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On the Probability of Rendezvous in Graphs

Martin Dietzfelbinger Hisao Tamaki

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Stuhlsatzenhausweg 85 66123 Saarbrücken Germany

Authors' Addresses

Martin Dietzfelbinger
Technische Universität Ilmenau
Fakultät für Informatik und Automatisierung
98684 Ilmenau, Germany
`martin.dietzfelbinger@tu-ilmenau.de`

Hisao Tamaki
Meiji University
Department of Computer Science
Kawasaki 214-8517, Japan
`tamaki@cs.meiji.ac.jp`

Abstract

In a simple graph G without isolated nodes the following random experiment is carried out: each node chooses one of its neighbors uniformly at random. We say a rendezvous occurs if there are adjacent nodes u and v such that u chooses v and v chooses u; the probability that this happens is denoted by $s(G)$. Métivier *et al.* (2000) asked whether it is true that $s(G) \geq s(K_n)$ for all n-node graphs G, where K_n is the complete graph on n nodes. We show that this is the case. Moreover, we show that evaluating $s(G)$ for a given graph G is a #P-complete problem, even if only d-regular graphs are considered, for any $d \geq 5$.

Note: Parts of the results of this paper were reported at ISAAC 2002.

Keywords

Network, Rendezvous experiment, Success probability, Negative correlation, #P-completeness

1 Introduction

The following random experiment was proposed and studied by Métivier, Saheb, and Zemmari [6]. Let $G = (V, E)$ be an undirected graph (which might represent a processor network) with $|V| = n \geq 2$ nodes. Each node $u \in V$ independently chooses one of its neighbors uniformly at random. The experiment does not make sense if G has a node without neighbors; thus throughout the paper we assume there are no isolated nodes in G. We say *there is a rendezvous*, and consider this as a "*success*", if there is an edge (u, v) in G such that u chooses v and v chooses u. Let $s(G) \in [0, 1]$ be the probability that there is a rendezvous if this experiment is carried out in G.

As usual, K_n denotes the complete graph with n nodes. The following is known [6]:

(i) $s(G) \geq 1 - e^{-n/2(n-1)}$ for all n-node graphs G;

(ii) $s(K_n) \to 1 - e^{-1/2}$ for $n \to \infty$.

Since obviously $e^{-n/2(n-1)} < e^{-1/2}$ for every n, asymptotically the complete graphs achieve the minimum rendezvous probability, namely $1 - e^{-1/2} \approx 0.39347$. It is natural to ask, as done by Métivier *et al.* [6, Remark 22], whether for *each* n the complete graph minimizes the rendezvous probability among all n-node graphs. The first purpose of this paper is to prove that this is true:

Theorem 1 *If G is a graph with $n \geq 2$ (non-isolated) nodes, then $s(G) \geq s(K_n)$.*

The randomized rendezvous protocol was introduced in [6]. The reader is referred to that paper for a thorough study of the rendezvous protocol, including its behaviour in some graph families, in particular trees, rings, and graphs with bounded degree. In Section 2, we will repeat the central definitions and the relevant results from [6], so as to make the present paper largely self-contained. On the basis of a preliminary version of [6], Austinat in his Diplom thesis [1] studied some aspects of the rendezvous experiment, in particular the problem dealt with in our Theorem 1.

Since the proof of Theorem 1 is rather involved, for motivation we will describe an obvious, but flawed proof strategy and give an outline of the proof in in Section 3.1. In Section 3.2 the main part of the proof is given; the proof of a technical lemma is supplied in Section 3.3. Although the applicability of the result may be limited, the proof of the theorem identifies some subtle dependencies that must be taken into account in random experiments in graphs, and it develops a novel way to deal with such dependencies.

The basic difficulty in proving Theorem 1 is that different graphs induce different probability spaces and that it takes some care to establish connections between these. The same difficulty is encountered when one looks at the following seemingly very simple question. It is easy to evaluate $s(K_n)$ for small values of n, and to obtain $s(K_2) = 1$, $s(K_3) = 0.75$, $s(K_4) = \frac{17}{27}$, and so on (see table in Section 2). One notices that the values $s(K_n)$, $n = 2, 3, \ldots$, are strictly decreasing for small values of n. We show that this is true in general:

Theorem 2 *$s(K_n) > s(K_{n+1})$, for all $n \geq 2$.*

The proof is given in Section 4. Even though the theorem may seem natural, the proof is by no means immediate. It combines manipulations of explicit formulae for $s(K_n)$, modifications of the original random experiments and basic calculus.

The last theme treated in this paper is the complexity of evaluating $s(G)$ if the graph G is given. It is not difficult to give a "closed" formula in the inclusion-exclusion style (see (2.5) in Section 2), but this formula involves a summation over all matchings in G, and hence usually exponentially many terms. Indeed, it is this connection to the problem of counting matchings in graphs that makes it possible to show our last result.

Recall ([7] and e. g., [4, ch. 9]) that #P is the class of all functions $f\colon \Sigma^* \to \mathbb{N}$ (for some alphabet Σ) so that there is a polynomially time bounded Turing machine M with

$$f(x) = \#(\text{accepting computations of } M \text{ on input } x) \; ,$$

and that a function f is #P-complete (w. r. t. Turing reductions) if it is in #P and if each $g \in \#\mathsf{P}$ can be computed by a polynomially time bounded oracle Turing machine with oracle f.

Theorem 3 *For any fixed integer $d \geq 5$, the problem of computing $s(G)$ for a given d-regular graph G is polynomially equivalent to some #P-complete problem.*

This theorem implies that it is unlikely that the rendezvous probability of graphs, even if restricted to regular graphs, can be calculated by a polynomial-time algorithm. The proof is given in Section 5. It involves a Turing reduction from the problem of counting perfect matchings in $(d-2)$-regular graphs to the problem of determining $s(G)$.

2 Basics

In this section, we give the basic definitions and list some known facts about the randomized rendezvous experiment. For further details, see [6].

Unless stated otherwise, we only consider graphs $G = (V, E)$, $n = |V| \geq 2$, without isolated nodes.

The set of neighbors of $u \in V$ is denoted by $N(u)$, its degree by $\deg(u) = |N(u)|$. (We have $1 \le \deg(u) \le n-1$.) We consider the random experiment of each node choosing one of its neighbors at random. The probability space induced by this experiment can be described as follows. An elementary event is a sequence $(c_u)_{u\in V}$, where $c_u \in N(u)$ for each u. This event occurs with (uniform) probability $\prod_{u\in V} \frac{1}{\deg(u)}$. Probabilities with respect to this experiment are denoted as $\mathbf{P}_G(\cdot)$ (or $\mathbf{P}(\cdot)$, if G is implied by the context). We say that *a rendezvous occurs at edge* $e = (u,v) \in E$ if $c_u = v$ and $c_v = u$; this event is denoted by R_e. We say that *a rendezvous occurs* if there is an edge at which a rendezvous occurs; this event is denoted by R or R_G. We let

$$s(G) = \mathbf{P}_G(R_G) = \mathbf{P}(R) = \mathbf{P}\Big(\bigcup_{e\in E} R_e\Big),$$

(the probability of a "success"), and $f(G) = 1 - s(G)$ (the probability of a "failure"). Let $p_e = \mathbf{P}(R_e)$. Clearly, for $e = (u,v)$ we have

$$p_e = \frac{1}{\deg(u)\cdot\deg(v)}.$$

An important measure is

$$m(G) = \sum_{e\in E} p_e,$$

the expected number of edges at which a rendezvous occurs. Clearly, $s(G) \le m(G)$. It is easy to establish a lower bound on $m(G)$:

$$m(G) = \frac{1}{2}\cdot\sum_{u\in V}\sum_{v\in N(u)} \frac{1}{\deg(u)\deg(v)} \ge \frac{1}{2}\cdot\sum_{u\in V}\sum_{v\in N(u)} \frac{1}{\deg(u)(n-1)} = \frac{1}{2}\cdot\frac{n}{n-1}. \tag{2.1}$$

For K_n, the complete graph on n nodes, this is an equality, since all nodes have degree $n-1$:

$$m(K_n) = \frac{1}{2}\cdot\sum_{u\in V}\sum_{v\in V-\{u\}} \frac{1}{(n-1)(n-1)} = \frac{1}{2}\cdot\frac{n}{n-1}.$$

The inequality $m(G) \ge m(K_n)$, which holds for all n-node graphs G, was a first indication for Theorem 1 to be true. In order to obtain a lower bound for $s(G)$ in terms of $m(G)$ we need the following basic fact, which is stated in [1] (with a proof by J. M. Robson) and also in [6, Proof of Prop. 16]. We give a simple proof without calculations.

Fact 4 *Let $B \subseteq E$ be an arbitrary set of edges, and let $e \in E - B$. Let C_B be the event that no rendezvous occurs along any edge in B. If $\mathbf{P}(C_B) > 0$, then*

$$\mathbf{P}(R_e \mid C_B) \ge \mathbf{P}(R_e) = p_e.$$

Proof. We must show:

$$\mathbf{P}(R_e \cap C_B) \geq \mathbf{P}(R_e) \cdot \mathbf{P}(C_B). \tag{2.2}$$

Let $e = (u, v)$. Let B' denote the set of those edges in B that are not incident with u or v, and let $C_{B'}$ denote the event that there is no rendezvous on an edge in B'. Observe that

(i) $C_B \subseteq C_{B'}$;

(ii) $R_e \cap C_B = R_e \cap C_{B'}$ (if there is a rendezvous on edge e, there cannot be a rendezvous on any edge in B incident with u or v);

(iii) the events $C_{B'}$ and R_e are independent,

and estimate

$$\mathbf{P}(R_e \cap C_B) \overset{\text{(ii)}}{=} \mathbf{P}(R_e \cap C'_B) \overset{\text{(iii)}}{=} \mathbf{P}(R_e) \cdot \mathbf{P}(C'_B) \overset{\text{(i)}}{\geq} \mathbf{P}(R_e) \cdot \mathbf{P}(C_B),$$

as desired. ■

From this fact it follows immediately that $\mathbf{P}(\overline{R}_e \mid C_B) \leq 1 - p_e$, whenever $\mathbf{P}(C_B) > 0$. An easy induction argument then shows that

$$1 - s(G) = f(G) = \mathbf{P}\Big(\bigcap_{e \in E} \overline{R}_e\Big) \leq \prod_{e \in E}(1 - p_e) \leq \exp(-\sum_{e \in E} p_e) = e^{-m(G)} \;,$$

where $\exp(x) = e^x$ is the exponential function. (We have used that $1 + y \leq e^y$ for all y.) We obtain

$$s(G) \geq 1 - e^{-m(G)}. \tag{2.3}$$

Together with (2.1) this implies a global constant lower bound for the rendezvous probability in graphs G with an arbitrary number n of nodes [6, Thm. 19, Cor. 20]:[1]

$$s(G) \geq 1 - e^{-n/2(n-1)} > 1 - e^{-1/2} \approx 0.39347 \;, \quad \text{for } n \geq 2 \,. \tag{2.4}$$

We proceed to note a "closed" formula for $s(G)$. In the general case, it has exponentially many terms, and hence is practically not very useful. However, it will be important in dealing with $s(K_n)$ in the proof of Theorem 2.

The general inclusion-exclusion formula for the probability of unions of events, when applied to $R_e, e \in E$, reads:

$$s(G) = \mathbf{P}\Big(\bigcup_{e \in E} R_e\Big) = \sum_{\emptyset \neq E' \subseteq E} (-1)^{|E'|+1} \cdot \mathbf{P}\Big(\bigcap_{e \in E'} R_e\Big).$$

[1] The first direct proof of this estimate was given by J. M. Robson.

In our case, if $e \neq e'$ have a node in common, then $\mathbf{P}(R_e \cap R_{e'}) = 0$. This means that in the sum all summands disappear where E' is not a set of node-disjoint edges — in other words, if E' is not a matching in G. Now if $M \subseteq E$ *is* a matching, then the events R_e, $e \in M$, are independent, hence we can write

$$\mathbf{P}(\bigcap_{e \in M} R_e) = \prod_{e \in M} p_e.$$

This immediately leads to the following formulation: Let $\mathcal{M}(G, j)$ denote the set of all matchings $M \subseteq E$ in G that contain exactly j edges, and let

$$S(G, j) = \sum_{M \in \mathcal{M}(G,j)} \prod_{e \in M} p_e.$$

Then

$$s(G) = \sum_{j \geq 1} (-1)^{j+1} S(G, j) , \tag{2.5}$$

Using (2.5), $s(G)$ is easily calculated for small graphs G. Moreover, (2.5) yields a useful formula for $s(K_n)$, for all $n \geq 2$. Namely, a moment's thought reveals that in K_n there are exactly

$$\binom{n}{2}\binom{n-2}{2} \cdots \binom{n-2j+2}{2} \Big/ j! = \frac{n^{\underline{2j}}}{j! \cdot 2^j}$$

matchings of cardinality j, where $n^{\underline{k}}$ denotes the "falling factorial" $n(n-1) \cdots (n-k+1)$ of k factors. If M is such a matching, then $\prod_{e \in M} p_e = 1/(n-1)^{2j}$. Thus,

$$s(K_n) = \sum_{j \geq 1} (-1)^{j+1} \cdot \frac{n^{\underline{2j}}}{j! \cdot 2^j \cdot (n-1)^{2j}} . \tag{2.6}$$

Equation (2.6) can be used to calculate $s(K_n)$ for small n, see Table 1.

n	2	3	4	5	6	7	8
$s(K_n)$	1	$\frac{3}{4}$	$\frac{17}{27}$	$\frac{145}{256}$	$\frac{1653}{3125}$	$\frac{7847}{15552}$	$\frac{401491}{823543}$
decimal value	1.0	0.75	0.6296...	0.56640625	0.52896	0.5045...	0.4875...

Table 1: $s(K_n)$ for small values of n

The first summand in the sum is $m(K_n) = n/2(n-1)$. If we fix $j \geq 2$, then $(-1)^{j+1} n^{\underline{2j}} / (j! \cdot 2^j \cdot (n-1)^{2j})$ tends to $(-1)^{j+1}/(j! \cdot 2^j)$ for $n \to \infty$ and has absolute value bounded by $1/(j! \cdot 2^j)$. This makes it easy to show that

$$s(K_n) \to \sum_{j \geq 1} (-1)^{j+1} \frac{1}{j! \cdot 2^j} = 1 - e^{-1/2}, \text{ for } n \to \infty. \tag{2.7}$$

From (2.4) we know that $s(K_n) \geq 1 - e^{-n/2(n-1)} > 1 - e^{-1/2}$ for all n, hence the limit in (2.7) is approached from above. This observation and Table 1 suggest that $s(K_n)$ is monotonically decreasing in n. This is the assertion of Theorem 2, to be proved later in the paper. For the time being, we establish the following rough estimate:

Lemma 5 *If $n \geq 5$ then $s(K_n) < 0.6$.*

Proof. By the table, $s(K_5) = \frac{145}{256} < 0.6$. If $n \geq 6$, we use that $s(K_n) < m(K_n) = n/2(n-1) \leq 6/10$. ∎

To conclude this section with basic observations, we prove Theorem 1 for graphs with very few nodes, by inspection. For $n = 2$, there is only one graph without isolated nodes, so there is nothing to prove. For $n = 3$ and $n = 4$, we recall one more fact from [6], namely that $s(G) = 1$ if G has a connected component that is a tree. (If each of the l nodes in a tree component chooses some outgoing edge, then — since there are only $l - 1$ edges — at least one edge must be chosen by its two endpoints, creating a rendezvous within the component, hence within G.) For $n = 3$, the only graph G without isolated nodes that is not complete is a path with 2 edges. Then $s(G) = 1 > \frac{3}{4} = s(K_3)$. For $n = 4$, we have $s(K_4) = \frac{17}{27}$, by formula (2.6). If G is not connected, it consists of two disjoint edges, thus has two tree components, and $s(G) = 1$. There are only five different incomplete connected graphs on four nodes, which are depicted in Figure 1.

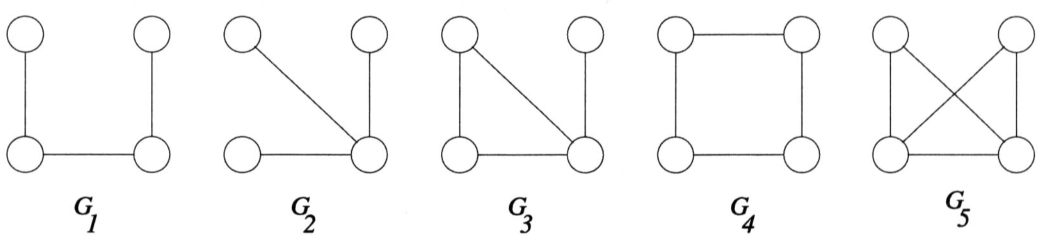

Figure 1: The incomplete connected graphs on 4 nodes

Graphs G_1 and G_2 have three edges and are trees, hence $s(G_1) = s(G_2) = 1$. For the other three graphs we use the inclusion-exclusion formula (2.5) to determine $s(G)$. Graph G_3 has four edges and one matching with two edges. Taking the different p_e's into account, we get

$$s(G_3) = \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 3} - \frac{1}{(2 \cdot 2)(1 \cdot 3)} = \frac{5}{6} > \frac{17}{27}.$$

Graph G_4 has four edges and two matchings with two edges; we get

$$s(G_4) = 4 \cdot \frac{1}{2 \cdot 2} - 2 \cdot \frac{1}{(2 \cdot 2)(2 \cdot 2)} = \frac{7}{8} > \frac{17}{27}.$$

Finally, graph G_5 has five edges and two matchings with two edges; formula (2.5) yields

$$s(G_5) = 4 \cdot \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 3} - 2 \cdot \frac{1}{(2 \cdot 3)(2 \cdot 3)} = \frac{13}{18} > \frac{17}{27}.$$

In the rest of the paper, we need to consider only graphs with $n \geq 5$ nodes.

3 The rendezvous probability is minimal in complete graphs

In this section, we prove our first theorem, which we recall here for the convenience of the reader.

Theorem 1 *If G is a graph with $n \geq 2$ (non-isolated) nodes, then $s(G) \geq s(K_n)$.*

As the proof is rather involved, we start by explaining the basic idea in a flawed approach, and show how to extend the random experiment underlying the rendezvous concept by allowing nonuniform probability distributions. Only then the formal proof is given.

3.1 Outline

A simple idea how the theorem might be proved is to consider the following step: to an incomplete graph G add some edge to obtain a graph G'. If this could be done in such a way that $s(G) \geq s(G')$, then starting from an arbitrary graph G_0 on n nodes we could iterate this step, until we finally reach the complete graph K_n, never increasing the rendezvous probability. Unfortunately, it can be shown that the operation of adding an edge is not generally monotone with respect to the rendezvous probability: there are graphs G and G' such that G' results from G by adding one edge, but $s(G) < s(G')$. This was observed by Austinat in his Diplom thesis [1]. (His example is described in [6].) Even worse, there are graphs G so that for *each* graph G' obtained from G by adding one edge we have $s(G) < s(G')$. In [1], the following example is described and attributed to J. M. Robson:

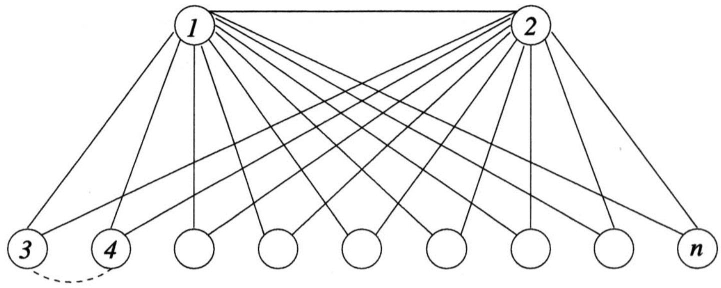

Figure 2: Adding any edge increases the rendezvous probability

Let G_n be the graph on node set $\{1,\ldots,n\}$, with edge set $\{(1,2)\} \cup \{(1,i),(2,i) \mid i = 3,\ldots,n\}$. It is not hard to see that $s(G_n) \to \frac{3}{4}$ for $n \to \infty$. (With probability $1 - O(\frac{1}{n})$, nodes 1 and 2 will choose two different nodes w_1 and w_2 from $\{3,\ldots,n\}$. In this case no rendezvous occurs if and only if w_1 chooses 2 and w_2 chooses 1. This happens with probability $\frac{1}{4}$.) On the other hand, adding one edge means adding (u,v) for some $u, v \in \{3,\ldots,n\}, u \neq v$. For symmetry reasons, we may assume that $u = 3, v = 4$. Call the resulting graph G'_n. Again, it is easy to see that $s(G'_n) \to \frac{7}{9}$ for $n \to \infty$. (With probability $1 - O(\frac{1}{n})$, nodes 1 and 2 will choose two different nodes w_1 and w_2 from $\{5,\ldots,n\}$. If this is the case, no rendezvous occurs if and only if w_1 chooses 2 and w_2 chooses 1 *and* there is no rendezvous on edge $(3,4)$. The probability for this to happen is $\frac{1}{4} \cdot \frac{8}{9} = \frac{2}{9}$.) Since $\frac{7}{9} - \frac{3}{4} = \frac{1}{36}$, we have $s(G_n) < s(G'_n)$ for n large enough.

In our proof of the theorem, we modifiy the idea just sketched as follows. We generalize the random experiments by admitting certain *nonuniform* distributions at the nodes.

As before, let a graph $G = (V,E)$ with n nodes be given, so that there are no isolated nodes. Associated with each node u is a probability distribution on the set $N(u) = \{v \mid (u,v) \in E\}$ of its neighbors in G, given by numbers $p_{uv} \in [0,1], v \in N(u)$, with

$$\sum_{v \in N(u)} p_{uv} = 1, \text{ for every } u \in V\ . \tag{3.1}$$

Node u chooses $v \in N(u)$ with probability p_{uv}, independently of the random choices of the other nodes. We assume

$$p_{uv} \geq \frac{1}{n-1}, \text{ for every } u \in V,\ v \in N(u). \tag{3.2}$$

This requirement has the effect that if u has maximal degree $n-1$ in G, then u chooses each of its neighbors with the same probability. In particular, if G is the complete graph K_n, all probabilities are equal, and we are back at the original experiment.

A combination of a graph and probability distributions for each node is denoted by $(G, (p_{uv})_{u \in V, v \in N(u)})$, or $(G, (p_{uv}))$ for short. If $(p_{uv})_{u \in V, v \in N(u)}$ is given by the context, we may also drop the (p_{uv}) part and simply write G. The probability of the elementary event $(c_u)_{u\in V}$ (as discussed at the beginning of Section 2) is $\prod_{u\in V} p_{uc_u}$. Probabilities in the resulting probability space are denoted as $\mathbf{P}_{(G,(p_{uv}))}(\,\cdot\,)$ or as $\mathbf{P}_G(\,\cdot\,)$. We say a rendezvous occurs at edge (u,v) if u has chosen v and v has chosen u (formally: $c_u = v$ and $c_v = u$). This event is denoted by $R_{(u,v)}$. The event $\bigcup_{e\in E} R_e$ (there is some rendezvous) is denoted by R_G. By $s(G,(p_{uv}))$ or $s(G)$ we denote the probability $\mathbf{P}_G(R_G)$ that the random experiment creates a rendezvous at one of the edges.

Given a graph $(G,(p_{uv}))$ with probabilities, we may add a new edge $(\hat{u},\hat{v})$ to G as follows. We arrange it so that $\hat{u}$ chooses $\hat{v}$ with probability exactly $\frac{1}{n-1}$, and so that the probabilities that $\hat{u}$ chooses one of its other neighbors are decreased accordingly, but so that each one remains at least $\frac{1}{n-1}$. At the other node $\hat{v}$ probabilities are rearranged similarly. Later we show that adding a new edge to $(G,(p_{uv}))$ in this way does not increase the probability of a rendezvous if $s(G,(p_{uv}))$ is not larger than $\alpha = (\sqrt{5}-1)/2 \approx 0.61803$. We already know that $s(K_n) \le 0.6$ for $n \ge 5$; this entails that it is easy to guarantee that $s(G,(p_{uv})) < \alpha$ in the relevant cases.

The overall argument now runs as follows. We start with some connected graph $G_0 = (V, E_0)$ on n nodes; we may assume $s(G_0) < \alpha$. The initial probabilities

$$p_{uv} := 1/|N(u)|, \text{ for } u \in V,\ (u,v) \in E_0\ ,$$

are chosen so that the uniform distribution is represented. Adding edges one by one, in the manner just described, we run through a sequence of graphs with *nonuniform* distributions, never increasing the probability for a rendezvous, until finally we reach the complete graph K_n with all probabilities being equal to $\frac{1}{n-1}$. This implies $s(G_0) \ge s(K_n)$, as desired.

In the rest of Section 3, the theorem is proved. In Section 3.2 we analyze the step of adding one edge with nonuniform probabilities and prove the theorem. The proof of a central, but technical lemma is supplied in Section 3.3.

3.2 Adding an edge in the nonuniform case

In this section we show that if $s(G)$ is not too large, then adding one edge in the careful manner described in Section 3.1 will not increase the rendezvous probability. For this section we assume that $n \ge 5$.

Assume a graph $G = (V, E)$ on node set $V = \{1, \ldots, n\}$ without isolated nodes is given together with a family $(p_{uv})_{u\in V, v\in N(u)}$ of probability distributions, which are assumed to satisfy equations (3.1) and inequalities (3.2).

We describe in detail the operation of adding one edge to $(G,(p_{uv}))$, in case G is not the complete graph. By renaming we may assume that $(1,2)$ is not in E. We form

$G' = (V, E')$ by $E' := E \cup \{(1,2)\}$ and fix new probabilities as follows. Let

$$p'_{12} = p'_{21} = \frac{1}{n-1}\ ; \tag{3.3}$$

for $u \in N(1)$, let $p'_{1u} = p_{1u} - \varepsilon_u$ for some $\varepsilon_u \geq 0$ such that

$$p'_{1u} \geq \frac{1}{n-1}, \text{ for } u \in N(1), \text{ and } \sum_{u \in N(1)} \varepsilon_u = \frac{1}{n-1}; \tag{3.4}$$

for $v \in N(2)$, let $p'_{2v} = p_{2v} - \delta_v$ for some $\delta_v \geq 0$ such that

$$p'_{2v} \geq \frac{1}{n-1}, \text{ for } v \in N(2), \text{ and } \sum_{v \in N(2)} \delta_v = \frac{1}{n-1}; \tag{3.5}$$

finally, let

$$p'_{uv} = p_{uv}, \text{ if } u, v \in V - \{1,2\}, v \in N(u). \tag{3.6}$$

Figure 3 illustrates this construction.

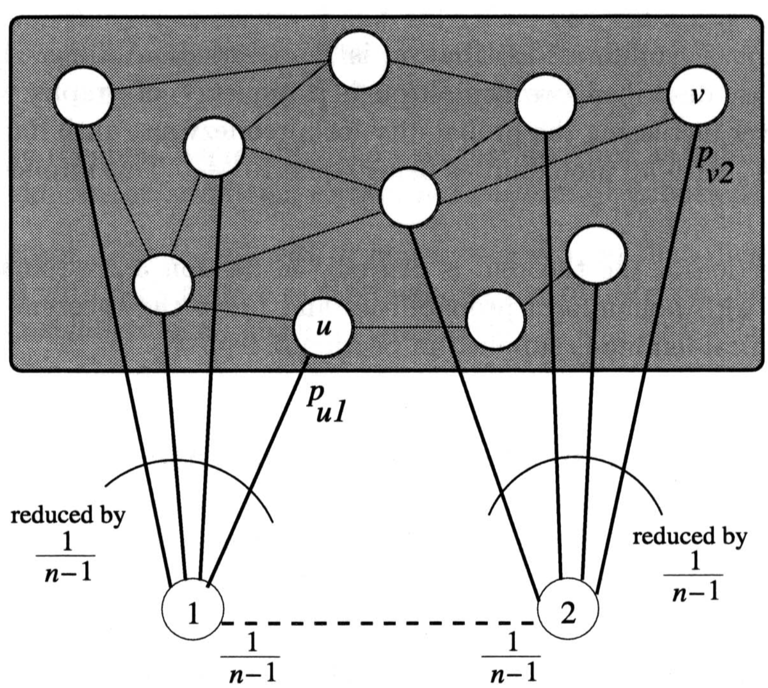

Figure 3: Inserting an edge with minimal probabilities

The remainder of this section is devoted to the proof of the following central lemma concerning $s(G) = s(G, (p_{uv}))$ and $s(G') = s(G', (p'_{uv}))$. It uses the number $\alpha = \frac{1}{2}(\sqrt{5} - 1) \approx 0.61803$, which is the unique solution of the equation $x + x^2 = 1$ in $[0, 1]$.

Lemma 6 (Main Lemma) *If $s(G) \le \alpha$, then $s(G') \le s(G)$.*

Proof. In the following, we write $\mathbf{P}(\,\cdot\,)$ for $\mathbf{P}_{(G,(p_{uv}))}(\,\cdot\,)$, $\mathbf{P}'(\,\cdot\,)$ for $\mathbf{P}_{(G',(p'_{uv}))}(\,\cdot\,)$, R for R_G and R' for $R_{G'}$. Consider the following event ("$\{3,\dots,n\}$ is clear of rendezvous"):

$$C = \{\text{no rendezvous occurs among any two of the nodes } 3,\dots,n\}. \tag{3.7}$$

The event C is not affected by the choices of nodes 1 and 2, hence its probability is not affected by the change of probabilities at these nodes:

$$\mathbf{P}(C) = \mathbf{P}'(C). \tag{3.8}$$

Now $\overline{C}$ implies R, hence

$$s(G) = \mathbf{P}(R) = \mathbf{P}(R \mid C)\cdot\mathbf{P}(C) + \mathbf{P}(\overline{C}) = \mathbf{P}(R \mid C)\cdot\mathbf{P}(C) + 1 - \mathbf{P}(C). \tag{3.9}$$

Similarly (using (3.8)),

$$s(G') = \mathbf{P}'(R') = \mathbf{P}'(R' \mid C)\cdot\mathbf{P}(C) + 1 - \mathbf{P}(C). \tag{3.10}$$

In case $\mathbf{P}(C) = 0$ the lemma is trivially true. Thus we may assume from now on that $\mathbf{P}(C) > 0$. By (3.9) and (3.10), to prove the lemma it is sufficient to show

$$\mathbf{P}'(R' \mid C) \le \mathbf{P}(R \mid C). \tag{3.11}$$

In the proof of (3.11), the following abbreviations are helpful:

$$\begin{aligned}
p_u &= p_{1u},\\
p_u^* &= \mathbf{P}(u \text{ chooses } 1 \mid C), \quad \text{for } u \in N(1);\\
q_v &= p_{2v},\\
q_v^* &= \mathbf{P}(v \text{ chooses } 2 \mid C), \quad \text{for } v \in N(2);\\
\beta_{uv}^* &= \mathbf{P}(u \text{ chooses 1 and } v \text{ chooses } 2 \mid C), \quad \text{for } u \in N(1), v \in N(2).
\end{aligned}$$

Note that the probabilities p_u^*, q_v^*, and β_{uv}^* would not change if $\mathbf{P}'(\,\cdot\,)$ instead of $\mathbf{P}(\,\cdot\,)$ was used.

In a way similar to Fact 4 condition C increases the probability that u chooses 1 (resp. that v chooses 2):

Lemma 7

(a) $p_u^* \ge p_{u1}$ *(hence $p_u^* \ge \frac{1}{n-1}$), for $u \in N(1)$.*

(b) $q_v^* \geq q_{v2}$ *(hence* $q_v^* \geq \frac{1}{n-1}$*), for* $v \in N(2)$.

Proof. It is sufficient to prove (a). Fix $u \in N(1)$. Let C' be the event that the choices of the nodes in $\{3, \ldots, n\} - \{u\}$ do not create a rendezvous among themselves (not regarding what u does). We use three simple observations, namely

(i) $\{u \text{ chooses } 1\} \cap C' = \{u \text{ chooses } 1\} \cap C$ (indeed, if u chooses 1, a rendezvous in $\{3, \ldots, n\}$ can only occur among nodes in $\{3, \ldots, n\} - \{u\}$);

(ii) the events C' and $\{u \text{ chooses } 1\}$ are independent;

(iii) $C \subseteq C'$,

to obtain:

$$
\begin{aligned}
p_u^* &= \mathbf{P}(u \text{ chooses } 1 \mid C) = \mathbf{P}(\{u \text{ chooses } 1\} \cap C)/\mathbf{P}(C) \\
&\overset{(i)}{=} \mathbf{P}(\{u \text{ chooses } 1\} \cap C')/\mathbf{P}(C) \\
&\overset{(ii)}{=} \mathbf{P}(u \text{ chooses } 1) \cdot \mathbf{P}(C')/\mathbf{P}(C) \\
&\overset{(iii)}{\geq} \mathbf{P}(u \text{ chooses } 1) = p_{u1}.
\end{aligned}
$$

This proves Lemma 7. ■

Define the events

$$R_1 = \bigcup_{u \in N(1)} R_{(1,u)} \quad \text{(there is a rendezvous involving node 1)}\ ,$$

and

$$R_2 = \bigcup_{v \in N(2)} R_{(2,v)} \quad \text{(there is a rendezvous involving node 2)}\ .$$

The corresponding events in the probability space for G' (not including the event that there is a rendezvous along the new edge $(1,2)$) are denoted by R_1' and R_2'. Now we write out the two probabilities in (3.11) in full. Using the obvious fact that for $u, \hat{u} \in N(1)$, $u \neq \hat{u}$ the events $\{1 \text{ chooses } u\}$ and $\{1 \text{ chooses } \hat{u}\}$ are disjoint, we obtain

$$
\begin{aligned}
\mathbf{P}(R \mid C) &= \mathbf{P}(R_1 \mid C) + \mathbf{P}(R_2 \mid C) - \mathbf{P}(R_1 \cap R_2 \mid C) \\
&= \sum_{u \in N(1)} p_u p_u^* + \sum_{v \in N(2)} q_v q_v^* - \sum_{u \in N(1), v \in N(2)} p_u q_v \beta_{uv}^*.
\end{aligned}
\tag{3.12}
$$

In $(G', (p'_{uv}))$, the choices made by nodes 1 and 2 are independent of C; further, if there is a rendezvous at $(1,2)$, there cannot be a rendezvous between 1 or 2 and any other node.

Hence we obtain

$$\mathbf{P}'(R' \mid C) = \mathbf{P}'(R_{(1,2)} \mid C) + \mathbf{P}'(R_1' \mid C) + \mathbf{P}'(R_2' \mid C) - \mathbf{P}'(R_1' \cap R_2' \mid C)$$

$$\begin{aligned}
&= p_{12}' p_{21}' + \sum_{u \in N(1)} (p_u - \varepsilon_u) p_u^* + \sum_{v \in N(2)} (q_v - \delta_v) q_v^* - \sum_{u \in N(1), v \in N(2)} (p_u - \varepsilon_u)(q_v - \delta_v) \beta_{uv}^* \\
&= \frac{1}{(n-1)^2} + \sum_{u \in N(1)} p_u p_u^* - \sum_{u \in N(1)} \varepsilon_u p_u^* + \sum_{v \in N(2)} q_v q_v^* - \sum_{v \in N(2)} \delta_v q_v^* - \\
&\qquad - \sum_{u \in N(1), v \in N(2)} (p_u q_v - p_u \delta_v - \varepsilon_u q_v + \varepsilon_u \delta_v) \beta_{uv}^*.
\end{aligned} \tag{3.13}$$

Subtracting (3.12) from (3.13), and using the obvious fact that $\varepsilon_u \delta_v \geq 0$, we get

$$\begin{aligned}
&\mathbf{P}'(R' \mid C) - \mathbf{P}(R \mid C) \leq \\
&\leq \frac{1}{(n-1)^2} - \sum_{u \in N(1)} \varepsilon_u p_u^* - \sum_{v \in N(2)} \delta_v q_v^* + \sum_{u \in N(1), v \in N(2)} (p_u \delta_v + \varepsilon_u q_v) \beta_{uv}^*.
\end{aligned} \tag{3.14}$$

At this point it becomes apparent that in order to proceed we must establish a relation between $\beta_{uv}^* = \mathbf{P}(u \text{ chooses } 1 \text{ and } v \text{ chooses } 2 \mid C)$ on the one hand and $p_u^* = \mathbf{P}(u \text{ chooses } 1 \mid C)$ and $q_v^* = \mathbf{P}(v \text{ chooses } 2 \mid C)$ on the other. In the unconditioned situation the events $\{u \text{ chooses } 1\}$ and $\{v \text{ chooses } 2\}$ are independent; in the probability space we obtained by conditioning on C this is no longer the case. The next lemma states that these two events are at least "negatively correlated" under the condition that C is true: if one of them happens, the other one becomes less likely. It is by no means obvious that this is so; the somewhat tricky proof will be supplied in the next section.

Lemma 8 (NegCorrelation Lemma) *$\beta_{uv}^* \leq p_u^* q_v^*$, for $u \in N(1), v \in N(2)$.*

Using Lemma 8 we may continue from inequality (3.14) as follows:

$$\begin{aligned}
&\mathbf{P}'(R' \mid C) - \mathbf{P}(R \mid C) \leq \\
&\leq \frac{1}{(n-1)^2} - \sum_{u \in N(1)} \varepsilon_u p_u^* - \sum_{v \in N(2)} \delta_v q_v^* + \sum_{u \in N(1), v \in N(2)} (p_u \delta_v + \varepsilon_u q_v) p_u^* q_v^* = \\
&= \frac{1}{(n-1)^2} - \sum_{u \in N(1)} \varepsilon_u p_u^* - \sum_{v \in N(2)} \delta_v q_v^* + \\
&\qquad + \Big(\sum_{u \in N(1)} p_u p_u^*\Big)\Big(\sum_{v \in N(2)} \delta_v q_v^*\Big) + \Big(\sum_{u \in N(1)} \varepsilon_u p_u^*\Big)\Big(\sum_{v \in N(2)} q_v q_v^*\Big) \\
&= \frac{1}{(n-1)^2} - \Big(\sum_{u \in N(1)} \varepsilon_u p_u^*\Big)\Big(1 - \sum_{v \in N(2)} q_v q_v^*\Big) - \Big(1 - \sum_{u \in N(1)} p_u p_u^*\Big)\Big(\sum_{v \in N(2)} \delta_v q_v^*\Big) \\
&= \frac{1}{(n-1)^2} - \Big(\sum_{u \in N(1)} \varepsilon_u p_u^*\Big) \mathbf{P}(\overline{R}_2 \mid C) - \mathbf{P}(\overline{R}_1 \mid C) \Big(\sum_{v \in N(2)} \delta_v q_v^*\Big).
\end{aligned} \tag{3.15}$$

By Lemma 7 we have $p_u^* \geq \frac{1}{n-1}$ and $q_v^* \geq \frac{1}{n-1}$, and by (3.4) and (3.5) we have $\sum_{u \in N(1)} \varepsilon_u = \frac{1}{n-1}$ and $\sum_{v \in N(2)} \delta_v = \frac{1}{n-1}$. Substituting this into (3.15) and simplifying we obtain

$$\begin{aligned}
&\mathbf{P}'(R' \mid C) - \mathbf{P}(R \mid C) \leq \\
&\leq \frac{1}{(n-1)^2} - \frac{1}{(n-1)^2} \cdot \mathbf{P}(\overline{R}_2 \mid C) - \mathbf{P}(\overline{R}_1 \mid C) \cdot \frac{1}{(n-1)^2} = \\
&= \frac{1}{(n-1)^2} \cdot (\mathbf{P}(R_1 \mid C) + \mathbf{P}(R_2 \mid C) - 1).
\end{aligned}$$

Thus, in order to prove (3.11) it is sufficient to show

$$\mathbf{P}(R_1 \mid C) + \mathbf{P}(R_2 \mid C) \leq 1,$$

or, equivalently,

$$\mathbf{P}(R_1 \cup R_2 \mid C) + \mathbf{P}(R_1 \cap R_2 \mid C) \leq 1. \tag{3.16}$$

We know from (3.9) that $\mathbf{P}(R) = \mathbf{P}(R \mid C)\mathbf{P}(C) + 1 \cdot (1 - \mathbf{P}(C))$. This means that $\mathbf{P}(R)$ is a convex combination of $\mathbf{P}(R \mid C)$ and 1, hence

$$\mathbf{P}(R_1 \cup R_2 \mid C) \leq \mathbf{P}(R \mid C) \leq \mathbf{P}(R).$$

Further, using Lemma 8 again, we get

$$\begin{aligned}
\mathbf{P}(R_1 \cap R_2 \mid C) &= \sum_{u \in N(1), v \in N(2)} p_u q_v \beta_{uv}^* && (3.17)\\
&\leq \sum_{u \in N(1), v \in N(2)} p_u q_v p_u^* q_v^* = \Big(\sum_{u \in N(1)} p_u p_u^* \Big) \Big(\sum_{v \in N(2)} q_v q_v^* \Big) && (3.18)\\
&= \mathbf{P}(R_1 \mid C) \cdot \mathbf{P}(R_2 \mid C) \leq \mathbf{P}(R \mid C)^2 \\
&\leq \mathbf{P}(R)^2.
\end{aligned}$$

Thus,

$$\mathbf{P}(R_1 \cup R_2 \mid C) + \mathbf{P}(R_1 \cap R_2 \mid C) \leq \mathbf{P}(R) + \mathbf{P}(R)^2 \leq \alpha + \alpha^2 = 1,$$

since we have assumed that $s(G) = \mathbf{P}(R) \leq \alpha = \frac{1}{2}(\sqrt{5} - 1)$, and $\alpha + \alpha^2 = 1$. Thus, (3.16) and hence (3.11) holds, and Lemma 6 is proved. ■

3.3 Proof of the NegCorrelation Lemma

This section is devoted to the proof of Lemma 8. In order to carry out a proof by induction, we formulate a more general statement.

Lemma 9 *Fix two distinct nodes in G, called 1 and 2 for simplicity, and a nonempty set $U \subseteq V - \{1,2\}$. Let $u, v \in U$ with $u \in N(1)$ and $v \in N(2)$, and consider the events*

$$\begin{aligned} A &= \{u \text{ chooses } 1\}, \\ B &= \{v \text{ chooses } 2\}, \text{ and} \\ C &= \{\text{there is no rendezvous among any two nodes in } U\}. \end{aligned}$$

Then

$$\mathbf{P}(A \cap B \mid C) \leq \mathbf{P}(A \mid C) \cdot \mathbf{P}(B \mid C).$$

(Lemma 8 follows by considering $U = V - \{1,2\}$.)

Before we start with the proof, we try to illustrate the difficulty. Figure 4 depicts an example situation.

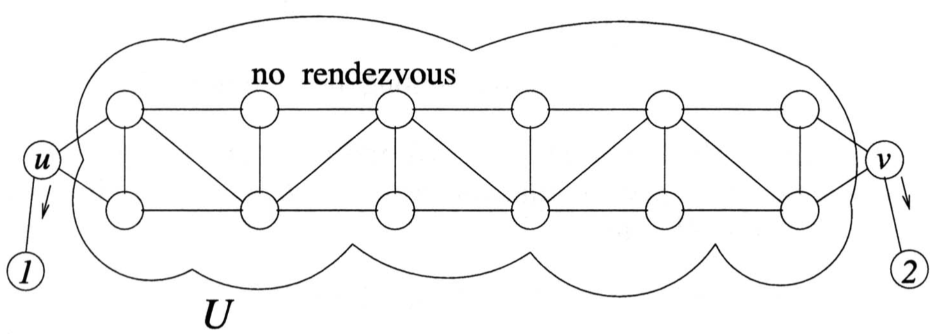

Figure 4: Long-distance influence of condition "no rendezvous in U"

We know that if u and v are distinct then A and B are independent, and if $u = v$ they exclude each other. Hence we always have $\mathbf{P}(A \cap B) \leq \mathbf{P}(A)\mathbf{P}(B)$. It is intuitively clear that the condition C influences the probabilities that u chooses 1 and that v chooses 2; in fact it is easy to see that $\mathbf{P}(A \mid C) \geq \mathbf{P}(A)$ and $\mathbf{P}(B \mid C) \geq \mathbf{P}(B)$ (just as in Lemma 7). However, since u and v are connected via a multitude of paths running through the graph, some of which may be very long (as in the figure), the interaction between A and B under condition C is quite unclear. The following proof deals with the situation efficiently by using an induction argument, formally on the size of U; but in essence the argument analyzes the "long-distance effects" by peeling off the graph starting from u and v in a breadth-first manner, thus shortening the paths between the distinguished nodes in each level of the induction.

Proof of Lemma 9. If $\Pr(C) = 0$, there is nothing to prove. Thus assume $\Pr(C) > 0$ and note that then the statement of the Lemma is equivalent to

$$\mathbf{P}(A \cap B \cap C)\mathbf{P}(C) \leq \mathbf{P}(A \cap C)\mathbf{P}(B \cap C). \tag{3.19}$$

We prove (3.19) by induction on $|U|$.

Base case $|U| = 1$: In this case u and v are identical, and hence A and B are disjoint events. Thus $\mathbf{P}(A \cap B \cap C) = 0$, and we are done.

Base case $|U| = 2$: In view of the argument in the previous case we may assume that $U = \{u, v\}$. If u and v are not adjacent, $\Pr(C) = 1$, and the claim follows from the independence of A and B. So assume edge (u, v) is present. The only place where a rendezvous can occur in $\{u, v\}$ is the edge (u, v). Hence $A = A \cap C$ and $B = B \cap C$, and we get

$$\mathbf{P}(A \cap B \cap C)\mathbf{P}(C) = \mathbf{P}(A \cap B)\mathbf{P}(C) = p_{u1}p_{v2}(1 - p_{uv}p_{vu}) \leq p_{u1}p_{v2} = \mathbf{P}(A \cap C)\mathbf{P}(B \cap C).$$

Induction step: Let $|U| \geq 3$. In view of the argument in case $|U| = 1$ we may assume that u and v are different. Let

$$\begin{aligned} W &= U - \{u, v\}, \text{ and} \\ D &= \{\text{there is no rendezvous among any two nodes in } W\}. \end{aligned}$$

Observe that $C \subseteq D$ and hence $\Pr(D) > 0$. We write out the probabilities that occur in (3.19) in more detail. It is obvious that the events A, B, and D are independent, and that $A \cap B \cap C = A \cap B \cap D$. Thus,

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A \cap B \cap D) = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(D) = p_{u1}p_{v2}\mathbf{P}(D). \tag{3.20}$$

We consider events that describe that u resp. v is involved in a rendezvous:

$$\begin{aligned} S &= \{\exists s \in N(u) \cap W : \text{ there is a rendezvous at } (u, s)\ \}, \\ T &= \{\exists t \in N(v) \cap W : \text{ there is a rendezvous at } (v, t)\ \}. \end{aligned}$$

Now note that for $B \cap D$ to occur there are two possibilities: either v chooses 2 and there is no rendezvous in U at all (event $B \cap C$) *or* v chooses 2 and there is a rendezvous in U, but none in W — but then u must be involved in some rendezvous (event $B \cap D \cap S$). Thus,

$$\begin{aligned} \mathbf{P}(B \cap C) &= \mathbf{P}(B \cap D) - \mathbf{P}(B \cap D \cap S) = \mathbf{P}(B)\mathbf{P}(D) - \mathbf{P}(B)\mathbf{P}(D \cap S) \\ &= p_{v2}(\mathbf{P}(D) - \mathbf{P}(D \cap S)). \end{aligned} \tag{3.21}$$

Similarly, we get

$$\begin{aligned} \mathbf{P}(A \cap C) &= \mathbf{P}(A \cap D) - \mathbf{P}(A \cap D \cap T) = \mathbf{P}(A)\mathbf{P}(D) - \mathbf{P}(A)\mathbf{P}(D \cap T) \\ &= p_{u1}(\mathbf{P}(D) - \mathbf{P}(D \cap T)). \end{aligned} \tag{3.22}$$

Finally, we note that for D to occur there are three possibilities: either there is no rendezvous in U (event C) *or* node u or node v are involved in some rendezvous with nodes in W and there is no rendezvous in W (event $(S \cup T) \cap D$) *or* there is a rendezvous at edge (u, v) and none in W (event $D \cap \{u \text{ chooses } v \text{ and } v \text{ chooses } u\}$). Thus,

$$
\begin{aligned}
\mathbf{P}(C) &= \mathbf{P}(D) - \mathbf{P}(D \cap S) - \mathbf{P}(D \cap T) + \mathbf{P}(D \cap S \cap T) \\
&\qquad - \mathbf{P}(D \cap \{u \text{ chooses } v \text{ and } v \text{ chooses } u\}) \\
&\leq \mathbf{P}(D) - \mathbf{P}(D \cap S) - \mathbf{P}(D \cap T) + \mathbf{P}(D \cap S \cap T).
\end{aligned}
\tag{3.23}
$$

Equations (3.20), (3.21), (3.22) and inequality (3.23) imply that in order to prove (3.19), it is sufficient to show that

$$
\begin{aligned}
&p_{u1}p_{v2}(\mathbf{P}(D)(\mathbf{P}(D) - \mathbf{P}(D \cap S) - \mathbf{P}(D \cap T) + \mathbf{P}(D \cap S \cap T)) \\
&\quad \leq\ p_{v2} \cdot (\mathbf{P}(D) - \mathbf{P}(D \cap S)) \cdot p_{u1} \cdot (\mathbf{P}(D) - \mathbf{P}(D \cap T))
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
&\mathbf{P}(D)(\mathbf{P}(D) - \mathbf{P}(D \cap S) - \mathbf{P}(D \cap T) + \mathbf{P}(D \cap S \cap T)) \\
&\quad \leq\ (\mathbf{P}(D) - \mathbf{P}(D \cap S))(\mathbf{P}(D) - \mathbf{P}(D \cap T).
\end{aligned}
\tag{3.24}
$$

By multiplying out and cancelling we see that (3.24) is equivalent to

$$
\mathbf{P}(D)\mathbf{P}(D \cap S \cap T) \leq \mathbf{P}(D \cap S)\mathbf{P}(D \cap T), \tag{3.25}
$$

or

$$
\mathbf{P}(S \cap T \mid D) \leq \mathbf{P}(S \mid D)\mathbf{P}(T \mid D). \tag{3.26}
$$

We can prove (3.26) by expanding the involved events and applying the induction hypothesis to distinguished nodes u and v and node set W. (For a similar calculation, cf. the proof of (3.17).)

$$
\begin{aligned}
\mathbf{P}(S \cap T \mid D) &= \sum_{\substack{s \in N(u) \cap W \\ t \in N(v) \cap W}} p_{us}p_{vt}\,\mathbf{P}(s \text{ chooses } v \text{ and } t \text{ chooses } u \mid D) \\
&\leq \sum_{s \in N(u) \cap W} \sum_{t \in N(v) \cap W} p_{us}p_{vt}\,\mathbf{P}(s \text{ chooses } v \mid D)\mathbf{P}(t \text{ chooses } u \mid D) \\
&= \Big(\sum_{s \in N(u) \cap W} p_{us}\,\mathbf{P}(s \text{ chooses } v \mid D)\Big)\Big(\sum_{t \in N(v) \cap W} p_{vt}\,\mathbf{P}(t \text{ chooses } u \mid D)\Big) \\
&= \mathbf{P}(S \mid D)\mathbf{P}(T \mid D).
\end{aligned}
$$

Thus, (3.19) holds, and the induction step is complete. ∎

3.4 Rendezvous and negative association

We conclude this section with a few remarks that places the rendezvous experiment into a wider context, and which may point the way to useful generalizations. The rendezvous experiment in graphs can be viewed as a game of "balls into bins with constraints". Assume we have n balls, numbered $1, \ldots, n$, that are thrown independently into bins numbered $1, \ldots, m$, according to the following rule: To every bin j a set $e_j \subseteq \{1, \ldots, n\}$ of balls is associated, so that each ball u occurs in at least one set e_j. Ball u lands in bin j with probability $1/|\{j' \mid u \in e_{j'}\}|$ if $u \in e_j$ and 0 otherwise. A "rendezvous" occurs if there is at least one bin j that is chosen by all u with $u \in e_j$. Obviously, our rendezvous model is obtained if each e_j contains exactly two balls and all the e_j are distinct. Then the set $\{e_1, \ldots, e_m\}$ can be interpreted as the edge set of a graph.

We note that to this setup the fundamental notion of "negatively associated random variables" is applicable. Namely, the random variables (for $1 \le u \le n$, $1 \le j \le m$, $u \in e_j$)

$$X_{uj} = \begin{cases} 1 & \text{, if ball } u \text{ falls into bin } j \\ 0 & \text{, otherwise .} \end{cases}$$

are *negatively associated*: if I and J are disjoint subsets of $\{(u,j) \mid 1 \le u \le n, 1 \le j \le m, u \in e_j\}$, then for arbitrary *monotone* events

$$A = A((X_{uj})_{(u,j) \in I}) \text{ , and } B = B((X_{uj})_{(u,j) \in J})$$

we have

$$\mathbf{P}(A \cap B) \le \mathbf{P}(A) \cdot \mathbf{P}(B) \ .$$

(This fundamental concept is studied in detail in [5]. The family (X_{uj}) is seen to be negatively associated by obvious application of some basic construction principles studied in that paper.) It is not hard to see that Fact 4 and Lemma 7 are direct consequences of this principle. Generalizing Fact 4, one can e. g. derive the following:
Claim: If in the standard setting for the rendezvous experiment A and B are disjoint edge sets then

$$\mathbf{P}(\text{no rendezvous in } A \cup B) \le \mathbf{P}(\text{no rendezvous in } A) \cdot \mathbf{P}(\text{no rendezvous in } B) \ .$$

Curiously, Lemma 8 does not seem to be such an easy consequence of knowledge about negatively associated random variables.

4 $s(K_n)$ is monotonically decreasing

In Section 2, we have seen by direct calculation that $s(K_2) > s(K_3) > \cdots > s(K_8)$. Thus it is natural to assume that the sequence $s(K_n), n \ge 2$, is strictly decreasing; and Theorem 2 claims that this is true. In this section, we prove that $s(K_n) > s(K_{n+1})$ for all $n \geq 2$ (where for the proof we may assume that some $n \geq 6$ is fixed). Surprisingly, there does not seem to be a straightforward proof, due to the fact that the probability spaces involved in the definition of $s(K_n)$ and $s(K_{n+1})$ are quite different and no direct transformation seems to be possible. The proof combines calculus (starting from formula (2.6)) with the analysis of random experiments "interpolating" between the rendezvous experiments in different complete graphs, by admitting other probabilities than $1/\deg v$.

For arbitrary $0 \leq p \leq \frac{1}{n-1}$, we consider the following modification of the rendezvous experiment on K_n: the probability that a particular node u chooses a node $v \neq u$ is p instead of $\frac{1}{n-1}$; there is a probability $1-(n-1)p$ that u does not choose any neighbor. Let $f_n(p)$ denote the rendezvous probability in this experiment. Clearly we have $f(\frac{1}{n-1}) = s(K_n)$. Adapting (2.6), we have

$$f_n(p) = \sum_{j\geq 1} (-1)^{j+1} \cdot \frac{n^{\underline{2j}} \cdot p^{2j}}{j! \cdot 2^j} . \tag{4.1}$$

It will turn out to be useful to study the function $f_n(p)$ by methods of standard calculus.

Lemma 10 (a) *$f_n'(p) = n(n-1)p \cdot (1 - f_{n-2}(p))$, for $0 \leq p \leq \frac{1}{n-1}$.*

(b) *$f_n'(p)$ is strictly increasing in $[\frac{1}{n}, \frac{1}{n-1}]$.*

Proof. (a) We calculate:

$$\begin{aligned} f_n'(p) &= \sum_{j\geq 1} (-1)^{j+1} \cdot \frac{n^{\underline{2j}} \cdot p^{2j-1}}{(j-1)! \cdot 2^{j-1}} \\ &= \sum_{j\geq 0} (-1)^{j} \cdot \frac{n^{\underline{2j+2}} \cdot p^{2j+1}}{j! \cdot 2^{j}} \\ &= n(n-1)p \cdot \sum_{j\geq 0} (-1)^{j} \frac{(n-2)^{\underline{2j}} \cdot p^{2j}}{j! \cdot 2^{j}} \qquad (4.2) \\ &= n(n-1)p \cdot (1 - f_{n-2}(p)). \end{aligned}$$

(b) We show that $g(p) = f_n''(p)/n(n-1)$ is positive in $[\frac{1}{n}, \frac{1}{n-1}]$. For this, we differentiate $f_n'(p)/n(n-1) = p \cdot (1 - f_{n-2}(p))$ and apply part (a) to f_{n-2}, to obtain

$$\begin{aligned} g(p) &= 1 - f_{n-2}(p) + p \cdot (1 - f_{n-2}'(p)) & (4.3) \\ &= 1 - f_{n-2}(p) + p \cdot (1 - (n-2)(n-3)p \cdot (1 - f_{n-4}(p))) . & (4.4) \end{aligned}$$

Now, we estimate $f_{n-2}(p)$ by considering the corresponding random experiment in K_{n-2}, which we view as being composed of a subgraph H with the structure of a K_{n-4} and two extra nodes u and v. (See figure 5.)

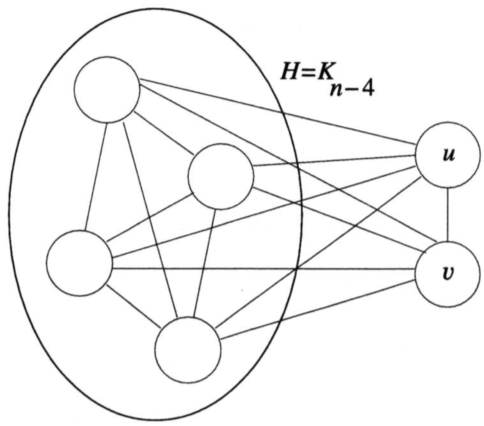

Figure 5: A K_{n-2} with two nodes u and v singled out

Let R_H denote the event that there is a rendezvous within H, and note that $\mathbf{P}(R_H) = f_{n-4}(p)$. Further, note that there are exactly $2(n-4)+1$ edges adjacent to u or v, and that the probability that a rendezvous occurs along one of these edges can be bounded above by $(2n-7)p^2$. This leads to the estimate

$$f_{n-2}(p) \leq f_{n-4}(p) + (2n-7)p^2. \tag{4.5}$$

If we substitute (4.5) into (4.4), we obtain

$$\begin{aligned} g(p) &\geq (1 - f_{n-4}(p)) - (2n-7)p^2 + p - (n-2)(n-3)p^2 \cdot (1 - f_{n-4}(p)) \\ &= (1 - f_{n-4}(p))(1 - (n-2)(n-3)p^2) + p - (2n-7)p^2 \,. \end{aligned} \tag{4.6}$$

The terms in (4.6) can be bounded as follows:

$$\begin{aligned} f_{n-4}(p) &\leq \binom{n-4}{2} \cdot p^2 \leq \frac{(n-4)(n-5)}{2(n-1)^2} < \frac{1}{2}\,; \\ 1 - (n-2)(n-3) \cdot p^2 &> \frac{(n-1)^2 - (n-2)(n-3)}{(n-1)^2} = \frac{3n-5}{(n-1)^2}\,; \\ p - (2n-7)p^2 &> \frac{1}{(n-1)^2} \cdot \left(\frac{(n-1)^2}{n} - (2n-7) \right) > \frac{5-n}{(n-1)^2}\,. \end{aligned}$$

Substituting these bounds in (4.6), we obtain

$$(n-1)^2 \cdot g(p) > \frac{3n-5}{2} + (5-n) = \frac{n+5}{2} > 0,$$

and hence $g(p) > 0$, which finishes the proof of (b). ■

Proof of Theorem 2. We analyze the random experiment that defines $s(K_{n+1})$ (with uniform probabilities $p = \frac{1}{n}$ at all nodes) in order to find an expression for $s(K_{n+1})$ in terms of $f_n(\frac{1}{n})$ and $f_{n-1}(\frac{1}{n})$, as follows. We view K_{n+1} as the union of a copy H of K_n and an extra node u. Let v be an arbitrary node in H. For symmetry reasons, we certainly have

$$s(K_{n+1}) = \mathbf{P}(\text{there is a rendezvous in } K_{n+1} \mid u \text{ chooses } v) .$$

In the following calculations, all probabilities are conditioned on the event that u chooses v. (Notation: $\mathbf{P}_v(\cdot) = \mathbf{P}_{K_{n+1}}(\cdot \mid u \text{ chooses } v)$.)

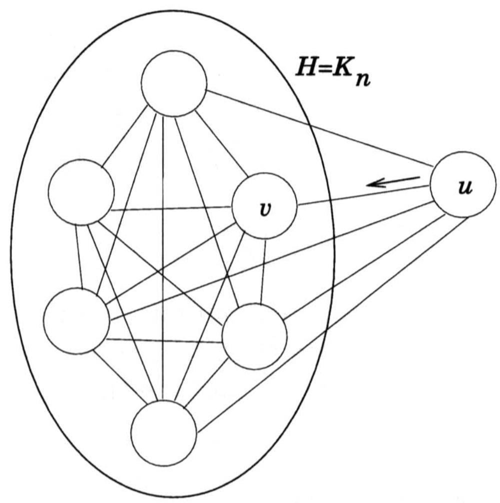

Figure 6: A K_{n+1} with a node u that has chosen a node v

We consider the following events:

$$\begin{aligned} A &= \{\text{ a rendezvous occurs at edge } (u,v)\}, \\ B &= \{\text{ a rendezvous occurs within } H\ \}, \\ C &= \{\text{ a rendezvous occurs within } H-v\}. \end{aligned}$$

Here $H - v$ denotes the graph obtained from H by removing node v, or K_{n+1} without nodes u and v. Note that there is a rendezvous in K_{n+1} if and only if there is one at edge (u, v) or one within H. Also note that $A \cap B = A \cap C$, and that A and C are independent with respect to $\mathbf{P}_v(\cdot)$. Thus we have

$$\begin{aligned} s(K_{n+1}) &= \mathbf{P}_v(A) + \mathbf{P}_v(B) - \mathbf{P}_v(A \cap B) \\ &= \mathbf{P}_v(A) + \mathbf{P}_v(B) - \mathbf{P}_v(A \cap C) \\ &= \mathbf{P}_v(A) + \mathbf{P}_v(B) - \mathbf{P}_v(A)\mathbf{P}_v(C). \end{aligned}$$

Noting that $\mathbf{P}_v(A) = \mathbf{P}_v(v \text{ chooses } u) = \frac{1}{n}$, that $\mathbf{P}_v(B) = f_n(\frac{1}{n})$, and that $\mathbf{P}_v(C) =$

$f_{n-1}(\frac{1}{n})$, we have

$$
\begin{aligned}
& s(K_n) - s(K_{n+1}) \\
= \quad & f_n\Big(\frac{1}{n-1}\Big) - \Big(\frac{1}{n} + f_n\Big(\frac{1}{n}\Big) - \frac{1}{n}\cdot f_{n-1}\Big(\frac{1}{n}\Big)\Big) \\
= \quad & f_n\Big(\frac{1}{n-1}\Big) - f_n\Big(\frac{1}{n}\Big) - \frac{1}{n}\cdot\Big(1 - f_{n-1}\Big(\frac{1}{n}\Big)\Big) \\
\overset{\text{(Le. 10(b))}}{>} \quad & f_n'\Big(\frac{1}{n}\Big)\cdot\Big(\frac{1}{n-1} - \frac{1}{n}\Big) - \frac{1}{n}\cdot\Big(1 - f_{n-1}\Big(\frac{1}{n}\Big)\Big) \\
\overset{\text{(Le. 10(a))}}{=} \quad & n(n-1)\cdot\frac{1}{n}\cdot\Big(1 - f_{n-2}\Big(\frac{1}{n}\Big)\Big)\cdot\frac{1}{n(n-1)} - \frac{1}{n}\cdot\Big(1 - f_{n-1}\Big(\frac{1}{n}\Big)\Big) \\
= \quad & \frac{1}{n}\cdot\Big(f_{n-1}\Big(\frac{1}{n}\Big) - f_{n-2}\Big(\frac{1}{n}\Big)\Big)\ .
\end{aligned}
$$

The last expression is positive, since carrying out our experiment in a K_{n-1} with $p = 1/n$ obviously has a larger success probability than in a subgraph of $n-2$ nodes (also with $p = 1/n$). Thus we obtain $s(K_{n+1}) < s(K_n)$, and the theorem is proved. ■

5 Computing $s(G)$ is #P-complete

In this section, we prove that the problem of computing $s(G)$ for given graph G is #P-complete. To state the result more formally, we define a corresponding counting problem. As described at the beginning of Section 2, the elementary events in the underlying probability space are sequences $(c_u)_{u\in V}$, where $c_u \in N(u)$ for each u, and each of these sequences has the same probability. Let $c(G)$ denote the number of elementary events in which a rendezvous occurs. Since $s(G) = c(G)/\prod_{v\in V}\deg(v)$, computing $s(G)$ is equivalent to computing $c(G)$. Thus our theorem can be rephrased as follows.

Theorem 3 *For any fixed integer $d \geq 5$, the problem of computing $c(G)$ for a given d-regular graph G is #P-complete.*

That computing $c(G)$ is in #P is trivial. The rest of this section is devoted to the proof that computing $c(G)$ (or equivalently $s(G)$) is #P-hard. We use the following result from Dagum and Luby [3], which refines the well-known result of Valiant [7] that counting the number of perfect matchings in a bipartite graph is #P-complete.

Theorem 11 (Dagum/Luby) *Let $d(n)$ be any integer valued function of n such that $3 \leq d(n) \leq n-3$. Then the problem of counting the number of perfect matchings in an n by n, $d(n)$-regular bipartite graph is #P-complete.*

Proof of Theorem 3. The proof consists in describing a Turing reduction from the problem in Theorem 11 to our problem. More specifically, suppose that for some fixed $d \geq 5$ we are given an oracle that on input G outputs $c(G)$ for arbitrary d-regular graphs G (from which we can readily calculate $s(G)$). Using this oracle, we construct a polynomial time algorithm that determines the number of perfect matchings for any given $(d-2)$-regular graph.

Fix $d \geq 5$ and suppose a $(d-2)$-regular graph $G = (V, E)$ with n vertices is given. We assume n is even and set $n = 2m$; otherwise the number of perfect matchings of G is trivially 0. Let X_i denote the number of matchings of size i in G, $1 \leq i \leq m$. Our plan of proof is as follows. We construct certain d-regular graphs G_k, $1 \leq k \leq m$, each containing G as a subgraph, such that $s(G_k)$ for each k can be expressed as a linear function of X_1, $\ldots$, X_m. This system can be viewed as a system of m linear equations in m unknowns $X_1, \ldots, X_m$. The right-hand side entries $s(G_1), \ldots, s(G_m)$ can be obtained by m oracle calls. We shall show that the coefficient matrix of this linear system is non-singular, and that the entries can be calculated in polynomial time. Therefore we can solve this system of equations for $X_1, \ldots, X_m$ and in particular determine the number of perfect matchings X_m of G, all in polynomial time.

We construct G_k as follows. Let H be a graph on $d+1$ nodes with two designated nodes $s \neq t$ that is obtained from the complete graph K_{d+1} by removing the edge (s, t). For $1 \leq k \leq m$, let H_k be the graph that results from "cascading" k copies of H through the nodes s and t. More precisely, H_k consists of k copies $H^1, \ldots, H^k$ and $k-1$ edges (t^i, s^{i+1}), $1 \leq i < k$, where s^i and t^i are the copies of s and t in H^i respectively. Note that all the nodes in H_k have degree d except for s_1 and t_k which have degree $d-1$. (For an example, see Fig. 7.)

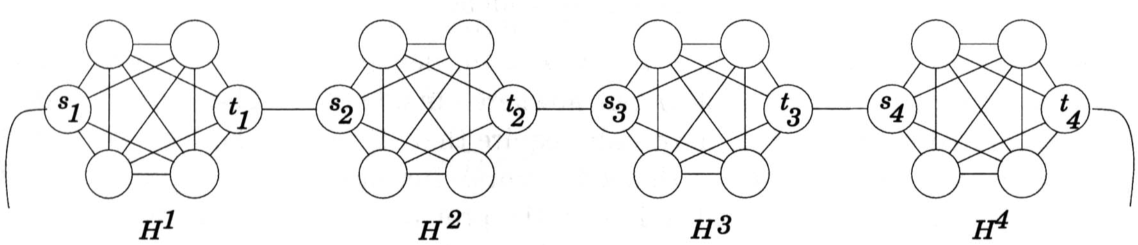

Figure 7: An H_4 with $d = 5$

Finally, G_k consists of G, a copy H_k^v of H_k for each $v \in V$, and edges (v, s^v) and (v, t^v) for each $v \in V$, where s^v and t^v are the copies of s^1 and t^k in H_k^v respectively. It is easy to verify that each G_k is d-regular.

We need to express $s(G_k)$ in terms of $X_1, \ldots, X_m$. For this, consider the following events for each $v \in V$.

A_k^v: no rendezvous occurs within H_k^v;

B_k^v: no rendezvous occurs within H_k^v and moreover s^v does not choose v;

C_k^v: no rendezvous occurs within H_k^v, on (v, s^v) or on (v, t^v).

It is clear that the probabilities of these events do not depend on the choice of v, so put $p_k = \mathbf{P}(A_k^v)$, $q_k = \mathbf{P}(B_k^v)$, and $r_k = \mathbf{P}(C_k^v)$. We postpone the proof of the following lemma.

Lemma 12 (a) $0 < q_k < p_k$ *for* $1 \leq k \leq m$,
(b) *the probabilities* p_k *and* q_k *can be computed in time polynomial in* k, *and*
(c) $q_{k+1}/p_{k+1} < q_k/p_k$ *for* $1 \leq k < m$.

Now consider $r_k = \mathbf{P}(C_k^v)$. When conditioned on the event that v chooses neither s^v nor t^v, the probability of C_k^v is p_k. When conditioned on the event that v chooses either s^v or t^v, by symmetry we may assume that s^v is chosen, and so the probability of C_k^v is q_k. Therefore we have

$$r_k = \frac{d-2}{d} p_k + \frac{2}{d} q_k. \tag{5.1}$$

Define D_k to be the event in the experiment on G_k that a rendezvous occurs within G but not outside of G. Then we have

$$s(G_k) = 1 - (r_k)^n + \mathbf{P}(D_k) \tag{5.2}$$

since the events C_k^v, $v \in V$, are mutually independent.

For each matching M of G with size i, $1 \leq i \leq m$, define D_k^M to be the event that a rendezvous occurs on every edge of M but nowhere outside of G. There are $2i$ nodes in V that are covered by M, and we must only require that A_k^v occurs. For the $n-2i$ nodes not covered by M we must require that C_k^v occurs. Because G_k is d-regular, all nodes choose each neighbor with probability $1/d$, so the probability for a rendezvous occurring at every edge in M is $1/d^{2i}$. Summing up, we have

$$\begin{aligned}
\mathbf{P}(D_k^M) &= d^{-2i}(p_k)^{2i}(r_k)^{n-2i} \\
&= (r_k)^n \left(d - 2 + \frac{2q_k}{p_k}\right)^{-2i},
\end{aligned}$$

where we used (5.1). By inclusion-exclusion (cf. (2.5)), we have

$$\mathbf{P}(D_k) = \sum_{1 \leq i \leq m} (-1)^{i+1} (r_k)^n \left(d - 2 + \frac{2q_k}{p_k}\right)^{-2i} X_i$$

where X_i is the number of matchings of size i in G. Together with equality (5.2), we obtain a linear equation

$$\sum_{1 \le i \le m} (a_k)^i X_i = b_k$$

where

$$a_k = -\left(d - 2 + \frac{2q_k}{p_k}\right)^{-2}$$

and

$$b_k = (1 - (r_k)^n - s(G_k))(r_k)^{-n},$$

for $k = 1, \ldots, m$. By Lemma 12(b) and using the assumed oracle we can compute all a_k and b_k in polynomial time. Moreover, the coefficients of this system form a Vandermonde matrix, which is nonsingular because, by Lemma 12(c), a_k, $1 \le k \le m$, are mutually distinct. Solving this system of equations, we obtain X_m, the number of perfect matchings of G. ■

Proof of Lemma 12. Part (a) follows from the trivial fact that event B_k^v is nonempty and is a proper subset of A_k^v. To prove (b) and (c), consider graph H' consisting of H and two distinct nodes $1, 2 \notin V(H)$ together with edges $(1, s)$ and $(2, t)$. In the uniform probability experiment on H', let A be the event that s chooses 1, B the event that t chooses 2 and C the event that there is no rendezvous in the subgraph H. By Lemma 9, we have

$$\mathbf{P}(A \cap B \cap C)\mathbf{P}(C) \le \mathbf{P}(A \cap C)\mathbf{P}(B \cap C). \tag{5.3}$$

Moreover, if we closely examine the induction step of the proof of Lemma 9, or (3.23) in particular, we see that (5.3) holds without equality for H':

$$\mathbf{P}(A \cap B \cap C)\mathbf{P}(C) < \mathbf{P}(A \cap C)\mathbf{P}(B \cap C). \tag{5.4}$$

Let $p = \mathbf{P}(C)$, $q = \mathbf{P}(\overline{A} \cap C) = \mathbf{P}(\overline{B} \cap C)$, and $r = \mathbf{P}(\overline{A} \cap \overline{B} \cap C)$. It is easy to verify that $p = p_1$ and $q = q_1$.

Claim: $pr < q^2$.

Proof of Claim.

$$\begin{aligned}
pr &= \mathbf{P}(C)\mathbf{P}(\overline{A} \cap \overline{B} \cap C) \\
&= \mathbf{P}(C)(P(C) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C) + \mathbf{P}(A \cap B \cap C)) \\
&< \mathbf{P}(C)(P(C) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C)) + \mathbf{P}(A \cap C)\mathbf{P}(B \cap C) \\
&= (\mathbf{P}(C) - \mathbf{P}(A \cap C))(\mathbf{P}(C) - \mathbf{P}(B \cap C)) \\
&= \mathbf{P}(\overline{A} \cap C)\mathbf{P}(\overline{B} \cap C) \\
&= q^2
\end{aligned}$$

where the inequality is due to (5.4). ■

We have the following recurrence.

$$\begin{aligned} p_{k+1} &= q_k p + (p_k - q_k) q \\ q_{k+1} &= q_k q + (p_k - q_k) r \end{aligned}$$

from which part (b) of the lemma follows. We prove (c) by induction on k. From the above recurrence, we have

$$\frac{q_{k+1}}{p_{k+1}} = \frac{q + (p_k/q_k - 1)r}{p + (p_k/q_k - 1)q}. \tag{5.5}$$

Let

$$g(x) = \frac{q + rx}{p + qx}. \tag{5.6}$$

Then, we have

$$g'(x) = \frac{pr - q^2}{(p + qx)^2} < 0, \tag{5.7}$$

so $g(x)$ is strictly decreasing. Therefore we have

$$\frac{q_2}{p_2} = g\Big(\frac{p}{q} - 1\Big) < g(0) = \frac{q_1}{p_1}, \tag{5.8}$$

the basis of the induction. For $k \geq 2$,

$$\frac{q_{k+1}}{p_{k+1}} = g\Big(\frac{p_k}{q_k} - 1\Big) < g\Big(\frac{p_{k-1}}{q_{k-1}} - 1\Big) = \frac{q_k}{p_k}$$

because $p_k/q_k > p_{k-1}/q_{k-1}$ by the induction hypothesis. ■

It might be interesting to view the above reduction method from the positive side and observe that it can be adapted to construct an unbiased estimator of the number of perfect matchings of an arbitrary given graph. Suppose a graph G of maximum degree $d > 0$ with $2n$ nodes is given. Choose n reals $0 < p_1 < \ldots < p_n \leq 1/d$ arbitrarily. For each $1 \leq i \leq n$, consider the rendezvous experiment in which the probability of each node choosing a particular neighbor is uniformly p_i and let $s_i(G)$ denote the success in which the probability of each node choosing a particular neighbor is uniformly p_i and let $s_i(G)$ denote the success probability of this experiment. Let X_j, $1 \leq j \leq n$ denote the number of distinct matchings in G of cardinality j. Then, adapting (2.5), we can express $s_i(G)$ as a linear combination of $X_1, \ldots, X_n$, where the coefficients are determined by p_i and do not depend on G. Solving this system of n linear equations for X_n, we can express X_n as a linear combination of $s_i(G)$, $1 \leq i \leq n$. Instead of using an oracle, we estimate each $s_i(G)$ by performing some number of experiments and taking the ratio of success.

The estimate of X_n obtained in this manner is a random variable whose expectation is the correct value of X_n, i.e., it is an unbiased estimator of X_n.

Unfortunately, this unbiased estimator does not seem to have any immediate algorithmic consequences, since it is unlikely to have a small variance when the number of experiments to estimate each $s_i(G)$ is polynomial in n. For more on unbiased estimators of the number of perfect machings, see [2].

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