

Simply Generated Trees, B-Series and Wigner Processes

Christian Mazza
 Section de Mathématiques
 2-4 Rue du Lièvre, CP 240
 CH-1211 Genève 24 (Suisse)
 christian.mazza@math.unige.ch

Abstract

We consider simply generated trees, like rooted plane trees, and consider the problem of computing generating functions of so-called bare functionals, like the tree factorial, using B-series from Butcher's theory. We exhibit a special class of functionals from probability theory: the associated generating functions can be seen as limiting traces of product of semi-circular elements.

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1 Introduction

Let \mathcal{F}_n denote the set of rooted plane trees of size n . Simply generated trees are families of trees obtained by assigning weights $\omega(t)$ to the elements $t \in \mathcal{F} = \cup_n \mathcal{F}_n$ using a degree function $\psi(z) = 1 + \sum_{k \geq 1} \psi_k z^k$ (see [20]). Basically, the weight $\omega(t)$ of some $t \in \mathcal{F}$ is obtained by multiplying the factors $\psi_{d(v)}$ over the nodes v of t , where $d(v)$ denotes the outdegree of v . Our main topic is the study of generating functions

$$Y(z) = \sum_{t \in \mathcal{F}} \omega(t) B(t) z^{|t|},$$

associated with multiplicative functions $B : \mathcal{F} \rightarrow \mathbb{R}$ defined recursively by using a sequence of real numbers $\{B_k\}_{k \in \mathbb{N}^+}$. We call such multiplicative functions *bare Green functions*: $\sum_{t \in \mathcal{F}_n} B(t) \omega(t)$ represents the sum of the

Feynman amplitudes associated to the relevant diagrams of size n in some field theory, and the generating function is then a part of the perturbative expansion of the solution of some equation describing the system (see [3, 6, 8, 15]).

In Section 4, we give an equation satisfied by Y when the weights B_k come from some master function $L(z) = \sum_{m \geq 0} L_m z^m$, with $B_k \equiv L(k)/k$, $\forall k \in \mathbb{N}^+$. We use series indexed by trees, the so-called B-series, as defined in [13, 14], to show in Theorem 1 that Y solves

$$Y' = L(1 + \theta)\Psi(Y),$$

where θ is the differential operator $\theta = z d/dz$. [1] considers a similar problem for *additive tree functionals* $s(t)$ defined on varieties of increasing trees, like $s(t) = \ln(B(t))$. Assuming some constraints on the degree function $\Psi(z)$, it is proven that the exponential generating function

$$S(z) = \sum_{t \in \mathcal{F}} \omega(t) s(t) z^{|t|} / |t|!,$$

is given by the formula

$$S(z) = W'(z) \int_0^z (F'(u)/W'(u)) du,$$

where $F(u) = \sum_{m \geq 0} \ln(B_m) W_m u^m / m!$ and $W(z) = \sum_{m \geq 0} W_m z^m / m!$ solves $W' = \Psi(W)$. We also consider a central functional called the *tree factorial*, denoted by $t!$ in the sequel, which is relevant in various fields, like algorithmics [9, 18], stochastics [11, 21], numerical analysis (see for example [5, 14]), and physics [6, 15]. We focus on its negative powers $1/(t!)^{l+1}$, $l \in \mathbb{N}$, which do not admit a master function when $l \geq 1$. [6] solved the case $l = 1$ by using the so-called Butcher's group (see for example [13, 14]). We provide in Theorem 2 a differential equation for the associated generating function, $\forall l \in \mathbb{N}$.

In Section 5, we define special multiplicative functionals for which the weights B_k are related to the covariance function r of some gaussian process, as $B_k = \beta^2 r(2k - 1)$, for some positive constant $\beta > 0$. We show that the generating function Y is related to the mean normalized trace of products of large symmetric random matrices having independent and identically distributed versions of the process as entries. Theorem 3 gives then a differential equation for the evolution of the trace of a stationary Wigner processes. It follows that most of the examples given in [3, 15] can be expressed in terms of traces of large random matrices. In Section 6, we show how B-series can be useful for studying traces of triangular operators appearing in free probability.

2 Basic notions

A rooted tree $t \in \mathcal{R}$ is a triple $t = (r, V, E)$ such that i) (V, E) is a non-empty directed tree with node set V and edge set E , ii) all edges are directed away from the root $r \in V$. The set of rooted trees of order n is denoted by \mathcal{R}_n , and the set of rooted trees is $\mathcal{R} = \cup_n \mathcal{R}_n$. A rooted plane tree $t \in \mathcal{F}$ is a quadruple $t = (r, V, E, L)$ satisfying i) and ii) and iii) $L := \{(\{w : vw \in E\}, L_v) : v \in V\}$ is a collection of $|V|$ linear orders. Given $v \in V$, let $\text{ch}(v) := \{w : vw \in E\}$ be the set of children of v . $d(v) := |\text{ch}(v)|$ is the outdegree of v . A rooted planar tree can be seen in the plane with the root in the lowest position, such that the orders L_v coincide with the left-right order. Next consider the partial ordering (V, \leq) defined by $u \leq v$ if and only if u lies on the path linking r and v . Given $v \in V$ and $t \in \mathcal{R}$ let t_v be the subtree of t rooted at v spanned by the subset $\{w; v \leq w\}$. A rooted labelled tree is a quadruple $t = (r, V, E, l)$ satisfying i) and ii), with a labelling $l : V \setminus \{r\} \rightarrow [|V|] := \{1, \dots, |V|\}$ such that $l(u) < l(v)$ when $u < v$. The set of rooted labelled trees of order n is denoted by \mathcal{L}_n . Let $\mathcal{L} = \cup_n \mathcal{L}_n$. This family is a special variety of increasing trees, as defined in [1, 12].

We next assign weights to the elements of \mathcal{F}_n , the set of rooted planar trees of order n : the resulting family of trees is said to be simply generated (see [21]). Given a sequence $\psi = \{\psi_k\}_{k \in \mathbb{N}}$ of real numbers with $\psi_0 = 1$, define recursively the weight $\omega(t)$ of $t \in \mathcal{F}$ as

$$\omega(t) = \psi_k \prod_{i=1}^k \omega(t_i), \quad k = d(r), \quad \omega(t) = \prod_{v \in V} \psi_{d(v)}.$$

where t_1, \dots, t_k are the $d(r)$ subtrees of t rooted at $\text{ch}(r)$. Let $\psi(z) := 1 + \sum_{k=1}^{\infty} \psi_k z^k$ be the generating function of the weight sequence ψ . Our favourite example is $\psi(z) = 1/(1 - z)$, with $\omega(t) = 1, \forall t$ (see [1, 20] for various interesting choices).

We will be concerned with functionals $B : \mathcal{F} \rightarrow \mathbb{R}$, where $\mathcal{F} = \cup_n \mathcal{F}_n$, called *bare Green functions*. This terminology is taken from quantum field theory where bare Green functions occur during the action of the renormalization group (see for example [3], § 4.2 or [6], § 6.1). Let \mathbf{B} denote the set of bare Green functions. Any element $B \in \mathbf{B}$ is given through a sequence of functions $B_k : \mathbb{R} \rightarrow \mathbb{R}, k \in \mathbb{N}^+$, which are usually Laurent series in some variable x (see for example [8]). In what follows, we simply write the sequence as $\{B_k\}_{k \in \mathbb{N}^+}$.

Definition 1 *The bare Green function $B \in \mathbf{B}, B : \mathcal{F} \rightarrow \mathbb{R}$, associated*

with the sequence of functions $\{B_k\}_{k \in \mathbf{N}_+}$ is defined recursively as

$$B(t) = B_{|t|} \prod_{i=1}^k B(t_i),$$

where t_1, \dots, t_k are the $d(r)$ subtrees of t rooted at $\text{ch}(r)$, and where $|t|$ denotes the number of nodes of t .

Notice that the value of B at $t \in \mathcal{F}$ does not depend on the linear orders and is independent of the labellings. When dealing with rooted trees, we will adopt the notation $t = B_+(t_1, \dots, t_k)$ for the operation of grafting the rooted trees t_1, \dots, t_k , that is by considering the tree t obtained by the creation of a new node r (the root) and then joining the roots of t_1, \dots, t_k to r . Bare Green functions appeared also in the probabilistic literature in specific situations. The basic example, in algorithmics [9, 18], in numerical analysis (see [4, 5, 14]), in stochastics [11, 20] and in physics (see for example [6]) is the *tree factorial*, defined by

Definition 2 Let $t \in \mathcal{R}$ with $t = B_+(t_1, \dots, t_k)$. Then the tree factorial $t!$ is the functional $B \in \mathbf{B}$ defined by $t! = |t| \prod_{i=1}^k t_i!$, associated with the sequence $\{B_k\}$ given by $B_k \equiv k$.

Remark 3 It should be pointed out that the functional acting on trees, given as $s(t) = \ln(B(t))$, for $B \in \mathbf{B}$ with $B_k > 0, \forall k \geq 1$, is an inductive map or an additive tree functional, as defined in [1]. Interestingly, $B(t) = 1/t!$ is used in [18] to define a probability measure on random search binary trees, and [9, 11] provide precise asymptotics for $\ln(t!)$.

3 Generating functions

We first give some basic results on tree factorials, symmetry factors, and generating functions associated with bare Green functions.

Definition 4 Let $t \in \mathcal{R}$. Then $\alpha(t)$ is the number of rooted labelled trees $t' \in \mathcal{L}$ of shape $t \in \mathcal{R}$, where the shape of a labelled tree (r, V, E, l) is (r, V, E) , $\kappa(t)$ is the number of rooted plane trees of shape t , and $\sigma(t)$ is the symmetry factor of the tree, to be defined later. Moreover, let $\omega_{\mathcal{L}}$ be the weight function associated with elements of \mathcal{L} , with weights given by $\psi_k \equiv 1/k!$.

Notice that $\alpha(t)$ is the Connes-Moscovici weight in quantum field theory (see [3, 7]). The symmetry factor satisfies the recursive definition:

$$\sigma(\{r\}) = 1,$$

$$\sigma(B_+(t_1^{n_1}, \dots, t_k^{n_k})) = n_1! \sigma(t_1)^{n_1} \dots n_k! \sigma(t_k)^{n_k},$$

where the indices n_i means that t is obtained by grafting n_1 times the tree t_1 , and so on, where we assume that the t_i are all different as rooted trees.

Lemma 5 *Let $t \in \mathcal{R}$. Then*

$$\alpha(t)\sigma(t) = \frac{|t|!}{t!}, \quad (1)$$

and

$$\alpha(t)t! = |t|!\omega_{\mathcal{L}}(t)\kappa(t). \quad (2)$$

Proof: (1) is well known (see for example [5]). Suppose that $t \in \mathcal{R}$ is such that $t = B_+(t_1^{n_1}, \dots, t_k^{n_k})$, the grafting of n_1 times the tree t_1 , and so on, where we set that the trees t_1, \dots, t_k are different as rooted trees. Then

$$\kappa(t) = \frac{(n_1 + \dots + n_k)!}{n_1! \dots n_k!} \kappa(t_1)^{n_1} \dots \kappa(t_k)^{n_k}.$$

Using the recursive definition of $\omega(t)$ and the definition of $\omega_{\mathcal{L}}$, we have

$$\omega_{\mathcal{L}}(t) = \frac{1}{(n_1 + \dots + n_k)!} \omega_{\mathcal{L}}(t_1)^{n_1} \dots \omega_{\mathcal{L}}(t_k)^{n_k}.$$

Therefore

$$\frac{1}{\omega_{\mathcal{L}}(t)\kappa(t)} = n_1! \dots n_k! \left(\frac{1}{\omega_{\mathcal{L}}(t_1)\kappa(t_1)} \right)^{n_1} \dots \left(\frac{1}{\omega_{\mathcal{L}}(t_k)\kappa(t_k)} \right)^{n_k},$$

and the results follows from the recursive definition of the symmetry factor. \square

Then

$$\sum_{t \in \mathcal{F}_n} B(t)\omega(t) = \sum_{t \in \mathcal{R}_n} B(t) \frac{\omega}{\omega_{\mathcal{L}}}(t) \alpha(t) \frac{t!}{|t|!} \quad (3)$$

where we have used (2) of Lemma 5.

Consider the generating function

$$Y(z) = \sum_{n \in \mathbb{N}^+} \frac{z^n}{n!} \sum_{t \in \mathcal{R}_n} \alpha(t) B(t) t! \omega(t) / \omega_{\mathcal{L}}(t). \quad (4)$$

Given $t \in \mathcal{R}$, the ratio $\omega/\omega_{\mathcal{L}}$ is associated with the weight sequence $\bar{\psi}_k \equiv \psi_k k!$; using the expansion $\psi(z) = 1 + \sum_{k \geq 1} \psi_k z^k = 1 + \sum_{k \geq 1} (\bar{\psi}_k / k!) z^k$, we see that $\bar{\psi}_k \equiv \psi^{(k)}(0)$. Consider the *elementary differentials* δ (see Section 4) defined by

Definition 6

$$\delta_{\{*\}} = 1, \quad \delta_t = \psi^{(k)}(0) \prod_{i=1}^k \delta_{t_i}, \quad \frac{\omega}{\omega_{\mathcal{L}}} = \delta,$$

when $t = B_+(t_1, \dots, t_l)$, where $*$ denotes the tree of a single node. For a map $a : \mathcal{R} \cup \{\emptyset\} \rightarrow \mathbb{R}$, a formal power series of the form $Y(z) = a(\emptyset)y_0 + \sum_{t \in \mathcal{R}} z^{|t|} a(t) \delta_t \alpha(t) / |t|!$ is called a B-series [13, 14].

Remark 7 When $B(t) = t!$, the series Y is given by

$$Y(z) = \sum_{t \in \mathcal{L}} (\omega(t) / \omega_{\mathcal{L}}(t)) z^{|t|} / |t|!.$$

Set $\phi_k = \psi_k k!$, $\forall k$, and consider the degree function $\phi(z) = 1 + \sum_{k \geq 1} (\phi_k / k!) z^k$. Following [1], Y solves $Y' = \phi(Y)$ (see also [20]). We shall see in the next section that it is a natural consequence of B-series expansions of solutions of ordinary differential equations.

4 Runge-Kutta methods for functionals over trees

Consider a dynamical system on \mathbb{R}

$$\frac{d}{ds} X(s) = F(X(s)), \quad X(s_0) = X_0,$$

for some smooth $F : \mathbb{R} \rightarrow \mathbb{R}$. The solution of this dynamical system has a B-series expansion of the form

$$X(s) = X_0 + \sum_{t \in \mathcal{R}} \frac{(s - s_0)^{|t|}}{|t|!} \alpha(t) \delta_t(s_0),$$

where the elementary differentials δ is defined recursively by

$$\delta_{\{r\}} = f(s_0), \quad \delta_t = \frac{\partial^k F}{\partial^k s} \delta_{t_1} \cdots \delta_{t_k},$$

when $t = B_+(t_1, \dots, t_k)$. These kinds of expansions have been treated in great detail in [4] and [5] and developed independently in combinatorics (see for example [16, 17]). Suppose that $s_0 = 0$ for simplicity. Butcher considered what happens with numerical approximations of the exact solution, the Runge-Kutta methods, which are themselves B-series [13, 14]; here we focus on specific B-series, which are associated to bare Green functions. Let $B \in \mathbf{B}$ be such that there exists a power series

$$L(z) = \sum_{m \geq 0} L_m z^m,$$

with

$$B_k = \frac{L(k)}{k}, \quad \forall k \in \mathbb{N}^+. \quad (5)$$

Bare Green functions satisfying (5) are used in practical situations in quantum field theory (see [3], § 4 and [6], § 6.1). Consider Euler's operator $\theta = z(d/dz)$, with $P(\theta)(z^n) = P(n)z^n$, $\forall n \in \mathbb{N}$, for any polynomial P , and consider the formal operator $L(\theta + 1)$ acting on monomials as

$$\begin{aligned} L(\theta + 1)(z^n) &= \sum_{m \geq 0} L_m(\theta + 1)^m(z^n) = \sum_{m \geq 0} L_m(n + 1)^m z^n \\ &= L(n + 1)z^n. \end{aligned}$$

Given a power series $Y(z) = \sum_{m \geq 0} a_m z^m$ converging for $|z| \leq 1$, we can define $L(\theta + 1)(Y)(z) := \sum_{m \geq 0} a_m L(m + 1)z^m$, which converges for $|z| \leq 1$ when the sequence $(L(k))_{k \geq 1}$ grows subexponentially. We will not focus on convergence questions here, and work at the formal level. Let B be a bare Green function with weights $(B_k)_{k \geq 1}$, such that (5) holds for some power series L . It should be pointed out that [3, 6, 15] deal with the master function L , but do not give explicitly an equation for Y . The next Theorem provides an equation; its proof uses explicitly B-series.

Theorem 1 *The formal power series*

$$Y(z) = \sum_{t \in \mathcal{R}} \frac{z^{|t|}}{|t|!} \alpha(t) t! B(t) \delta_t, \quad (6)$$

solves $Y' = L(1 + \theta)\psi(Y)$.

Proof:

$$\psi(Y(z)) = \sum_{k \geq 0} \frac{\psi^{(k)}(0)}{k!} \sum_{(t_1 \dots t_k) \in \mathcal{R}^k} \frac{z^{\sum_i |t_i|}}{|t_1|! \dots |t_k|!} \prod_{i=1}^k \alpha(t_i) B(t_i) t_i! \delta_{t_i}.$$

For given $(t_1 \dots t_k) \in \mathcal{R}^k$, set $t = B_+(t_1, \dots, t_k)$. Then $\sum_i |t_i| = |t| - 1$, $\psi^{(k)}(0) \delta_{t_1} \dots \delta_{t_k} = \delta_t$, and $B(t_1) \dots B(t_k) = B(t)/B_{|t|}$. The associated term becomes

$$z^{|t|-1} \frac{B(t)}{B_{|t|}} \delta_t \frac{\alpha(t_1) \dots \alpha(t_k)}{|t_1|! \dots |t_k|!} \frac{t!}{|t|}.$$

Next, every rooted tree $t \in \mathcal{R}$ can be decomposed uniquely as $t = B_+(t_1^{n_1}, \dots, t_m^{n_m})$, meaning that t is obtained by grafting n_1 times t_1 and so on, where the t_i are different as rooted trees, with $k = n_1 + \dots + n_m$. Collecting the terms associated with t , we get the contribution

$$\frac{z^{|t|-1}}{k!} \frac{B(t)}{B_{|t|}} \delta_t \frac{t!}{|t|} \sum_{(t'_1 \dots t'_k) \in \mathcal{R}^k}^* \frac{\alpha(t'_1) \dots \alpha(t'_k)}{|t'_1|! \dots |t'_k|!},$$

where $*$ means that the sum is taken over all the collections $(t'_1 \cdots t'_k) \in \mathcal{R}^k$ such that $t = B_+(t'_1, \dots, t'_k)$. The above sum reduced then to

$$\frac{(n_1 + \cdots + n_m)!}{n_1! \cdots n_m!} \frac{\alpha(t_1)^{n_1} \cdots \alpha(t_m)^{n_m}}{k! |t_1|!^{n_1} \cdots |t_m|!^{n_m}} = \frac{1}{n_1!} \left(\frac{1}{\sigma(t_1) t_1!} \right)^{n_1} \cdots \frac{1}{n_m!} \left(\frac{1}{\sigma(t_m) t_m!} \right)^{n_m},$$

where we use the first identity of Lemma 5. Using the recursive definition of the symmetry factor σ , we obtain

$$\begin{aligned} \sum_{(t'_1 \cdots t'_k) \in \mathcal{R}^k}^* \frac{\alpha(t'_1) \cdots \alpha(t'_k)}{k! |t'_1|! \cdots |t'_k|!} &= \frac{1}{t_1!^{n_1} \cdots t_m!^{n_m} \sigma(B_+(t_1, \dots, t_m))} \\ &= \frac{|t|}{t!} \frac{1}{\sigma(t)} = \frac{|t| \alpha(t)}{|t|!}. \end{aligned}$$

We thus get that the contribution associated with $t \in \mathcal{R}$ is given by

$$\frac{z^{|t|-1}}{k!} \frac{B(t)}{B_{|t|}} \delta_t \frac{t!}{|t|} \frac{|t| \alpha(t)}{|t|!} = \frac{z^{|t|-1}}{|t|!} \frac{B(t)}{B_{|t|}} \alpha(t) \delta_t t!.$$

Therefore

$$\begin{aligned} L(\theta + 1) \psi(Y) &= \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \alpha(t) t! \delta_t \frac{L(\theta + 1) (z^{|t|-1})}{|t|!} \\ &= \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \alpha(t) t! \delta_t \frac{L(|t|) z^{|t|-1}}{|t|!} \\ &= \sum_{t \in \mathcal{R}} \frac{B(t)}{B_{|t|}} \alpha(t) t! \delta_t \frac{B_{|t|} |t| z^{|t|-1}}{|t|!} \\ &= \sum_{t \in \mathcal{R}} \frac{z^{|t|-1}}{(|t| - 1)!} B(t) t! \delta_t = \frac{dY}{dz}. \end{aligned}$$

□

Remark 8 As we have observed in Remark 3, $s(t) = \ln(B(t))$ is an inductive map when the weights B_k are positive. It turns out that the exponential generating function associated with s can be given as an integral transform for varieties of increasing trees (see for example Section 1). This is the topic of [1].

Example 9

When $L(z) = z$, with $B_k \equiv 1$, and $\psi(z) = 1/(1-z)$, one has $\sum_{t \in \mathcal{F}_{n+1}} B(t) = C_n$, the Catalan number of order n , with $C_n = \binom{2n}{n}/(n+1)$. Then $Y(z) =$

$z \sum_{n \geq 0} z^n C_n$ is solution of the differential equation $Y'(z) = L(1 + \theta)(1/(1 - Y(z)))$, that is $Y'(z) = (z/(1 - Y(z)))'$. The unique solution with $Y(0) = 0$ satisfies $Y(z) = z/(1 - Y(z))$, or $Y(z) = (1 - \sqrt{1 - 4z})/2$, corresponding to a well known result.

□

[6] , § 5.3, considers the case where $B(t) = (1/t!)^2$, which is not of the form given in (5): in this situation, $B_k = 1/k^2$, with $L(z) = 1/z$. The solution is obtained by using the stucture of the so-called Butcher's group of B-series (that is series of the form (6), where the group structure is given in [13, 14]) by tensoring known B-series:

Example 10

Consider the bare functional given by $B_k \equiv 1/k^2$, with $B(t) = 1/t^2$. Following Brouder, the associated B-series, as given in (6), is solution of the second order differential equation

$$zY'' + Y' = \psi(Y).$$

When $\psi(z) = \exp(z)$, the solution is given by

$$Y(z) = -2 \ln(1 - z/2) = \sum_{n=1}^{\infty} \frac{z^n}{n} \frac{1}{2^{n-1}},$$

giving

$$\sum_{t \in \mathcal{R}_n} \frac{\alpha(t)}{t!} = \frac{(n-1)!}{2^{n-1}}.$$

□

We study the general moment problem $B(t) = (1/t!)^{l+1}$, $l \in \mathbb{N}$, by working directly on a suitable differential equation as follows: the operator $L(\theta + 1)$ takes the form $L(\theta + 1) = 1/(\theta + 1)^l$. Assume that the differential operator $L(\theta + 1)$ is invertible. Then the formal systems becomes

$$L(\theta + 1)^{-1} \frac{d}{dz} Y = \psi(Y). \quad (7)$$

Consider again the second moment problem for the tree factorial, with $B_k \equiv 1/k^2$ and $L(k) \equiv 1/k$. Choose L such that $L(z) = 1/z$; the inverse operator might be equal to $L(\theta + 1)^{-1} = \theta + 1$, and, if this is the case,

$$(\theta + 1) \frac{d}{dz} Y = \psi(Y),$$

with $(\theta + 1)(d/dz) = z(d^2/dz^2) + (d/dz)$, see Example 10.

More generally, if one considers the moment of order $l + 1 \in \mathbb{N}$ of the inverse tree factorial, the choice $L(k) = 1/k^l$ should give $(\theta + 1)^l \frac{d}{dz} Y(z) = \psi(Y)$. Our result, Theorem 2 below shows that the formalism of inversion is correct in term of power series. This result sheds some light and extends the computations done in [6] for the second moment, and its proof avoids computations in the Butcher's group.

Theorem 2 *The B-series $Y(z)$ associated with the moment of order $(l + 1)$ of the inverse tree factorial satisfies the differential equation*

$$(\theta + 1)^l \frac{d}{dz} Y = \psi(Y). \quad (8)$$

Proof: Let

$$Y(z) = \sum_{t \in \mathcal{R}} \frac{z^{|t|}}{|t|!} \alpha(t) \frac{1}{t^{l+1}} t! \delta_t.$$

Then

$$\psi(Y(z)) = \sum_{k \geq 0} \frac{\psi^{(k)}(0)}{k!} \sum_{(t_1 \dots t_k) \in \mathcal{R}^k} \frac{z^{\sum_i |t_i|}}{|t_1|! \dots |t_k|!} \frac{\alpha(t_1) \dots \alpha(t_k)}{(t_1! \dots t_k!)^l} \delta_{t_1} \dots \delta_{t_k}.$$

For given $(t_1 \dots t_k) \in \mathcal{R}^k$, set $t = B_+(t_1, \dots, t_k)$. Then $\sum_i |t_i| = |t| - 1$, $\psi^{(k)}(0) \delta_{t_1} \dots \delta_{t_k} = \delta_t$, and $(t_1! \dots t_k!)^l = t^l / |t|^l$. The associated term becomes

$$\frac{z^{|t|-1} |t|^l}{t^l} \delta_t \frac{\alpha(t_1) \dots \alpha(t_k)}{|t_1|! \dots |t_k|!}.$$

Proceeding as in the proof of Theorem 1, we get that the contribution associated with $t \in \mathcal{R}$ is given by

$$\frac{z^{|t|-1} |t|^l}{t^l} \delta_t \frac{|t| \alpha(t)}{|t|!} = \frac{z^{|t|-1}}{(|t| - 1)!} \frac{|t|^l}{t^l} \alpha(t) \delta_t.$$

On the other hand,

$$\begin{aligned} (\theta + 1)^l \frac{d}{dz} Y(z) &= (\theta + 1)^l \sum_{t \in \mathcal{R}} \frac{z^{|t|-1}}{(|t| - 1)!} \alpha(t) \frac{1}{t^l} \delta_t \\ &= \sum_{t \in \mathcal{R}} \frac{|t|^l z^{|t|-1}}{(|t| - 1)!} \alpha(t) \frac{1}{t^l} \delta_t. \end{aligned}$$

□

In the next section, we show that traces of certain products of Wigner matrices (see for example [24]) provide natural examples of bare Green functions.

5 Wigner processes

Definition 11 *The N -dimensional random matrices $\Gamma_N := (\gamma_{i,j})_{1 \leq i,j \leq N}$ are called Wigner matrices of variance β^2 if the following holds.*

- *Each Γ_N is symmetric, that is, $\gamma_{i,j} = \gamma_{j,i}$.*
- *For $i \leq j$, the random variables $\gamma_{i,j}$ are independent and centered.*
- *For $i \neq j$, $\mathbb{E}(\gamma_{i,j}^2) = \beta^2$.*
- *For any $k \geq 2$, $\mathbb{E}(|\gamma_{i,j}|^k) \leq c_k$, where c_k is independent of $i \leq j$.*

Definition 12 *The sequence $\Gamma_N(k) := (\gamma_{i,j}(k))_{1 \leq i,j \leq N}$ of N -dimensional random matrices, indexed by $k \geq 1$, is called a Wigner process of variance β^2 and correlation function r , $r(k,k) = 1$, $|r(k,m)| \leq 1$ and $r(k,m) = r(m,k)$ if the following holds.*

- *Each $\Gamma_N(k)$ is a Wigner matrix of variance β^2 in the sense of definition 11.*
- *For $i \leq j$, each process $(\gamma_{i,j}(k))_k$ is independent of the others.*
- *For $i \neq j$, the process $(\gamma_{i,j}(k))_k$ is r -correlated, that is, for any $k \geq m$,*

$$\mathbb{E}(\gamma_{i,j}(k)\gamma_{i,j}(m)) := \beta^2 r(k,m). \quad (9)$$

A Wigner process is stationary when r is such that $r(k,m) = r(|k-m|)$.

Let D_N be a sequence of random diagonal matrices, with independent and identically distributed entries of law μ , having finite moment $\mu_k = \mu(X^k)$, $k \geq 1$, with $\mu_1 = 1$. Let

$$Q_k^N := N^{-k/2} D_N \prod_{m=1}^k (\Gamma_N(m) D_N),$$

and set

$$B_k^N(r) = N^{-1} \mathbb{E}(\text{tr}(Q_k^N)).$$

Involutions, Dyck paths and rooted plane trees

For $k \geq 1$, $[k] := \{1, 2, \dots, k\}$, $\mathcal{I}(k)$ is the set of the involutions of $[k]$ with no fixed point, $\mathcal{J}(k)$ is the subset of $\mathcal{I}(k)$ of the involutions σ with

no crossing. This means that the configurations $i < j < \sigma(i) < \sigma(j)$ do not appear in $\sigma \in \mathcal{J}(k)$. Let $i \in \text{cr}(\sigma)$ denote the fact that $i < \sigma(i)$. Let $\mathcal{D}(2k)$ be the set of the Dyck paths of length $2k$, that is, of the sequences $c := (c_n)_{0 \leq n \leq 2k}$ of nonnegative integers such that $c_0 = c_{2k} = 0$, $c_n - c_{n-1} = \pm 1$, $n \in [2k]$. Thus, exactly k indices $n \in [2k]$ correspond to ascending steps (c_{n-1}, c_n) , that is, to steps when $c_n = c_{n-1} + 1$. We denote this by $n \in \text{asc}(c)$. The k others indices correspond to descending steps, that is, to steps when $c_n = c_{n-1} - 1$, and we denote this by $n \in \text{desc}(c)$. We make use of bijections between $\mathcal{D}(2k)$ and $\mathcal{J}(2k)$ [2]. If $c \in \mathcal{D}(2k)$, $\phi(c) := \sigma \in \mathcal{J}(2k)$ is an involution which maps each element of $\text{desc}(c)$ to a smaller element of $\text{asc}(c)$. Thus, $\text{cr}(\sigma) = \text{asc}(c)$. More specifically, if $n \in \text{desc}(c)$, $\sigma(n)$ is the greatest $m \leq n$ such that $(c_{m-1}, c_m) = (c_n, c_{n-1})$. Finally, the set $\mathcal{D}(2k)$ is in bijection with \mathcal{F}_{k+1} , the set of rooted plane trees on $k+1$ nodes, where the bijection is given by the walk on the tree from the right to the left (see for example [25]). Let σ_t denote the involution of $\mathcal{J}(2(|t| - 1))$ corresponding to $t \in \mathcal{F}$. Given $t \in \mathcal{F}_{k+1}$, consider the walk on t from the right to the left: every edge (v, w) with $w \in \text{ch}(v)$, is crossed at some instant $s_v \in [2k]$ as $(v \rightarrow w)$ and at a later time $s_w \in [2k]$ as $(w \rightarrow v)$. Clearly, $s_w = s_v + 2(|t_w| - 1) + 1$, where t_w is the subtree of t rooted at node w , that is the subgraph of t induced by the nodes u with $u \geq w$. σ_t is such that $\sigma_t(s_v) = s_w$ and vice versa.

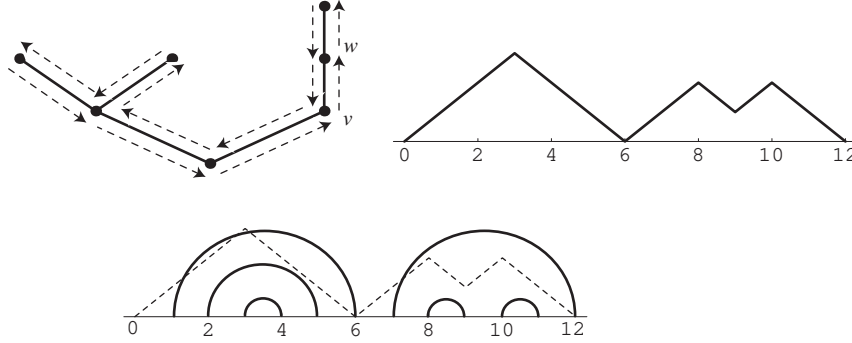


Fig 1. Bijections between \mathcal{F}_{k+1} , $\mathcal{D}(2k)$ and $\mathcal{J}(2k)$ ($s_v = 2$ and $s_w = 5$).

Proposition 13 Assume that the covariance r is such that $r(l, m) = r(|l - m|)$. Then, the functional $B^r \in \mathbf{B}$ given by the weights

$$B_k^r = \beta^2 r(2k - 1), \quad \forall k \geq 1,$$

is such that

$$\frac{B^r(t)}{B_{|t|}^r} = \prod_{i \in \text{cr}(\sigma_t)} (\beta^2 r(i, \sigma_t(i))).$$

Proof: Let $t \in \mathcal{F}$. Let $s_v < s_w$ be the instants where the oriented edges $(v \rightarrow w)$ and $(w \rightarrow v)$, $w \in \text{ch}(v)$, are crossed during the walk on the tree. $r(s_v, \sigma_t(s_v)) = r(s_w - s_v) = r(2|t_w| - 1)$, and thus $\beta^2 r(s_v, \sigma_t(s_v)) = B_{|t_w|}^r$. Finally, $\prod_{i \in \text{cr}(\sigma_t)} \beta^2 r(i, \sigma_t(i)) = \prod_{w \neq \text{root}} B_{|t_w|}^r = B^r(t)/B_{|t|}^r$, as required. \square

As we have just seen, every Wigner process with covariance r such that $r(l, m) = r(|l - m|)$ produces a bare Green function $B^r \in \mathbf{B}$. The converse is not true, that is, there exists $B \in \mathbf{B}$ such that B is not of the form $B = B^r$ for some covariance function r . Set $\mathbf{B}^w = \{B \in \mathbf{B}; \exists \text{ a covariance } r \text{ with } B = B^r\}$.

Let ψ_μ be the generating function of the weight sequence $\psi_k = \mu_{k+1}$, and let $\omega_\mu(t)$, $t \in \mathcal{F}$ be the associated weight function.

Theorem 3 *Let $(\Gamma_N(k))_{k \geq 1}$ be a stationary Wigner process of covariance function r and variance β^2 , and let D_N be a sequence of random diagonal matrices, independent of the Wigner process, with i.i.d. entries λ_j of law μ , with $\mu_1 = \mu(\lambda) = 1$ and finite moments $\mu_k = \mu(\lambda^k)$, $\forall k$. Then*

$$B_{2k}^N(r) \longrightarrow B_{2k}(r) = \frac{1}{B_{k+1}^r} \sum_{t \in \mathcal{F}_{k+1}} B^r(t) \omega_\mu(t),$$

and $B_{2k+1}^N(r) \longrightarrow 0$, $N \rightarrow \infty$. Assume that the covariance is such that there exists a power series $L^r(z)$ with $B_k^r = L^r(k)/k$, $\forall k$. Then the formal power series

$$Y(z) = \sum_{k \geq 1} z^k B_k^r B_{2(k-1)}(r),$$

solves

$$Y' = L^r(\theta + 1) \psi_\mu(Y).$$

Moreover

$$\sum_{k \geq 1} z^k B_{2(k-1)}(r) = z \psi_\mu(Y). \quad (10)$$

Example 14

Let $B(t) = 1/t!$. If a tree t has n nodes and $n - 1$ edges, then the requirement $B_n = 1/n$ is satisfied iff $\beta^2 r(2n - 1) = 1/n$, that is r must be such that $\beta^2 r(k) = 2/(k + 1)$, $k \in 2\mathbb{N} + 1$. By construction, $r(0) = 1$ and therefore $\beta^2 = 2$. $1/(x + 1)$ is positive definite, which implies that $B(t) = 1/t!$ is element of \mathbf{B}^w . Next, from Theorem 2, the generating function $Y(z) = \sum_{t \in \mathcal{F}} z^{|t|} B^r(t) \omega_\mu(t)$ is solution of the system $(d/dz)Y(z) = \psi_\mu(Y)$.

Assume that μ is the point mass δ_1 , that is each matrix D_N is the identity matrix of size N , with $\psi_\mu(z) = 1/(1-z)$. The solution of the system is $Y(z) = 1 - \sqrt{1-2z} = 2\tilde{Y}(z/2)$, where \tilde{Y} is the series given in Example 9. On the other hand, Proposition 13 and Theorem 3 show that $Y(z) = \sum_{k \geq 1} z^k B_k^r B_{2(k-1)}(r)$. Therefore the limiting mean normalized trace $B_{2k}(r)$ of the product of correlated random matrices $\prod_{m=1}^{2k} \Gamma_N(m)$ is such that $B_{2k}(r) = E(Z^{2k})/k!$, where Z denotes a normal $N(0,1)$ random variable.

□

Example 15

Consider as in Example 9 the special case where $L(z) = z$. The associated inductive parameter (see Remark 8) is the tree size. The covariance r is constant with $r(k) \equiv 1$, and $B_k^r \equiv 1$. Then the generating function Y is solution of the fixed point equation $Y(z) = z\psi_\mu(Y(z))$ (either by Theorem 1 or by (10)). Notice that in this situation, $\Gamma_N(m) \equiv \Gamma_N(1)$, and thus $B_k^N(r)$ describes the mean normalized moment of the spectral measure of the random matrix $D_N(\Gamma_N(1)D_N)^k$. This example can be extended by considering $L(z) = z\rho^z$, for some $0 < \rho \leq 1$. When D_N is the identity matrix, $Y(z)$ is related to the Rogers-Ramanujan continued fraction [19], and corresponds to the generating function associated with path length, see [1, 25].

□

Proof of Theorem 3: The first part is a generalization of Theorem 1 of [19]. Set $\tilde{\gamma}_{ij}(m) = \gamma_{ij}(m)\lambda_j$, and $\tilde{\Gamma}_N(m) = \Gamma_N(m)D_N$. The mean normalized trace adds the contributions $E(i) = E(\lambda_{i_0}\tilde{\gamma}_{i_0i_1}\cdots\tilde{\gamma}_{i_{k-1}i_k})$, for paths $i = (i_l)_{0 \leq l \leq k}$, with $i_l \in [N]$ and $i_0 = i_k$. The $\tilde{\gamma}_{ij}$ are centered, so that any edge (i, j) appearing once appears at least twice. Given i , define $\varepsilon_1 = 1$ and $\varepsilon_l = +1$ when $i_l \notin \{i_0, \dots, i_{l-1}\}$, and $\varepsilon_l = -1$ otherwise, and consider the walk $c = (c_l)$ defined by $c_l = \sum_{j=1}^l \varepsilon_j$, with $c_k \leq 0$. The support of i is $s(i) = \{i_l; 0 \leq l \leq k\}$, of size $s = |s(i)|$, with $s \leq 1 + k/2$. The contribution $E(i)$ is independent of the labels i_l ; they are $N(N-1)\cdots(N-s+1)$ labellings giving the same walk c , with the same contribution. Thus, the normalization $N^{-(1+k/2)}$ shows that the only walks surviving in the large N limit are those with $s = 1 + k/2$. This shows that $B_k^N(r) \rightarrow 0$ when k is odd. Concerning $B_{2k}^N(r)$, $s = 1 + k$ means that every edge occurring in the path occurs exactly twice, in opposite directions. c is a Dyck path of $\mathcal{D}(2k)$; let $t \in \mathcal{F}$ be the associated rooted plane tree, with involution σ_t . Using the right to left walk on t and the independence of the random variables, the contribution $E(i)$ of any path leading to c or t is $E(i) = \prod_{m \in cr(\sigma_t)} E(\gamma(m)\gamma(\sigma_t(m))) E(\prod_v \lambda_v^{d(v)+1})$ where $d(v) = |\text{ch}(v)|$.

From Proposition 13, one obtains $E(i) = (B^r(t)/B_{k+1}^r) \prod_v \mu_{d(v)+1}$, with $B_{2k}(r) = \sum_{t \in \mathcal{F}_{k+1}} (B^r(t)/B_{k+1}^r) \prod_v \mu_{d(v)+1}$, as required. (10) is a consequence of the multiplicative form of bare Green functions and of Lemma 1.9, chap. III.1 of [14].

□

These results show that the elements of \mathbf{B}^w appear naturally in the computation of normalized traces of products of large random matrices (see for example [23]). In the next Section we illustrate B-series by considering triangular operators from free probability.

6 On Dykema-Haagerup triangular operator

Let \mathcal{B} be an algebra and \mathcal{A} be a \mathcal{B} bi-module. Let $\kappa : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$ be a bilinear map. We follow [22] by defining the product $a_1 \bullet_\kappa a_2 = \kappa(a_1, a_2)$, $a_1, a_2 \in \mathcal{A}$, and setting

$$i) (ba_1) \bullet_\kappa a_2 = b(a_1 \bullet_\kappa a_2),$$

$$ii) (a_1 b) \bullet_\kappa a_2 = a_1 \bullet_\kappa (ba_2),$$

$$iii) a_1 \bullet_\kappa (a_2 b) = (a_1 \bullet_\kappa a_2) b.$$

Let $\sigma \in \mathcal{J}(2n)$ be an involution of $[2n]$ without fixed point and without crossing. Given a word $a = a_1 \cdots a_{2n}$ in \mathcal{A} , σ induces parentheses on a , and the precedings rules permit the evaluation of this parenthized word. This extends to a map κ_σ on \mathcal{A}^{2n} . Sniady defines such maps to prove a conjecture of Dykema and Haagerup on generalized circular elements. Let $(\mathcal{B} \subset A, E)$ be an operator valued probability space, that is \mathcal{A} is a unital $*$ -algebra, $\mathcal{B} \subset A$ an unital $*$ -subalgebra and $E : \mathcal{A} \rightarrow \mathcal{B}$ be a conditional expectation (linear, $E(1) = 1$, and $E(b_1 a b_2) = b_1 E(a) b_2$, $\forall b_1, b_2 \in \mathcal{B}$, $a \in \mathcal{A}$).

Definition 16 *$T \in \mathcal{A}$ is a generalized circular element if there is a bilinear map κ satisfying the rules i), ii) and iii) such that*

$$E(b_1 T^{s_1} b_2 T^{s_2} \cdots b_{2n} T^{s_{2n}}) = \sum_{\sigma \in \mathcal{J}(2n)} \kappa_\sigma(b_1 T^{s_1}, \dots, b_{2n} T^{s_{2n}}),$$

$$E(b_1 T^{s_1} b_2 T^{s_2} \cdots b_{2n+1} T^{s_{2n+1}}) = 0,$$

$\forall b_1, \dots, b_{2n+1} \in \mathcal{B}$ and $\forall s_1, \dots, s_{2n+1} \in \{1, *\}$.

The triangular operator T of Dykema and Haagerup is obtained from $\mathcal{B} = \mathbb{C}[x]$, the $*$ -algebra of complex polynomials of one variable by setting

$$[\kappa(T, bT^*)](x) = \int_x^1 b(s) ds,$$

$$[\kappa(T^*, bT)](x) = \int_0^x b(s)ds,$$

$$[\kappa(T, bT)](x) = [\kappa(T^*, bT^*)](x) = 0.$$

T is the limit for the convergence of $*$ -moments of large upper triangular random matrices T_N ([10]). Define a trace τ as (see [22])

$$\tau(a) = \tau(E(a)), \quad \tau(b) = \int_0^1 b(s)ds.$$

In what follows, we use P-series (where P stands for partitioned differential systems, see [13]). We follow [6], and adapt his notations to P-series. Given some function ψ , and two kernels $(a^x(u, v))_{u, v \in [0, 1]}$ and $(a^y(u, v))_{u, v \in [0, 1]}$, consider the iterated integrals ϕ_u^x and ϕ_u^y which are functionals over \mathcal{R} defined by $\phi_u^x(*) = \phi_u^y(*) = 1$, and, for $t = B_+(t_1, \dots, t_k)$,

$$\phi_u^x(t) = \prod_{i=1}^k \int_0^1 a^x(u, v) \phi_v^y(t_i) dv,$$

$$\phi_u^y(t) = \prod_{i=1}^k \int_0^1 a^y(u, v) \phi_v^x(t_i) dv.$$

Lemma 17 *Let $a^x(u, v) = I_{[0, u]}(v)$ and $a^y(u, v) = I_{[u, 1]}(v)$. Then*

$$\tau(TT^*)^n = \sum_{t \in \mathcal{F}_{n+1}} \int_0^1 \phi_v^x(t) dv = \sum_{t \in \mathcal{F}_{n+1}} \int_0^1 \phi_v^y(t) dv.$$

Proof: The word $W = (TT^*) \cdots (TT^*)$ is of the generic form with $b_1 = \cdots b_{2n} = 1$ (Definition 16). Let $t \in \mathcal{F}_{n+1}$ with associated involution σ_t (see Section 5). Let s_v and s_w be the instants where the walk on t crosses the oriented edges $(v \rightarrow w)$ and $(w \rightarrow v)$, with $w \in \text{ch}(v)$. We colour these edges by giving colour '1' to $(v \rightarrow w)$ when the symbol in W located at position s_v is T , and give the colour '*' otherwise. Clearly, both edges have different colours, and the elements of the set of edges $\{(v \rightarrow w); w \in \text{ch}(v)\}$ (the children of v in t) have the same colour. The result is then a consequence of the definition of the product with the rules i), ii) and iii). □

Remark 18 *Iterated integrals are naturel objects to consider in the setting of Butcher's Theory. For example, in the framework of Theorem 1, the iterated integrals $\phi_u(t)$ defined by $\phi_u(t) = \prod_{i=1}^k \int_0^u L(\theta + 1)(\phi_v(t_i)) dv$, when $t = B_+(t_1, \dots, t_k)$, are such that $\phi_1(t) = B(t)$, $\forall t \in \mathcal{F}$.*

Proposition 19 *The P-series*

$$X_u(s) = X_0 + \sum_{t \in \mathcal{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta_t \int_0^1 a^x(u, v) \phi_v^y(t) dv,$$

and

$$Y_u(s) = Y_1 + \sum_{t \in \mathcal{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta_t \int_0^1 a^y(u, v) \phi_v^x(t) dv,$$

are solutions of the integral system

$$\begin{aligned} X_u(s) &= X_0 + s \int_0^1 a^x(u, v) \psi(Y_v(s)) dv, \\ Y_u(s) &= Y_1 + s \int_0^1 a^y(u, v) \psi(X_v(s)) dv. \end{aligned}$$

Proof: This is consequence of Butcher's general theory (see [4]). To prove it more directly, proceed as in the proof of Theorem 1

□

Corollary 20 *Let $X_0 = Y_1 = 0$. Assume that $a^x(u, v) = \mathbf{I}_{[0, u]}(v)$ and $a^y(u, v) = \mathbf{I}_{[u, 1]}(v)$. Suppose that $\psi(z) = 1/(1 - z)$. Then*

$$Y_0(s) = \sum_{t \in \mathcal{R}} \frac{s^{|t|}}{|t|!} \alpha(t) t! \delta_t \int_0^1 \phi_v^x(t) dv = \sum_{t \in \mathcal{F}} s^{|t|} \tau(TT^*)^{|t|-1}.$$

This result shows that the generating function of the *-moments of the operator TT^* can be obtained by solving the system given in Proposition 19. We recover in this way a result of [10], Lemmas 8.5 and 8.8.

Lemma 21 *In the setting of Corollary 20, the generating function $Y_0(s)$ solves*

$$G\left(\frac{s}{1 - Y_0(s)}\right) = s, \tag{11}$$

where $G(z) = z \exp(-z)$, that is, $L(s) = s/(1 - Y_0(s))$ and G are inverse with respect to composition. Moreover $\tau(TT^*)^n = n^n/(n+1)!$.

Proof: We solve the integral system by looking for solutions of the form $X_u(s) = 1 - \exp(\lambda u)$ and $Y_u(s) = 1 - \exp(\lambda'(u - 1))$, with $(d/du)X_u(s) = s/(1 - Y_u(s))$ and $(d/du)Y_u(s) = -s/(1 - X_u(s))$. We deduce that $\lambda' = -\lambda$ is solution of the equation $\lambda + s \exp(-\lambda) = 0$. The formula for the moments of TT^* is a consequence of Lagrange's inversion formula.

□

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