

A Residual Based Sparse Approximate Inverse Preconditioning Procedure for Large Sparse Linear Systems*

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Abstract

The SPAI algorithm, a sparse approximate inverse preconditioning technique for large sparse linear systems, proposed by Grote and Huckle [SIAM J. Sci. Comput., 18 (1997), pp. 838–853.], is based on the F-norm minimization and computes a sparse approximate inverse M of a large sparse matrix A adaptively. However, SPAI may be costly to seek the most profitable indices at each loop and M may be ineffective for preconditioning. In this paper, we propose a residual based sparse approximate inverse preconditioning procedure (RSAI), which, unlike SPAI, is based on only the *dominant* rather than all information on the current residual and augments sparsity patterns adaptively during the loops. RSAI is less costly to seek indices and is more effective to capture a good approximate sparsity pattern of A^{-1} than SPAI. To control the sparsity of M and reduce computational cost, we develop a practical RSAI(*tol*) algorithm that drops small nonzero entries adaptively during the process. Numerical experiments are reported to demonstrate that RSAI(*tol*) is at least competitive with SPAI and can be considerably more efficient and effective than SPAI. They also indicate that RSAI(*tol*) is comparable to the PSAI(*tol*) algorithm proposed by one of the authors in 2009.

Keywords. RSAI, SPAI, PSAI, sparse approximate inverse, F-norm minimization, preconditioning, sparsity pattern, adaptive, Krylov solver.

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1 Introduction

Consider the iterative solution of large sparse linear system

$$Ax = b, \tag{1.1}$$

where A is an $n \times n$ real nonsingular and nonsymmetric matrix, and b is a given n -dimensional vector. This kind of problem is a core problem in scientific and engineering computing. Krylov iterative solvers, such as the generalized minimal residual method (GMRES) and the biconjugate gradient stabilized method (BiCGStab) [2, 32], have been commonly used in nowadays for solving (1.1). However, when A has bad spectral property or is ill conditioned, the convergence of Krylov solvers are generally extremely slow [32]. In order to accelerate the convergence of Krylov solvers, one must utilize preconditioning techniques to improve the conditioning of (1.1), so that Krylov solvers applied to resulting preconditioned systems converge fast. Sparse approximate

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inverse (SAI) preconditioning has been one major class of general-purpose preconditioning [5, 17, 32], and it aims at computing a sparse approximate inverse $M \approx A^{-1}$ or factorized $M = M_1 M_2 \approx A^{-1}$ directly. With such an M available, the right and left preconditioned systems are

$$AMy = b, x = My \quad \text{and} \quad MAx = Mb, \quad (1.2)$$

respectively, and the factorized preconditioned system is

$$M_1 A M_2 y = M_1 b, \quad x = M_2 y. \quad (1.3)$$

We then use a Krylov solver to solve (1.2) or (1.3), depending on the way that M is applied. Since the coefficient matrices in the above preconditioned systems are roughly the identity matrix I , Krylov solvers are expected to converge quickly.

The success of SAI preconditioning is based on the underlying hypothesis that the majority of entries of A^{-1} are small, which means that A , indeed, has good sparse approximate inverses. A good preconditioner M should be as sparse as possible, and it should be constructed efficiently and applied within Krylov solvers cheaply. There are two kinds of approaches to computing M . One of them gets a factorized $M = M_1 M_2$ and applies M_1 and M_2 to (1.3). Efficient algorithms of this kind are approximate inverse (AINV) type algorithms, which are derived from the incomplete biconjugation procedure [7, 8]. Their stabilized and block variations are developed in [6]. An alternative is the balanced incomplete factorization (BIF) algorithm, which computes an incomplete LU (ILU) factorization and its inverse simultaneously [10, 11]. The other kind of approach is based on F-norm minimization, which is inherently parallelizable and constructs M by minimizing $\|AM - I\|_F$ with certain sparsity constraints on M , where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. We will introduce more on this approach in the next paragraph. Kolotilina and Yeremin [31] have proposed a factorized sparse approximate inverse (FSAI) preconditioning procedure, which is a mixture of the above two kinds. FSAI has been generalized to block form, called BFSAI, in [23, 24]. An adaptive algorithm that generates the pattern of the BFSAI preconditioner M can be found in [22, 23, 25]. For a comprehensive survey and comparison of SAI preconditioning procedures, we refer the reader to [5, 9].

We focus on F-norm minimization based SAI preconditioning and revisit the SPAI algorithm [19] in this paper. A key of this kind of preconditioning is the determination of an effective approximate sparsity pattern of A^{-1} . There are two approaches to doing this, one of which is static and the other is adaptive. A static SAI preconditioning procedure first prescribes a sparsity pattern of M and then computes M by solving n least squares (LS) problems independently [3, 4]. The main difficulty of this approach is how to choose an effective approximate sparsity pattern of A^{-1} . A lot of research has been done on this issue, and some priori envelope patterns for effective approximate sparsity patterns of A^{-1} have been established; see, e.g., [13, 18, 21]. For a general irreducible sparse A , however, these envelope patterns are often quite dense, so that it is expensive to use them as patterns of M directly. To this end, several researchers have proposed and developed adaptive procedures, which start with a simple initial sparsity pattern and successively augment or adjust it until either the resulting M satisfies a prescribed accuracy, or the maximum loops are performed, or the maximum number of nonzero entries in M is reached. Such idea was first advocated in [15]. Grote and Huckle [19] have proposed the SPAI algorithm aimed at augmenting the sparsity pattern of M adaptively by adding the small number of most profitable indices at each loop. It has been generalized to block form, called BSPAI, in [1]. Gould and Scott [20] have given a number of enhancements which may improve the performance of the

SPAI algorithm. Chow and Saad [14] have put forward a minimal residual based (MR) algorithm that uses the sparse-sparse iteration with dropping strategies. SPAI is more robust than the MR algorithm [5]. Motivated by the Cayley–Hamilton theorem, Jia and Zhu [29] have proposed an adaptive Power SAI (PSAI) preconditioning procedure and developed a practical PSAI(tol) algorithm with tol dropping tolerance, which has been shown to be at least competitive with SPAI and can outperform SPAI considerably for some difficult problems. Jia and Zhang [27] have recently established a mathematical theory on dropping tolerances tol for PSAI(tol) and all the static F-norm minimization based SAI preconditioning procedures. Based on the theory, they have designed robust adaptive dropping criteria. With the criteria applied, PSAI(tol) and the mentioned static SAI preconditioning procedures can make M as sparse as possible and as equally effective as the possibly much denser one generated by the basic PSAI or the static SAI procedures without dropping small entries.

Remarkably, the unique fundamental mathematical distinction of all the adaptive F-norm minimization based SAI preconditioning procedures consists in the way that the sparsity pattern of M is augmented or adjusted. Practically, both static and adaptive SAI procedures must control the sparsity of M . We mention that there is a prefiltration, which shrinks the pattern of A by dropping small entries of A before implementing a SAI preconditioning algorithm [12, 30, 33, 34].

Jia and Zhang [28] have made an analysis on SPAI and PSAI(tol) and shown why PSAI(tol) can be more effective than SPAI for preconditioning (1.1), accompanied by detailed numerical comparisons of the two algorithms on a lot of regular and irregular sparse problems arising from applications. Here the meaning of ‘regular sparse’ is that all columns of A are sparse, and that of ‘irregular sparse’ is that A has at least one relatively dense column, whose number of nonzero entries is substantially more than the average number of nonzero entries per column of A . Empirically and numerically, a column is declared irregular sparse if it has at least $10p$ nonzero entries, where p is the average number of nonzero entries per column of A [28]. For A irregular sparse, Jia and Zhang [28] have shown that SPAI must be costly to seek and add indices at each loop and, moreover, the resulting M may be ineffective for preconditioning, while PSAI(tol), though also very costly, can produce an effective preconditioner M . To this end, they have proposed an approach that first transforms the irregular sparse (1.1) into certain new regular ones and then uses SPAI and PSAI(tol) to construct M for the regular problems. Such approach greatly improves the computational efficiency of SPAI and PSAI(tol) as well as the preconditioning effectiveness of SPAI applied to irregular sparse (1.1) directly.

In this paper, suppose that the current M is not good enough, and we need to augment or adjust the sparsity pattern of M in order to get a better M . We will propose a new adaptive F-norm minimization based SAI preconditioning procedure, called the residual based SAI (RSAI) algorithm, and develop a practical RSAI algorithm with dropping criteria exploited. The RSAI algorithm may greatly improve the computational efficiency and the preconditioning effectiveness of SPAI, especially when A is irregular sparse. The basic idea of RSAI algorithm is as follows: Differently from SPAI, for each column m_k , $k = 1, 2, \dots, n$, of the current M , by only selecting a few dominant indices that correspond to the largest entries in the residual of m_k , we augment the sparsity pattern of m_k using a new approach that avoids some possibly expensive computation and logical comparisons in SPAI when determining the most profitable indices based on the whole residual of m_k . As it will turn out, the RSAI procedure is not only (considerably) less costly to seek and add indices but also more effective to capture a good approximate sparsity pattern of A^{-1} than SPAI, which is

especially true for an irregular sparse A . We derive a quantitative estimate for the number of nonzero entries in M , demonstrating how it depends on the sparsity pattern of A , the number of indices exploited that correspond to the largest entries of the residual at each loop, and the number l_{\max} of loops. To control the sparsity of M and improve computational efficiency, we develop a practical $\text{RSAI}(tol)$ algorithm by making use of the adaptive dropping criterion established in [27], which guarantees that two M obtained by $\text{RSAI}(tol)$ and the basic RSAI without dropping small entries have comparable preconditioning effects. We show that the positions of large entries in M are automatically adjusted in a global sense during the loops. It is known [27, 29] that SPAI retains the already occupied positions of nonzero entries in M in subsequent loops and adds new positions of nonzero entries in M at each loop. As a result, the $\text{RSAI}(tol)$ algorithm captures an approximate sparsity pattern of A^{-1} in a globally optimal sense, while the SPAI algorithm achieves this goal only in a locally optimal sense. This difference may make $\text{RSAI}(tol)$ advantageous over SPAI. Numerical experiments will confirm that $\text{RSAI}(tol)$ is at least competitive with and can be substantially more efficient and effective than SPAI, and they will also illustrate that $\text{RSAI}(tol)$ is as comparably effective as $\text{PSAI}(tol)$.

The paper is organized as follows. In Section 2, we review the F-norm minimization based SAI preconditioning and the SPAI procedure, and introduce the notation to be used. In Section 3, we propose the basic RSAI procedure. In Section 4, we make a theoretical analysis and give some practical considerations, based on which we develop a practical $\text{RSAI}(tol)$ algorithm with dynamic dropping strategy in [27] exploited. Finally, we make numerical experiments on a number of real-world problems to confirm our assertions on $\text{RSAI}(tol)$, SPAI and $\text{PSAI}(tol)$ in Section 5.

2 The F-norm minimization SAI preconditioning and the SPAI procedure

A F-norm minimization based SAI preconditioning procedure solves the problem

$$\min_{M \in \mathcal{M}} \|AM - I\|_F, \quad (2.1)$$

where \mathcal{M} is the set of matrices with a given sparsity pattern \mathcal{J} . Denote by \mathcal{M}_k the set of n -dimensional vectors whose sparsity pattern is $\mathcal{J}_k = \{i \mid (i, k) \in \mathcal{J}\}$, and let $M = (m_1, m_2, \dots, m_n)$. Then (2.1) can be separated into n independent constrained LS problems

$$\min_{m_k \in \mathcal{M}_k} \|Am_k - e_k\|, \quad k = 1, 2, \dots, n, \quad (2.2)$$

where $\|\cdot\|$ denotes the 2-norm of a matrix or vector, and e_k is the k -th column of the identity matrix I of order n . For each k , let \mathcal{I}_k be the set of indices of nonzero rows of $A(\cdot, \mathcal{J}_k)$. Define $A_k = A(\mathcal{I}_k, \mathcal{J}_k)$, the reduced size vector $\tilde{m}_k = m_k(\mathcal{J}_k)$, and $\tilde{e}_k = e_k(\mathcal{I}_k)$. Then (2.2) amounts to solving the smaller unconstrained LS problems

$$\min_{\tilde{m}_k} \|A_k \tilde{m}_k - \tilde{e}_k\|, \quad k = 1, 2, \dots, n, \quad (2.3)$$

which can be solved by QR decompositions in parallel.

If M is not good enough, an adaptive SAI preconditioning procedure, such as SPAI [19] and PSAI [29], improves it by augmenting or adjusting the sparsity pattern \mathcal{J}_k dynamically and updating \tilde{m}_k for $k = 1, 2, \dots, n$ efficiently. We describe SPAI below.

Denote by $\mathcal{J}_k^{(l)}$ the sparsity pattern of m_k after l loops starting with an initial pattern $\mathcal{J}_k^{(0)}$, and by $\mathcal{I}_k^{(l)}$ the set of indices of nonzero rows of $A(\cdot, \mathcal{J}_k^{(l)})$. Let $A_k =$

$A(\mathcal{I}_k^{(l)}, \mathcal{J}_k^{(l)})$, $\tilde{e}_k = e_k(\mathcal{I}_k^{(l)})$ and \tilde{m}_k the solution of (2.3). Denote the residual of (2.2) by

$$r_k = Am_k - e_k, \quad (2.4)$$

whose norm is exactly equal to the residual norm $\|\tilde{r}_k\|$ of (2.3) defined by $\tilde{r}_k = A_k\tilde{m}_k - \tilde{e}_k$. If $r_k \neq 0$, define \mathcal{L}_k to be the set of indices i for which $r_k(i) \neq 0$ and \mathcal{N}_k the set of indices of nonzero columns of $A(\mathcal{L}_k, \cdot)$. Then

$$\hat{\mathcal{J}}_k = \mathcal{N}_k \setminus \mathcal{J}_k^{(l)} \quad (2.5)$$

constitutes the new candidates for augmenting $\mathcal{J}_k^{(l)}$ in the next loop of SPAI. For each $j \in \hat{\mathcal{J}}_k$, SPAI solves the one-dimensional problem

$$\min_{\mu_j} \|r_k + \mu_j Ae_j\|, \quad (2.6)$$

and the 2-norm ρ_j of the new residual $r_k + \mu_j Ae_j$ satisfies

$$\rho_j^2 = \|r_k\|^2 - \frac{(r_k^T Ae_j)^2}{\|Ae_j\|^2}. \quad (2.7)$$

SPAI takes l_a , $1 \sim 5$, indices from $\hat{\mathcal{J}}_k$ corresponding to the smallest ρ_j , called the most profitable indices, and adds them to $\mathcal{J}_k^{(l)}$ to obtain $\mathcal{J}_k^{(l+1)}$. Let $\hat{\mathcal{I}}_k$ be the set of indices of new nonzero rows corresponding to the most profitable indices added. SPAI gets the new row indices $\mathcal{I}_k^{(l+1)} = \mathcal{I}_k^{(l)} \cup \hat{\mathcal{I}}_k$, and by it and $\mathcal{J}_k^{(l+1)}$ we form an updated LS problem (2.3), which is cheaply solved by updating the previous \tilde{m}_k . Proceed in such way until $\|r_k\| = \|Am_k - e_k\| \leq \varepsilon$ or $l \geq l_{\max}$, where ε is a prescribed tolerance, usually $0.1 \sim 0.4$.

Each loop of SPAI consists of two main steps, which include the selection of the most profitable indices and the solution of the resulting new LS problem, respectively. For the selection of the most profitable indices, one first determines \mathcal{L}_k through r_k and $\hat{\mathcal{J}}_k$ through \mathcal{L}_k and \mathcal{J}_k , computes the ρ_j , then orders them, and finally selects the most profitable indices. Clearly, whenever the cardinality of $\hat{\mathcal{J}}_k$ is big, this step is time consuming. It has been shown [28] that the cardinality of $\hat{\mathcal{J}}_k$ is always big for $l = 1, 2, \dots, l_{\max}$ when the k -th column of A is relatively dense and the initial pattern $\mathcal{J}^{(0)}$ is that of I , causing that SPAI is very costly to select the most profitable indices. In addition, we can easily see that SPAI is also costly to seek the most profitable indices for A row irregular sparse. This is the case that an index in \mathcal{L}_k corresponds to a relatively dense row of A . For detailed numerical evidence on this, we refer to [26, 28], where it has been clearly shown that SPAI is too costly and impractical since it spends too much time seeking the most profitable indices when A is irregular sparse.

3 The RSAI preconditioning procedure

Our motivation for proposing a new SAI preconditioning procedure is that SPAI may be costly to select the most profitable indices and may be ineffective for preconditioning, which is definitely the case when A has some relatively dense columns. Our new approach to augmenting $\mathcal{J}_k^{(l)}$ and $\mathcal{I}_k^{(l)}$ is based on the partially *dominant* other than all information on the current residual r_k , and picks up new indices at each loop directly and cheaply that avoids possibly expensive computation and logical comparisons needed by SPAI. Importantly, the new SAI preconditioning procedure is more effective to capture a good approximate sparsity pattern of A^{-1} . Since our procedure critically

depends on the *sizes* of entries of r_k , we call the resulting procedure the Residual based Sparse Approximate Inverse (RSAI) preconditioning procedure. In what follows we present a basic RSAI procedure.

Suppose that M is the one generated by RSAI after l loops starting with the initial sparsity pattern $\mathcal{J}^{(0)}$. Consider the residual $\|r_k\|$ defined by (2.4) for $k = 1, 2, \dots, n$. If $\|r_k\| \leq \varepsilon$ for a prescribed tolerance ε , then m_k satisfies the required accuracy, and we do not improve m_k further. If $\|r_k\| > \varepsilon$, we must augment or adjust $\mathcal{J}_k^{(l)}$ to update m_k so as to reduce $\|r_k\|$.

Denote by $r_k(i)$ the i -th entry of r_k , and by \mathcal{L}_k the set of indices i for which $r_k(i) \neq 0$. *Heuristically*, the indices corresponding to the largest entries $r_k(i)$ of r_k in magnitude are the most important and dominate \mathcal{L}_k in the sense that these large entries contribute most to the size of $\|r_k\|$. Therefore, in order to reduce $\|r_k\|$ both substantially and cheaply, these most important or dominant indices should have priority, that is, we should take precedence to reduce the large entries $r_k(i)$ of r_k by augmenting or adjusting the pattern of m_k based on the dominant indices described above. Therefore, unlike SPAI, as a starting point, rather than using the whole \mathcal{L}_k , RSAI will exploit only the dominant subset of it and augment or adjust the pattern of m_k in a completely new manner, as will be detailed below.

At loop l , denote by $\hat{\mathcal{R}}_k^{(l)}$ the set of the dominant indices i corresponding to the largest $|r_k(i)|$. Let $\hat{\mathcal{J}}_k$ be the set of all new column indices of A that correspond to $\hat{\mathcal{R}}_k^{(l)}$ of row indices but do not appear in $\mathcal{J}_k^{(l)}$. We then add $\hat{\mathcal{J}}_k$ to $\mathcal{J}_k^{(l)}$ to obtain a new sparsity pattern $\mathcal{J}_k^{(l+1)}$. Denote by $\hat{\mathcal{I}}_k$ the set of indices of new nonzero rows corresponding to the added column indices $\hat{\mathcal{J}}_k$. Then we update $\mathcal{I}_k^{(l+1)} = \mathcal{I}_k^{(l)} \cup \hat{\mathcal{I}}_k$ and form the new LS problem

$$\min \|A(\mathcal{I}_k^{(l+1)}, \mathcal{J}_k^{(l+1)})m_k(\mathcal{J}_k^{(l+1)}) - e_k(\mathcal{I}_k^{(l+1)})\|, \quad (3.1)$$

whose solution can be updated from $m_k(\mathcal{J}_k^{(l)})$ in the way described in [19, 29], and the updated m_k is a better approximation to the k -th column of A^{-1} . We repeat this process until $\|r_k\| \leq \varepsilon$ or l reaches l_{\max} . The above RSAI procedure effectively suppresses the effects of the large entries $|r_k(i)|$ and reduces $\|r_k\|$.

Now we give more insight into $\hat{\mathcal{R}}_k^{(l)}$. When choosing it from \mathcal{L}_k in the above way, we may encounter $\hat{\mathcal{R}}_k^{(l)} = \hat{\mathcal{R}}_k^{(l-1)}$. If so, we cannot augment the sparsity pattern of m_k . In this case, we set

$$\mathcal{R}_k^{(l)} = \bigcup_{i=0}^{l-1} \hat{\mathcal{R}}_k^{(i)}, \quad (3.2)$$

and choose $\hat{\mathcal{R}}_k^{(l)}$ from the set whose elements are in \mathcal{L}_k but not in $\mathcal{R}_k^{(l)}$. Obviously, the resulting $\hat{\mathcal{R}}_k^{(l)}$ is always non-empty unless m_k is exactly the k -th column of A^{-1} . On the other hand, if $\hat{\mathcal{J}}_k$ happens to be empty, then $\mathcal{J}_k^{(l)} = \mathcal{J}_k^{(l+1)}$ and we just skip to loop $l+2$, and so forth. Since $r_k \neq 0$, there must exist a $\tilde{l} \geq l+1$ such that $\hat{\mathcal{J}}_k$ is not empty. Otherwise, $\mathcal{J}_k^{(l)}$ is the set of indices of all nonzero columns of $A(\mathcal{L}_k, \cdot)$, and

$$r_k(\mathcal{L}_k)^T A(\mathcal{L}_k, \mathcal{J}_k^{(l)}) = r_k(\mathcal{L}_k)^T A(\mathcal{L}_k, \cdot) = 0, \quad (3.3)$$

which means that $r_k(\mathcal{L}_k) = 0$, i.e., $r_k = 0$, and m_k is exactly the k -th column of A^{-1} since $A(\mathcal{L}_k, \cdot)$ has row full rank.

The above RSAI procedure can be described as Algorithm 1, named as the basic RSAI algorithm.

Algorithm 1 The basic RSAI Algorithm

For $k = 1, 2, \dots, n$ compute m_k :

- 1: Choose an initial pattern $\mathcal{J}_k^{(0)}$ of m_k , and give a user-prescribed tolerance ε and the maximum number l_{\max} of loops. Set $l = 0$ and $\mathcal{R}_k^{(0)} = \emptyset$.
 - 2: Determine $\mathcal{I}_k^{(0)}$, the set of indices of nonzero rows of $A(\cdot, \mathcal{J}_k^{(0)})$, solve (2.3) for \tilde{m}_k by the QR decomposition of $A_k = A(\mathcal{I}_k^{(0)}, \mathcal{J}_k^{(0)})$, recover m_k from \tilde{m}_k , and compute $r_k = Am_k - e_k$.
 - 3: **while** $\|r_k\| > \varepsilon$ and $l < l_{\max}$ **do**
 - 4: Set \mathcal{L}_k to be the set of indices i for which $r_k(i) \neq 0$, sort $|r_k(i)|$ in decreasing order, and let $\hat{\mathcal{R}}_k^{(l)}$ be the set of dominant indices i that correspond to a few largest $|r_k(i)|$ appearing in \mathcal{L}_k but not in $\mathcal{R}_k^{(l)}$. Augment $\mathcal{R}_k^{(l+1)} = \mathcal{R}_k^{(l)} \cup \hat{\mathcal{R}}_k^{(l)}$.
 - 5: Set $\hat{\mathcal{J}}_k$ equal to the set of all new column indices of $A(\hat{\mathcal{R}}_k, \cdot)$ but not in $\mathcal{J}_k^{(l)}$. Let $\hat{\mathcal{I}}_k$ be the set of row indices corresponding to $\hat{\mathcal{J}}_k$ that do not appear in $\mathcal{I}_k^{(l)}$, and update $\mathcal{I}_k^{(l+1)} = \mathcal{I}_k^{(l)} \cup \hat{\mathcal{I}}_k$.
 - 6: Set $l = l + 1$; if $\hat{\mathcal{J}}_k = \emptyset$, then go to step 3.
 - 7: For each $j \in \hat{\mathcal{J}}_k$, update m_k and $\|r_k\|$ using the approach in [19, 29], respectively.
 - 8: **end while**
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Two key differences between SPAI and RSAI are clear now. Firstly, for RSAI we order the nonzero entries in r_k , pick up a few dominant indices $\hat{\mathcal{R}}_k^{(l)}$ with the largest entries of r_k in magnitude that do not appear in $\mathcal{R}_k^{(l)}$. Using $\hat{\mathcal{R}}_k^{(l)}$, we determine the new indices $\hat{\mathcal{J}}_k$ and add them to $\mathcal{J}_k^{(l)}$ to get the new column indices $\mathcal{J}_k^{(l+1)}$, by which we identify the new row indices $\hat{\mathcal{I}}_k$ to be added and form the set $\mathcal{I}_k^{(l+1)}$ of row indices. Secondly, for RSAI we do not perform possibly expensive steps (2.6) and (2.7) and the ordering and sorting followed. Since $\hat{\mathcal{R}}_k^{(l)}$ is a subset of \mathcal{L}_k and we assume that it has only a few elements, its cardinality can be much smaller than that of \mathcal{L}_k , which is typically true for A irregular sparse. As a result, the determination of $\mathcal{I}_k^{(l+1)}$ and $\mathcal{J}_k^{(l+1)}$ is less costly, and it can be substantially less time consuming than SPAI, especially when A is irregular sparse.

Similar to SPAI, we need to provide an initial sparsity pattern of M for the RSAI algorithm, which is usually chosen to be that of I when A has nonzero diagonals. We also need to provide a stopping criterion ε , the number of the dominant indices and the maximum number l_{\max} of loops. For them, we take $\varepsilon = 0.1 \sim 0.4$, similarly to that used in F-norm based SAI preconditioning procedures including SPAI and PSAI. We suggest taking the cardinality c of $\hat{\mathcal{R}}_k^{(l)}$ to be 3 or so at each loop. As for outer loops l_{\max} , we take it to be small, say 10.

In the next section, we make some theoretical analysis and develop a more practical RSAI algorithm with some dropping strategy used.

4 Theoretical analysis and a practical RSAI algorithm

We cannot guarantee that M obtained by the basic RSAI algorithm is nonsingular without additional requirements. Grote and Huckle [19] present several results, showing how the non-singularity of M is related to ε and how the eigenvalues and the singular values of the preconditioned matrix AM distribute. Their results are general and apply to M obtained by any F-norm minimization based SAI preconditioning procedure.

Huckle [21] shows that the patterns of $(A^T A)^{\mu-1} A^T$ for small μ are effective upper bounds for the sparsity pattern of M by the SPAI algorithm. Note that both RSAI and SPAI augment the sparsity pattern of M based on the indices of nonzero entries of residuals r_k , $k = 1, 2, \dots, n$. Consequently, in the same way as [21], we can justify that the patterns of $(A^T A)^{\mu-1} A^T$ for small μ are also upper bounds for the sparsity pattern of M obtained by the basic RSAI algorithm.

Let us get insight into the pattern of M by the basic RSAI algorithm. Obviously, the dominant indices at each loop and the maximum number l_{\max} of loops directly affect the sparsity of M . Now we present a quantitative upper bound for the number of nonzero entries in M and show how the sparsity of M depends on that of A , the number c of the dominant indices at each loop and l_{\max} .

Theorem 4.1. *Assume that RSAI runs l_{\max} loops to generate $M = (m_1, m_2, \dots, m_n)$. Denote by g_k the number of nonzero entries in $A(k, :)$, $1 \leq k \leq n$, by $g = \max_{1 \leq k \leq n} g_k$, by c the number of the dominant indices at each loop, and by $nnz(m_k)$ the number of nonzero entries in m_k . Then for $\mathcal{J}_k^{(0)} = \{k\}$ we have*

$$nnz(m_k) \leq \min\{gcl_{\max} + 1, n\}, k = 1, 2, \dots, n \quad (4.1)$$

and

$$nnz(M) \leq \min\{(gcl_{\max} + 1)n, n^2\}. \quad (4.2)$$

Proof. By assumption, it is known that the number of nonzero entries added to m_k at each loop does not exceed gc . Therefore, we have

$$nnz(m_k) \leq gcl_{\max} + 1,$$

which, together with the trivial bound $nnz(m_k) \leq n$, establishes (4.1). Note that the number of nonzero entries of M is $\sum_{k=1}^n nnz(m_k)$. (4.2) is direct from (4.1). \square

Theorem 4.1 shows that if all the rows of A are sparse, i.e., g is small, then M must be sparse for l_{\max} and m small. On the other hand, if A has one relatively dense row, some columns of M may become dense quickly with increasing l . This is the case once an index in $\hat{\mathcal{R}}_k^{(l)}$ at some loops corresponds to a relatively dense row of A . In this case, the basic RSAI is inefficient since a relatively large LS problem will emerge in the next loop, causing that solving it is expensive. This is a shortcoming of the basic RSAI algorithm for A row irregular sparse.

Under the underlying hypothesis that the majority of the entries of A^{-1} are small, we know that whenever M becomes relatively dense in the course of construction, the majority of its entries must be small and make little contribution to A^{-1} . Therefore, in order to control the sparsity of M and improve the efficiency of the basic RSAI algorithm, we should drop those small entries promptly during the process for practical purposes. This asks us to introduce some reasonable dropping strategies into the basic RSAI algorithm so as to develop practical RSAI algorithms for constructing an effective and sparse M .

We should be aware that SPAI implicitly uses a dropping strategy to ensure that all the columns of M constructed by it are sparse. Precisely, suppose that SPAI is run l_{\max} loops starting with the pattern of I and a few, say, l_a most profitable indices are added to the pattern of m_k at each loop. Then the number of nonzero entries of the final m_k does not exceed $1 + l_a l_{\max}$, which is fixed and small as l_a and l_{\max} are both fixed and small. Such M may not be robust since the number of large entries in the k -th column of A^{-1} is unknown in practice. Moreover, for a general sparse A , the numbers

of large entries in the columns of A^{-1} may have great differences, that is, good sparse approximations of A^{-1} may well be irregular sparse. This is particularly true for A irregular sparse and even for A regular sparse [28]. Consequently, SPAI may generate a poor preconditioner M since some columns of it are too sparse for given small l_a and l_{\max} .

The above analysis suggests that we should not fix the number of large entries of each column of M for RSAI in advance. In the spirit of PSAI(tol) [29], a more robust and general-purpose dropping strategy is to retain all the large entries of m_k produced and drop those small ones below a prescribed tolerance tol during the loops. This kind of dropping strategy should better capture a good approximate sparsity pattern of A^{-1} and produce an effective M more possibly.

Dropping tolerances tol used in SAI preconditioning procedures had been empirically chosen as some small quantities, say 10^{-3} . Recently, Jia and Zhang [27] have shown that such dropping criteria are not robust and may lead to a sparser but ineffective preconditioner M or a possibly quite dense M , which, though effective for preconditioning, is very costly to construct and apply. Jia and Zhang [27] have established a mathematical theory on robust dropping tolerances for PSAI(tol) and all the static F-norm minimization based SAI preconditioning procedures. Based on the theory, they have designed robust and adaptive selection criteria for dropping tolerances, which adapt to the RSAI algorithm directly: an entry m_{jk} is dropped whenever

$$|m_{jk}| \leq tol_k = \frac{\varepsilon}{nnz(m_k)\|A\|_1}, j = 1, 2, \dots, n \quad (4.3)$$

during the loops, where m_{jk} is the j -th entry of m_k and $nnz(m_k)$ is the number of nonzero entries in m_k at the current loop, and $\|\cdot\|_1$ is the 1-norm of a matrix. In terms of the theory in [27], the RSAI(tol) equipped with the above dropping criterion will generate a SAI preconditioner M that has comparable preconditioning quality to the possibly much denser one generated by the basic RSAI algorithm without dropping small nonzero entries; see [27, Theorem 3.5].

Introducing (4.3) into Algorithm 1, we have developed a practical algorithm, called the RSAI(tol) algorithm. As a key comparison of RSAI(tol) and SPAI, we notice that, for RSAI(tol), the positions of large entries of m_k are adjusted dynamically during the loops, while the already occupied positions of nonzero entries in m_k by SPAI retain unchanged in subsequent loops and we simply add a few new indices to the pattern of m_k at each loop. Remarkably, some entries of m_k are well likely to change from large to small during the loops, so that the final M may have some small entries that contribute little to A^{-1} . In other words, RSAI(tol) seeks the positions of large entries of A^{-1} in a globally optimal sense, while the SPAI algorithm achieves this goal in a locally optimal sense. Consequently, RSAI(tol) captures the sparsity pattern of A^{-1} more effectively than SPAI.

5 Numerical experiments

In this section we test a number of real-world problems coming from applications, which are described in Table 1¹. We also list some useful information about the test matrices in Table 1. For each matrix, we give the number s of irregular columns as defined in the introduction, the average number p of nonzero entries per column and the

¹All of these matrices are either from the Matrix Market of the National Institute of Standards and Technology at <http://math.nist.gov/MatrixMarket/> or from the University of Florida Sparse Matrix Collection at <http://www.cise.ufl.edu/research/sparse/matrices/>

number p_d of nonzero entries in the densest column. To make the efficiency comparison as fair as possible, we have written the Fortran codes of SPAI, RSAI(tol) and PSAI(tol). After M is constructed by one of them, we then apply it within Krylov solvers. We shall demonstrate that RSAI(tol) works well. In the meantime, we compare RSAI(tol) with SPAI and PSAI(tol), illustrating that RSAI(tol) is at least competitive with SPAI and can be considerably more efficient and effective than the latter for some problems, and it is as comparably effective as PSAI(tol) for preconditioning $Ax = b$.

Table 1: The description of test matrices, where s is the number of irregular columns, p the average number of nonzero entries per column, and p_d the number of the nonzero entries in the densest column. An irregular column means that the number of its nonzero entries exceeds $10p$.

matrices	n	nnz	s	p	p_d	Description
fs_541_3	541	4282	1	8	538	2D/3D problem
orsirr_1	1030	6858	0	7	13	Oil reservoir simulation
orsirr_2	886	5970	0	7	14	Oil reservoir simulation
sherman1	1000	3750	0	4	7	Oil reservoir simulation
sherman2	1080	23094	0	21	34	Oil reservoir simulation
sherman5	3312	20793	0	6	17	Oil reservoir simulation
saylr4	3564	22316	0	6	7	3D reservoir simulation
cavity11	2597	71601	0	27	62	Subsequent computational fluid dynamics problem
ex36	3079	53099	0	17	37	Computational fluid dynamics problem
e20r0100	4241	131556	0	31	62	Computational fluid dynamics problem
e30r0000	9661	306356	0	32	62	Computational fluid dynamics problem
e40r0100	17281	553562	0	32	62	Computational fluid dynamics problem
powersim	15838	64424	2	4	41	Power network problem
raefsky3	21200	1488768	0	70	80	computational fluid dynamics problem
scircuit	170998	958936	104	6	353	Circuit, many parasitics

We perform numerical experiments on an Intel Core 2 Quad CPU E8400@ 3.00GHz with 2GB memory under the Linux operating system. The computations of constructing M are done using Fortran 90 with the machine precision $\epsilon_{\text{mach}} = 2.2 \times 10^{-16}$. We take the initial sparsity pattern as that of I for SPAI and RSAI(tol). We apply row Dulmage-Mendelsohn permutations to the matrices having zero diagonals so as to make their diagonals nonzero [16]. The related Matlab commands are $j = \mathbf{dmperm}(A)$ and $A = A(j, :)$. We applied **demp** to cavity11, ex36, e20r0100, e30r0000 and e40r0100. We use the M generated by RSAI(tol), SPAI and PSAI(tol) as right preconditioners, and use BiCGStab as the Krylov solver, whose code is from Matlab 7.8.0. The initial guess on the solution of $Ax = b$ is always $x_0 = 0$, and the right-hand side b is formed by choosing the solution $x = (1, 1, \dots, 1)^T$. The stopping criterion is

$$\frac{\|b - A\tilde{x}\|}{\|b\|} < 10^{-8}, \tilde{x} = M\tilde{y},$$

where \tilde{y} is the approximate solution obtained by BiCGStab applied to the preconditioned linear system $AMy = b$.

In all the tables, ϵ , l_{max} and c stand for the accuracy requirements, the maximum loops that RSAI(tol) allows, and the numbers of the dominant indices exploited by RSAI(tol), respectively. $spar = \frac{nnz(M)}{nnz(A)}$ denotes the sparsity of M relative to A , $iter$ stands for the number of iterations used by BiCGStab, n_c is the number of columns of M whose residual norms do not drop below ϵ , and $ptime$ and $stime$ denote the CPU timings (in seconds) of constructing M and of solving the preconditioned linear systems by BiCGStab, respectively. † indicates that convergence is not attained within 1000 iterations, ‡ indicates that the Krylov solver stagnates and – means that we do not count CPU timings when BiCGStab fails to converge within 1000 iterations.

5.1 The effectiveness of the $\text{RSAI}(tol)$ algorithm

First of all, we illustrate that $\text{RSAI}(tol)$ can capture an effective approximate sparsity pattern of A^{-1} by taking the 886×886 regular sparse matrix `orsirr_2` as an example. We take the number $c = 3$ of the dominant indices exploited at each loop, the prescribed accuracy $\varepsilon = 0.2$ and $l_{\max} = 15$. We have found that all columns of M satisfy the desired accuracy. We use the Matlab code `inv(A)` to compute A^{-1} directly and then retain the $\text{nnz}(M)$ largest entries of A^{-1} in magnitude. Figure 1 depicts the patterns of M and A^{-1} that drops its small nonzero entries. We see that the pattern of M matches that of A^{-1} quite well, demonstrating that the $\text{RSAI}(tol)$ algorithm can indeed capture an effective approximate sparsity pattern of A^{-1} . Importantly, it is clear from the figure that the sparsity patterns of M and A^{-1} are irregular and the numbers of large entries in the columns and rows of A^{-1} vary greatly, although the matrix `orsirr_2` itself is regular sparse. As we have counted, the number of irregular columns in A^{-1} is 63, each of which has more than $10 \frac{\text{nnz}(M)}{n}$ nonzero entries. It means that about 8% columns in A^{-1} are irregular sparse. This confirms the fact addressed in [28] that a good sparse approximate inverse of a regular sparse matrix can be irregular sparse. Such fact also implies that SPAI cannot guarantee to capture an effective approximate sparse pattern even when A is regular sparse and thus may be less effective for preconditioning an regular sparse problem (1.1) since all the columns of M constructed by SPAI are sparse and each of them has at most $1 + l_a l_{\max}$ nonzero entries, where l_a is the number of most profitable indices added at each loop.

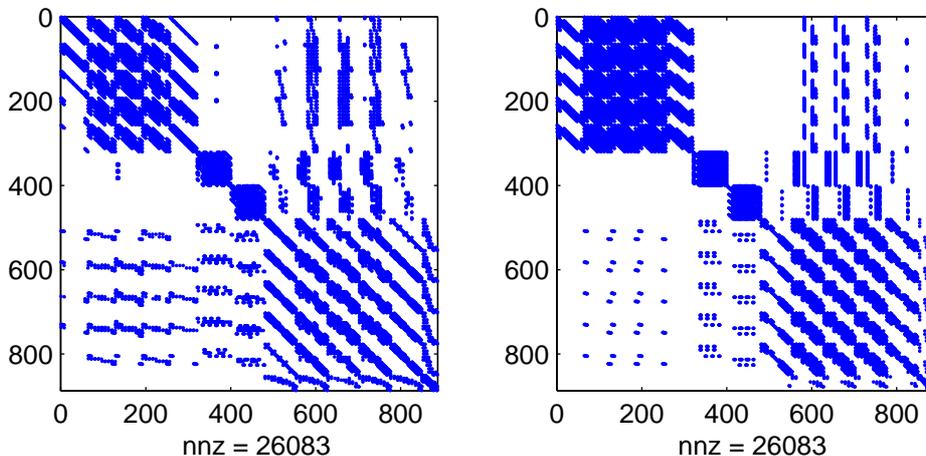


Figure 1: `orsirr_2`: left: the pattern of M ; right: the sparsity pattern of A^{-1} .

Now we take some difficult problems from Table 1 which make BiCGStab without preconditioning fail to converge within 1000 iterations. We precondition these problems by $\text{RSAI}(tol)$. The preconditioners are all computed by setting $\varepsilon = 0.4$, $c = 3$ and $l_{\max} = 10$. Table 2 lists the results obtained.

We observe that the $\text{RSAI}(tol)$ algorithm is effective to precondition these linear systems and accelerates the convergence of BiCGStab dramatically in all cases, as *iter* indicates. At the same time, we also find that the cost of constructing M is dominant.

Table 2: The RSAI(tol) algorithm for difficult problems

matrices	$spar$	$ptime$	n_c	$iter$	$stime$
orsirr_1	2.14	0.18	0	29	0.01
orsirr_2	2.14	0.14	0	28	0.01
sherman2	2.76	1.59	0	6	0.01
sherman5	1.15	0.32	0	38	0.02
saylr4	1.22	0.92	0	672	0.49
ex36	2.40	4.82	10	90	0.11
e20r0100	2.77	20.42	148	148	0.45
raefsky3	1.46	104.70	0	206	4.78

This is similar to SPAI and PSAI as well as all the other non-factorized and factorized sparse approximate inverse preconditioning procedures [5, 9]. For all the problems, we see that the n_c are equal to zero except for ex36 and e20r0100, for which n_c is very small relative to n . This means that for given parameters the RSAI(tol) algorithm generally construct effective preconditioners, as also reflected by the $iter$, which are much smaller than 1000. Therefore, we conclude from Table 2 that the RSAI(tol) algorithm is generally effective for preconditioning (1.1).

Next we vary the stopping criterion ε to see how it affects the sparsity of M and convergence of BiCGStab. We still set $c = 3$ and $l_{\max} = 10$. Table 3 reports the results obtained for sherman1, saylr4 and orsirr_2. As expected, BiCGStab used fewer iterations in all cases and M becomes denser as ε decreases. However, we find that, for the three problems, though a smaller ε makes BiCGStab converge faster, it is more costly to construct a denser M . Compared with the cost of constructing M , the cost of applying M and solving the preconditioned systems by BiCGStab is negligible for general problems, provided that M is an effective preconditioner. The choice of $\varepsilon = 0.4$ is the best as far as the total costs are concerned. The table also implies that $l_{\max} = 10$ is conservative for the four given ε since actual loops used does not achieve it and whether or RSAI(tol) terminated is up to ε for the test problems.

Table 3: The results for sherman1, saylr4 and orsirr_2 with different ε

sherman1: $c = 3$ and $l_{\max} = 10$				
ε	$spar$	$ptime$	$iter$	$stime$
0.4	2.04	0.06	29	0.01
0.3	2.38	0.07	29	0.01
0.2	3.96	0.18	18	0.01
0.1	11.46	1.83	10	0.01
saylr4: $c = 3$ and $l_{\max} = 10$				
ε	$spar$	$ptime$	$iter$	$stime$
0.4	1.22	0.92	672	0.49
0.3	1.95	1.90	267	0.21
0.2	3.08	3.84	85	0.08
0.1	6.79	12.50	45	0.05
orsirr_2: $c = 3$ and $l_{\max} = 10$				
ε	$spar$	$ptime$	$iter$	$stime$
0.4	2.14	0.14	28	0.01
0.3	2.75	0.21	23	0.01
0.2	4.36	0.52	16	0.01
0.1	8.23	2.35	11	0.01

Finally, we vary c to investigate how it affects the sparsity of M and the convergence of BiCGStab. We set $\varepsilon = 0.4$ and $l_{\max} = 20$, and take sherman2, ex36 and raefsky3 as

examples. Table 4 lists the results. We find that the M gradually become denser but the CPU time of constructing them does not necessarily become more as c increases. This should be expected since at each loop the main cost of $\text{RSAI}(tol)$ is the solutions of LS problems other than the ordering of entries of the current r_k and picking up indices. By comparison, as far as the overall performance, measured by $ptime$, n_c and $iter$, is concerned, we find that $c = 3$ may be a very best choice.

Table 4: The results for `sherman2`, `ex36` and `raefsky3` with different c

sherman2: $\varepsilon = 0.4$ and $l_{\max} = 20$					
c	$spar$	$ptime$	n_c	$iter$	$stime$
1	1.99	1.70	2	8	0.01
2	2.13	1.55	0	7	0.01
3	2.76	1.59	0	6	0.01
4	3.22	1.70	0	7	0.01
5	3.55	1.94	0	7	0.01
ex36: $\varepsilon = 0.4$ and $l_{\max} = 20$					
c	$spar$	$ptime$	n_c	$iter$	$stime$
1	1.83	7.12	59	103	0.11
2	2.22	5.75	0	85	0.11
3	2.40	4.86	0	91	0.11
4	2.67	4.94	0	83	0.11
5	2.93	5.00	0	86	0.11
raefsky3: $\varepsilon = 0.4$ and $l_{\max} = 20$					
c	$spar$	$ptime$	n_c	$iter$	$stime$
1	1.17	120.49	0	†	–
2	1.31	111.59	0	206	4.20
3	1.46	104.70	0	206	4.78
4	1.59	120.85	0	91	2.03
5	1.74	125.84	0	62	1.46

5.2 The $\text{RSAI}(tol)$ algorithm versus the SPAI algorithm

In this subsection, we compare the performance of $\text{RSAI}(tol)$ and SPAI. For $\text{RSAI}(tol)$ we take $\varepsilon = 0.3$, the number $c = 3$ of the dominant indices exploited at each loop and $l_{\max} = 10$. For SPAI we take the same ε and add three most profitable indices to the pattern of m_k at each loop for $k = 1, 2, \dots, n$. We set $l_{\max} = \lfloor 10 \times nnz(A) / (3 \times n) \rfloor$ to control $nnz(M)/nnz(A) \leq 5$ for the SPAI algorithm, where $\lfloor x \rfloor$ is the Gaussian function. Table 5 presents the results.

From the table, we see that the M constructed by the $\text{RSAI}(tol)$ algorithm are (almost) equally sparse as the counterparts by the SPAI algorithm except powersim. However, the $\text{RSAI}(tol)$ algorithm uses substantially less CPU time and more efficient than the SPAI algorithm except powersim, especially when a given A is irregular sparse. For a dense column of A , the cardinalities of the corresponding $\hat{\mathcal{J}}_k$ used by SPAI are big during the loops, causing that it is time consuming to determine the most profitable indices added to the patterns of m_k at each loop. In contrast, $\text{RSAI}(tol)$ overcomes this shortcoming by quickly finding new indices added at each loop and is considerably more efficient. For `sherman2`, `raefsky3` and some others, we also find that the $\text{RSAI}(tol)$ algorithm spends much less CPU timings than the SPAI algorithm when the given problem is relatively dense, i.e., the average number d of nonzero entries per column is relatively large. This is because SPAI spent much more time seeking the most profitable indices for d bigger than $\text{RSAI}(tol)$.

Table 5: The RSAI(tol) algorithm versus the SPAI algorithm

matrices	The RSAI(tol) algorithm					The SPAI algorithm				
	Constructing M			BiCGStab		Constructing M			BiCGStab	
	$spar$	$ptime$	n_c	$iter$	$stime$	$spar$	$ptime$	n_c	$iter$	$stime$
fs_541_3	1.52	0.30	1	16	0.01	1.45	0.49	6	18	0.01
orsirr_1	2.67	0.23	0	24	0.01	1.58	0.38	2	29	0.01
orsirr_2	2.75	0.21	0	23	0.01	1.63	0.36	2	26	0.01
sherman1	2.38	0.07	0	29	0.01	1.72	0.12	7	31	0.01
sherman2	2.97	2.19	1	5	0.01	2.69	66.7	102	†	—
sherman5	1.65	0.50	0	30	0.02	1.18	2.00	3	34	0.02
saylr4	1.95	1.90	0	267	0.21	1.97	2.09	15	271	0.20
cavity11	3.13	45.9	243	172	0.28	2.79	635.8	534	215	0.34
ex36	2.58	6.45	160	77	0.10	1.63	28.01	191	93	0.10
e20r0100	2.90	26.21	413	149	0.47	2.98	980.3	895	212	0.68
e30r0000	2.79	76.8	916	168	1.28	3.09	2727	2606	211	1.69
e40r0100	2.90	138.7	1664	425	7.99	3.06	5181	4649	477	6.70
powersim	4.92	24.2	1277	839	3.02	2.63	21.7	1414	†	—
raefsky3	2.31	355	0	75	2.12	1.10	10655	94	†	—
scircuit	2.82	1283	4	849	46.41	2.19	21134	7535	†	—

For the preconditioning effectiveness, we observe that the n_c in $\text{RSAI}(tol)$ are considerably smaller than those in SPAI in all cases, especially when A is irregular sparse. The reason is that, as we have explained, $\text{RSAI}(tol)$ is more effective to capture a good approximate sparsity pattern of A^{-1} than SPAI. Our results illustrate that with the roughly same number of nonzero entries allowed, the M by $\text{RSAI}(tol)$ are more effective than their counterparts by SPAI, that is, $\text{RSAI}(tol)$ indeed captures the positions of large entries more reliably than SPAI does, and the latter may retain positions of some small nonzero entries but misses some large entries. As a result, the M generated by SPAI are less effective than those by $\text{RSAI}(tol)$. This is also confirmed by the number $iter$ of iterations, where the $iter$ preconditioned by $\text{RSAI}(tol)$ are considerably fewer than those by SPAI for all the test problems. Particularly, for `sherman2`, `powersim`, `raefsky3` and `scircuit`, SPAI fails to make BiCGStab converge within 1000 iterations, while $\text{RSAI}(tol)$ works well, and this is even true for the regular sparse matrix `sherman2` and `raefsky3`. Actually, for `powersim` and `scircuit`, BiCGStab preconditioned by SPAI reduces the relative residual to the level of 10^{-3} after several iterations, and afterwards it does not decrease further.

In view of the above, the $\text{RSAI}(tol)$ algorithm is at least competitive with and can be considerably more efficient and effective than the SPAI algorithm, especially for the problems where A is irregular sparse or has relatively more nonzero entries.

5.3 The $\text{RSAI}(tol)$ algorithm versus the $\text{PSAI}(tol)$ algorithm

Keeping in mind the results of Section 5.2, we now compare the $\text{RSAI}(tol)$ algorithm with the $\text{PSAI}(tol)$ algorithm proposed in [29] and improved in [27]. We also take $\varepsilon = 0.3$ and $l_{\max} = 10$ for $\text{PSAI}(tol)$. Table 6 reports the results obtained by $\text{PSAI}(tol)$.

Table 6: The results by the $\text{PSAI}(tol)$ algorithm

matrices	Preconditioning			BiCGStab	
	<i>spar</i>	<i>ptime</i>	n_c	<i>iter</i>	<i>stime</i>
fs_541_3	1.87	0.13	0	5	0.01
orsirr_1	5.36	1.69	0	25	0.01
orsirr_2	5.66	1.55	0	23	0.01
sherman1	2.89	0.12	0	27	0.01
sherman2	2.74	1.59	0	4	0.01
sherman5	1.57	0.46	0	29	0.02
saylr4	1.86	12.26	0	307	0.26
cavity11	11.84	155.9	0	55	0.35
ex36	6.45	13.50	0	55	0.12
e20r0100	9.44	177.4	0	91	0.79
e30r0000	13.19	1670	29	†	—
e40r0100	18.38	8779	494	‡	—
powersim	9.91	72.9	399	52	0.41
raefsky3	5.03	1281	0	63	3.19

We observe that the n_c by the $\text{PSAI}(tol)$ algorithm are no more than those by the $\text{RSAI}(tol)$ algorithm for all the problems. Actually, $\text{PSAI}(tol)$ obtains the M with the desired accuracy satisfied for all the problems except `e30r0000`, `e40r0100` and `powersim`. This means that $\text{PSAI}(tol)$ can construct effective preconditioners for most general sparse problems. We do not list the test problem `scircuit` in the table because $\text{PSAI}(tol)$ was found out of memory, which is due to the occurrence of some large sized LS problems during the process. We refer the reader to [28, 29] for explanations on some typical features of $\text{PSAI}(tol)$. Compared with $\text{RSAI}(tol)$, it is seen that, except `powersim`,

e30r0000 and e40r0100, for the other test problems, there is no obvious winner between it and $\text{PSAI}(tol)$ in term of $iter$. Moreover, the setup time of M by two algorithms are also comparable for regular problems. For some problem which has relatively more nonzero entries, such as raefsky3 and so on, the setup time of M by $\text{PSAI}(tol)$ is more than those by $\text{RSAI}(tol)$. This is because we get a denser M by $\text{PSAI}(tol)$. However, it is more effective than those by $\text{RSAI}(tol)$ in term of n_c . Based on these observations, we conclude that the $\text{RSAI}(tol)$ algorithm is as comparably effective as the $\text{PSAI}(tol)$ algorithm.

6 Conclusions

SPAI may be costly to seek approximate sparsity patterns of A^{-1} and ineffective for preconditioning a large sparse linear system, which is especially true when A is irregular sparse or is not very sparse. To this end, we have proposed a basic RSAI algorithm that only uses the dominant information on residual and can adaptively determine a good approximate sparsity pattern of A^{-1} . We have derived an estimate for the number of nonzero entries of M . In order to control the sparsity of M and improve efficiency, we have developed a practical $\text{RSAI}(tol)$ algorithm with the robust adaptive dropping strategy [27] exploited. We have tested a number of real-world problems to illustrate the efficiency and preconditioning effectiveness of $\text{RSAI}(tol)$. Meanwhile, we have numerically compared it with SPAI and $\text{PSAI}(tol)$, showing that $\text{RSAI}(tol)$ is at least competitive with and can be substantially more efficient and effective than SPAI, and it is also comparably effective as $\text{PSAI}(tol)$.

Some other research is significant. As we have seen, $\text{RSAI}(tol)$ may be costly for A row irregular sparse, which is also the case for SPAI. In order to make $\text{RSAI}(tol)$ efficient for both column and row irregular sparse problems, we have exploited the idea proposed in [28] to transform such a problem into certain regular ones, so that $\text{RSAI}(tol)$ can be much more efficient to construct effective preconditioners [26].

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