*On Characterizing Hypergraph Regularity**

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ABSTRACT: Szemerédi's Regularity Lemma is a well-known and powerful tool in modern graph theory. This result led to a number of interesting applications, particularly in extremal graph theory. A regularity lemma for 3-uniform hypergraphs developed by Frankl and Rödl [8] allows some of the Szemerédi Regularity Lemma graph applications to be extended to hypergraphs. An important development regarding Szemerédi's Lemma showed the equivalence between the property of ϵ-regularity of a bipartite graph G and an easily verifiable property concerning the neighborhoods of its vertices (Alon et al. [1]; cf. [6]). This characterization of ϵ-regularity led to an algorithmic version of Szemerédi's lemma [1]. Similar problems were also considered for hypergraphs. In [2], [9], [13], and [18], various descriptions of quasi-randomness of k-uniform hypergraphs were given. As in [1], the goal of this paper is to find easily verifiable conditions for the hypergraph regularity provided by [8]. The hypergraph regularity of [8] renders quasi-random "blocks of hyperedges" which are very sparse. This situation leads to technical difficulties in its application. Moreover, as we show in this paper, some easily verifiable conditions analogous to those considered in [2] and [18] fail to be true in the setting of [8]. However, we are able to find some necessary and sufficient

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conditions for this hypergraph regularity. These conditions enable us to design an algorithmic version of a hypergraph regularity lemma in [8]. This algorithmic version is presented by the authors in [5]. Random Struct. Alg., 21: 293–335, 2002

1. INTRODUCTION

In 1975, E. Szemerédi proved a beautiful result concerning the coarse structure of every graph [29]. His lemma helped in proving many theorems in extremal graph theory (see, e.g. [20], [25], and [26]). The essential concept central to Szemerédi's Lemma is that of an ϵ-regular pair.

1.1. Graphs and ϵ-Regular Pairs

Let a graph $G = (V, E)$ be given. For two nonempty disjoint sets $X, Y \subseteq V$, we denote by $E(X, Y)$ the set of edges between X and Y (i.e. $E(X, Y) = \{\{x, y\} : x \in X, y \in Y\}$). We set $d(X, Y) = d(G_{XY}) = |E(X, Y)||X|^{-1}|Y|^{-1}$ as the *density* of the bipartite graph $G_{XY} = (X \cup Y, E(X, Y))$. We state the following definition.

Definition 1.1. *Let positive constant ϵ be fixed. We say that the pair X, Y is* ϵ-regular *if $|d(X, Y) - d(X_0, Y_0)| < \epsilon$ holds whenever $X_0 \subseteq X$, $Y_0 \subseteq Y$ satisfy $|X_0| \geq \epsilon|X|$, $|Y_0| \geq \epsilon|Y|$.*

With $G = (V, E)$ fixed, we call a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ *equitable* if it satisfies $|V_1| = |V_2| = \cdots = |V_t|$ and $|V_0| < t$; we call an equitable partition *ϵ-regular* if all but $\epsilon\binom{t}{2}$ pairs V_i, V_j are ϵ-regular. Szemerédi's Regularity Lemma is stated precisely as follows.

Theorem 1.2 (Szemerédi's Regularity Lemma [29]). *Let $\epsilon > 0$ be given and let k_0 be a positive integer. There exist positive integers $N = N(\epsilon, k_0)$ and $K = K(\epsilon, k_0)$ so that any graph $G = (V, E)$ with $|V| = n \geq N$ vertices admits an ϵ-regular equitable partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ with k satisfying $k_0 \leq k \leq K$.*

For related topics, see [19].

1.1.1. A Local Characterization of an ϵ-Regular Pair.

In all that follows, we consider a fixed bipartite graph G with bipartition $X \cup Y$. For fixed positive constants α and ϵ, we assume $d(X, Y) \sim_\epsilon \alpha$, where by $a \sim_\gamma b$, we mean $(1 + \gamma)^{-1} \leq a/b \leq 1 + \gamma$. We denote by $\deg_G(x)$ the number of vertices that are neighbors of x in the graph G, and by $\deg_G(x_1, x_2)$ the number of vertices that are neighbors of both x_1 and x_2 in G.

The property of ϵ-regularity of G is a "global" property in the sense that it asserts a fact about every pair of reasonably large subsets of its vertex classes X and Y. An important development regarding Szemerédi's Lemma showed the equivalence between this global regularity property of G and a fairly simple "local" property concerning the neighborhoods of the vertices in X. Given positive reals α, ϵ and ϵ', consider the following two properties:

$$\mathbf{G_1} = \mathbf{G_1}(\epsilon), \qquad G \text{ is } \epsilon\text{-regular with density } d(X, Y) \sim_\epsilon \alpha.$$

$$\mathbf{G_2} = \mathbf{G_2}(\epsilon'):$$

(i) $\deg_G(x) \sim_{\epsilon'} \alpha|Y|$ for all but $\epsilon'|X|$ vertices $x \in X$,
(ii) $\deg_G(x_1, x_2) \sim_{\epsilon'} \alpha^2|Y|$ for all but $\epsilon'|X|^2$ pairs $x_1, x_2 \in X$.

It was shown in [1] (cf. [6]) that properties $\mathbf{G_1}$ and $\mathbf{G_2}$ are equivalent in the following sense.

Theorem 1.3 (Alon, Duke, Lefmann, Rödl, and Yuster [1]). *For any $\epsilon' > 0$ there exists $\epsilon > 0$ such that*

$$\mathbf{G_1}(\epsilon) \Rightarrow \mathbf{G_2}(\epsilon').$$

Similarly, for any $\epsilon > 0$, there exists $\epsilon' > 0$ such that

$$\mathbf{G_2}(\epsilon') \Rightarrow \mathbf{G_1}(\epsilon).$$

The equivalence of Properties $\mathbf{G_1}$ and $\mathbf{G_2}$ tells us that the notion of ϵ-regularity is equivalent to a condition concerning uniformity of degrees and codegrees. Since degrees and codegrees concern only vertices and pairs of vertices, and not large subsets as in the definition of ϵ-regularity, Property $\mathbf{G_2}$ is a "local" criterion for the regularity of graphs. For related topics, see [10]. For a problem with related concepts but a different flavor, see [7].

1.1.2. An Algorithmic Version of Theorem 1.2. The original proof of Theorem 1.2 was nonconstructive. Theorem 1.3 played the crucial role in developing the algorithmic version of Szemerédi's Regularity Lemma [1]. We state the algorithmic version of Theorem 1.2 precisely.

Theorem 1.4 (Constructive Regularity Lemma, [1]). *For every $\epsilon > 0$ and every positive integer k, there exists an integer $Q = Q(\epsilon, k)$ such that every graph G with $n > Q$ vertices admits an ϵ-regular partition into $t + 1$ classes for some $k < t < Q$ and such a partition can be found in $O(M(n))$ sequential time, where $M(n)$ denotes the time needed for the multiplication of two $(0, 1)$ matrices of size n.*

For related topics, see [3], [11], and [12]. For an additional problem using Szemerédi's Regularity Lemma, see [25] and [26].

1.2. Hypergraphs and (δ, r)-Regular Triads

One of the main reasons for the wide applicability of Szemerédi's Regularity Lemma is that it enables one to find small subgraphs in "regular situations." In [8], Frankl and Rödl developed a regularity lemma for 3-uniform hypergraphs which admits the analogous result that one can find small subsystems in "regular situations" (see [14], [17], [22–24], [27], and [28]). We give a precise presentation of their hypergraph regularity lemma in the Appendix. At this time, we focus on outlining the goals of this paper.

Just as the essential concept of Szemerédi's Lemma is that of an ϵ-regular pair, the essential concept in the Frank-Rödl Regularity Lemma is that of a (δ, r)-*regular triad*. The definition of a (δ, r)-regular triad is, unfortunately, rather technical. We therefore defer precise discussion of it until the section to follow. (The need for this more technical concept, and additional parameter r describing the regularity, arises from the fact that these regular triads obtained by the hypergraph regularity lemma of [8] are very sparse hypergraphs).

In [2] and later in [18], equivalent conditions describing quasirandomness of hypergraphs were considered. These conditions were similar to $\mathbf{G_1}$ and $\mathbf{G_2}$ for graphs and corresponded to a special case of $(\delta, 1)$-regularity. To cover the full case of (δ, r)-regularity, we consider analogous conditions to those studied in [2] and [18]. For general r, we find, quite surprisingly, that these analogous conditions are no longer equivalent to the concept of (δ, r)-regularity. However, we are able to develop some implications among these conditions, and these implications are used in [5] to develop an algorithmic version of a special case of the hypergraph regularity lemma of [8]. We discuss these implications as well as some open questions in detail in Section 3.

In the section to follow, we give definitions of necessary concepts. In Section 3, we give a precise account of our theorems as well as some open questions. The remaining sections contain details of proofs.

2. DEFINITIONS AND NOTATION

In this section, we give background definitions and notation that we use in this paper.

2.1. Graph Concepts

We begin with the following definitions.

Definition 2.1. *We say that the bipartite graph G is (α, ϵ)-*regular *if*

$$\alpha(1 - \epsilon) < d_G(X_0, Y_0) < \alpha(1 + \epsilon)$$

for every pair of subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $|X_0| > \epsilon|X|$ and $|Y_0| > \epsilon|Y|$. Here the density $d_G(X_0, Y_0)$ is as defined in the Introduction.

Definition 2.2. *A 3-partite graph G with a fixed 3-partition (V_1, V_2, V_3) is referred to as a* 3-partite cylinder. *We write $G = \cup_{1 \le i < j \le 3} G^{ij}$, where $G^{ij} = G[V_i, V_j] = \{\{v_i, v_j\} \in G : v_i \in V_i, v_j \in V_j\}$. Let $\ell > 0$, $\epsilon > 0$ be given. We call G an $(\ell, \epsilon, 3)$-*cylinder *if each bipartite graph G^{ij}, $1 < i < j \le 3$, is $(1/\ell, \epsilon)$-regular. We also sometimes refer to an $(\ell, \epsilon, 3)$-cylinder as a* triad.

We frequently use the following notation. For a graph G, let $\mathcal{K}_3(G) = \{\{x, y, z\} : \{x, y, z\}$ is the vertex set of a triangle in $G\}$. While we will state it precisely in Section 4 (cf. Fact 4.1), note that when G is an $(\ell, \epsilon, 3)$-cylinder with 3-partition satisfying $|V_1| = |V_2| = |V_3| = n$, $|\mathcal{K}_3(G)|$ is about n^3/ℓ^3.

2.2. Hypergraph Concepts

We begin with the following definitions.

Definition 2.3. *We refer to any 3-partite, 3-uniform hypergraph $\mathcal{H}$ with a fixed 3-par-*

tition (V_1, V_2, V_3) *as a* 3-partite 3-cylinder. *If* G *is a* 3*-partite cylinder with the same vertex partition, then we say* G underlies $\mathcal{H}$ *if* $\mathcal{H} \subseteq \mathcal{K}_3(G)$.

Definition 2.4. *Let* $\mathcal{H}$ *be a* 3*-partite* 3*-cylinder with underlying* 3*-partite cylinder* $G = G^{12} \cup G^{23} \cup G^{13}$. *Let* $\vec{Q} = (Q(1), \ldots, Q(r))$ *be an r-tuple of* 3*-partite cylinders* $Q(s) = Q^{12}(s) \cup Q^{23}(s) \cup Q^{13}(s)$ *satisfying that for every* $s \in \{1, 2, \ldots, r\}$, *for each* $\{i, j\}$, $1 \leq i < j \leq 3$, $Q^{ij}(s) \subseteq G^{ij}$. *We define the* density $d_{\mathcal{H}}(\vec{Q})$ *of* $\vec{Q}$ *as*

$$d_{\mathcal{H}}(\vec{Q}) = \begin{cases} \dfrac{\left|\mathcal{H} \cap \cup_{s=1}^{r} \mathcal{K}_3(Q(s))\right|}{\left|\cup_{s=1}^{r} \mathcal{K}_3(Q(s))\right|} & \textit{if } \left|\cup_{s=1}^{r} \mathcal{K}_3(Q(s))\right| > 0, \\ 0 & \textit{otherwise}. \end{cases} \tag{1}$$

Definition 2.5. *Let a positive integer r and a real* $\delta > 0$ *be given. We say that a* 3*-cylinder* $\mathcal{H}$ *is* (α, δ, r)-regular with respect to G *if for any r-tuple of* 3*-partite cylinders* $\vec{Q} = (Q(1), \ldots, Q(r))$ *as above, if*

$$\left|\bigcup_{s=1}^{r} \mathcal{K}_3(Q(s))\right| > \delta\left|\mathcal{K}_3(G)\right|, \tag{2}$$

then

$$\left|d_{\mathcal{H}}(\vec{Q}) - \alpha\right| < \delta. \tag{3}$$

We say $\mathcal{H}$ *is* (δ, r)-regular with respect to G *if it is* (α, δ, r)*-regular for some* α. *If the regularity condition fails to be satisfied for any* α, *we say that* $\mathcal{H}$ *is* (δ, r)-irregular with respect to G.

2.3. Links, Colinks and More Graph Regularity

We make precise some final concepts and notation.

Definition 2.6. *Let* $\mathcal{H}$ *be a* 3*-partite* 3*-cylinder with underlying* 3*-partite cylinder* $G = G^{12} \cup G^{23} \cup G^{13}$ *on* 3*-partition* V_1, V_2, V_3. *Let* $x \in V_1$. *We define the* link graph L_x^{23} *of x to be the subgraph of* G^{23} *with vertex set* $N_{G^{12}}(x) \cup N_{G^{13}}(x)$ *and edge set*

$$L_x^{23} = \{\{y, z\} \in G^{23} : \{x, y, z\} \in \mathcal{H}\}. \tag{4}$$

For two vertices $x, y \in V_1$, *we define the* colink graph L_{xy}^{23} *of x and y to be the subgraph of* G^{23} *with vertex set* $N_{G^{12}}(x, y) \cup N_{G^{13}}(x, y)$ *and edge set*

$$L_{xy}^{23} = L_x^{23} \cap L_y^{23}. \tag{5}$$

We now define a concept of (δ, r)-graph-regularity.

Definition 2.7. *Let* γ, δ *be positive reals, let r be a positive integer, and let L be a* *bipartite graph with bipartition* (U, V). *We say that* L *is* (γ, δ, r)-regular *if for any* r*-tuple of pairs of subsets* $(\{U_j, V_j\})_{j=1}^r$, $U_j \subseteq U$, $V_j \subseteq V$, $1 \leq j \leq r$, *satisfying*

$$\left| \bigcup_{j=1}^{r} (U_j \times V_j) \right| > \delta |U||V|, \tag{6}$$

we have

$$\gamma(1 - \delta) < \frac{\left|L \cap \bigcup_{j=1}^{r} (U_j \times V_j)\right|}{\left|\bigcup_{j=1}^{r} (U_j \times V_j)\right|} < \gamma(1 + \delta). \tag{7}$$

Note that $(\gamma, \delta, 1)$-regularity of a bipartite graph is essentially the same concept as (γ, δ)-regularity (see Definition 2.1).

3. MAIN RESULTS

In this section, we present the main results of this paper.

3.1. The Local-Global Conditions and Their Implications

We begin by considering the setup we use in this paper.

Setup. *Let real number* $\epsilon > 0$ *and positive integers* ℓ *and* n *be given, where we always assume* $n > n_0(\ell, \epsilon)$. *Suppose*

(i) $\mathcal{H}$ *is a 3-partite 3-cylinder with 3-partition* $\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3$, *where* $|\mathrm{V}_1| = |\mathrm{V}_2| = |\mathrm{V}_3| = \mathrm{n}$,
(ii) G *is an underlying* $(\ell, \epsilon, 3)$*-cylinder of* $\mathcal{H}$.

We consider the following two properties for a hypergraph $\mathcal{H}$ and graph G as in the Setup. In what follows, $\mathbf{H_1}$ is the "global condition" similar to $\mathbf{G_1}$ (as in the Introduction) and $\mathbf{H_2}$ is the "local condition" similar to $\mathbf{G_2}$. Let α, δ_A, $\delta_B > 0$ be given and let r_A, r_B be given positive integers.

$\mathbf{H_1} = \mathbf{H_1}(\delta_A, r_A)$, $\mathcal{H}$ is (α, δ_A, r_A)-regular with respect to G.

$\mathbf{H_2} = \mathbf{H_2}(\delta_B, r_B)$:

(i) the link graph L_x^{23} is $(\alpha/\ell, \delta_B, r_B)$-regular for all but at most $\delta_B n$ vertices $x \in V_1$,
(ii) the colink graph L_{xy}^{23} is $(\alpha^2/\ell, \delta_B, r_B)$-regular for all but at most $\delta_B n^2$ pairs $x, y \in V_1$.

Remark 3.1. For the special case $\ell = r_A = r_B = 1$, conditions $\mathbf{H_1}$ and $\mathbf{H_2}$ were proved to be equivalent in [2] (for $\alpha = 1/2$) and in [18] (for arbitrary α). Schematically, this equivalence may be stated as follows:

1. $\forall\alpha$, $\forall\delta_B$, $\exists\delta_A$: for $\ell = r_A = r_B = 1$,

$$\mathbf{H_1}(\delta_A, 1) \Rightarrow \mathbf{H_2}(\delta_B, 1). \tag{8}$$

2. $\forall\alpha$, $\forall\delta_A$, $\exists\delta_B$: for $\ell = r_A = r_B = 1$,

$$\mathbf{H_2}(\delta_B, 1) \Rightarrow \mathbf{H_1}(\delta_A, 1). \tag{9}$$

Observe that in the case $\ell = 1$, the underlying graph G is the complete 3-partite graph $K(V_1, V_2, V_3)$ (which, for any $\epsilon > 0$, is a $(1, \epsilon, 3)$-cylinder).

The object of this paper is to decide how (8) and (9) extend to the general setting of Property $\mathbf{H_1}(\delta_A, r_A)$.

For arbitrary ℓ and r_B, the following lemma was given in [8] and [22].

Lemma 3.2 (Regularity of Links). *For all positive α, δ_B, there exists δ_A (viz. $\delta_A = \delta_B^2/9$) so that for all positive integers ℓ and r_B, setting $r_A = r_B$, there exists $\epsilon > 0$ so that, in the context of the Setup, $\mathbf{H_1}(\delta_A, r_B)$ implies statement (i) of $\mathbf{H_2}(\delta_B, r_B)$.*

In Section 5, we prove the following accompaniment to Lemma 3.2.

Lemma 3.3 (Regularity of Colinks). *For all positive α and δ_B there exists δ_A such that for all positive integers ℓ and r_B, there exist positive integer r_A and $\epsilon > 0$ so that, in the context of the Setup, $\mathbf{H_1}(\delta_A, r_A)$ implies statement (ii) of $\mathbf{H_2}(\delta_B, r_B)$.*

Note that in Lemma 3.3, unlike Lemma 3.2, we have that $r_A > r_B$ (cf. Question 3.7).

Lemmas 3.2 and 3.3 immediately lead to the first main theorem of this paper.

Theorem 3.4. *For all positive α and δ_B there exists δ_A such that for all positive integers ℓ and r_B, there exist positive integer r_A and $\epsilon > 0$ so that, in the context of the Setup, $\mathbf{H_1}(\delta_A, r_A)$ implies $\mathbf{H_2}(\delta_B, r_B)$.*

Partly due to the equivalence discussed in Remark 3.1, the authors initially thought that Theorem 3.4 would be reversible. Schematically, observe that Theorem 3.4 says $\forall\alpha$, $\forall\delta_B$, $\exists\delta_A$: $\forall\ell$, $\forall r_B$, $\exists r_A$, $\exists\epsilon$:

$$\mathbf{H_1}(\delta_A, r_A) \Rightarrow \mathbf{H_2}(\delta_B, r_B).$$

It seemed likely that $\forall\alpha$, $\forall\delta_A$, $\exists\delta_B$: $\forall\ell$, $\forall r_A$, $\exists r_B$, $\exists\epsilon$: $\mathbf{H_2}(\delta_B, r_B) \Rightarrow \mathbf{H_1}(\delta_A, r_A)$. We show in this paper that, surprisingly, this is indeed *not* the case. Schematically, we show $\exists\alpha$, $\exists\delta_A$: $\forall\delta_B$: $\exists\ell$, $\exists r_A$, $\forall r_B$, $\forall\epsilon$,

$$\mathbf{H_2}(\delta_B, r_B) \not\Rightarrow \mathbf{H_1}(\delta_A, r_A).$$

In other words, $\mathbf{H_1}$ and $\mathbf{H_2}$ are not, in general, equivalent. A (technical) probabilistic construction is given in Section 6 which proves the following theorem, the second main result of this paper.

Theorem 3.5. *There exists $\alpha > 0$ and $\delta_A > 0$ so that for all $\delta_B > 0$, there exist integers ℓ and r_A so that for all integers r_B, for all $\epsilon > 0$, in the context of the Setup,* $\mathbf{H_2}(\delta_B, r_B) \not\Rightarrow \mathbf{H_1}(\delta_A, r_A)$. *In other words, with these constants, there exist $\mathcal{H}$ and G as in the Setup which satisfy* $\mathbf{H_2}(\delta_B, r_B)$ *but not* $\mathbf{H_1}(\delta_A, r_A)$.

We mention that Theorems 3.4 and 3.5 are the two main results of this paper. In what follows, however, we will include discussion of some related results from [5].

Despite the inequivalence discussed above of properties $\mathbf{H_1}$ and $\mathbf{H_2}$, the following theorem, proved in [5], gives at least a partial context in which property $\mathbf{H_2}$ may ensure property $\mathbf{H_1}$.

Theorem 3.6 [5]. *For all $\alpha > 0$ and $\delta_B > 0$ there exists $\delta_A > 0$ such that for every integer ℓ there exists $\epsilon > 0$ such that, in the context of the Setup,* $\mathbf{H_2}(\delta_B, 1)$ *implies* $\mathbf{H_1}(\delta_A, 1)$.

In [5], Lemmas 3.2 and 3.3, together with Theorem 3.6, are used to design an algorithmic version of a special case of the hypergraph regularity lemma in [8].

3.2. Open Problems and Summary of Results

We emphasize the following open problems associated with the conditions $\mathbf{H_1}$ and $\mathbf{H_2}$ above.

Question 3.7. *Is it true that for any α, $\delta_B > 0$, there exists $\delta_A > 0$ so that for any integers ℓ and r_B, setting $r = r_A = r_B$, there exists $\epsilon > 0$ so that, in the context of the Setup,* $\mathbf{H_1}(\delta_A, r) \Rightarrow \mathbf{H_2}(\delta_B, r)$?

Observe from Lemma 3.2, to answer Question 3.7, it suffices to answer whether $\mathbf{H_1}(\delta_A, r)$ implies statement (*ii*) of $\mathbf{H_2}(\delta_B, r)$. Recall from Remark 3.1, Question 3.7 has an affirmative answer when $\ell = r = r_A = r_B = 1$.

As a special case of Question 3.7, the authors believe the following is true.

Conjecture 3.8. *For any α, $\delta_B > 0$, there exists $\delta_A > 0$ so that for any integer ℓ, for $r_B = 1$, setting $r_A = 1$, there exists $\epsilon > 0$ so that, in the context of the Setup,* $\mathbf{H_1}(\delta_A, 1) \Rightarrow \mathbf{H_2}(\delta_B, 1)$.

Observe that Conjecture 3.8, if true, together with Theorem 3.6, would provide an equivalence between properties $\mathbf{H_1}$ and $\mathbf{H_2}$ in the case that $r_A = r_B = 1$. In other words, a validation of Conjecture 3.8, together with Theorem 3.6, would provide a characterization of $(\delta, 1)$-regularity in terms of purely local conditions.

We conclude this subsection with a summary of the theorems and questions concerning implications among $\mathbf{H_1}$ and $\mathbf{H_2}$.

Summary 3.9 (Summary for $\mathbf{H_1}$ and $\mathbf{H_2}$). *When $r = 1$, we have the following statements.*

1. $\forall\alpha$, $\forall\delta_A$, $\exists\delta_B$: $\forall\ell$, for $r_A = r_B = 1$, $\exists\epsilon$:

$$\mathbf{H_2}(\delta_B, 1) \Rightarrow \mathbf{H_1}(\delta_A, 1) \qquad \textit{Theorem } 3.6.$$

2. $\forall\alpha$, $\forall\delta_{\mathrm{B}}$, $\exists\delta_{\mathrm{A}}$: $\forall\ell$, for $\mathrm{r_A} = \mathrm{r_B} = 1$, $\exists\epsilon$:

$$\mathbf{H_1}(\delta_A, 1) \overset{?}{\Rightarrow} \mathbf{H_2}(\delta_B, 1) \qquad \textit{Conjecture } 3.8.$$

For general r, we have the following statements.

3. $\forall\alpha$, $\forall\delta_{\mathrm{B}}$, $\exists\delta_{\mathrm{A}}$: $\forall\ell$, $\forall\mathrm{r_B}$, $\exists\mathrm{r_A}$, $\exists\epsilon$:

$$\mathbf{H_1}(\delta_A, r_A) \Rightarrow \mathbf{H_2}(\delta_B, r_B) \qquad \textit{Theorem } 3.4.$$

4. $\exists\alpha$, $\exists\delta_{\mathrm{A}}$, $\forall\delta_{\mathrm{B}}$: $\exists\ell$, $\exists\mathrm{r_A}$, $\forall\mathrm{r_B}$, $\forall\epsilon$,

$$\mathbf{H_2}(\delta_B, r_B) \not\Rightarrow \mathbf{H_1}(\delta_A, r_A) \qquad \textit{Theorem } 3.5.$$

5. $\forall\alpha$, $\forall\delta_{\mathrm{B}}$, $\exists\delta_{\mathrm{A}}$: $\forall\ell$, $\forall\mathrm{r_B}$, setting $\mathrm{r} = \mathrm{r_A} = \mathrm{r_B}$, $\exists\epsilon$:

$$\mathbf{H_1}(\delta_A, r) \overset{?}{\Rightarrow} \mathbf{H_2}(\delta_B, r) \qquad \textit{Question } 3.7.$$

In Section 5, we prove Lemma 3.3. In Section 4, we supply some additional facts and definitions we need in our proof of Lemma 3.3. In Section 6, we prove Theorem 3.5.

3.3. An Algorithmic Hypergraph Regularity Lemma

As mentioned in the Introduction, the equivalence $\mathbf{G_1} \Leftrightarrow \mathbf{G_2}$ was crucial in developing the algorithmic version of Szemerédi's Regularity Lemma [1]. Despite the inequivalence of properties $\mathbf{H_1}$ and $\mathbf{H_2}$, in [5], we use Lemmas 3.2 and 3.3, together with Theorem 3.6, to prove an algorithmic version of a special case of Frankl and Rödl's Hypergraph Regularity Lemma. A precise formulation of this hypergraph regularity lemma is given in the Appendix. The following theorem is proved in [5].

Theorem 3.10. *For every δ and γ with $0 < \gamma \le 2\delta^4$, for all integers t_0 and ℓ_0 and all functions $\epsilon(\ell) > 0$, there exist T_0, L_0, and N_0 such that any 3-uniform hypergraph $\mathcal{H} \subseteq [N]^3$, $N \ge N_0$, admits a $(\delta, 1)$-regular, $(\ell, t, \gamma, \epsilon(\ell))$-partition for some t and ℓ satisfying $t_0 \le t \le T_0$ and $\ell_0 \le \ell \le L_0$. Moreover, such a partition can be found in time polynomial in N.*

4. SUPPLEMENTAL FACTS AND CONCEPTS

We begin this section with facts which pertain to graphs.

4.1. Some Graph Facts

The following fact gives precise estimates on $|\mathcal{K}_3(G)|$ when G is an $(\ell, \epsilon, 3)$-cylinder.

Fact 4.1. *For any positive integer ℓ and positive real θ, there exists ϵ so that whenever G is an $(\ell, \epsilon, 3)$-cylinder with vertex partition (V_1, V_2, V_3), where $|V_1| = |V_2| = |V_3| = n$, then*

$$(1-\theta)\frac{n^3}{\ell^3} < |\mathcal{K}_3(G)| < (1+\theta)\frac{n^3}{\ell^3}.$$

Fact 4.1 is easy to prove using Definition 2.2.

The following fact is a slight variation of a fact in [8]. Let positive reals ϵ, $\sigma \le 1$ and integers ℓ, M be given. Let G be a 3-partite graph satisfying the following properties:

(a) G has 3-partition $V(G) = W_0 \cup W_1 \cup W_2$, where $|W_0| = \sigma M$ and $|W_1| = |W_2| = M$.
(b) Both bipartite graphs G^i induced by $W_0 \cup W_i$, $i = 1, 2$, have the property that the density of any subgraph induced on $W_0' \cup W_i'$, $W_0' \subseteq W_0$, $W_i' \subseteq W_i$, $|W_0'| \ge \epsilon M$, $|W_i'| \ge \epsilon M$, is between $(1/\ell)(1-\epsilon)$ and $(1/\ell)(1+\epsilon)$.
(c) for each $x \in W_0$ and for each $i = 1, 2$, we have $(1/\ell)(1-\epsilon)M \le |N_{G^i}(x)| \le (1/\ell)(1+\epsilon)M$.

The following fact holds for graphs with these properties.

Fact 4.2. *For all $\sigma > 0$, for all integers ℓ, for all $0 < \epsilon < \frac{1}{2\ell}$, there exist $M_0 = M_0(\sigma, \epsilon, \ell)$, so that whenever G is a graph satisfying properties (a), (b), and (c) above with constants σ, ϵ, ℓ, and $M \ge M_0$, then there are at least $b \ge \sigma/4\epsilon$ vertices $x_1, x_2, \ldots, x_b \in W_0$ such that*

$$\frac{M}{\ell^2}(1-\epsilon)^2 \le |N_{G^i}(x_u, x_v)| \le \frac{M}{\ell^2}(1+\epsilon)^2 \tag{10}$$

for all $\binom{b}{2}$ pairs u, v, $1 \le u < v \le b$, and for $i = 1, 2$.

Remark 4.3. We mention the following algorithmic version of Fact 4.2. Under the hypothesis of Fact 4.2, one may construct the promised set $x_1, \ldots, x_b \in W_0$ satisfying (10) for $i = 1, 2$ in time $O(M^3)$.

Indeed, under the hypothesis above, define graph Γ to be the set of all pairs $\{x, y\}$ from W_0 satisfying that, for some $i = 1, 2$, $|N_{G^i}(x, y)|$ does not satisfy (10). Observe that we may construct Γ in time $O(M^3)$. It is easy to see that the maximum degree in Γ, $\Delta(\Gamma)$, satisfies

$$\Delta(\Gamma) < 4\epsilon M.$$

Indeed, fix vertex $x \in W_0$. For $i = 1, 2$, by property (c),

$$|N_{G^i}(x)| \ge \frac{M}{\ell}(1-\epsilon) \ge \frac{M}{2\ell} > \epsilon M.$$

Using property (b), the inequalities above imply that for each $i = 1, 2$, less than $2\epsilon M$ vertices $y \in W_0$ fail to satisfy

$$\frac{1}{\ell}(1-\epsilon)|N_{G^i}(x)| \le |N_{G^i}(y) \cap N_{G^i}(x)| \le \frac{1}{\ell}(1+\epsilon)|N_{G^i}(x)|. \tag{11}$$

Using property (c) to bound $|N_{G^i}(x)|$, over both $i = 1, 2$, we see from (11) that less than $4\epsilon M$ vertices $y \in W_0$ are neighbors of x in Γ.

Observe that the promised set $x_1, \ldots, x_b \in W_0$ guaranteed by Fact 4.2 is just an independent set in Γ.

For a given graph F having maximum degree Δ, it is easy to see that one may construct in time $O(|V(F)|^2)$ an independent set in F of size $|V(F)|/(\Delta + 1)$. Indeed, applying the greedy coloring algorithm to F yields a decomposition of $V(F)$ into at most $\Delta + 1$ independent sets. Consequently, one such set must be at least as large as promised.

Applying these observations to given Γ, we see that in time $O(M^2)$ we may construct an independent set of size

$$\frac{|W_0|}{\Delta(\Gamma) + 1} \geq \frac{\sigma M}{4\epsilon M} \geq \frac{\sigma}{4\epsilon}.$$

Since we may construct the graph Γ in time $O(M^3)$, we conclude the promised set $x_1, \ldots, x_b \in W_0$ guaranteed by Fact 4.2 may be found in time $O(M^3)$.

4.2. Some Hypergraph Comments

We begin with the following definition.

Definition 4.4. *Let $\alpha, \gamma, \epsilon > 0$ and let r, ℓ, n be given positive integers. Let $\mathcal{H}$ and G be as in the Setup. Define a vertex $x \in V_1$ to be a* good vertex *if it satisfies the following two properties*:

(a) $\frac{n}{\ell}(1 - \epsilon) \leq |N_{G^{12}}(x)| \leq \frac{n}{\ell}(1 + \epsilon)$ *and* $\frac{n}{\ell}(1 - \epsilon) \leq |N_{G^{13}}(x)| \leq \frac{n}{\ell}(1 + \epsilon)$,
(b) L_x^{23} *is* $(\alpha/\ell, \gamma, r)$*-regular.*

Set $V_1^{\mathrm{good}}(\alpha, \gamma, \ell, r, \epsilon, n)$ *to be the set of good vertices.*

With parameters $\alpha, \delta, \ell, r, \epsilon$ and n, suppose $\mathcal{H}$ and G as in the Setup satisfy $\mathbf{H_1}(\delta, r)$. We may conclude from Lemma 3.2 that for $\epsilon = \epsilon(\alpha, \delta, r, \ell)$ sufficiently small, the set of good vertices $V_1^{\mathrm{good}}(\alpha, 3\sqrt{\delta}, \ell, r, \epsilon, n)$ satisfies $|V_1^{\mathrm{good}}(\alpha, 3\sqrt{\delta}, \ell, r, \epsilon, n)| > (1 - 3\sqrt{\delta})n$. We use the notation V_1^{good} for $V_1^{\mathrm{good}}(\alpha, 3\sqrt{\delta}, \ell, r, \epsilon, n)$.

One crucial step in the proof of Lemma 3.3 will be to establish the regularity of *mixed links*. For two vertices $x, y \in V_1$, we define the *mixed link graph* $M_{(x,y)}^{23}$ of x and y to be the subgraph of L_x^{23} induced by the vertex set $N_{G^{12}}(y) \cup N_{G^{13}}(y)$. Note that, unlike L_{xy}^{23}, the mixed link graph $M_{(x,y)}^{23}$ is not the same as $M_{(y,x)}^{23}$, and we emphasize this using the ordered pair subscript notation. The proof of the following lemma can be found in [4].

Lemma 4.5 (Regularity of Mixed Links). *For all positive reals α, γ, and for all positive integers ℓ and r, there exists $\epsilon > 0$ so that whenever $\mathcal{H}$ and G are as in the Setup, for each good vertex $x \in V_1^{\mathrm{good}}(\alpha, \gamma, \ell, r, \epsilon, n)$, all but at most $\sqrt{\epsilon}n$ vertices $y \in V_1^{\mathrm{good}}$ satisfy that $M_{(x,y)}^{23}$ is $(\alpha/\ell, \gamma^{1/3}, r)$-regular.*

We make the following comment.

Remark 4.6. We make the following comment concerning both Definitions 2.5 and 2.7. For a hypergraph $\mathcal{H}$ (bipartite graph L), note that, to show the (α, δ, r)-regularity of $\mathcal{H}$ ((γ, δ, r)-regularity of L), one must show that any appropriately given r-tuple of 3-partite cylinders $\vec{Q}$ (pairs of subsets $(U_i, V_i)_{i=1}^r$) satisfies (3) ((7)). In both (3) and (7), one must show the corresponding density satisfies both a lower and an upper bound. In this paper, when we assert a hypergraph $\mathcal{H}$ (bipartite graph L) satisfies Definition 2.5 (Definition 2.7), we often only verify that the lower bound of (3) ((7)) is satisfied. The corresponding proof that the upper bound is also satisfied is entirely symmetric to the lower bound confirmation. We therefore often omit these details.

We state the following definition related to Definitions 2.5 and 2.7.

Definition 4.7. *If hypergraph $\mathcal{H}$ is not (α, δ, r)-regular with respect to graph G, then any r-tuple $\vec{Q} = (Q(1), \ldots, Q(r))$ satisfying* (2) *but failing* (3) *is said to be a* witness of the (α, δ, r)-irregularity *of $\mathcal{H}$ with respect to G. Similarly, if bipartite graph L is not (γ, δ, r)-regular, then any r-tuple $(\{U_j, V_j\})_{j=1}^r$ satisfying* (6) *but failing* (7) *is said to be a* witness of the (γ, δ, r)-irregularity *of L.*

5. REGULARITY OF COLINKS

We prove the following version of Lemma 3.3.

Lemma 5.1. *For all positive α, δ', β there exists δ such that for all positive integers ℓ and r', there exist positive integer r, real $\epsilon > 0$ and n_0 so that with these constants, whenever $\mathcal{H}$ and G as in the Setup satisfy property $\mathbf{H_1}(\delta, r)$, then for all but $\sqrt{\epsilon}n$ vertices $x \in V_1^{\text{good}}$ there are at most βn vertices $y \in V_1^{\text{good}}$ such that L_{xy}^{23} is not $(\alpha^2/\ell, \delta', r')$-regular.*

In what follows the constants we define always satisfy the following hierarchy:

$$\alpha,\ \delta',\ \beta \gg \delta \gg 1/\ell,\ 1/r' \gg 1/r \gg \epsilon \gg 1/n. \tag{12}$$

Proof of Lemma 5.1. We begin by first defining the constants involved.

Definitions of the Constants. Let α, δ', β be given. Let δ be such that

$$(1 - \delta^{1/6}) > 1/2, \tag{13}$$

$$\frac{16\sqrt{\delta}}{\beta^2\alpha^2(\delta')^2} < \frac{1}{2}, \tag{14}$$

and

$$\alpha - 2\delta \geq \alpha\left(\frac{1-\delta'}{1-3^{1/3}\delta^{1/6}}\right)\frac{1}{1-\dfrac{16\sqrt{\delta}}{\beta^2\alpha^2(\delta')^2}}. \tag{15}$$

Observe that the last inequality is satisfied for sufficiently small δ.

Let ℓ be a given integer, let r' be given. Set[1]

$$f = \frac{2\delta^{1/2}\ell^2}{\beta\alpha\delta'}, \tag{16}$$

and define

$$r = fr'. \tag{17}$$

Set $\theta = 1/2$. Let $\epsilon_{4.1} = \epsilon_{4.1}(\theta, \ell)$ be that constant guaranteed by Fact 4.1. Let $\epsilon_{4.5} = \epsilon_{4.5}(\alpha, \delta, \ell, r)$ be that constant guaranteed by Lemma 4.5. Let $\epsilon > 0$ satisfy

$$\epsilon < \min\{\epsilon_{4.1}, \epsilon_{4.5}\}, \tag{18}$$

and

$$f \leq \frac{1}{4\epsilon^{1/2}}, \qquad \epsilon^{1/4} \leq \beta/2, \qquad \epsilon \leq \frac{1}{8\ell^7}, \qquad (1+\epsilon)^{10} \leq 1.1. \tag{19}$$

With the constant α given above, δ given in (13), (15), and (14), integers ℓ, r' given above, r and ϵ given in (17), (18), and (19) respectively, and $n \geq n_0$ for a sufficiently large $n_0(\alpha, \delta', \beta, \delta, \ell, r', r, \epsilon)$, let $\mathcal{H}$ and G as in the Setup satisfy the property $\mathbf{H_1}$. We show that for all but $\epsilon^{1/2}n$ vertices $x \in V_1^{\text{good}}$ there are at most βn vertices $y \in V_1^{\text{good}}$ such that L_{xy}^{23} is not $(\alpha^2/\ell, \delta', r')$-regular.

On the contrary, assume $X \subseteq V_1^{\text{good}}$, $|X| \geq \epsilon^{1/2}n$, satisfies that for each $x \in X$ there exists $Y = Y_x \subseteq V_1^{\text{good}}$, $|Y| \geq \beta n$, such that for each $y \in Y$, L_{xy}^{23} is not $(\alpha^2/\ell, \delta', r')$-regular. We show that as a consequence of our assumption, there exists an r-tuple of triads $\vec{Q} = (Q_1, Q_2, \ldots, Q_r)$ such that

$$\left| \bigcup_{s=1}^{r} \mathcal{K}_3(Q_s) \right| > \delta |\mathcal{K}_3(G^{12} \cup G^{23} \cup G^{13})|, \tag{20}$$

but

$$\frac{\left| \mathcal{H} \cap \bigcup_{s=1}^{r} \mathcal{K}_3(Q_s) \right|}{\left| \bigcup_{s=1}^{r} \mathcal{K}_3(Q_s) \right|} \leq \alpha - \delta. \tag{21}$$

Thus, the proof of Lemma 5.1 will be complete since the existence of $\vec{Q}$ satisfying (20) and (21) contradicts the (α, δ, r)-regularity of $\mathcal{H}$ with respect to G.

---

[1]More precisely, in the proof of Lemma 5.1, we take $f = \lceil 2\delta^{1/2}\ell^2/(\beta\alpha\delta') \rceil$ so that f will be an integer. However, for simplicity of calculations which follow, we drop the ceiling notation $\lceil \quad \rceil$.

We make the following remark.

Remark 5.2. We actually show the following slightly stronger algorithmic assertion with regard to (20) and (21) which is needed in [5]. We assume that we are given a set $X \subseteq V_1^{\rm good}$, $|X| \geq \epsilon^{1/2}n$, satisfying that for each $x \in X$, we are given a set $Y_x \subseteq V_1^{\rm good}$, $|Y_x| \geq \beta n$, such that for each $y \in Y_x$, L_{xy}^{23} is not $(\alpha^2/\ell, \delta', r')$-regular. Assume, moreover, that for each $x \in X$, $y \in Y_x$, we are given a witness $(\{U_j^{xy}, V_j^{xy}\})_{j=1}^{r'}$ of the $(\alpha^2/\ell, \delta', r')$-irregularity of L_{xy}^{23} (cf. Definition 4.7). We show there exists an algorithm **A** which converts $\{(\{U_j^{xy}, V_j^{xy}\})_{j=1}^{r'} : x \in X, y \in Y_x\}$ into a witness $\vec{Q} = (Q_1, Q_2, \ldots, Q_r)$ of the (α, δ, r)-irregularity of $\mathcal{H}$ with respect to G (cf. Definition 4.7). Moreover, we show **A** converts $\{(\{U_j^{xy}, V_j^{xy}\})_{j=1}^{r'} : x \in X, y \in Y_x\}$ into $\vec{Q} = (Q_1, Q_2, \ldots, Q_r)$ in time $O(n^4)$. For future reference, we display our algorithmic assertion.

Algorithm A.
Given: Sets $X \subseteq V_1^{\rm good}$ and $\{Y_x : x \in X\}$ and witnesses

$$\mathcal{W} = \{(\{U_j^{xy}, V_j^{xy}\})_{j=1}^{r'} : x \in X, y \in Y_x\}. \tag{22}$$

Output:
In time $O(n^4)$, a witness

$$\vec{Q} = (Q_1, Q_2, \ldots, Q_r) \tag{23}$$

of the (α, δ, r)-irregularity of $\mathcal{H}$ with respect to G is produced.
We further comment on this algorithm in Remarks 5.6 and 5.7 and in Lemma 5.8. ■

In what follows, we first produce the promised r-tuple $\vec{Q}$. In Claim 5.4, we prove that $\vec{Q}$ satisfies (20). In Claim 5.5, we prove that $\vec{Q}$ satisfies (21). This concludes our proof of Lemma 5.1.

We begin by defining r-tuple $\vec{Q}$. To that end, we use the following claim.

Claim 5.3. *For f given in* (16), *there exists $X_0 = \{x_1, x_2, \ldots, x_f\} \subseteq X$, $|X_0| = f$, satisfying that, for each x_i, $x_j \in X_0$,*

$$\frac{n}{\ell^2}(1-\epsilon)^2 < |N_{G^{12}}(x_i) \cap N_{G^{12}}(x_j)| < \frac{n}{\ell^2}(1+\epsilon)^2 \tag{24}$$

and

$$\frac{n}{\ell^2}(1-\epsilon)^2 < |N_{G^{13}}(x_i) \cap N_{G^{13}}(x_j)| < \frac{n}{\ell^2}(1+\epsilon)^2. \tag{25}$$

Proof of Claim 5.3. We verify that Fact 4.2 applies to G induced on X, V_2, and V_3. In the context of Fact 4.2, set $W_0 = X$, $W_1 = V_2$, $W_2 = V_3$, and set $\sigma = \epsilon^{1/2}$. As G is an $(\ell, \epsilon, 3)$-cylinder, the density of the subgraph induced on $X' \cup V_i'$, $X' \subseteq X$, $V_i' \subseteq V_i$, $|X'| \geq \epsilon n$, $|V_i'| \geq \epsilon n$, $i = 2, 3$, is between $(1/\ell)(1-\epsilon)$ and $(1/\ell)(1+\epsilon)$. As X is a set of good vertices (cf. Definition 4.4), for each $i = 2, 3$ we have $(n/\ell)(1-\epsilon) \leq |N_{G^{1i}}(x)|$

$\leq (n/\ell)(1+\epsilon)$ for each $x \in X$. As we assume n is sufficiently large and $\epsilon < 1/(8\ell^7) < 1/(2\ell)$ [cf. (19)], Fact 4.2 applies to produce a set of vertices $\{x_1, x_2, \ldots, x_b\} \subseteq X$, $b \geq \epsilon^{1/2}/4\epsilon$ satisfying (24) and (25). Since $f \leq 1/(4\epsilon^{1/2})$ [cf. (19)], we may extract the subset $\{x_1, x_2, \ldots, x_f\}$. ∎

Fix $x_i \in X_0$. Fix $y_k^{x_i} \in Y_{x_i}$. By our assumption that $L^{23}_{x_i y_k^{x_i}}$ is not $(\alpha^2/\ell, \delta', r')$-regular, there exist sets $U_j^{(i,k)} \subseteq N_{G^{12}}(x_i, y_k^{x_i})$, $V_j^{(i,k)} \subseteq N_{G^{13}}(x_i, y_k^{x_i})$, $1 \leq j \leq r'$ such that

$$\left|\bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right| > \delta' |N_{G^{12}}(x_i, y_k^{x_i})||N_{G^{13}}(x_i, y_k^{x_i})|,$$

$$> \delta' \left(\frac{n}{\ell^2}\right)^2 (1-\epsilon)^4, \tag{26}$$

but without loss of generality (see Remark 4.6)

$$\left| L^{23}_{x_i y_k^{x_i}} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right| \leq \frac{\alpha^2}{\ell}(1-\delta')\left|\bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|. \tag{27}$$

Fix $x_i \in X_0$. We invoke Lemma 4.5 to infer that with $x_i \in X_0$ fixed, all but $\epsilon^{1/2} n$ vertices $y_k^{x_i} \in Y_{x_i}$ satisfy that $M^{23}_{(x_i, y_k^{x_i})}$ is $(\alpha/\ell, 3^{1/3}\delta^{1/6}, r)$-regular. In particular, as $r' \leq r$ [cf. (17)], we see that all but $\epsilon^{1/2} n$ vertices $y_k^{x_i} \in Y_{x_i}$ satisfy that $M^{23}_{(x_i, y_k^{x_i})}$ is $(\alpha/\ell, 3^{1/3}\delta^{1/6}, r')$-regular. Set Z_{x_i} to be the set of all vertices $y_k^{x_i}$ for which $M^{23}_{(x_i, y_k^{x_i})}$ is not $(\alpha/\ell, 3^{1/3}\delta^{1/6}, r')$-regular. Observe from (18) that $\epsilon < \epsilon_{4.5}$, and Lemma 4.5 applies. We then conclude from Lemma 4.5 that

$$|Z_{x_i}| < \sqrt{\epsilon} n. \tag{28}$$

Let $\widetilde{Y_{x_i}} = Y_{x_i} \backslash Z_{x_i} = \{y_1^{x_i}, y_2^{x_i}, \ldots, y_t^{x_i}\}$ be the set of $t \geq |Y_{x_i}| - \epsilon^{1/2} n \geq (\beta - \epsilon^{1/2}) n \geq \frac{\beta}{2} n$ vertices guaranteed by Lemma 4.5.

Now, fix $y_k^{x_i} \in \widetilde{Y_{x_i}}$. Note that by the definition of $\widetilde{Y_{x_i}}$, every $y_k^{x_i} \in \widetilde{Y_{x_i}}$ satisfies

$$\left| M^{23}_{(x_i, y_k^{x_i})} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right| > \frac{\alpha}{\ell}(1 - 3^{1/3}\delta^{1/6})\left|\bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|. \tag{29}$$

As

$$M^{23}_{(x_i, y_k^{x_i})} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)}) = L^{23}_{x_i} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)}), \tag{30}$$

we further infer from (27) and (29) that

$$\left| L^{23}_{x_i y_k^{x_i}} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right| \leq \alpha \frac{1-\delta'}{1 - 3^{1/3}\delta^{1/6}} \left| L^{23}_{x_i} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|. \tag{31}$$

With $x_i \in X_0$, $y_k^{x_i} \in Y_{x_i}$, fix $j \in \{1, 2, \ldots, r'\}$. Define

$$Q^{12}(i, k, j) = \{\{y_k^{x_i}, u\} : u \in U_j^{(i,k)}\},$$

$$Q^{13}(i, k, j) = \{\{y_k^{x_i}, v\} : v \in V_j^{(i,k)}\},$$

For $i \in \{1, 2, \ldots, f\}$, $j \in \{1, 2, \ldots, r'\}$ we set

$$Q^{12}(i, j) = \bigcup_{k=1}^{t} Q^{12}(i, k, j), \tag{32}$$

$$Q^{13}(i, j) = \bigcup_{k=1}^{t} Q^{13}(i, k, j), \tag{33}$$

and

$$Q^{23}(i, j) = L_{x_i}^{23}. \tag{34}$$

Note, in particular, that in both (32) and (33), we are only taking the union over $y_k^{x_i} \in \widetilde{Y_{x_i}}$ (cf. Remark 5.6).

Set

$$Q(i, j) = Q^{12}(i, j) \cup Q^{13}(i, j) \cup Q^{23}(i, j).$$

Observe from (17) that $fr' = r$. Define the r-tuple of triads

$$\vec{Q} = (Q_1, Q_2, \ldots, Q_r) = (Q(i, j) : 1 \le i \le f, 1 \le j \le r'). \tag{35}$$

Note that if

$$Q(i, k, j) \overset{\text{def}}{=} Q^{12}(i, k, j) \cup Q^{13}(i, k, j) \cup L_{x_i}^{23},$$

then

$$Q(i, j) = \bigcup_{k=1}^{t} Q(i, k, j).$$

Observe also that

$$\mathcal{K}_3(Q(i, j)) = \bigcup_{k=1}^{t} \mathcal{K}_3(Q(i, k, j)). \tag{36}$$

We now prove the following two claims.

Claim 5.4.

$$\left|\bigcup_{i=1}^{f}\bigcup_{j=1}^{r'}\mathcal{K}_3(Q(i,j))\right| > \delta|\mathcal{K}_3(G^{12}\cup G^{23}\cup G^{13})|.$$

Claim 5.5.

$$\frac{\left|\mathcal{H}\cap\bigcup_{i=1}^{f}\bigcup_{j=1}^{r'}\mathcal{K}_3(Q(i,j))\right|}{\left|\bigcup_{i=1}^{f}\bigcup_{j=1}^{r'}\mathcal{K}_3(Q(i,j))\right|}\le\alpha-\delta.$$

Note that these two claims prove Lemma 5.1.

We make the following remark.

Remark 5.6. Recall in (20) and (21) that we promised to produce a witness of the (α, δ, r)-irregularity of $\mathcal{H}$ with respect to G (cf. Definition 4.7). Claims 5.4 and 5.5 show that $\vec{Q} = (Q(1), \ldots, Q(r))$ defined in (35) is such a witness.

In order to show the existence of the algorithm **A** promised in Remark 5.2, we will construct in time $O(n^4)$ a witness $(\hat{Q}(1), \ldots, \hat{Q}(r))$ of the (α, δ, r)-irregularity of $\mathcal{H}$ with respect to G.

In order to avoid the discussion of how to construct the bipartite graphs $Q^{1p}(i, j)$, $p = 2, 3$, $1 \le i \le f$, $1 \le j \le r'$, from (32) and (33) [recall that $Q(i, j) = Q^{12}(i, j) \cup Q^{13}(i, j) \cup Q^{23}(i, j)$], we will consider "easily constructible" graphs $\hat{Q}(i, j) = \hat{Q}^{12}(i, j) \cup \hat{Q}^{13}(i, j) \cup \hat{Q}^{23}(i, j)$. The new graphs $\hat{Q}(i, j)$, $1 \le i \le f$, $1 \le j \le r'$, differ from the earlier graphs $Q(i, j)$, $1 \le i \le f$, $1 \le j \le r'$, very slightly. Consequently, $\hat{Q}(i, j)$, $1 \le i \le f$, $1 \le j \le r'$, will serve the same purpose as $Q(i, j)$, $1 \le i \le f$, $1 \le j \le r'$, concerning Claims 5.4 and 5.5.

More precisely, recall from (22) in Remark 5.2 that we assume we are given a set $X \subseteq V_1^{\mathrm{good}}$, $|X| \ge \sqrt{\epsilon}n$, so that for each $x \in X$, we are given a set $Y_x \subseteq V_1^{\mathrm{good}}$, $|Y_x| \ge \beta n$, so that, for each $y \in Y_x$, we are given a witness $(\{U_j^{xy}, V_j^{xy}\})_{j=1}^{r'}$ of the (α^2/ℓ, δ', r')-irregularity of L_{xy}^{23}. Using Claim 4.2, from the given set X we extracted $X_0 = \{x_1, x_2, \ldots, x_f\} \subseteq X$, where f is given in (16). We then extracted the set $\widetilde{Y_{x_i}} = Y_{x_i} \setminus Z_{x_i} = \{y_1^{x_i}, y_2^{x_i}, \ldots, y_t^{x_i}\}$ guaranteed by Lemma 4.5. Recall from (28) that $|Z_{x_i}| < \sqrt{\epsilon}n$.

For $p = 2, 3$, $1 \le i \le f$, $1 \le j \le r'$, we defined in (32) and (33)

$$Q^{1p}(i, j) = \bigcup_{k=1}^{t} Q^{1p}(i, k, j).$$

Note that $Q^{1p}(i, j)$ can also be written as

$$Q^{1p}(i, j) = \bigcup_{\{k: y_k^{x_i}\in\widetilde{Y_{x_i}}\}} Q^{1p}(i, k, j).$$

We now define

$$\hat{Q}^{1p}(i, j) = \bigcup_{\{k : y_k^{x_i} \in Y_{x_i}\}} Q^{1p}(i, k, j) = Q^{1p}(i, j) \cup \bigcup_{\{k : y_k^{x_i} \in Z_{x_i}\}} Q^{1p}(i, k, j). \tag{37}$$

Since $|Z_{x_i}| < \sqrt{\epsilon} n$, we only slightly extend $Q^{1p}(i, j)$ to $\hat{Q}^{1p}(i, j)$.

We set

$$\hat{Q}(i, j) = \hat{Q}^{12}(i, j) \cup \hat{Q}^{13}(i, j) \cup \hat{Q}^{23}(i, j).$$

We consider the r-tuple

$$(\hat{Q}(i, j) : 1 \le i \le f,\ 1 \le j \le r'). \tag{38}$$

The r-tuple $(\hat{Q}(i, j) : 1 \le i \le f, 1 \le j \le r')$ also satisfies Claims 5.4 and 5.5. Indeed, Claim 5.4 is easily seen to hold for $(\hat{Q}(i, j) : 1 \le i \le f, 1 \le j \le r')$ since

$$\left| \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i, j)) \right| \ge \left| \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i, j)) \right|.$$

It is not too hard to see that Claim 5.5 holds for $(\hat{Q}(i, j) : 1 \le i \le f, 1 \le j \le r')$. Since $|Z_{x_i}| < \sqrt{\epsilon} n$, it easily follows that

$$\left| \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i, j)) \right| \le (1 + \epsilon^{1/4}) \left| \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i, j)) \right|.$$

Consequently,

$$\frac{\left| \mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i, j)) \right|}{\left| \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i, j)) \right|} \le (1 + \epsilon^{1/4}) \frac{\left| \mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i, j)) \right|}{\left| \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i, j)) \right|}. \tag{39}$$

As we will easily show in the upcoming inequality (52) of the upcoming Remark 5.7, the right-hand side of (39) may be bounded from above by $\alpha - \delta$. Consequently, $(\hat{Q}(i, j) : 1 \le i \le f, 1 \le j \le r')$ is also a witness of the (α, δ, r)-irregularity of $\mathcal{H}$.

It remains to show that the witness $(\hat{Q}(i, j) : 1 \le i \le f, 1 \le j \le r')$ is easily constructible. We first discuss why the new graphs $\hat{Q}(i, j)$, $1 \le i \le f$, $1 \le j \le r'$, are easily constructible. Observe from (37) that, for fixed $x_i \in X_0$, $1 \le j \le r'$, all that is needed to construct $\hat{Q}(i, j)$ is the set Y_{x_i} and the witness $\{U_j^{(i,k)}, V_j^{(i,k)}\}$. However, as discussed in Remark 5.2 and (22), with $x = x_i$, we assume that both

$$Y_x = Y_{x_i}$$

and

$$\{\{U_j^{xy}, V_j^{xy}\} : y \in Y_x\} = \{\{U_j^{(i,k)}, V_j^{(i,k)}\} : y_k^{x_i} \in Y_{x_i}\}$$

are given to us.[2] Consequently, all that is needed to construct the witness $(\hat{Q}(i, j) : 1 \le i \le f,\ 1 \le j \le r')$ is a construction of the set $X_0 = \{x_1, x_2, \ldots, x_f\} \subseteq X$ that we extracted from X. However, according to Remark 4.3, X_0 may be found in time $O(n^3)$.

Since $(\hat{Q}(i, j) : 1 \le i \le f,\ 1 \le j \le r')$ given in (38) may be constructed in time $O(n^3)$ and since $(\hat{Q}(i, j) : 1 \le i \le f,\ 1 \le j \le r')$ satisfies Claims 5.4 and 5.5, our algorithmic assertion of Remark 5.2 is proved.

We begin by proving Claim 5.5.

Proof of Claim 5.5. We now produce an upper bound for the quantity

$$\frac{\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i, j))\right|}{\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i, j))\right|}. \tag{40}$$

We first consider the numerator of (40). Observe

$$\begin{aligned}
\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i, j))\right| &= \left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{H} \cap \mathcal{K}_3(Q(i, j))\right| \\
&= \left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{H} \cap \mathcal{K}_3\left(\bigcup_{k=1}^{t} Q(i, k, j)\right)\right| \\
&\le \sum_{i=1}^{f} \left|\bigcup_{j=1}^{r'} \mathcal{H} \cap \mathcal{K}_3\left(\bigcup_{k=1}^{t} Q(i, k, j)\right)\right|.
\end{aligned} \tag{41}$$

From (36), we infer that for each $1 \le i \le f,\ 1 \le j \le r'$

$$\mathcal{H} \cap \mathcal{K}_3\left(\bigcup_{k=1}^{t} Q(i, k, j)\right) = \bigcup_{k=1}^{t} \mathcal{H} \cap \mathcal{K}_3(Q(i, k, j)).$$

Observe from (34) that for $1 \le i \le f$,

$$\begin{aligned}
\left|\bigcup_{j=1}^{r'} \bigcup_{k=1}^{t} \mathcal{H} \cap \mathcal{K}_3(Q(i, k, j))\right| &= \sum_{k=1}^{t} \left|\bigcup_{j=1}^{r'} L^{23}_{y_k^{x_i}} \cap Q^{23}(i, j) \cap (U_j^{(i,k)} \times V_j^{(i,k)})\right| \\
&= \sum_{k=1}^{t} \left|\bigcup_{j=1}^{r'} L^{23}_{y_k^{x_i}} \cap L^{23}_{x_i} \cap (U_j^{(i,k)} \times V_j^{(i,k)})\right| \\
&= \sum_{k=1}^{t} \left|L^{23}_{x_i y_k^{x_i}} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|.
\end{aligned} \tag{42}$$

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[2]Observe that we are not given the set $\widetilde{Y_{x_i}}$ that is used in the original graphs $Q(i, j)$, $1 \le i \le f$, $1 \le j \le r'$.

Thus, from (41) and (42), we infer

$$\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| \leq \sum_{i=1}^{f} \sum_{k=1}^{t} \left| L^{23}_{x_i y_k^{x_i}} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|.$$

From (31), we further conclude

$$\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| \leq \alpha \frac{1-\delta'}{1-\delta^{1/6}} \sum_{i=1}^{f} \sum_{k=1}^{t} \left| L^{23}_{x_i} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|. \quad (43)$$

We now consider the denominator of (40). Observe

$$\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| \geq \sum_{i=1}^{f} \left|\bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| - \sum_{1 \leq i,i' \leq f} \left|\bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j)) \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i',j))\right|. \quad (44)$$

To bound the second term in (44), we first consider the quantity

$$\left|\bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j)) \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i',j))\right|$$

for a fixed choice of $1 \leq i < i' \leq f$. Observe

$$\left|\bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j)) \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i',j))\right| \leq \sum_{y \in \widetilde{Y_{x_i}} \cap \widetilde{Y_{x_{i'}}}} |G^{23}[N_{G^{12}}(x_i, x_{i'}, y), N_{G^{13}}(x_i, x_{i'}, y)]|. \quad (45)$$

Indeed, for x_i, $x_{i'}$, $1 \leq i < i' \leq f$, $\{y, u, v\} \in \cup_{j=1}^{r'} \mathcal{K}_3(Q(i,j)) \cap \cup_{j=1}^{r'} \mathcal{K}_3(Q(i',j))$ implies that $u \in N_{G^{12}}(x_i, x_{i'}, y)$, $v \in N_{G^{13}}(x_i, x_{i'}, y)$, and $\{u, v\} \in L^{23}_{x_i} \cap L^{23}_{x_{i'}} \subseteq G^{23}$.

Continuing, as x_i, $x_{i'}$ satisfy (24) and (25), we see from (19) that

$$|N_{G^{12}}(x_i) \cap N_{G^{12}}(x_{i'})| > \epsilon n,$$

$$|N_{G^{13}}(x_i) \cap N_{G^{13}}(x_{i'})| > \epsilon n.$$

Consequently, as G is an $(\ell, \epsilon, 3)$-cylinder, all but $4\epsilon n$ vertices $y \in \widetilde{Y_{x_i}} \cap \widetilde{Y_{x_{i'}}}$ satisfy

$$\frac{n}{\ell^3}(1-\epsilon)^3 < |N_{G^{12}}(x_i, x_{i'}, y)| < \frac{n}{\ell^3}(1+\epsilon)^3,$$

$$\frac{n}{\ell^3}(1-\epsilon)^3 < |N_{G^{13}}(x_i, x_{i'}, y)| < \frac{n}{\ell^3}(1+\epsilon)^3.$$

From (19), we see that both quantities above are larger than ϵn. We therefore conclude that, for all but $4\epsilon n$ vertices $y \in \widetilde{Y_{x_i}} \cap \widetilde{Y_{x_{i'}}}$,

$$|G[N_{G^{12}}(x_i, x_{i'}, y), N_{G^{13}}(x_i, x_{i'}, y)]| < \left(\frac{n}{\ell^3}\right)^2 \frac{1}{\ell} (1+\epsilon)^7.$$

Consequently, we further infer from (45) that

$$\begin{aligned}\left|\bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j)) \cap \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i',j))\right| &\leq 4\epsilon n^3 + |\widetilde{Y_{x_i}} \cap \widetilde{Y_{x_{i'}}}| \frac{n^2}{\ell^7} (1+\epsilon)^7 \\ &\leq 4\epsilon n^3 + \frac{n^3}{\ell^7}(1+\epsilon)^7 \\ &\leq 2\frac{n^3}{\ell^7}, \end{aligned} \tag{46}$$

where the last inequality follows from our choice of ϵ in (19).

Using the bound in (46), we further infer from (44) that

$$\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| \geq \sum_{i=1}^{f} \left|\bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| - f^2 \frac{n^3}{\ell^7}. \tag{47}$$

To bound the first order term above, consider $|\cup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))|$ for a fixed choice $1 \leq i \leq f$. We infer from (34) that

$$\left|\bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| = \left|\bigcup_{k=1}^{t} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,k,j))\right| = \sum_{k=1}^{t} \left|L_{x_i}^{23} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|.$$

Consequently, we see from (47) that

$$\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| \geq \sum_{i=1}^{f} \sum_{k=1}^{t} \left|L_{x_i}^{23} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right| - f^2 \frac{n^3}{\ell^7}. \tag{48}$$

Returning to (40), we infer from (43) and (48) that

$$\frac{\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|}{\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|} \leq \alpha\left(\frac{1-\delta'}{1-3^{1/3}\delta^{1/6}}\right) \frac{\sum_{i=1}^{f}\sum_{k=1}^{t}\left|L_{x_i}^{23} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|}{\sum_{i=1}^{f}\sum_{k=1}^{t}\left|L_{x_i}^{23} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right| - f^2 \frac{n^3}{\ell^7}}$$

$$= \alpha\left(\frac{1-\delta'}{1-3^{1/3}\delta^{1/6}}\right) \frac{1}{1 - \dfrac{f^2 \frac{n^3}{\ell^7}}{\sum_{i=1}^{f}\sum_{k=1}^{t}\left|L_{x_i}^{23} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right|}}. \tag{49}$$

To further bound (49) from above, we infer from (29), (30), and (26) that for fixed $1 \leq i \leq f$, $1 \leq k \leq t$,

$$\left|L_{x_i}^{23} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)})\right| > \frac{\alpha}{\ell}(1-\delta^{1/6})\delta' \frac{n^2}{\ell^4}(1-\epsilon)^4 > \frac{\alpha}{4\ell^5}\delta' n^2, \tag{50}$$

where the last inequality follows from (13) and (19). Consequently, we further infer from (49) that, with $t \geq \frac{\beta}{2} n$,

$$\frac{\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|}{\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|} \leq \alpha\left(\frac{1-\delta'}{1-3^{1/3}\delta^{1/6}}\right) \frac{1}{1-8f/\beta\alpha\delta'\ell^2}.$$

Using (16), we see

$$\frac{\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|}{\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|} \leq \alpha\left(\frac{1-\delta'}{1-3^{1/3}\delta^{1/6}}\right) \frac{1}{1-16\delta^{1/2}/\beta^2\alpha^2(\delta')^2}.$$

As our choice of δ in (15) guarantees

$$\alpha\left(\frac{1-\delta'}{1-3^{1/3}\delta^{1/6}}\right) \frac{1}{1-16\delta^{1/2}/\beta^2\alpha^2(\delta')^2} \leq \alpha - 2\delta \leq \alpha - \delta, \tag{51}$$

the inequality (21) holds. Thus our proof of Claim 5.5 is complete. ∎

We note the following remark.

Remark 5.7. Recall the definition of $(\hat{Q}(i, j) : 1 \le i \le f,\ 1 \le j \le r')$ from (38) discussed in Remark 5.6. Recall from (39) that

$$\frac{\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i,j))\right|}{\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i,j))\right|} \le (1+\epsilon^{1/4}) \frac{\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|}{\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right|}.$$

Consequently, from (51), we see

$$\frac{\left|\mathcal{H} \cap \bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i,j))\right|}{\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(\hat{Q}(i,j))\right|} \le (\alpha - 2\delta)(1+\epsilon^{1/4}) \le \alpha - \delta. \tag{52}$$

This concludes the proof of our algorithmic assertion in Remark 5.6. ∎

We now prove Claim 5.4.

Proof of Claim 5.4. We show that

$$\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| > \delta |\mathcal{K}_3(G^{12} \cup G^{23} \cup G^{13})|.$$

Using (48), we see

$$\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| \ge \sum_{i=1}^{f} \sum_{k=1}^{t} \left| L_{x_i}^{23} \cap \bigcup_{j=1}^{r'} (U_j^{(i,k)} \times V_j^{(i,k)}) \right| - f^2 \frac{n^3}{\ell^7}.$$

Using (50), we further conclude from $t \ge \beta n/2$ that

$$\begin{aligned}\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| &\ge f \frac{\beta}{8} \alpha\delta' \frac{n^3}{\ell^5} - f^2 \frac{n^3}{\ell^7}, \\ &= \frac{n^3}{\ell^3}\left[f\frac{\beta}{8}\alpha\delta' \frac{1}{\ell^2} - f^2 \frac{1}{\ell^4}\right].\end{aligned}$$

From (16), we infer

$$\left|\bigcup_{i=1}^{f} \bigcup_{j=1}^{r'} \mathcal{K}_3(Q(i,j))\right| \ge \frac{n^3}{\ell^3}\left[1/4\sqrt{\delta} - \frac{4\delta}{\beta^2\alpha^2(\delta')^2}\right] = 1/4\sqrt{\delta}\frac{n^3}{\ell^3}\left[1 - \frac{16\sqrt{\delta}}{\beta^2\alpha^2(\delta')^2}\right] > 2\delta\frac{n^3}{\ell^3},$$

where the last inequality follows from our choice of δ in (14). We further conclude

$$\left|\bigcup_{i=1}^{f}\bigcup_{j=1}^{r'}\mathcal{K}_3(Q(i,j))\right| \geq \delta\frac{n^3}{\ell^3}(1+\theta) \geq \delta|\mathcal{K}_3(G^{12}\cup G^{23}\cup G^{13})|,$$

where the last inequality follows from Fact 4.1 and our choice of ϵ in (18). Thus, Claim 5.4 is proved. ■

For future reference in [5], we state that the following variant of Lemma 3.3 discussed in Remarks 5.2 and 5.6.

Lemma 5.8 (Regularity of Colinks—Constructive Version). *For all positive reals α and δ_B there exists δ_A such that, for all positive integers ℓ and r_B, there exist positive integer r_A and real $\epsilon > 0$ so that, in the context of the Setup, the following holds: Suppose statement (ii) of $\mathbf{H_2}(\delta_B, r_B)$ fails to hold. Moreover, suppose for each $x, y \in V_1$, $x \neq y$, where L_{xy}^{23} is not $(\alpha^2/l, \delta_B, r_B)$-regular, a witness $(\{U_j^{xy}, V_j^{xy}\})_{j=1}^{r_B}$ of the $(\alpha^2/\ell, \delta_B, r_B)$-irregularity of L_{xy}^{23} is given (cf. Definition 4.7). Then there exists an algorithm $\mathbf{A}$ which in time $O(n^3)$ converts the witnesses $\{(\{U_j^{xy}, V_j^{xy}\})_{j=1}^{r_B} : x \neq y$, where L_{xy}^{23} is not $(\alpha^2/l, \delta_B, r_B)$-regular$\}$, into a witness $\vec{Q} = (Q(1), \ldots, Q(r_A))$ of the (α, δ_A, r_A)-irregularity of $\mathcal{H}$ with respect to G (cf. Definition 4.7).*

6. COUNTEREXAMPLE

In this section, we prove Theorem 3.5, showing that the implication $\mathbf{H_2} \Rightarrow \mathbf{H_1}$ is not true in general. Our goal is to construct a 3-partite 3-cylinder $\mathcal{H}$ with an underlying cylinder G, as in the Setup, such that all links and co-links of $\mathcal{H}$ are regular but $\mathcal{H}$ is not.

In order to define the promised constants and construct the counterexample, we need some auxiliary lemmas. In the next subsection, we provide these lemmas.

6.1. Some Auxiliary Lemmas

Our first two lemmas are straightforward applications of the Chernoff inequality [16]. We omit their details here and refer the reader to [4] for the complete details.

Lemma 6.1. *For any positive reals α and δ_B there exists an integer $r_{6.1}$ so that for all $r \geq r_{6.1}$, there exists a 3-uniform 3-partite hypergraph Γ satisfying the following properties:*

(0) Γ has 3-*partition*

$$W = W_1 \cup W_2 \cup W_3,\ |W_1| = |W_2| = |W_3| = r.$$

(i)

$$\alpha r^3(1-\delta_B^3) \leq |\Gamma| \leq \alpha r^3(1+\delta_B^3).$$

(ii) For any $i \in W_1$ and $j \in W_2$, $\alpha r(1-\delta_B^3) \leq |N_\Gamma(i,j)| \leq \alpha r(1+\delta_B^3)$.

(*iii*) *For all* $i, j \in W_1$ *and* $k, l \in W_2$ *such that* $(i, k) \neq (j, l)$,

$$\alpha^2 r(1 - \delta_B^3) \leq |N_\Gamma(i, k) \cap N_\Gamma(j, l)| \leq \alpha^2 r(1 + \delta_B^3).$$

Lemma 6.2. *For all positive reals* δ_B *and* ϵ *and positive integers* ℓ *and* r *there exist* n *and a* 3*-partite graph* $G = G^{12} \cup G^{23} \cup G^{13}$, *together with edge partitions*

$$G^{ij} = G_1^{ij} \cup \cdots \cup G_r^{ij},$$

$1 \leq i < j \leq 3$, *satisfying the following properties*:

(0) G *is an* $(\ell, \epsilon, 3)$*-cylinder.*

(*i*) *For any vertex* $x \in V_1$ *and for any* $i \in \{1, 2, \ldots, r\}$,

$$\frac{n}{r\ell}(1 - \delta_B^3) \leq |N_{G_i^{12}}(x)| \leq \frac{n}{r\ell}(1 + \delta_B^3)$$

and

$$\frac{n}{r\ell}(1 - \delta_B^3) \leq |N_{G_i^{13}}(x)| \leq \frac{n}{r\ell}(1 + \delta_B^3).$$

(*ii*) *For any two vertices* $x, y \in V_1$, $x \neq y$ *and for any* $i, j \in \{1, 2, \ldots, r\}$,

$$\frac{n}{r^2\ell^2}(1 - \delta_B^3) \leq |N_{G_i^{12}}(x) \cap N_{G_j^{12}}(y)| \leq \frac{n}{r^2\ell^2}(1 + \delta_B^3)$$

and

$$\frac{n}{r^2\ell^2}(1 - \delta_B^3) \leq |N_{G_i^{13}}(x) \cap N_{G_j^{13}}(y)| \leq \frac{n}{r^2\ell^2}(1 + \delta_B^3).$$

(iii) *For any* $s \in \{1, 2, \ldots, r\}$ *and for any* $X \subseteq V_i$ *and any* $Y \subseteq V_j$, $1 \leq i < j \leq 3$, *such that* $|X| \geq \sqrt{n}$, $|Y| \geq \sqrt{n}$, *we have*

$$\left|\frac{1}{r\ell} - d_{G_s^{ij}}(X, Y)\right| < \frac{\delta_B^3}{r\ell}.$$

For future purposes, we state the following fact related to Fact 4.1.

Fact 6.3. *Suppose* $G_{s_1}^{12} \cup G_{s_2}^{13} \cup G_{s_3}^{23}$ *is a* 3*-partite graph such that each* $G_{s_k}^{ij}$, $1 \leq i < j \leq 3$, $1 \leq k \leq 3$ *satisfies the property in* (*iii*) *from Lemma* 6.2. *Then*

$$\frac{1}{2}\left(\frac{n}{r\ell}\right)^3 \leq |\mathcal{K}_3(G_{s_1}^{12} \cup G_{s_2}^{13} \cup G_{s_3}^{23})| \leq 2\left(\frac{n}{r\ell}\right)^3.$$

For our final lemma, we need the following definition.

Definition 6.4. *Let* ϵ, η, *and* α *be positive reals*, *and let* F *be a bipartite graph with bipartition* (U, V). *We say that* F *is* $(\alpha, (\epsilon, \eta))$-regular *if for any* $U' \subseteq U$, $|U'| \geq \epsilon|U|$ *and for any* $V' \subseteq V$, $|V'| \geq \epsilon|V|$, *we have*

$$\alpha(1 - \eta) \le d_F(U', V') \le \alpha(1 + \eta).$$

Although in a slightly different language, the following lemma essentially appeared as Claim 4.10 in [21].

Lemma 6.5. *Let α, δ be positive reals and let r be a positive integer. There exists $\epsilon > 0$ so that whenever F is an $(\alpha, (\epsilon, \delta))$-regular bipartite graph with bipartition (U, V), then F is $(\alpha, 2\delta, r)$-regular.*

We omit the proof of Lemma 6.5 and encourage the reader to see [4] and [21].

6.2. Construction of $\mathcal{H}$

6.2.1. Definitions of the Constants. Let $\alpha = 1/2$ and $\delta_A = 1/16$, and let $\delta_B > 0$ be given. Without loss of generality, we may assume that δ_B satisfies

$$\delta_B < \frac{1}{20}, \tag{53}$$

$$(1 - \delta_B^3)^2 > 1 - 3\delta_B^2, \tag{54}$$

$$(1 + \delta_B^3)^2 < 1 + 3\delta_B^2. \tag{55}$$

For our promised value ℓ, we choose any integer $\ell > 1/\delta$. Let

$$r_{6.1} = r_{6.1}(\alpha, \delta_B)$$

be that constant guaranteed by Lemma 6.1. Let integer $r' > \ell$ satisfy

$$20\left(1 - \frac{1}{5\delta_B^2 r'}\right) \ge 6, \tag{56}$$

$$\left(1 + \frac{16}{6\delta_B^2 r'}\right)^{-1} \ge 1 - \delta_B^2, \tag{57}$$

$$1 + \frac{64}{6\alpha^2\delta_B^2 r'} \le 1 + \delta_B^2. \tag{58}$$

Let

$$r = \max\{r_{6.1}, r'\}, \tag{59}$$

and set

$$r_A = 2\alpha r^3. \tag{60}$$

Let r_B, ϵ and n_0 be given. Let

$$\epsilon_{6.5} = \epsilon_{6.5}(\alpha, 3\delta_B^2, r_B)$$

be that constant guaranteed to exist by Lemma 6.5. Let

$$\epsilon_{4.1} = \epsilon_{4.1}(\ell, 1/2)$$

be that constant guaranteed to exist by Fact 4.1. Let $\epsilon' > 0$ satisfy

$$(1 - \delta_B^3)^2(1 - 16\epsilon') \geq 1 - 3\delta_B^2, \tag{61}$$

$$(1 + \delta_B^3)^2(1 + 32\epsilon'\ell) \leq 1 + 3\delta_B^2, \tag{62}$$

$$(1 + \delta_B^3)^2\left(1 + \frac{64\epsilon'}{\ell}\right) \leq 1 + 3\delta_B^2, \tag{63}$$

$$\epsilon' < \frac{1}{2r\ell^2}. \tag{64}$$

[Note that (61), (62) and (63) are possible due to (54) and (55).] Note that we may assume, without loss of generality, that

$$\epsilon \leq \min\{\epsilon_{6.5}, \epsilon_{4.1}, \epsilon'\}. \tag{65}$$

Let

$$n_{6.2} = n_{6.2}(\delta_B, \ell, r)$$

be that constant guaranteed by Lemma 6.2. Take

$$n \geq n_{6.2} \tag{66}$$

to be sufficiently large (wherever needed).

6.2.2. Construction of $\mathcal{H}$. We define hypergraph $\mathcal{H}$ with underlying cylinder G as promised by Theorem 3.5. To that end, with $\alpha = 1/2$ and δ_B given above and r given in (59), we first define an auxiliary structure. Let Γ be a 3-uniform hypergraph guaranteed by Lemma 6.1 satisfying:

(0) Γ has 3-partition

$$W = W_1 \cup W_2 \cup W_3, \qquad |W_1| = |W_2| = |W_3| = r. \tag{67}$$

(i)

$$\alpha r^3(1 - \delta_B^3) \leq |\Gamma| \leq \alpha r^3(1 + \delta_B^3). \tag{68}$$

(ii) For any $i \in W_1$ and $j \in W_2$,

$$\alpha r(1 - \delta_B^3) \leq |N_\Gamma(i, j)| \leq \alpha r(1 + \delta_B^3). \tag{69}$$

(iii) For all $i, j \in W_1$ and $k, l \in W_2$ such that $(i, k) \neq (j, l)$,

$$\alpha^2 r(1 - \delta_B^3) \leq |N_\Gamma(i, k) \cap N_\Gamma(j, l)| \leq \alpha^2 r(1 + \delta_B^3). \tag{70}$$

Continuing, we now define the promised underlying cylinder G. With δ_B given, ℓ given, and r given in (59), ϵ satisfying (65) and n given in (66), let 3-partite graph $G = G^{12} \cup G^{23} \cup G^{13}$, together with edge partitions

$$G^{ij} = G_1^{ij} \cup \cdots \cup G_r^{ij}, \tag{71}$$

$1 \leq i < j \leq 3$, be guaranteed by Lemma 6.2 satisfying the following properties:

(0) G is an $(\ell, \epsilon, 3)$-cylinder.
(i) For any vertex $x \in V_1$ and for any $i \in \{1, 2, \ldots, r\}$,

$$\frac{n}{r\ell}(1 - \delta_B^3) \leq |N_{G_i^{12}}(x)| \leq \frac{n}{r\ell}(1 + \delta_B^3) \tag{72}$$

and

$$\frac{n}{r\ell}(1 - \delta_B^3) \leq |N_{G_i^{13}}(x)| \leq \frac{n}{r\ell}(1 + \delta_B^3). \tag{73}$$

(ii) For any two vertices $x, y \in V_1$ and for any $i, j \in \{1, 2, \ldots, r\}$,

$$\frac{n}{r^2\ell^2}(1 - \delta_B^3) \leq |N_{G_i^{12}}(x) \cap N_{G_j^{12}}(y)| \leq \frac{n}{r^2\ell^2}(1 + \delta_B^3) \tag{74}$$

and

$$\frac{n}{r^2\ell^2}(1 - \delta_B^3) \leq |N_{G_i^{13}}(x) \cap N_{G_j^{13}}(y)| \leq \frac{n}{r^2\ell^2}(1 + \delta_B^3). \tag{75}$$

(iii) For any $s \in \{1, 2, \ldots, r\}$ and for any $X \subseteq V_i$ and any $Y \subseteq V_j$, $1 \leq i < j \leq 3$, such that $|X| \geq \sqrt{n}$, $|Y| \geq \sqrt{n}$ we have

$$\left|\frac{1}{r\ell} - d_{G_s^{ij}}(X, Y)\right| < \frac{\delta_B^3}{r\ell}. \tag{76}$$

We now define the promised hypergraph $\mathcal{H}$ having underlying cylinder G defined above. For convenience in what follows, we use the following notation concerning the graph G defined above. For $x, y, z \in V_1 \cup V_2 \cup V_3$ and $1 \leq i, j, k \leq r$, we use $\{x, y, z\} = \{x, y, z\}_{ijk}$ to denote that $\{x, y, z\}$ satisfies $x \in V_1$, $y \in V_2$ and $z \in V_3$ and $\{x, y\} \in G_i^{12}$, $\{x, z\} \in G_j^{13}$ and $\{y, z\} \in G_k^{23}$. Then we define the hypergraph $\mathcal{H}$ as

$$\mathcal{H} = \{\{x, y, z\} : \{x, y, z\} = \{x, y, z\}_{ijk} \quad \text{where } (i, j, k) \in \Gamma\}. \tag{77}$$

This completes our definition of $\mathcal{H}$ and G.

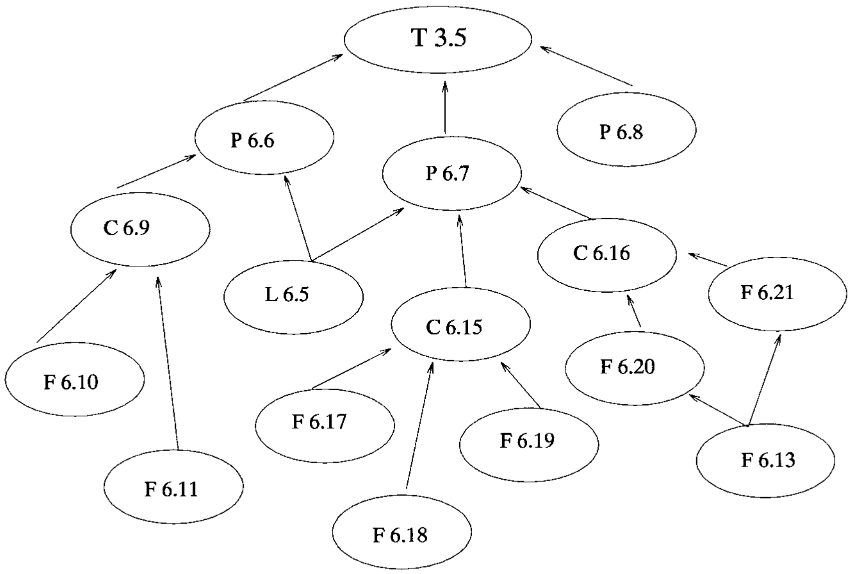

Fig. 1. A flow chart of proving Theorem 3.5.

We note the following remark. Observe from (77) that $d_{\mathscr{H}}(G) = |\Gamma|/r^3$. Then by (68), we see

$$\alpha(1 - \delta_B^3) \leq d_{\mathscr{H}}(G) \leq \alpha(1 + \delta_B^3). \tag{78}$$

6.3. Proof of Theorem 3.5

Figure 1 illustrates a flow chart of proving Theorem 3.5. The letter T represents "Theorem," the letter L represents "Lemma," the letter C represents "Claim," and the letter F represents "Fact."

We break our proof of Theorem 3.5 into three parts according to the following propositions. Note that each of the following propositions refers to the hypergraph $\mathscr{H}$ and graph G constructed above:

Proposition 6.6. *For each $x \in V_1$, L_x^{23} is $(\alpha/\ell, \delta_B, r_B)$-regular.*

Proposition 6.7. *For each $x, y \in V_1$, $x \neq y$, L_{xy}^{23} is $(\alpha^2/\ell, \delta_B, r_B)$-regular.*

Proposition 6.8. *$\mathscr{H}$ is not (α, δ_A, r_A)-regular.*

Then Propositions 6.6, 6.7, and 6.8 together immediately imply Theorem 3.5.

The proof of Proposition 6.8 is easy so we present it immediately. We then comment on our strategy for proving Propositions 6.6 and 6.7.

Proof of Proposition 6.8. Define

$$\vec{Q} = (G_i^{12} \cup G_j^{13} \cup G_k^{23} : (i, j, k) \in \Gamma).$$

Observe from (77) that all triangles of $\vec{Q}$ coincide with triples of $\mathcal{H}$. Consequently, $d_{\mathcal{H}}(\vec{Q}) = 1$.

We prove that $\vec{Q}$ is an r_A-tuple of triads which satisfies

$$\left| \bigcup_{(i,j,k)\in\Gamma} \mathcal{K}_3(G_i^{12} \cup G_j^{13} \cup G_k^{23}) \right| > \delta_A |\mathcal{K}_3(G)|, \tag{79}$$

but

$$\left| d_{\mathcal{H}}(\vec{Q}) - d_{\mathcal{H}}(G) \right| \geq \delta_A. \tag{80}$$

Observe by (68) and (60) and the fact that $\delta_B < 1$ that

$$|\Gamma| \leq \alpha r^3 (1 + \delta_B^3) \leq 2\alpha r^3 = r_A. \tag{81}$$

Therefore, $\vec{Q}$ is an r_A-tuple of triads.

To prove (79), observe that

$$\left| \bigcup_{(i,j,k)\in\Gamma} \mathcal{K}_3(G_i^{12} \cup G_j^{13} \cup G_k^{23}) \right| = \sum_{(i,j,k)\in\Gamma} |\mathcal{K}_3(G_i^{12} \cup G_j^{13} \cup G_k^{23})|. \tag{82}$$

Applying Fact 6.3 to each $G_i^{12} \cup G_j^{13} \cup G_k^{23}$, $(i, j, k) \in \Gamma$, we see

$$|\mathcal{K}_3(G_i^{12} \cup G_j^{13} \cup G_k^{23})| > \frac{1}{2}\left(\frac{n}{r\ell}\right)^3. \tag{83}$$

Additionally, we see from (68) that

$$|\Gamma| > \frac{\alpha r^3}{2}. \tag{84}$$

We then infer from (82), (83), and (84) that

$$\left| \bigcup_{(i,j,k)\in\Gamma} \mathcal{K}_3(G_i^{12} \cup G_j^{13} \cup G_k^{23}) \right| > \frac{\alpha}{4}\frac{n^3}{\ell^3}. \tag{85}$$

On the other hand, we see from (65) and Fact 4.1 that

$$|\mathcal{K}_3(G)| < 2\left(\frac{n}{\ell}\right)^3. \tag{86}$$

From (85) and (86) we infer that

$$\left| \bigcup_{(i,j,k)\in\Gamma} \mathcal{K}_3(G_i^{12} \cup G_j^{13} \cup G_k^{23}) \right| > \frac{\alpha}{8} |\mathcal{K}_3(G)|.$$

Then (79) follows from the facts that $\alpha = 1/2$ and $\delta_A = 1/16$.

We see (80) follows immediately by the construction of $\vec{Q}$. Recall $d_{\mathcal{H}}(\vec{Q}) = 1$. On the other hand, from (78), we see $\alpha(1 - \delta_B) \le d_{\mathcal{H}}(G) \le \alpha(1 + \delta_B)$. Consequently, with $\alpha = 1/2$, $\delta_A = 1/16$ and $\delta_B < 1/4$, $|d_{\mathcal{H}}(\vec{Q}) - d_{\mathcal{H}}(G)| \ge \delta_A$ follows. ∎

We now return to proving Propositions 6.6 and 6.7. To prove Proposition 6.6, we use the following claim, proved below.

Claim 6.9. *For each $x \in V_1$, L_x^{23} is $(\alpha/\ell, (\epsilon, 3\delta_B^2))$-regular.*

We then see that Proposition 6.6 follows immediately from Claim 6.9, Lemma 6.5, and our choice of ϵ in (65). Indeed, by Claim 6.9, Lemma 6.5, and our choice of ϵ in (65), we see that for each $x \in V_1$, L_x^{23} is $(\alpha/\ell, 6\delta_B^2, r_B)$-regular. As $6\delta_B^2 \le \delta_B$ from (53), Proposition 6.6 follows.

Proof of Claim 6.9. Fix $x \in V_1$ and let $A \subseteq N_{G^{12}}(x)$, $|A| > \epsilon|N_{G^{12}}(x)|$ and $B \subseteq N_{G^{13}}(x)$, $|B| > \epsilon|N_{G^{13}}(x)|$ be given. We show

$$\frac{\alpha}{\ell}(1 - 3\delta_B^2)|A||B| \le |L_x^{23}(A, B)| \le \frac{\alpha}{\ell}(1 + 3\delta_B^2)|A||B|. \tag{87}$$

The upcoming Facts 6.10 and 6.11 essentially prove (87). Before presenting these facts, we prepare the following notation and terminology. For $1 \le i, j \le r$, set

$$A_i = A \cap N_{G_i^{12}}(x),$$

$$B_j = B \cap N_{G_j^{13}}(x),$$

where graphs G_i^{12}, G_j^{13} are given in (71). Since $\cup_{i=1}^r N_{G_i^{12}}(x) = N_{G^{12}}(x)$, we see $\cup_{i=1}^r A_i = A$. Similarly, $\cup_{i=1}^r B_i = B$. For $1 \le i \le r$, we call A_i *big* if

$$|A_i| \ge \epsilon^3 |N_{G_i^{12}}(x)| \tag{88}$$

and *small* otherwise. Similarly, for $1 \le j \le r$, we call B_j *big* if

$$|B_j| \ge \epsilon^3 |N_{G_j^{13}}(x)| \tag{89}$$

and *small* otherwise. Set

$$T_B = \{(i, j) : A_i \text{ and } B_j \text{ are big}\},$$

$$T_S = \{(i, j) : A_i \text{ or } B_j \text{ is small}\}.$$

Observe that in the language above,

$$|L_x^{23}(A, B)| = \sum_{(i,j)\in T_B} |L_x^{23}(A_i, B_j)| + \sum_{(i,j)\in T_S} |L_x^{23}(A_i, B_j)|. \tag{90}$$

Facts 6.10 and 6.11 may be given as follows.

Fact 6.10. *For each* $(i, j) \in T_B$,

$$\frac{\alpha}{\ell}(1 - \delta_B^3)^2|A_i||B_j| \le |L_x^{23}(A_i, B_j)| \le \frac{\alpha}{\ell}(1 + \delta_B^3)^2|A_i||B_j|.$$

Fact 6.11.

$$\sum_{(i,j)\in T_S} |A_i||B_j| \le 4\epsilon^3 \frac{n^2}{\ell^2}.$$

We prove Fact 6.10 momentarily. Fact 6.11 follows trivially from the definition of T_S; thus, we omit it.

We now use Facts 6.10 and 6.11 to conclude our proof of Claim 6.9. We begin with the lower bound in (87).

Observe

$$|L_x^{23}(A, B)| = \sum_{(i,j)\in T_B} |L_x^{23}(A_i, B_j)| + \sum_{(i,j)\in T_S} |L_x^{23}(A_i, B_j)| \ge \sum_{(i,j)\in T_B} |L_x^{23}(A_i, B_j)|.$$

By Fact 6.10,

$$|L_x^{23}(A, B)| \ge \frac{\alpha}{\ell}(1 - \delta_B^3)^2 \sum_{(i,j)\in T_B} |A_i||B_j|.$$

As $|A||B| = \sum_{(i,j)\in T_B} |A_i||B_j| + \sum_{(i,j)\in T_S} |A_i||B_j|$, Fact 6.11 implies

$$\begin{aligned} |L_x^{23}(A, B)| &\ge \frac{\alpha}{\ell}(1 - \delta_B^3)^2\left[|A||B| - 4\epsilon^3 \frac{n^2}{\ell^2}\right], \\ &\ge \frac{\alpha}{\ell}(1 - \delta_B^3)^2|A||B|\left[1 - \frac{4\epsilon^3 n^2/\ell^2}{|A||B|}\right]. \end{aligned}$$

Since $|A| \ge \epsilon|N_{G^{12}}(x)| > \epsilon\frac{n}{2\ell}$ and $|B| \ge \epsilon|N_{G^{13}}(x)| > \epsilon\frac{n}{2\ell}$ were given, we see

$$|L_x^{23}(A, B)| \ge \frac{\alpha}{\ell}(1 - \delta_B^3)^2|A||B|(1 - 16\epsilon) \ge \frac{\alpha}{\ell}(1 - 3\delta_B^2)|A||B|,$$

where the last inequality follows from (61).

In a way similar to above, one may show

$$|L_x^{23}(A, B)| \leq \frac{\alpha}{\ell}(1 + \delta_B^3)^2(1 + 32\epsilon\ell)|A||B|.$$

Then the upper bound of (87) follows immediately from (62). This concludes our proof of Claim 6.9. ∎

Proof of Fact 6.10. Fix $(i, j) \in T_B$. Observe

$$L_x^{23}(A_i, B_j) = L_x^{23} \cap \bigcup_{k=1}^{r} G_k^{23}(A_i, B_j),$$

and thus

$$|L_x^{23}(A_i, B_j)| = \sum_{k=1}^{r} |L_x^{23} \cap G_k^{23}(A_i, B_j)|. \tag{91}$$

Observe from (77) that, for $1 \leq k \leq r$,

$$L_x^{23} \cap G_k^{23}(A_i, B_j) = \begin{cases} G_k^{23}(A_i, B_j) & \text{if } k \in N_\Gamma(i, j), \\ \emptyset & \text{else.} \end{cases}$$

Then we may rewrite (91) as

$$|L_x^{23}(A_i, B_j)| = \sum_{k \in N_\Gamma(i,j)} |G_k^{23}(A_i, B_j)|. \tag{92}$$

Fix $k \in N_\Gamma(i, j)$. Since $(i, j) \in T_B$, we see from (88), (89), (72), and (73) that

$$|A_i| \geq \epsilon^3 |N_{G_i^{12}}(x)| \geq \epsilon^3 \frac{n}{2r\ell} > \sqrt{n}, \tag{93}$$

and

$$|B_j| \geq \epsilon^3 |N_{G_j^{13}}(x)| \geq \epsilon^3 \frac{n}{2r\ell} > \sqrt{n}. \tag{94}$$

Using (93), (94), and (76), we infer

$$\frac{|A_i||B_j|}{r\ell}(1 - \delta_B^3) \leq |G_k^{23}(A_i, B_j)| \leq \frac{|A_i||B_j|}{r\ell}(1 + \delta_B^3). \tag{95}$$

Combining (92) and (95), we see

$$|N_\Gamma(i, j)| \frac{|A_i||B_j|}{r\ell}(1 - \delta_B^3) \leq |L_x^{23}(A_i, B_j)| \leq |N_\Gamma(i, j)| \frac{|A_i||B_j|}{r\ell}(1 + \delta_B^3).$$

Then Fact 6.10 follows immediately from (69). ∎

Consequently, Claim 6.9 and then Proposition 6.6 are proved.

To prove Proposition 6.7, we take a similar approach to that suggested above for links in Claim 6.9. Unfortunately, technical reasons prohibit us from merely copying Claim 6.9 for co-links (these reasons will be seen in context). To prove Proposition 6.7, we therefore must develop the following extended strategy below. We begin with the following definition.

Definition 6.12. *For a pair of distinct vertices $x, y \in V_1$, define the* 2-diagonal *of x and y as*

$$\mathrm{DIAG}_2(x, y) = \bigcup_{i=1}^{r} (N_{G_i^{12}}(x) \cap N_{G_i^{12}}(y)),$$

where graphs G_i^{12}, $1 \leq i \leq r$ are given in (71).

For future purposes, we give the following easy estimation that follows immediately from (74).

Fact 6.13. *For a pair of distinct vertices $x, y \in V_1$,*

$$\frac{1}{2}\frac{n}{r\ell^2} < |\mathrm{DIAG}_2(x, y)| < 2\frac{n}{r\ell^2}.$$

We proceed with the following definition.

Definition 6.14. *For a pair of distinct vertices $x, y \in V_1$, define*

$$\widetilde{L_{xy}^{23}} = L_{xy}^{23}[N_{G^{12}}(x, y)\backslash \mathrm{DIAG}_2(x, y), N_{G^{13}}(x, y)].$$

We remark that only later in context will it be clear why we removed the 2*-diagonal.*

To prove Proposition 6.7, we prove the following claims.

Claim 6.15. *For each pair $x, y \in V_1$, $\widetilde{L_{xy}^{23}}$ is $(\alpha^2/\ell, (\epsilon, 3\delta_B^2))$-regular.*

Claim 6.16. *For a pair $x, y \in V_1$, if $\widetilde{L_{xy}^{23}}$ is $(\alpha^2/\ell, 6\delta_B^2, r_B)$-regular, then L_{xy}^{23} is $(\alpha^2/\ell, 20\delta_B^2, r_B)$-regular.*

We then see that Proposition 6.7 follows from Claim 6.15, Lemma 6.5 [and our choice of ϵ in (65)], and Claim 6.16. Indeed, by Claim 6.15 and Lemma 6.5 [and our choice of ϵ in (65)], we see that for each pair $x, y \in V_1$, $\widetilde{L_{xy}^{23}}$ is $(\alpha^2/\ell, 6\delta_B^2, r_B)$-regular. By Claim 6.16, we then conclude that each L_{xy}^{23} is $(\alpha^2/\ell, 20\delta_B^2, r_B)$-regular. As $20\delta_B^2 \leq \delta_B$ from (53), Proposition 6.7 follows.

The proofs of Claims 6.9 and 6.15 are essentially identical. We sketch an outline for the proof of Claim 6.15 (indicating precisely why we removed the 2-diagonal). We then conclude this section with a proof of Claim 6.16.

Proof of Claim 6.15 (Outline). Fix $x, y \in V_1$ and let $A \subseteq N_{G^{12}}(x, y) \setminus \mathrm{DIAG}_2(x, y)$, $|A| > \epsilon |N_{G^{12}}(x, y) \setminus \mathrm{DIAG}_2(x, y)|$, $B \subseteq N_{G^{13}}(x, y)$, $|B| > \epsilon |N_{G^{13}}(x, y)|$ be given. We must show

$$\frac{\alpha^2}{\ell}(1 - 3\delta_B^2)|A||B| \leq |\widetilde{L_{xy}^{23}}(A, B)| \leq \frac{\alpha^2}{\ell}(1 + 3\delta_B^2)|A||B|. \tag{96}$$

We proceed similarly to before. For $1 \leq i, i' \leq r$, $1 \leq j, j' \leq r$, set

$$A_{i,i'} = A \cap N_{G_i^{12}}(x) \cap N_{G_{i'}^{12}}(y),$$

$$B_{j,j'} = B \cap N_{G_j^{13}}(x) \cap N_{G_{j'}^{13}}(y).$$

For $1 \leq i, i' \leq r$, define $A_{i,i'}$ to be *big* if $|A_{i,i'}| > \epsilon^3 |N_{G_i^{12}}(x) \cap N_{G_{i'}^{12}}(y)|$ and *small* otherwise. For $1 \leq j, j' \leq r$, define $B_{j,j'}$ to be *big* if $|B_{j,j'}| > \epsilon^3 |N_{G_j^{13}}(x) \cap N_{G_{j'}^{13}}(y)|$ and *small* otherwise. Set

$$\widetilde{T_B} = \{(i, i', j, j') : A_{i,i'} \text{ and } B_{j,j'} \text{ are big}\},$$

$$\widetilde{T_S} = \{(i, i', j, j') : A_{i,i'} \text{ or } B_{j,j'} \text{ is small}\}.$$

We proceed with the following facts.

Fact 6.17. *Suppose $(i, i', j, j') \in \widetilde{T_B}$. Then $i \neq i'$.*

Proof of Fact 6.17. Fix $(i, i', j, j') \in \widetilde{T_B}$. We show $i \neq i'$. Recall that by Definition 6.12,

$$\mathrm{DIAG}_2(x, y) = \bigcup_{i=1}^{r} (N_{G_i^{12}}(x) \cap N_{G_i^{12}}(y)).$$

where graphs G_i^{12}, $1 \leq i \leq r$ are given in (71). Recall $A \subseteq N_{G^{12}}(x, y) \setminus \mathrm{DIAG}_2(x, y)$. Consequently, $A_{i,i} = A \cap N_{G_i^{12}}(x) \cap N_{G_i^{12}}(y) = \emptyset$. As $(i, i', j, j') \in \widetilde{T_B}$, $|A_{i,i'}| > 0$, thus, $i \neq i'$. ∎

Fact 6.18. *For each $(i, i', j, j') \in \widetilde{T_B}$,*

$$\frac{\alpha^2}{\ell}(1 - \delta_B^3)^2 |A_{i,i'}||B_{j,j'}| \leq |\widetilde{L_{xy}^{23}}(A_{i,i'}, B_{j,j'})| \leq \frac{\alpha^2}{\ell}(1 + \delta_B^3)^2 |A_{i,i'}||B_{j,j'}|.$$

The proof of Fact 6.18 is similar to the proof of Fact 6.10, and we sketch it at the end of the proof of Claim 6.15.

Fact 6.19.

$$\sum_{(i,i',j,j')\in \widetilde{T_S}} |A_{i,i'}||B_{j,j'}| \leq 4\epsilon^3 \frac{n^2}{\ell^4}.$$

Similar to Fact 6.11, Fact 6.19 is trivial, and we omit its proof here.
We proceed with our outline of the proof of Claim 6.15. Note that, similar to (90),

$$|\widetilde{L_{xy}^{23}}(A, B)| = \sum_{(i,i',j,j')\in \widetilde{T_B}} |\widetilde{L_{xy}^{23}}(A_{i,i'}, B_{j,j'})| + \sum_{(i,i',j,j')\in \widetilde{T_S}} |\widetilde{L_{xy}^{23}}(A_{i,i'}, B_{j,j'})|.$$

Using Facts 6.18 and 6.19, we conclude

$$|\widetilde{L_{xy}^{23}}(A, B)| \leq \frac{\alpha^2}{\ell}(1-\delta_B^3)^2|A||B| + 4\epsilon^3 \frac{n^2}{\ell^4}.$$

Then the upper bound of (96) follows from (61). In a similar way, one can show the lower bound of (96).

In order to conclude the proof of Claim 6.15, it remains to give (a sketch of) the proof of Fact 6.18.

Proof of Fact 6.18 (Sketch). Fix $(i, i', j, j') \in \widetilde{T_B}$. As with (91), observe

$$|\widetilde{L_{xy}^{23}}(A_{i,i'}, B_{j,j'})| = \sum_{k=1}^{r} |\widetilde{L_{xy}^{23}} \cap G_k^{23}(A_{i,i'}, B_{j,j'})|. \tag{97}$$

Observe from (77) that, for $1 \leq k \leq r$,

$$\widetilde{L_{xy}^{23}} \cap G_k^{23}(A_{i,i'}, B_{j,j'}) = \begin{cases} G_k^{23}(A_{i,i'}, B_{j,j'}) & \text{if } k \in N_\Gamma(i,j) \cap N_\Gamma(i',j'), \\ \varnothing & \text{else.} \end{cases}$$

Then we may rewrite (97) as

$$|\widetilde{L_{xy}^{23}}(A_{i,i'}, B_{j,j'})| = \sum_{k\in N_\Gamma(i,j)\cap N_\Gamma(i',j')} |G_k^{23}(A_{i,i'}, B_{j,j'})|. \tag{98}$$

Fix $k \in N_\Gamma(i, j) \cap N_\Gamma(i', j')$. In precisely the same way as before, we may conclude

$$|A_{i,i'}| \geq \epsilon^3 |N_{G_i^{12}}(x) \cap N_{G_{i'}^{12}}(y)| \geq \epsilon^3 \frac{n}{2r^2\ell^2} > \sqrt{n},$$

and

$$|B_{j,j'}| \geq \epsilon^3 |N_{G_j^{13}}(x) \cap N_{G_{j'}^{13}}(y)| \geq \epsilon^3 \frac{n}{2r^2\ell^2} > \sqrt{n}.$$

Consequently, we infer

$$\frac{|A_{i,i'}||B_{j,j'}|}{r\ell}(1-\delta_B^3) \le |G_k^{23}(A_{i,i'}, B_{j,j'})| \le \frac{|A_{i,i'}||B_{j,j'}|}{r\ell}(1+\delta_B^3). \tag{99}$$

Combining (98) and (99), we see

$$\frac{|A_{i,i'}||B_{j,j'}|}{r\ell}(1-\delta_B^3) \le \frac{|\widetilde{L_{xy}^{23}}(A_{i,i'}, B_{j,j'})|}{|N_\Gamma(i, j) \cap N_\Gamma(i', j')|} \le \frac{|A_{i,i'}||B_{j,j'}|}{r\ell}(1+\delta_B^3).$$

As $(i, i', j, j') \in \widetilde{T_B}$, Fact 6.17 implies $i \neq i'$. Consequently, $(i, j) \neq (i', j')$. Therefore, Fact 6.18 follows immediately from (70). ■

We proved that, for each pair $x, y \in V_1$, $\widetilde{L_{xy}^{23}}$ is $(\alpha^2/\ell, (\epsilon, 3\delta_B^2))$-regular. Recall that by Lemma 6.5 and our choice of ϵ in (65), we see that for each pair $x, y \in V_1$, $\widetilde{L_{xy}^{23}}$ is $(\alpha^2/\ell, 6\delta_B^2, r_B)$-regular. To finish the proof of Proposition 6.7, we need to show that for each pair $x, y \in V_1$, if $\widetilde{L_{xy}^{23}}$ is $(\alpha^2/\ell, 6\delta_B^2, r_B)$-regular, then L_{xy}^{23} is $(\alpha^2/\ell, 20\delta_B^2, r_B)$-regular, that is to prove Claim 6.16.

Proof of Claim 6.16. Let $x, y \in V_1$ be fixed. Let $V_{2,j}$, $V_{3,j}$, $V_{2,j} \subseteq N_{G^{12}}(x, y)$, $V_{3,j} \subseteq N_{G^{13}}(x, y)$, $1 \le j \le r_B$, be an r_B-tuple of pairs of subsets satisfying

$$\left|\bigcup_{j=1}^{r_B}(V_{2,j} \times V_{3,j})\right| > 20\delta_B^2 |N_{G^{12}}(x, y)||N_{G^{13}}(x, y)|. \tag{100}$$

We are going to show

$$\frac{\alpha^2}{\ell}(1-20\delta_B^2) \le \frac{|L_{xy}^{23} \cap \cup_{j=1}^{r_B}(V_{2,j} \times V_{3,j})|}{|\cup_{j=1}^{r_B}(V_{2,j} \times V_{3,j})|} \le \frac{\alpha^2}{\ell}(1+20\delta_B^2). \tag{101}$$

For $1 \le j \le r_B$, set

$$V'_{2,j} = V_{2,j} \setminus \mathrm{DIAG}_2(x, y).$$

Observe from Fact 6.13 that

$$|V'_{2,j}| > |V_{2,j}| - 2\frac{n}{r\ell^2}.$$

We prove the following two facts.

Fact 6.20.

$$\left|\bigcup_{j=1}^{r_B}(V'_{2,j} \times V_{3,j})\right| > 6\delta_B^2 |N_{G^{12}}(x, y) \setminus \mathrm{DIAG}_2(x, y)||N_{G^{13}}(x, y)|.$$

Fact 6.21.

$$(1-\delta_B^2)\frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}\le\frac{|L_{xy}^{23}\cap\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}$$
$$\le(1+\delta_B^2)\frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}.$$

Before proving Facts 6.20 and 6.21, we show how they imply (101).

Indeed, by Fact 6.20 and the $(\alpha^2/\ell,\ 6\delta_B^2,\ r)$-regularity of $\widetilde{L_{xy}^{23}}$, we see

$$\frac{\alpha^2}{\ell}(1-6\delta_B^2)\le\frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}\le\frac{\alpha^2}{\ell}(1+6\delta_B^2). \tag{102}$$

By Fact 6.21, we thus conclude

$$\frac{\alpha^2}{\ell}(1-6\delta_B^2)(1-\delta_B^2)\le\frac{|L_{xy}^{23}\cap\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}\le\frac{\alpha^2}{\ell}(1+6\delta_B^2)(1+\delta_B^2).$$

Then (101) follows from (53).

Thus, to complete the proof of Claim 6.16, we prove Facts 6.20 and 6.21.

Proof of Fact 6.20. Observe

$$\left|\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right|\ge\left|\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right|-|N_{G^{13}}(x,y)||\mathrm{DIAG}_2(x,y)|.$$

Using Fact 6.13, we thus conclude

$$\left|\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right|\ge\left|\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right|-2\frac{n}{r\ell^2}|N_{G^{13}}(x,y)|.$$

From (100), we conclude that

$$\left|\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right|\ge 20\delta_B^2|N_{G^{12}}(x,y)||N_{G^{13}}(x,y)|-2\frac{n}{r\ell^2}|N_{G^{13}}(x,y)|$$
$$=20\delta_B^2|N_{G^{13}}(x,y)||N_{G^{13}}(x,y)|\left(1-\frac{n}{10\delta_B^2r\ell^2}\frac{1}{|N_{G^{12}}(x,y)|}\right).$$

Since $|N_{G^{12}}(x,y)|>\frac{n}{2\ell^2}$, we see

$$\left|\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| \geq 20\delta_B^2\left(1-\frac{1}{5\delta_B^2 r}\right)|N_{G^{12}}(x,y)||N_{G^{13}}(x,y)|$$
$$\geq 20\delta_B^2\left(1-\frac{1}{5\delta_B^2 r}\right)|N_{G^{12}}(x,y)\backslash \mathrm{DIAG}_2(x,y)||N_{G^{13}}(x,y)|. \quad (103)$$

Fact 6.20 then follows from (56). ∎

Proof of Fact 6.21. Observe that

$$\left|L_{xy}^{23}\cap\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right| \geq \left|\widetilde{L_{xy}^{23}}\cap\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| \quad (104)$$

and

$$\left|L_{xy}^{23}\cap\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right| \leq \left|\widetilde{L_{xy}^{23}}\cap\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| + |G[\mathrm{DIAG}_2(x,y),N_{G^{13}}(x,y)]|. \quad (105)$$

We first bound (105) from above, and begin by estimating $|G[\mathrm{DIAG}_2(x,y),N_{G^{13}}(x,y)]|$.

From $|N_{G^{13}}(x,y)| > \frac{n}{2\ell^2}$ and (64), we infer $|N_{G^{13}}(x,y)| \geq \epsilon n$. From Fact 6.13 and (64), we infer $|\mathrm{DIAG}_2(x,y)| > \epsilon n$. Since G^{23} is $(1/\ell,\epsilon)$-regular, $\epsilon < 1/2$, we infer

$$|G[\mathrm{DIAG}_2(x,y),N_{G^{13}}(x,y)]| < 2\,\frac{|\mathrm{DIAG}_2(x,y)||N_{G^{13}}(x,y)|}{\ell}. \quad (106)$$

Using (106) in (105), we conclude that

$$\left|L_{xy}^{23}\cap\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right| \leq \left|\widetilde{L_{xy}^{23}}\cap\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| + 2\,\frac{|\mathrm{DIAG}_2(x,y)||N_{G^{13}}(x,y)|}{\ell}. \quad (107)$$

We further bound (107). Observe from $|N_{G^{13}}(x,y)| < 2\frac{n}{\ell^2}$ and Fact 6.13 that

$$|\mathrm{DIAG}_2(x,y)||N_{G^{13}}(x,y)| < 4\,\frac{n^2}{r\ell^4}. \quad (108)$$

We may therefore conclude from (107) that

$$\left|L_{xy}^{23}\cap\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right| \leq \left|\widetilde{L_{xy}^{23}}\cap\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| + 8\,\frac{n^2}{r\ell^5}. \quad (109)$$

Observe

$$\left|\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right| \geq \left|\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| \tag{110}$$

and

$$\left|\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right| \leq \left|\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| + |\mathrm{DIAG}_2(x,y)||N_{G^{13}}(x,y)|. \tag{111}$$

Using (108), we infer from (111) that

$$\left|\bigcup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})\right| \leq \left|\bigcup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})\right| + 4\frac{n^2}{r\ell^4}. \tag{112}$$

We now prove Fact 6.21. We infer from (104) and (112) that

$$\begin{aligned}\frac{|L_{xy}^{23}\cap\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|} &\geq \frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})| + 4\dfrac{n^2}{r\ell^4}}\\ &= \frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}\left(1+\frac{4n^2/r\ell^4}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}\right)^{-1}.\end{aligned}$$

From (103) we see

$$\begin{aligned}\frac{|L_{xy}^{23}\cap\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|} &\geq \frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}\\ &\quad\times\left(1+\frac{16n^2/r\ell^4}{20\delta_B^2\left(1-\dfrac{1}{5\delta_B^2 r}\right)|N_{G^{12}}(x,y)||N_{G^{13}}(x,y)|}\right)^{-1}.\end{aligned}$$

Since $|N_{G^{12}}(x,y)|, |N_{G^{13}}(x,y)| > \frac{n}{2\ell^2}$ and by (56), we further conclude

$$\frac{|L_{xy}^{23}\cap\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|} \geq \frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}\left(1+\frac{16}{6\delta_B^2 r}\right)^{-1}.$$

The lower bound of Fact 6.21 then follows from (57).

Using (109) and (110), we see

$$\frac{|L_{xy}^{23}\cap\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|}{|\cup_{j=1}^{r_B}(V_{2,j}\times V_{3,j})|} \leq \frac{|\widetilde{L_{xy}^{23}}\cap\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})| + 8n^2/r\ell^5}{|\cup_{j=1}^{r_B}(V'_{2,j}\times V_{3,j})|}.$$

Using (103) and (56), we infer

$$\begin{aligned}\frac{\left|L^{23}_{xy} \cap \cup_{j=1}^{r_B} (V_{2,j} \times V_{3,j})\right|}{\left|\cup_{j=1}^{r_B} (V_{2,j} \times V_{3,j})\right|} &\le \frac{\left|\widetilde{L^{23}_{xy}} \cap \cup_{j=1}^{r_B} (V'_{2,j} \times V_{3,j})\right|}{\left|\cup_{j=1}^{r_B} (V'_{2,j} \times V_{3,j})\right|} \\ &\quad + \frac{32n^2/r\ell^5}{20\delta_B^2\left(1 - \dfrac{1}{5\delta_B^2 r}\right)\left|N_{G^{12}}(x, y)\right|\left|N_{G^{13}}(x, y)\right|} \\ &\le \frac{\left|\widetilde{L^{23}_{xy}} \cap \cup_{j=1}^{r_B} (V'_{2,j} \times V_{3,j})\right|}{\left|\cup_{j=1}^{r_B} (V'_{2,j} \times V_{3,j})\right|} + \frac{32}{6\delta_B^2 r\ell}. \end{aligned} \tag{113}$$

Rewriting (113) and using (102), we obtain

$$\frac{\left|L^{23}_{xy} \cap \cup_{j=1}^{r_B} (V_{2,j} \times V_{3,j})\right|}{\left|\cup_{j=1}^{r_B} (V_{2,j} \times V_{3,j})\right|} \le \frac{\left|\widetilde{L^{23}_{xy}} \cap \cup_{j=1}^{r_B} (V'_{2,j} \times V_{3,j})\right|}{\left|\cup_{j=1}^{r_B} (V'_{2,j} \times V_{3,j})\right|}\left(1 + \frac{64}{6\alpha^2\delta_B^2 r}\right).$$

The upper bound of Fact 6.21 then follows from (58). ∎

APPENDIX

A.1. The Frankl-Rödl Hypergraph Regularity Lemma

In this section, we state the Frankl-Rödl Hypergraph Regularity Lemma. First, we state a number of supporting definitions. By $K(U, V)$ we denote the complete bipartite graph on bipartition (U, V).

Definition 7.1. *Let V be a set with $|V| = N$. An $(\ell, t, \gamma, \epsilon)$*-partition *$\mathcal{P}$ of $[V]^2$ is an (auxiliary) partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ of V, together with a system of edge-disjoint bipartite graphs $\mathcal{B} = \{P^{ij}_\alpha: 1 \le i < j \le t, 0 \le \alpha \le \ell_{ij} \le \ell\}$, such that:*

(i) $|V_0| < t$ *and* $|V_1| = |V_2| = \cdots = |V_t| = \lfloor \frac{N}{t} \rfloor =^{def} n$.
(*ii*) $\cup_{\alpha=0}^{\ell_{ij}} P^{ij}_\alpha = K(V_i, V_j)$ *for all i, j, $1 \le i < j \le t$.*
(*iii*) *All but $\gamma\binom{t}{2}m^2$ pairs $\{v_i, v_j\}$, $v_i \in V_i$, $v_j \in V_j$, $1 \le i < j \le t$, are edges of ϵ-regular bipartite graphs P^{ij}_α.*
(*iv*) *For all but at most $\gamma\binom{t}{2}$ pairs i, j, $1 \le i < j \le t$, we have that $|P^{ij}_0| \le \gamma n^2$ and P^{ij}_α is $(1/\ell, \epsilon)$-regular for all $\alpha = 1, \ldots, \ell_{ij}$.*

Definition 7.2. *If $\mathcal{P}$ is an $(\ell, t, \gamma, \epsilon)$-partition of V and the bipartite graphs P^{ij}_α, P^{is}_β, and P^{js}_μ are all $(1/\ell, \epsilon)$-regular, then a* triad *$P = (P^{ij}_\alpha, P^{is}_\beta, P^{js}_\mu)$ of $\mathcal{P}$ is the 3-partite graph with vertex set $V_i \cup V_j \cup V_s$ formed by the union of these three bipartite graphs.*

Note that a triad of $\mathcal{P}$ is an $(\ell, \epsilon, 3)$-cylinder. Also, each triad P underlies a subhypergraph of $\mathcal{H}$, which we denote by $\mathcal{H}(P)$, consisting of all triples in $\mathcal{H}$ that are triangles in P. We call $\mathcal{H}(P)$ the 3-*partite* 3-*cylinder* of $\mathcal{H}$ *on* P.

The Hypergraph Regularity Lemma will basically say that the vertex set of any large enough 3-uniform hypergraph $\mathcal{H}$ has an $(\ell, t, \gamma, \epsilon)$-partition $\mathcal{P}$ such that most 3-partite

3-cylinders $\mathcal{H}(P)$ of $\mathcal{H}$ will be "very regular." To make this concept precise, we need the following definition.

Definition 7.3. *Let $\mathcal{H}$ be a 3-uniform hypergraph with vertex set V, where $|V| = N$. We say that an $(\ell, t, \gamma, \epsilon)$-partition $\mathcal{P}$ of V is (δ, r)-*regular *for $\mathcal{H}$ if*

$$\sum \{|\mathcal{K}_3(P)| : P \text{ is a triad of } \mathcal{P},\ \mathcal{H}(P) \text{ is } (\delta, r)\text{-irregular w.r.t. } P\} < \delta N^3.$$

This implies, in particular, that if $\mathcal{H}$ is a dense hypergraph [that is, $|\mathcal{H}| = \Omega(N^3)$], then most of the triples of $\mathcal{H}$ belong to (δ, r)-regular triads of the partition $\mathcal{P}$. We may now state the Regularity Lemma of [8].

Theorem 7.4. *For every δ and γ with $0 < \gamma \le 2\delta^4$, for all integers t_0 and ℓ_0 and for all integer-valued functions $r(t, \ell)$ and all functions $\epsilon(\ell) > 0$, there exist T_0, L_0, and N_0 such that any 3-uniform hypergraph $\mathcal{H} \subseteq [N]^3$, $N \ge N_0$, admits a $(\delta, r(t, \ell))$-regular, $(\ell, t, \gamma, \epsilon(\ell))$-partition for some t and ℓ satisfying $t_0 \le t \le T_0$ and $\ell_0 \le \ell \le L_0$.*

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