,|\, 2k - 1 \leq p < 4k - 1, gcd(k - 1,p - 4) = 1\}$.

In each case, a corresponding optimal hop is given by $h^* = \min\{h,n - h\}$, where

(a) $h = \overline{2s(k + 1) - t + 1}$; (s,t) is an integer solution to $sp - tk = 1$,
(b) $h = \overline{2sk - t + 1}$; (s,t) is an integer solution to $t(k + 1) - sp = 1$,
(c) $h = \overline{2s(k + 2) - t + 1}$; (s,t) is an integer solution to $s(p - 4) - t(k - 1) = 1$,

and where $\overline{m}$ denotes the smallest nonnegative integer such that $m \equiv \overline{m} (\text{mod}\, n)$.

Proof. (a) By Lemma 2(a), n is optimal if $gcd(n,k) = 1$. Letting $n = n_k - 1 - p = 2k(k + 1) - p$, this condition is equivalent to $gcd(p,k) = 1$ and, hence, to the existence of nonnegative integers s,t such that $sp - tk = 1$. Let $q = 2s(k + 1) - t$, then $qk = 2sk(k + 1) - tk = s(n + p) - tk \equiv 1 \bmod n$, and according to Lemma 2(a), $h = \overline{q + 1}$ is an optimal hop. Then, taking $h^* = \min\{h,n - h\}$ ensures that $0 < h^* \leq n/2$. n_k is already known to the optimal by Example 1. The proofs of parts (b) and (c) are similar to that of part (a); for part (c), note that $n = 2k^2 + 2k - p = 2(k + 2)(k - 1) - (p - 4)$. ■

Example 2. Let $k = 11$, $n = 257$. Here, $n_{11} = 265$ and $p = 7$.
Since $gcd(p,k) = 1$, n is in Ψ_1. The equation $7s - 11t = 1$ has the smallest positive integer solution $s = 8$, $t = 5$, and so $h = 16 \cdot 12 - 5 + 1 = 188$. $h^* = \min\{188,257 - 188\} = 69$. In this case we also have $gcd(p,k + 1) = 1$; hence, $n \in \Psi_2$. The smallest solution to the equation $12t - 7s = 1$ is $s = 5$, $t = 3$, and, thus, $h^* = 108$ is also an optimal hop for $n = 257$.

For the subsequent discussion, we shall distinguish in each range $R[k]$ the "quartile points":

$$q_1[k] = 2k^2 - k; \qquad q_2[k] = 2k^2; \qquad q_3[k] = 2k^2 + k;$$
$$q_4[k] = 2k^2 + 2k = n_k - 1.$$

Example 3.

(a) Let k be arbitrary, $n = q_2[k] - 1 = 2k^2 - 1$ corresponding to $p = 2k + 1$. Here, $gcd(p,k) = gcd(p,k + 1) = gcd(p - 4,k - 1) = 1$; hence, all parts of Theorem 4 are applicable. The solution $(s,t) = (1,2)$ will produce the values $2k + 1$ and $2k - 1$ for h^*, corresponding to parts (a) and (b) of the theorem. The value $h^* = 2k + 1$ is also obtained from part (c), using $s = k - 2$, $t = 2k - 5$.

(b) For each $k \geq 1$, $q_i[k]$ is not in $\Psi_1[k]$, $i = 1,2,3,4$, since those values of n correspond to $p = 3k$, $2k$, k, 0, respectively. Yet, $q_1[k] \in \Psi_3[k]$ and $q_3[k] \in \Psi_2[k]$. If k is even, then $q_2[k] \in \Psi_2[k] \cap \Psi_3[k]$.

For some values of k, the families $\Psi_i[k]$ cover large continuous segments of $R[k]$, and in some cases, the determination of optimal values in $R[k]$ is completely settled, as is shown next.

Corollary.

(a) If k is prime, then every $n \in R[k]$, $n \neq n_k - 1$, is optmal.

(b) If $k + 1$ is prime, then every $n \in R[k]$, $n \neq n_k - 1$ is optimal with the possible exception of $q_2[k] - 2 = 2k^2 - 2$.

Proof. (a) If k is prime, $\Psi_1[k] = R[k] - \{q_1[k],q_2[k],q_3[k],q_4[k]\}$. But $q_1[k]$ and $q_3[k]$ are optimal, according to Example 3(b); $q_2[k]$ was shown to be optimal in [9]; and $q_4[k] = n_k - 1$ is not optimal by Theorem 3(b).

(b) In this case, $\Psi_2[k] = R[k] - \{q_1[k] - 3;\ q_2[k] - 2;\ q_3[k] - 1;\ q_4[k]\}$, but $q_3[k] - 1 \in \Psi_1[k]$ (corresponding to $p = k + 1$) and $q_1[k] - 3$ was proven in [9] to be optimal. ■

3. RELATIONSHIPS BETWEEN THE OPTIMAL FAMILIES

- The families $\Psi_1[k]$, $\Psi_2[k]$, and $\Psi_3[k]$ are neither comparable nor exclusive, as shown in the following examples (compare to Example 3):

$$q_2[k] - 1 \in \Psi_1[k] \cap \Psi_2[k] \cap \Psi_3[k], \quad \text{for each } k \geq 1;$$
$$q_3[k] \in \Psi_2[k] - \Psi_1[k]; \quad q_3[k] - 1 \in \Psi_1[k] - \Psi_2[k];$$
$$q_1[k] \in \Psi_3[k] - \Psi_1[k]; \quad q_1[k] - 1 \in \Psi_1[k] - \Psi_3[k];$$
$$q_3[k] \in \Psi_2[k] - \Psi_3[k]; \quad q_1[k] \in \Psi_3[k] - \Psi_2[k] \quad \text{if } k + 1 \equiv 0 \bmod 3.$$

- Both optimal and suboptimal values of n exist that are not in $\Psi = \Psi_1 \cup \Psi_2 \cup \Psi_3$.

  (a) By Theorem 3 we know that for all $k \geq 1$, $n_k - 1$ is not optimal.

  (b) $q_2[k] - 2$ is not in Ψ whenever $k \geq 4$, k even; this value of n can be optimal (e.g., for $k=4$, $q_2[4]-2=30$ is optimal with $h=8$; for $k=8$, $q_2[8]-2=126$ is optimal with $h=12$), or nonoptimal (e.g., for $k=6$, $q_2[6]-2=70$, see algorithm in [9]).

- Although Ψ is considerably larger than the optimal families described in [9], there are infinitely many values of n that are covered by those families but are not in Ψ. (See Example in [9].)
- Table 1 illustrates membership in the various dense families, for $k=9$.

TABLE I. Membership in dense optimal families for $k=9$.

n	Ψ_1	Ψ_2	Ψ_3	Optimal	Suboptimal
146	*			*	
147		*	*	*	
148	*			*	
149	*	*	*	*	
150				x	
151	*	*	*	*	
152	*			*	
153		*	*	*	
154	*			*	
155	*		*	*	
156				x	
157	*	*	*	*	
158	*			*	
159		*	*	*	
160	*			*	
161	*	*	*	*	
162				x	
163	*	*	*	*	
164	*			*	
165				x	
166	*			*	
167	*	*		*	
168				x	
169	*	*		*	
170	*			*	
171		*		*	
172	*			*	
173	*	*		*	
174					*
175	*			*	
176	*			*	
177		*		*	
178	*			*	
179	*	*		*	
180					*
181	*	*		*	

An "x" in the "optimal" column indicates that this value of n, although not in Ψ, is covered by the optimal families in [9].

4. THE RELATIVE SIZE OF THE Ψ FAMILIES

- The **density** of a family Θ is defined by

$$f_k(\Theta = \frac{|\Theta[k]|}{|R[k]|} = \frac{|\Theta[k]|}{4k},$$

where $\Theta[k] = \Theta \cap R[k]$. In our case, we get

$$f_k(\Psi_1) = \frac{\phi(k)}{k}, \qquad f_k(\Psi_2) \approx \frac{\phi(k+1)}{k}, \qquad f_k(\Psi_3) \approx \frac{1}{2} \cdot \frac{\phi(k-1)}{k},$$

where ϕ denotes Euler's phi-function, i.e.,

$$\phi(k) = |\{m \in Z | 1 \leq m \leq k \text{ and } (m,k) = 1\}|.$$

- Define the **cumulative density** of a family Θ by

$$F_k(\Theta) = \frac{1}{n_k} \sum_{j=1}^{k} f_j(\Theta) \cdot |R[j]|.$$

Numerical computation of the **actual** value of $F_k(\Psi)$ corresponding to the range $n \leq 10^6$ ($k = 708$) shows

$$F_{708}(\Psi_1 \cup \Psi_2) = 0.89, \qquad F_{708}(\Psi) = 0.92.$$

Hence, Ψ covers 92% of all values of n up to 1,000,000!
- All values of n that were not optimal, up to ~8,000,000, were found to be **suboptimal**, i.e., $D_n^* = k + 1$, where $n \in R[k]$. (See [10] and the Conjecture in [9]).

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REFERENCES

[1] J.-C. Bermond, F. Comellas, and D. F. Hsu, Distributed loop computer networks: A survey, to appear in the *J. of Parallel Distributed Comput.*

[2] J.-C. Bermond, C. Dolorme, and J.-J. Quisquater, Strategies for interconnection-networks: Some methods from graph theory. *J. Parallel Distributed Comput.* **3** (1986) 433–449.

[3] J.-C. Bermond, O. Favaron, and M. Maheo, Hamiltonian decomposition of Cayley graphs of degree four. *J. Combinatorial Theory, Ser.* B **46** (1989) 142–153.

[4] J.-C. Bermond, G. Illiades, and C. Peyrat, An optimization problem in distributed loop computer networks. *Proceedings of the 3rd International Conference on Combinatorial Mathematics, New York, June 1985, Ann. N.Y. Acad. Sci.* (1989).

[5] F. T. Boesch and J. K. Wang, Reliable circulant-networks with minimum transmission delay. *IEEE Trans. Circuits Syst.* **CAS-32** (1985) 1286–1291.
[6] F. R. K. Chung, *Diameters of Communication Networks: AMS Short Course on the Mathematics of Information Processing,* Vol. 34, AMS, Providence, (1986) 1–18.
[7] D. Z. Du, D. F. Hsu, Q. Li, and J. Xu, A combinatorial problem related to distributed loop networks. *Networks,* Vol. **20**(2) (1990) 173–180.
[8] M. T. Liu, Distributed loop computer networks. *Adv. Comput.* **17** (1978) 163–221.
[9] D. Tzvieli, "Minimal-diameter double-loop networks: Large infinite optimal families," *Networks,* to appear.
[10] D. Tzvieli, Double-loop interconnection networks with minimal transmission delay, PhD Dissertation, Louisiana State University, Baton Rouge, LA, August, 1988.
[11] C. K. Wong and D. Coppersmith, A combinatorial problem related to multimodule memory organization, *J. Assoc. Comput. Machin.* **21**(3) (1974) 392–401.
[12] J. L. A. Yebra, M. A. Fiol, P. Morillo, and I. Alegre, The Diameter of undirected graphs associated to plane tesselations. *Tenth British Combinatorial Conference, Ars Combinatoria,* Vol. **20B** (1985) 159–171.