Optimal design for step-stress accelerated test with random discrete stress elevating times based on gamma degradation process

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Abstract

Recently, a step-stress accelerated degradation test (SSADT) plan, in which the stress level is elevated when the degradation value of a product crosses a pre-specified value, was proposed. The times of stress level elevating are random and vary from product to product. In this paper we extend this model to a more economic plan. The proposed extended model has two economical advantages compared with the previous one. The first is that the times of stress level elevating in the new model are identical for all products, which enable us to use only one chamber (oven) for testing all test units. The second is that, the new method does not require continuous inspection and to elevate the stress level, it is not necessary for the experimenter to inspect the value of the degradation continually. The new method decrease the cost of measurement and also there is no need to use electronic sensors to detect the first passage time of the degradation to the threshold value in the new method. We assume that the degradation path follows a gamma process. The stress level is elevated as soon as the measurement of the degradation of one of the test units, at one of the specified times, exceeds the threshold value. Under the constraint that the total experimental cost does not exceed a pre-specified budget, the optimal settings including the optimal threshold value, sample size, measurement frequency and termination time are obtained by minimizing the asymptotic variance of an estimated quantile of the lifetime distribution of the product. A case study is presented to illustrate the proposed method.

Keywords: Discrete times, Fisher information matrix, Gamma process, Inverse Gaussian distribution.

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1 Introduction

Due to the rapid improvement of quality of products of today, it is difficult to assess the failure information of a product using traditional life testing procedures, which record only failure times. An alternative approach is to collect the degradation data at higher levels of stress, thus yielding more information about the lifetime distribution in a reasonable amount of time compared to traditional life tests. Such a life testing plan is called an accelerated degradation test (ADT). Mathematical models for analyzing ADT data have been proposed by Meeker & Escobar [7]. Meeker *et al.* [8] described accelerated degradation models that relate to physical failure mechanisms. Yu & Tseng [13] proposed a stopping rule for terminating an ADT. Chen & Zheng [2] proposed an approach for degradation analysis which makes inference directly on the lifetime distribution. Huang & Dietrich [3], used maximum likelihood estimation to estimate the model's parameters. Joseph & Yu [4], developed a methodology for quality improvement when degradation data are available as the response in the experiments.

ADT is usually very costly, since it requires destroying a considerable number of products at each level of stress. Moreover, determining suitable levels of stress for the experiment is not straightforward. To handle these problems, step stress accelerated degradation tests (SSADT) were proposed by Tseng & Wen [12] in a case study of LEDs. In a SSADT framework, each product is first tested, subject to a pre-determined stress level for a specified duration, and the degradation data are collected. A product which survives until the end of the first step is again tested at a higher stress level and for a different time duration. The stress level is elevated step by step until an appropriate termination time is reached. The advantage of the SSADT is that the number of test units needed for conducting this test is relatively small.

To conduct a SSADT efficiently, special attention is paid to the number of tested units (sample size), measurement frequency and termination time. For ADT, Boulanger & Escobar [1] addressed the problem of optimal determination of the stress levels as well as the sample size for each stress level for a specified termination time. The problem of optimizing the test design have been extensively studied in recent years. Three commonly used optimization criteria are the minimum approximated variance (Avar) of the maximum likelihood estimators (MLE) of reliability, mean time to failure (MTTF) and the quantiles of the lifetime distribution. Tang *et al.* [11] discussed the optimal planning of a simple SSADT and minimized the test cost while achieving a requisite level of estimates, from which the optimal number of inspections of the whole test can be determined. Liao & Tseng [6] used a Wiener process to model a typical SSADT problem and then, under the additional constraint that the total experimental cost does not exceed a predetermined budget, obtained the optimal settings of the SSADT for minimizing the Avar of the estimated $100p^{\text{th}}$ percentile of the product's lifetime distribution. They derived the optimal test plans for the sample size, measurement frequency and termination time. Tseng *et al.*

[14] discussed the SSADT problem for a gamma process and gave the optimal settings for minimizing the Avar of the estimated MTTF of the lifetime distribution of the product.

Recently, Pan & Balakrishnan [9] proposed a step-stress accelerated degradation test (SSADT) plan in which the stress level is elevated when the degradation value of a product crosses a pre-specified value. In such a plan the times of elevating the stress levels are random and vary from product to product. However, such a plan can be extended to a more economic plan in two ways. First one can extend the model such that the times of stress level elevating in the new model be identical for all products, which enable us to use only one chamber (oven) for testing all test units. Second, one can conduct the plan such that it does not require continuous inspection of the degradation by the experimenter (or may be electronic sensors). In the later we don't need to spend any money on electronic sensors and also we have decreased the cost of measurement.

To extend the model, we assume that the measurements are made at the end of consecutive intervals of fixed duration. The experimenter elevates the stress level as soon as a new measurement of at least one of the test units exceeds the threshold value.

In this paper, we propose a SSADT model when the degradation path follows a gamma process. The measurements are made at times kf, for k = 1, ..., M, where f is the measurement frequency in unit of time and M is the number of measurements. The stress level is elevated as soon as a new measurement of the degradation of one at least of the test units exceeds the threshold value. Next, under the constraint that the total experimental cost does not exceed a pre-specified budget, the optimal settings including the optimal threshold value, sample size, measurement frequency and termination time are obtained by minimizing the Avar of an estimated quantile of the lifetime distribution of the product.

The paper is organized as follows. The model is described in Section II. An optimization algorithm is presented in Section III to obtain the optimal settings under budget constraints. Section IV presents a real data example for illustrating the proposed algorithm and some concluding remarks are made in Section V.

2 Model description

Suppose that n test units are subject to a degradation test. Let L(t|S) denote the degradation path of the product under a level of stress S at time t. Denote the use stress level (under the normal conditions) by S_0 . The lifetime of the i^{th} unit under the normal conditions, $T_0^{(i)}$, is defined the first time that $L(t|S_0)$ crosses the critical value D, that is

$$T_0^{(i)} = \inf\{t | L^{(i)}(t|S_0) \ge D\},\tag{1}$$

in which $L^{(i)}(t|S_0)$, i = 1, ..., n, denotes the degradation value of the i^{th} unit under the stress S_0 at time t.

Pan & Balakrishnan [9], proposed a SSADT model in which the j^{th} stress level of the i^{th} unit, $i = 1, \ldots, n, j = 1, \ldots, m$, is elevated as soon as the degradation of the unit

passes the threshold value ω_j ($\omega_0 = 0 < \omega_1 < \cdots < \omega_m < D$). Hence, the testing stress of the *i*th unit, $i = 1, \ldots, n$, under such a SSADT model can be expressed for the stress values $S_0 < S_1 < \cdots < S_m$ as follows

$$S^{(i)} = \begin{cases} S_1 & \tau_{i,0} = 0 \le t < \tau_{i,1} \\ S_2 & \tau_{i,1} \le t < \tau_{i,2}, \\ \vdots \\ S_m & \tau_{i,m-1} \le t < T, \end{cases}$$
(2)

where T is the termination time of the test and

$$\tau_{i,j} = \inf\{t | L^{(i)}(t | S_j) \ge \omega_j\}, \quad j = 1, \dots, m-1.$$
(3)

in which $L^{(i)}(t|S_j)$, i = 1, ..., n, j = 1, ..., m, denotes the degradation value of the i^{th} unit under the stress levels S_j at time t. We should keep in mind that the degradation values do not depend on the random variables $\tau_{i,j}$, j = 1, ..., m - 1, and the random variables $\tau_{i,j}$, j = 1, ..., m - 1, depend on the degradation values. In other words, we have

$$S^{(i)} = \begin{cases} S_1 & -\infty \leq L^{(i)}(t|S_1) \leq \omega_1, \\ S_2 & \omega_1 \leq L^{(i)}(t|S_2) \leq \omega_2, \\ \vdots \\ S_m & \omega_{m-1} \leq L^{(i)}(t|S_m) \leq D & \& t < T. \end{cases}$$
(4)

Also, it is worth noting the difference of notations in this paper with the paper of Pan and Balakrishnan [9]. In this paper, $L^{(i)}(t|S_j)$ denotes the cumulative degradation value of i^{th} item during the test, under the j^{th} stress level, while in Pan and Balakrishnan [9], it denotes the cumulative degradation value of i^{th} item during the j^{th} stress level.

They considered both Wiener and Gamma processes for the degradation path of the products. In the following, we assume that the independent increments of the degradation path of the products follows a gamma process (Lawless & Crowder [5]); that is, for fixed t and Δt ,

$$\Delta L(t|S_j) = L(t + \Delta t|S_j) - L(t|S_j) \sim Ga(\alpha_j \Delta t, \beta),$$
(5)

where $Ga(\alpha_j \Delta t, \beta)$ stands for the gamma distribution with shape and scale parameter $\alpha_j \Delta t$ and β , respectively. Thus, we have, for t > 0

$$L^{(i)}(t|S_j) \sim Ga\left(\alpha_j(t-\tau_{i,j}) + \sum_{k=1}^{j-1} \alpha_i(\tau_{i,k} - \tau_{i,k-1})\right).$$

Park and Padgett [10] provided the approximate distribution of iid random differences $\tau_{i,k} - \tau_{i,k-1}$, for $k = 1, \ldots, m-1$ as Birnbaum-Saunders distribution with cdf

$$G_{\tau_{i,k}-\tau_{i,k-1}}(t) \approx \Phi\left(\frac{1}{\delta_1}\left(\sqrt{\frac{t}{\gamma_1}} - \sqrt{\frac{\gamma_1}{t}}\right)\right).$$
(6)

Under the above settings and using (6), Pan and Balakrishnan [9] derived the approximate joint pdf of $\tau_{i,1}, \ldots, \tau_{i,m-1}$ as

$$g_{\tau_{i,1},\dots,\tau_{i,m-1}}(t_1,\dots,t_{m-1}) \approx \prod_{k=1}^{m-1} \left\{ \frac{\alpha_k}{2\sqrt{2\pi\beta(\omega_k-\omega_{k-1})}} \left[\left(\frac{\alpha_k(t_k-t_{k-1})}{\beta(\omega_k-\omega_{k-1})} \right)^{-1/2} + \left(\frac{\alpha_k(t_k-t_{k-1})}{\beta(\omega_k-\omega_{k-1})} \right)^{-3/2} \right] \cdot \exp\left[-\frac{\beta(\omega_k-\omega_{k-1})}{2} + \left(\frac{\alpha_k(t_k-t_{k-1})}{\beta(\omega_k-\omega_{k-1})} + \frac{\beta(\omega_k-\omega_{k-1})}{2} - 2 \right) \right] \right\}.$$
(7)

The idea of using random time for elevating the stress level of SSADT is interesting. However, implementing the above SSADT model with stress model (2) is not economic, since the main results for conducting a SSADT is that, we can economically use only one chamber (oven), which allows us to elevate gradually the stress levels at specified (or random) times for all testing units. From equation (2), it is clear that n chambers should be used simultaneously, since with current metrology technology it is not possible to conduct such an SSADT plan within a single chamber.

A more economic model with random stress-elevating times can be conducted with the common stress

$$S = \begin{cases} S_1 & 0 \le t < \tau_{(1),1} \\ S_2 & \tau_{(1),1} \le t < \tau_{(1),2}, \\ \vdots \\ S_m & \tau_{(1),m-1} \le t < T, \end{cases}$$
(8)

for all test units, in which

$$\tau_{(1),j} = \min\{\tau_{i,j}; \ i = 1, \dots, n\}, \quad j = 1, \dots, m-1.$$
(9)

Under the SSADT plan with stress model (8), the j^{th} stress level is elevated for all units as soon as the degradation of at least one of the units exceeds the threshold value w_j . The test terminated at time T.

Using (7), one can obtain the joint distribution of $\tau_{(1),1}, \ldots, \tau_{(1),m-1}$, which is a rather complicated formula.

For the case m = 2, since $\tau_{1,1}, \ldots, \tau_{n,1}$ are independent and identically distributed, using (6) $\tau_{(1),1}$ has the approximate cdf

$$G_{\tau_{(1),1}}(t;\theta) \simeq 1 - \left(1 - \Phi\left(\frac{1}{\delta_1}\left(\sqrt{\frac{t}{\gamma_1}} - \sqrt{\frac{\gamma_1}{t}}\right)\right)\right)^n.$$
(10)

Also, for the case m = 3, using (6), the approximate joint survival function of $\tau_{(1),1}$ and $\tau_{(1),2}$ is obtained as follows

$$\Pr(\tau_{(1),1} > t_1, \tau_{(1),2} > t_2; \theta) = \left(\Pr(\tau_{1,1} > t_1, \tau_{1,2} > t_2; \theta)\right)^n,$$

where

$$\Pr(\tau_{1,1} > t_1, \tau_{1,2} > t_2; \theta) = \Pr(\tau_{1,1} > t_1; \theta) - \Pr(\tau_{1,1} > t_1, \tau_{1,2} \le t_2; \theta)$$

= $\Pr(\tau_{1,1} > t_1; \theta)(1 - \Pr(\tau_{1,2} - \tau_{1,1} \le t_2 - t_1; \theta))$
 $\simeq \left(1 - \Phi\left(\frac{1}{\delta_1}\left(\sqrt{\frac{t_1}{\gamma_1}} - \sqrt{\frac{\gamma_1}{t_1}}\right)\right)\right)$
 $\times \Phi\left(\frac{1}{\delta_1}\left(\sqrt{\frac{t_2 - t_1}{\gamma_1}} - \sqrt{\frac{\gamma_1}{t_2 - t_1}}\right)\right).$ (11)

It is easy but a bit tedious to extend the above discussion for $m \ge 4$. For instance, for m = 4, we have

$$\begin{aligned} \Pr(\tau_{1,1} > t_1, \tau_{1,2} > t_2, \tau_{1,3} > t_3; \theta) &= \Pr(\tau_{1,1} > t_1; \theta) - \Pr(\tau_{1,1} > t_1, \tau_{1,2} \le t_2; \theta) \\ &- \Pr(\tau_{1,2} > t_2, \tau_{1,3} \le t_3; \theta) \\ &+ \Pr(\tau_{1,1} \le t_1, \tau_{1,2} > t_2, \tau_{1,3} \le t_3; \theta) \\ &= \Pr(\tau_{1,1} > t_1; \theta) \Pr(\tau_{1,2} - \tau_{1,1} \le t_2 - t_1; \theta) \\ &- \Pr(\tau_{1,2} > t_2; \theta) \Pr(\tau_{1,3} - \tau_{1,2} \le t_3 - t_2; \theta) \\ &+ \Pr(\tau_{1,1} \le t_1; \theta) \Pr(\tau_{1,2} - \tau_{1,1} > t_2 - t_1; \theta) \\ &\times \Pr(\tau_{1,3} - \tau_{1,2} \le t_3 - t_2; \theta). \end{aligned}$$

The second step for conducting a more economic plan is to extend the model with test stress model (8) such that it does not require continuous inspection of the degradation level of test units by the experimenter or electronic sensors. In the following, we describe the extended model with discrete inspection times.

Suppose we want to make M measurements for each unit with a measurement frequency per f unit of time. Note that for the stress model (8), since the stress elevating time is random, we can only determine the total number of measurements and the number of measurements under each stress level is random. Under the j^{th} stress level, $j = 1, \ldots, m$, we elevate the stress level, as soon as a new measurement of the degradation of at least one of the test units exceeds the threshold value ω_j . The testing stress of the i^{th} unit, $i = 1, \ldots, n$, under such a model can be expressed as follows

$$S = \begin{cases} S_1 & 0 \le t < \kappa_1 f \\ S_2 & \kappa_1 f \le t < \kappa_2 f, \\ \vdots \\ S_m & \kappa_{m-1} f \le t < M f, \end{cases}$$
(12)

where

$$\kappa_j = \min\{M, \left[\left[\frac{\tau_{(1),j}}{f}\right]\right]\}, \quad j = 1, \dots, m-1,$$
(13)

in which [[x]] stands for the smallest integer greater than x and $\tau_{(1),i}$ is defined in (9).

The random variables κ_j are discrete with the obvious property that $1 \leq \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_{m-1} \leq M$, with probability 1. Also, it is straightforward that, if for some j^* , $\kappa_{j^*} = M$, we have $\kappa_j = M$, for all $j > j^*$ and $\min\{T_{(1),m}/f, M\} = M$. Therefore, if for some j^* , $\kappa_{j^*} = M$, the testing stress takes only the values $S_1 < \cdots < S_{j^*}$ during the total time interval, [0, Mf].

For the case m = 2, we have

$$\Pr(\kappa_1 = k; \theta) = \Pr((k-1)f < \tau_{(1),1} \le kf; \theta)$$

$$\approx G_{\tau_{(1),1}}(kf; \theta) - G_{\tau_{(1),1}}((k-1)f; \theta), \quad k = 1, \dots, M-1,$$
(14)

and

$$\Pr(\kappa_1 = M; \theta) = \Pr(\tau_{(1),j} > (M-1)f; \theta) \approx 1 - G_{\tau_{(1),1}}((M-1)f; \theta),$$
(15)

where $G_{\tau_{(1),1}}(t;\theta)$ is given in (10).

Also, for the case m = 3, the joint pmf of κ_1 and κ_2 is

$$\Pr(\kappa_1 = k_1, \kappa_2 = k_2; \theta) = P((k_1 - 1)f < \tau_{(1),1} \le k_1 f, (k_2 - 1)f < \tau_{(1),2} \le k_2 f; \theta),$$

$$k_1, k_2 = 1, \dots, M - 1,$$
(16)

and simillar expressions are derived easily for $Pr(\kappa_1 = M, \kappa_1 = k_2; \theta)$, $Pr(\kappa_1 = k_1, \kappa_1 = M; \theta)$ and $Pr(\kappa_1 = M, \kappa_1 = M; \theta)$, which are all approximated using the joint survival function given in (11). Similar formulas can also be obtained for the joint pmf of $\kappa_1, \ldots, \kappa_{m-1}$, for $m \ge 4$.

Suppose that the relation between α_j and the stress level S_j is modeled by the Arrhenius reaction rate model as

$$\alpha_j = exp(a + \frac{b}{273 + S_j}), \ j = 0, 1, \dots, m,$$

for two unknown parameters a and b.

For k = 0, ..., M - 1 and i = 1, ..., n, let G_{ki} denote the degradation increment of the i^{th} unit at time (k+1)f relative to time kf, that is

$$G_{ki} = L^{(i)}((k+1)f) - L^{(i)}(kf),$$

where $L^{(i)}(kf)$ is the level of degradation of the i^{th} unit at time kf. We have $L^{(i)}(t) = L^{(i)}(t|S_j)$ for all $t \in [kf, (k+1)f]$ for $k = \kappa_{j-1}, \ldots, \kappa_j - 1$ with $\kappa_0 = 0, \ \kappa_m = M$, and $j = 1, \ldots, m$.

Using (5), the conditional distribution of G_{ki} given κ_1 , for $k = \kappa_{j-1}, \ldots, \kappa_j - 1$, is $Ga(\alpha_j f, \beta), j = 1, \ldots, m$, in which we assume $\kappa_0 = 0$ and $\kappa_m = M$. Therefore, given

 $G_{ki} = g_{ki}$ and $(\kappa_1 = k_1, \ldots, \kappa_{m-1} = k_{m-1})$, the likelihood function of $\theta = (a, b, \beta)$ is given by

$$L(\theta) = P_{\theta}(\kappa_1 = k_1, \dots, \kappa_{m-1} = k_{m-1}) \prod_{i=1}^n \prod_{j=1}^m \prod_{k=k_{j-1}}^{k_j-1} \frac{g_{ki}^{f\alpha_j - 1} e^{-g_{ki}/\beta}}{\Gamma(f\alpha_j)\beta^{f\alpha_j}},$$
(17)

where $P_{\theta}(\kappa_1 = k_1, \ldots, \kappa_{m-1} = k_{m-1})$ is given in (14) to (16) for m = 2, 3 and similar formulas can also be obtained for the joint pmf of $\kappa_1, \ldots, \kappa_{m-1}$, for $m \ge 4$.

3 The Optimal Design

We develope the optimization algorithm for the case m = 2. The generalization of the proposed algorithm for larger values of m is straightforward. For the case m = 2, the optimization of the SSADT model in (12) consists of finding the optimal values of n, M, f, and the optimal value of ω_1 . We concern the problem of minimizing the Avar of an estimate of a quantile of the product's lifetime distribution as an optimization criterion. The approximated cdf of the lifetime of the product under the use stress level, S_0 , which is defined as in (1), is (see Park and Padgett, [10])

$$F_0(t) \simeq \Phi\left(\frac{1}{\alpha_0^*}\left(\sqrt{\frac{t}{\beta_0^*}} - \sqrt{\frac{\beta_0^*}{t}}\right)\right),\tag{18}$$

where

$$\alpha_0^* = \sqrt{\frac{\beta}{D}}, \text{ and } \beta_0^* = \frac{D}{\beta \alpha_0}.$$

Using (18), one can obtain the $100p^{\text{th}}$ percentile of T_0 as $\xi_p = F_0^{-1}(p)$, $0 . The MLE of <math>\xi_p$, $\hat{\xi_p} = \hat{F}_0^{-1}(p)$, is obtained by substituting the MLEs of β and α_0 into $F_0^{-1}(p)$. These values can be obtain by maximizing the likelihood function of $\theta = (a, b, \beta)$ in (17).

The Avar of ξ_p can be obtained as a function of ω_1 , based on Avar of the MLE (The inverse of the Fisher information matrix). Using the delta method we have

$$\operatorname{Avar}(\hat{\xi}_p;\omega_1,n,f,M) = \frac{1}{(\hat{f}_0(\hat{\xi}_p))^2} h^T I^{-1}(\hat{\theta}(\omega_1))h,$$

where f_0 is the corresponding pdf of the cdf in (18), the transpose of vector h is

$$h^{T} = \left(\frac{\partial F_{0}(\hat{\xi}_{p})}{\partial a}, \frac{\partial F_{0}(\hat{\xi}_{p})}{\partial b}, \frac{\partial F_{0}(\hat{\xi}_{p})}{\partial \beta}\right),$$

and $I(\theta)$ is the Fisher information matrix of the likelihood in (17) which is calculated and given in Appendix and $\hat{\theta}(\omega_1)$ is the MLE of θ for a fixed ω_1 .

We have

$$\frac{\partial F_0(t)}{\partial a} = \frac{1}{2\alpha_0^*} \left(\sqrt{\frac{t}{\beta_0^*}} + \sqrt{\frac{\beta_0^*}{t}} \right) \phi_0(t)$$
$$\frac{\partial F_0(t)}{\partial b} = \frac{1}{273 + S_j} \frac{\partial F_0(t)}{\partial a},$$
$$\frac{\partial F_0(t)}{\partial \beta} = \frac{1}{\beta \alpha_0^*} \sqrt{\frac{\beta_0^*}{t}} \phi_0(t),$$

and

$$f_0(t) = \frac{1}{2\alpha_0^* \sqrt{t\beta_0^*}} (1 + \frac{\beta_0^*}{t})\phi_0(t),$$

where $\phi_0(t) = \phi\left(\frac{1}{\alpha_0^*}\left(\sqrt{\frac{t}{\beta_0^*}} - \sqrt{\frac{\beta_0^*}{t}}\right)\right)$ and ϕ stands for the pdf of the standard normal distribution.

The total cost of the experiment TC(n, f, M) is given by

$$TC(n, f, M) = C_{op} \cdot f \cdot M + C_{mea} \cdot n \cdot M + C_{it} \cdot n$$

where C_{op} is the unit cost of operation per unit of time, C_{mea} is the unit cost of measurement, and C_{it} is the unit cost of items. The optimization criterion is to find the values $n^*, f^*, M^*, \omega_1^*$, which minimize $\operatorname{Avar}(\hat{\xi}_p; \omega_1, n, f, M)$, subject to $TC(n^*, f^*, M^*) \leq C_b$, where C_b is the total budget for conducting the degradation experiment.

Since the parameter spaces $\mathcal{N} = \{(n, f, M) : TC(n, f, M) \leq C_b\}$ and $\Omega = \{\omega_1 : \omega_1 \in (0, D)\}$ are independent we have

$$\inf_{(n,f,M,\omega_1)\in\mathcal{N}\times\Omega}\operatorname{Avar}(\hat{\xi}_p;\omega_1,n,f,M) = \inf_{\omega_1\in\Omega}\inf_{(n,f,M)\in\mathcal{N}}\operatorname{Avar}(\hat{\xi}_p;\omega_1,n,f,M).$$
(19)

Thus, although there is no analytic expression for the solution of the optimization problem of finding $\inf_{(n,f,M)\in\mathcal{N}} \operatorname{Avar}(\hat{\xi}_p;\omega_1,n,f,M)$, due to the integer restriction on the optimality parameters n, f and M, the global and unique minimum of $\operatorname{Avar}(\hat{\xi}_p;\omega_1,n,f,M)$ can be found by searching the minimum through all possible values of n, f and M (see for example Tseng *et al.* [14]), for a fixed value of ω_1 and then applying the common minimization algorithms for minimization of the continuous function $\min_{(n,f,M)}(\operatorname{Avar}(\hat{\xi}_p;\omega_1,n,f,M))$ of ω_1 .

Summing up, the optimal solution of the above optimization problem can be determined by the Algorithm I below.

Algorithm I:

Step 1) Define the function $\varphi(\omega_1)$ with domain (0, D) as follows:

Step 1-1) Compute the largest possible number for n, when f = 1 and M = 2 (one measurement for each stress level), which is equal to

$$n_{max} = \left[\frac{C_b - 2C_{op}}{2C_{mea} + C_{it}}\right].$$

Step 1-2) Set n = 1.

Step 1-3) Compute the largest possible number for f, for the fixed n, and M = 2, which is

$$f_{max} = \left[\frac{C_b - 2C_{mea} \cdot n - nC_{it}}{2C_{op}}\right].$$

Step 1-4) Set f = 1.

- Step 1-5) Let $M = \left[\frac{C_b nC_{it}}{nC_{mea} + fC_{op}}\right]$.
- Step 1-6) Compute Avar $(\hat{\xi}_p; \omega_1, n, f, M)$.

Step 1-7) Set f = f + 1, and repeat steps 1-5 and 1-6 until $f = f_{max}$.

Step 1-8) Set n = n + 1 and repeat steps 1-3 through 1-7 until $n = n_{max}$.

Step 1-9) Let $(n^*(\omega_1), f^*(\omega_1), M^*(\omega_1)) = \arg\min_{(n,f,M)}(\operatorname{Avar}(\hat{\xi}_p; \omega_1, n, f, M)).$

Step 1-10) Return

$$\varphi(\omega_1) = \min_{(n,f,M)} (\operatorname{Avar}(\hat{\xi}_p; \omega_1, n, f, M)).$$

Step 2) Let $\omega_1^* = \arg \inf_{\omega_1} [\varphi(\omega_1)]$ and $(n^*, F^*, M^*) = (n^*(\omega_1^*), f^*(\omega_1^*), M^*(\omega_1^*)).$

It is worth noting that the above optimization algorithm can easily be extended for larger values of m. For example generalization to the case m = 3, can be easily made by replacing $\varphi(\omega_1)$ and (0, D) in Step 1 of Algorithm I with $\varphi(\omega_1, \omega_2)$ and $(0, D)^2$, respectively.

4 Numerical illustration

We consider the data from the carbon-film-resistor problem described by Meeker & Escobar [7], for the purpose of illustration of the proposed procedure. Increasing the resistance value of the carbon-films resistors over time reduces the performance quality of the products and leads to failure. The failure occurs as soon as the percent increase in resistance hits a threshold value D = 5, under the use operating temperature $S_0 = 50^{\circ}$ C. Suppose that the cost of operation, C_{op} is \$1.9 per unit of time, the cost of measurement,

p	$\hat{\xi}_p$	minimum C.V.	ω_1^*	$G_{\tau_{(1),1}}(T^*;\hat{\theta})$
0.1	292795.6	0.1389	0.0507	0.9999950
0.2	307152.4	0.1358	0.0505	0.9999951
0.3	317950.3	0.1335	0.0504	0.9999952
0.4	327481.6	0.1318	0.0503	0.9999953
0.5	336650.3	0.1300	0.0502	0.9999954
0.6	346075.7	0.1283	0.0501	0.9999955
0.7	356450.0	0.1266	0.0500	0.9999956
0.8	368981.1	0.1245	0.0499	0.9999957
0.9	387073.5	0.1218	0.0497	0.9999959

Table 1: Optimal SSADT plan for minimizing $\operatorname{Avar}(\hat{\xi}_p; \omega_1, n, f, M)$ for different values of p, under the budget constraint.



Figure 1: Plot of ω_1^* versus p.



Figure 2: Plot of minimum C.V. as a function of p.

 C_{mea} is \$1.3 for each measurement, the cost of each item, C_{it} is \$53, the unit time is 4 hours and the total budget is \$1500.

For obtaining the optimal design we need some initial values for the parameters. These initial values are usually obtained from a pilot study. The pilot study need not to be even a step stress test. Tseng *et al.* [14] used this data set and obtained an initial estimate of the parameters as follows

 $(\hat{a}, \hat{b}, \hat{\beta}) = (4.11, -4006.46, 0.0594).$

We use the above estimates as the true parameter vector of the SSADT model with gamma degradation process and $S_1 = 83^{\circ}$ C and $S_2 = 133^{\circ}$ C. Algorithm I is used with software R.2.14 for the optimization process. Interestingly, the optimal sampling setting is obtained as

$$(n^*, f^*, M^*) = (13, 52, 7),$$

for all values of p = 0.1(0.1)0.9. Hence, the optimal total time of experiment is equal to $T^* = 4fM = 1456$. The optimal threshold values ω_1^* , are given in Table 1 which also includes the estimated ξ_p , minimum C.V. (the minimum coefficient of variation of $\hat{\xi}_p$ equal to $\sqrt{\varphi(\omega_1^*)}/\hat{\xi}_p$, where φ is defined in Algorithm I) and the probability of stress level elevation before the end of the experiment, that is $G_{\tau_{(1),1}}(T^*;\hat{\theta})$, for p = 0.1(0.1)0.9. For example, for estimating the median time to failure of the product (p = 0.5), the optimal setting is to elevate the stress level as soon as a new measurement of the degradation of

Table 2: Stability analysis of parameter estimation for different values of (n, f, M, ω_1) .

(n, f, M, ω_1)	$\operatorname{Bias}(\hat{a})$	$MSE(\hat{a})$	$\operatorname{Bias}(\hat{b})$	$MSE(\hat{b})$	$\operatorname{Bias}(\hat{\beta})$	$MSE(\hat{\beta})$
(13, 52, 7, 0.0504)	0.125667	0.503172	0.460590	0.212146	-0.001530	9.228e-05
(13, 52, 7, 0.0502)	0.204601	0.912977	0.460766	0.212310	-0.003645	9.973 e-05
(13,62,6,0.0818)	0.153269	0.647985	0.460649	0.212201	-0.002000	0.000117
(13, 62, 6, 0.0912)	0.133570	0.535452	0.460604	0.212159	-0.004489	0.000113
(13, 204, 2, 0.6181)	0.188061	0.595449	0.460837	0.212375	-0.005635	0.000303

at least one of the test units exceeds the threshold value $\omega_1^* = 0.0502$, which occurs with probability 0.9999954.

Figure 1 shows the plot of ω_1^* versus p. The optimal value of ω_1^* decreases as p increases. The reason is that the degradation values under stress level S_2 provide more information about upper quantiles (ξ_p for greater values of p) than those under S_1 .

Also the minimum C.V. is plotted as a function of p in Figure 2. As we can see from Figure 2, the estimation precision increases for upper quantiles. This means that the SSADT plan is more suitable for estimation of the upper quantiles of the product's lifetime.

5 Sensitivity and stability analysis

The optimal design depends on the initial estimates of the parameters. Hence, it is appropriate for the optimization algorithm to be rather robust for departures from the real values of the parameters. In order to check the effect of variation of $\hat{\theta}$ on the optimal setting $(n^*, f^*, M^*, \omega_1^*)$, we consider a pilot run on the SSADT plan for various combinations of $((1 + \epsilon_1)a, (1 + \epsilon_2)b, (1 + \epsilon_3)\beta)$. Table 3 presents some values of ϵ_1 , ϵ_2 and ϵ_3 which cause a change in the optimal setting (n^*, f^*, M^*) and the corresponding ω_1^* , for p = 0.3, 0.5, 0.7.

From the values of Table 3, it is clear that the optimization algorithm is quite robust for small departures from θ , since the changes in decision variables $(\omega_1^*, n^*, f^*, M^*)$ occur for relatively large values of $(\epsilon_1, \epsilon_2, \epsilon_3)$. In fact, the variation of $\hat{\theta}$ affect on the value of ω_1^* , and this leads to a more robust structure for the remaining decision variables (n^*, f^*, M^*) .

To examine the stability of the optimal test plan, we perform a simulation study with $\hat{\theta} = (4.11, -4006.46, 0.0594)$ taken as the true parameter of the SSADT model. A Monte Carlo simulation study with 10,000 iterations is performed and the ML estimate of the θ is obtained for each iteration using the log-likelihood function in (17), for some values of the decision vector (n, f, M, ω_1) . The bias and the mean square error (MSE) of the estimates are obtained and tabulated in Table 2. From the values of Table 2 it can be observed that the estimates of the parameters are quite stable, for small departures from the optimal plan.

				p = 0.3			
ϵ_1	ϵ_2	ϵ_3	min.C.V.	ω_1^*	n^*	f^*	M^*
0	0	0	0.1335	0.0504	13	52	7
48.5%	0	-3.5%	0.0519	0.1517	13	62	6
-7.9%	0	4.9%	0.2959	0.6181	13	204	2
0	1.9%	-1.9%	0.2797	0.6181	13	204	2
0	-6.6%	4.3%	0.0992	0.0818	13	62	6
-1.8%	1.8%	0	0.2887	0.6181	13	204	2
6.5%	-6.5%	0	0.0877	0.0893	13	62	6
2.8%	2.8%	-1.9%	0.2778	0.6181	13	204	2
-3.9%	-1.8%	5.8%	0.1224	0.0744	13	62	6
				p = 0.5			
ϵ_1	ϵ_2	ϵ_3	min.C.V.	ω_1^*	n^*	f^*	M^*
0	0	0	0.1300	0.0502	13	52	7
41.5%	0	-2.5%	0.0519	0.1517	13	62	6
-7.8%	0	4.7%	0.2868	0.6181	13	204	2
0	1.8%	-1.8%	0.2704	0.6181	13	204	2
0	-8.0%	8.0%	0.0903	0.0912	13	62	6
-1.6%	1.6%	0	0.2763	0.6181	13	204	2
7.2%	-7.0%	0	0.08219	0.0923	13	62	6
2.7%	2.7%	-2.2%	0.2690	0.6181	13	204	2
-9.3%	-9.4%	9.3%	0.1012	0.0832	13	62	6
				p = 0.7			
ϵ_1	ϵ_2	ϵ_3	min.C.V.	ω_1^*	n^*	f^*	M^*
0	0	0	0.1266	0.0500	13	52	7
75%	0	-67%	0.0289	0.1090	13	62	6
-31.5%	0	31.5%	0.4516	0.9999	13	204	2
0	0.9%	-7%	0.2500	0.6181	13	204	2
0	-9.9%	9.0%	0.0802	0.1020	13	62	6
-1.5%	1.5%	0	0.2669	0.6181	13	204	2
9.9%	-9.9%	0	0.0660	0.1164	13	62	6
2.7%	2.7%	-2.3%	0.2616	0.8181	13	204	2
-0.12	-0.12	0.12	0.0917	0.0931	13	62	6

Table 3: Sensitivity analysis for p = 0.3, 0.5, 0.7 and different values of ϵ_1, ϵ_2 and ϵ_3 . p = 0.3

6 Conclusion

This paper presents an approach for performing an SSADT experiment in which the stress level is elevated on the basis of the degradation value. In the proposed method, there is no need to observe the value of the degradation continually. The stress level is elevated as soon as a new measurement of the degradation exceeds the threshold value. We consider this threshold value as a decision variable in the optimization problem of the plan as well as the usual variables such as sample size, measurement frequency and number of measurements. An algorithm is proposed for minimizing the approximated variance of the ML estimator of the p^{th} quantile of the lifetime of the products, under the budget constraints. It is observed that the proposed optimization algorithm is quite robust for departures from the values of the parameters. Also the estimates of the parameters are quite stable for small departures from the optimized plan.

Appendix [The Fisher information matrix]

Since, the likelihood function in (17) satisfies the regularity conditions, specially that the parameter space does not depend on the observations, the Fisher information matrix of the SSADT data about the parameter θ is as follows

$$I(\theta) = -\mathbf{E} \begin{bmatrix} \frac{\partial^2}{\partial a^2} l(\theta) & \frac{\partial^2}{\partial a \partial b} l(\theta) & \frac{\partial^2}{\partial a \partial \beta} l(\theta) \\ \frac{\partial^2}{\partial a \partial b} l(\theta) & \frac{\partial^2}{\partial b^2} l(\theta) & \frac{\partial^2}{\partial b \partial \beta} l(\theta) \\ \frac{\partial^2}{\partial a \partial \beta} l(\theta) & \frac{\partial^2}{\partial b \partial \beta} l(\theta) & \frac{\partial^2}{\partial \beta^2} l(\theta) \end{bmatrix},$$

where, $l(\theta)$ is the log-likelihood of θ which, for the case m = 2 of (17), is

$$l(\theta) = \log \mathcal{P}_{\kappa_1}(\kappa_1) + \sum_{i=1}^n \sum_{j=1}^2 \sum_{k=\kappa_{j-1}}^{\kappa_j-1} \left\{ (f\alpha_j - 1) \log g_{ki} - g_{ki}\beta - \log \Gamma(f\alpha_j) - f\alpha_j \log \beta \right\},$$

in which $\kappa_0 = 0$ and $\kappa_2 = M$.

For the sake of brivity we define the following notations:

$$c_1(t) = \frac{1}{\delta_1} \left(\sqrt{\frac{t}{\gamma_1}} - \sqrt{\frac{\gamma_1}{t}} \right),$$
$$c_2(t) = \frac{1}{\delta_1} \left(\sqrt{\frac{t}{\gamma_1}} + \sqrt{\frac{\gamma_1}{t}} \right),$$
$$\Delta[\varphi(kf)] = \varphi(kf) - \varphi((k-1)f)$$

for any function φ , and

$$E_{\kappa} = \sum_{k=1}^{M-1} k \mathbf{P}(\kappa_i = k).$$

Also, let

$$g_1(t) = n[1 - \Phi(c_1(t))]^{n-1}\phi(c_1(t))$$

and

$$g_1'(t) = -n(n-1)[1 - \Phi(c_1(t))]^{n-2}\phi^2(c_1(t)) - nc_1(t)[1 - \Phi(c_1(t))]^{n-1}\phi(c_1(t))$$

denote the differentiation of $G_{\tau_{(1),1}}(t)$ and $g_1(t)$ with respect to $c_1(t)$, respectively. Since we have for j = 1, 2 and $i = 1, \ldots, n$

$$E(\log g_{ki}|\kappa_{i,1}) = \sum_{j=1}^{2} \psi_0(f\alpha_j) I_{\{\kappa_{j-1},\dots,\kappa_j-1\}}(k) I_{1,\dots,M-1}(\kappa_1) + \psi_0(f\alpha_1) I_M(\kappa_1) + \log \beta,$$

we obtain

$$E\left[\frac{-\partial^2}{\partial a^2}l(\theta)\right] = nf^2 \left\{ [\alpha_1^2\psi_1(f\alpha_1) - \alpha_2^2\psi_1(f\alpha_2)] \cdot E_{\kappa} + M\alpha_1^2\psi_1(f\alpha_1)[1 - G_{\tau_{(1),1}}((M-1)f;\theta)] + M\alpha_2^2\psi_1(f\alpha_2)G_{\tau_{(1),1}}((M-1)f;\theta) \right\} + \frac{A}{4},$$

where $\psi_0(t) = \frac{d}{dt} \log \Gamma(t)$ and $\psi_1(t) = \frac{d^2}{dt^2} \log \Gamma(t)$ are the digamma and trigamma functions, respectively, and

$$A = \sum_{k=2}^{M-1} \frac{\left[\Delta [c_2(kf)g_1(kf)]\right]^2}{\Delta [G_{\tau_{(1),1}}(kf;\theta)]} + \frac{[c_2(f)g_1(f)]^2}{G_{\tau_{(1),1}}(f;\theta)} + \frac{[c_2((M-1)f)g_1((M-1)f)]^2}{1 - G_{\tau_{(1),1}}((M-1)f;\theta)} \\ - \sum_{k=2}^{M-1} \left[\Delta [c_1(kf)g_1(kf) + (c_2(kf))^2 g_1'(kf)]\right] - \left[c_1(f)g_1(f) + (c_2(f))^2 g_1'(f)\right] \\ + \left[c_1((M-1)f)g_1((M-1)f) + (c_2((M-1)f))^2 g_1'((M-1)f)\right].$$

Similarly, we have

$$E\left[\frac{-\partial^2}{\partial b^2}l(\theta)\right] = nf^2 \left\{ \left[\left(\frac{\alpha_1}{273 + S_1}\right)^2 \psi_1(f\alpha_1) - \left(\frac{\alpha_2}{273 + S_2}\right)^2 \psi_1(f\alpha_2) \right] \cdot E_{\kappa} + M \left(\frac{\alpha_1}{273 + S_1}\right)^2 \psi_1(f\alpha_1)[1 - G_{\tau_{(1),1}}((M-1)f;\theta)] + M \left(\frac{\alpha_2}{273 + S_2}\right)^2 \psi_1(f\alpha_2)G_{\tau_{(1),1}}((M-1)f;\theta) \right\} + \left(\frac{1}{273 + S_1}\right)^2 \frac{A}{4}$$

and

$$\begin{split} E\left[\frac{-\partial^2}{\partial a\partial b}l(\theta)\right] &= nf^2 \left\{ \left[\left(\frac{\alpha_1^2}{273 + S_1}\right)\psi_1(f\alpha_1) - \left(\frac{\alpha_2^2}{273 + S_2}\right)\psi_1(f\alpha_2)\right].E_{\kappa} \right. \\ &+ M\left(\frac{\alpha_1^2}{273 + S_1}\right)\psi_1(f\alpha_1)[1 - G_{\tau_{(1),1}}((M-1)f;\theta)] \\ &+ M\left(\frac{\alpha_2^2}{273 + S_2}\right)\psi_1(f\alpha_2)G_{\tau_{(1),1}}((M-1)f;\theta) \right\} \\ &+ \left(\frac{1}{273 + S_1}\right)\frac{A}{4}. \end{split}$$

Since

$$E(g_{ki}|\kappa_{i,1}) = \sum_{j=1}^{2} f \alpha_{j} \beta I_{\{\kappa_{j-1},\dots,\kappa_{j}-1\}}(k) I_{1,\dots,M-1}(\kappa_{1}) + f \alpha_{1} \beta I_{M}(\kappa_{1}) + \log \beta,$$

we have similarly

$$E\left[\frac{-\partial^2}{\partial\beta\partial a}l(\theta)\right] = \frac{C\gamma_1}{2\beta} + \frac{nf}{\beta}\left\{ [\alpha_1 - \alpha_2] \cdot E_{\kappa} + M\alpha_1 [1 - G_{\tau_{(1),1}}((M-1)f;\theta)] + M\alpha_2 G_{\tau_{(1),1}}((M-1)f;\theta) \right\},$$

where

$$\begin{split} C &= \sum_{k=2}^{M-1} \frac{\Delta[\sqrt{\frac{\alpha_1}{kf}}g_1(kf)]\Delta[c_2(kf)g_1(kf)]}{\Delta[G_{\tau_{(1),1}}(kf;\theta)]} - \sum_{k=2}^{M} \Delta\left[\sqrt{\frac{\alpha_1}{kf}}g_1(kf) + \gamma_1\frac{\alpha_1}{kf}g_1'(kf)\right] \\ &+ \frac{\sqrt{\frac{\alpha_1}{f}}g_1(f)c_2(f)g_1(f)}{G_{\tau_{(1),1}}(f;\theta)} - \sqrt{\frac{\alpha_1}{f}}g_1(f) - \gamma_1\frac{\alpha_1}{f}g_1'(f) \\ &+ \sqrt{\frac{\alpha_1}{(M-1)f}}g_1((M-1)f)\frac{c_2((M-1)f)g_1((M-1)f)}{1 - G_{\tau_{(1),1}}((M-1)f;\theta)} \\ &+ \sqrt{\frac{\alpha_1}{(M-1)f}}g_1((M-1)f) + \gamma_1\frac{\alpha_1}{(M-1)f}g_1'((M-1)f) \end{split}$$

Also

$$E\left[\frac{-\partial^2}{\partial\beta\partial b}l(\theta)\right] = \frac{1}{273 + S_1}\frac{C\gamma_1}{2\beta} + \frac{nf}{\beta}\left\{ \left[\frac{\alpha_1}{273 + S_1} - \frac{\alpha_2}{273 + S_2}\right] \cdot E_{\kappa} + M\frac{\alpha_1}{273 + S_1}\left[1 - G_{\tau_{(1),1}}((M-1)f;\theta)\right] + M\frac{\alpha_2}{273 + S_2}G_{\tau_{(1),1}}((M-1)f;\theta)\right\}.$$

Finally, we obtain

$$E\left[\frac{-\partial^2}{\partial\beta^2}l(\theta)\right] = \frac{D}{\beta^2} + \frac{nf}{\beta^2}\left\{ [\alpha_1 - \alpha_2] \cdot E_{\kappa} + M\alpha_1 [1 - G_{\tau_{(1),1}}((M-1)f;\theta)] + M\alpha_2 G_{\tau_{(1),1}}((M-1)f;\theta) \right\},$$

where

$$D = \sum_{k=2}^{M-1} \frac{\left[\Delta[\gamma_1 \sqrt{\frac{\alpha_1}{kf}} g_1(kf)]\right]^2}{\Delta[G_{\tau_{(1),1}}(kf;\theta)]} + \frac{\left[\gamma_1 \sqrt{\frac{\alpha_1}{f}} g_1(f)\right]^2}{G_{\tau_{(1),1}}(f;\theta)} + \frac{\left[\gamma_1 \sqrt{\frac{\alpha_1}{(M-1)f}} g_1((M-1)f)\right]^2}{1 - G_{\tau_{(1),1}}((M-1)f;\theta)} \\ + \sum_{k=2}^{M-1} \left[\Delta[2\gamma_1 \sqrt{\frac{\alpha_1}{kf}} g_1(kf) + g_1'(kf)(\gamma_1 \sqrt{\frac{\alpha_1}{kf}})^2]\right] + 2\gamma_1 \sqrt{\frac{\alpha_1}{f}} g_1(f) + g_1'(f)(\gamma_1 \sqrt{\frac{\alpha_1}{f}})^2 \\ - 2\gamma_1 \sqrt{\frac{\alpha_1}{(M-1)f}} g_1((M-1)f) - g_1'((M-1)f)(\gamma_1 \sqrt{\frac{\alpha_1}{(M-1)f}})^2$$

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