The Klee-Minty random edge chain moves with linear speed

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Abstract: An infinite sequence of 0's and 1's evolves by flipping each 1 to a 0 exponentially at rate one. When a 1 flips, all bits to its right also flip. Starting from any configuration with finitely many 1's to the left of the origin, we show that the leftmost 1 moves right with linear speed. Upper and lower bounds are given on the speed.

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1 Introduction

1.1 Motivation

The simplex algorithm is widely used to solve linear programs. It works well in practice, though often one cannot prove that it will. A parameter in the algorithm is the rule that selects one move among possible moves ("pivots") that decrease the objective function. Deterministic pivot rules are known to be possibly very far from optimal. For example, consider problems with the number m of constraints of the same order as the dimension, n. For virtually every deterministic pivot rule there is a problem for which the algorithm will take exponential time, although it is conjectured that there exists a descent path whose length O(n). Variants of the original argument by Klee and Minty [KM72] are cited in [GHZ98, page 2].

Randomized pivot rules appear to do better. According to [GHZ98], several of the most popular randomized pivot rules appear to have polynomial – even quadratic – running time. Rigorous and general results on these, however, have been hard to come by. When one restricts to a narrow class of test problems, it becomes possible to obtain some rigorous results. Gärtner, Hank and Ziegler [GHZ98] consider three randomized pivot rules. Relevant to the present paper are their results on the *random edge* rule, in which the next move is chosen uniformly among edges leading to decrease the objective function. They analyze the performance of this rule on a class of linear programs, the feasible polyhedra for which are called *Klee-Minty cubes*, after [KM72]. Such cubes are good benchmarks because they cause some pivot rules to pass through a positive fraction of the vertices. They prove that the expected run time is quadratic, up to a possible log factor in the lower bound:

Theorem 1 (GHZ) The expected number, E_n of steps taken by the random edge rule, started at a random vertex of a Klee-Minty cube, is bounded by

$$\frac{n^2}{4H_{n+1}-1} \le E_n \le \binom{n+1}{2}.$$

Here, $H_n = \sum_{j=1}^n \frac{1}{j} \sim \log n$ is the n^{th} harmonic number.

Their lower bound rules out the possibility that $E_n \sim n \operatorname{polylog}(n)$ which was twice conjectured by previous researchers [PS82, page 29], [Kel81]. They guess that the upper bound is

the correct order of magnitude, and state an improvement in the upper bound from $(1/2)n^2$ to $0.27...n^2$, whose proof is omitted.

The method of analysis in [GHZ98] is via a combinatorial model due to [AC78] which describes the progress of the algorithm as a random walk on an acyclic directed graph. In their model, vertices are bijectively mapped to sequences of 0's and 1's of length n, and each move consists of flipping a 1 (chosen uniformly at random) to a 0, and simultaneously flipping all bits to the right of the chosen bit. It was in this form that the problem came to our attention. Indeed, the remainder of the paper is framed in terms of a variant of this model, which we find to be an intrinsically interesting model. Our main result, Theorem 2, closes the gap left open in [GHZ98], proving that the upper bound is sharp to within a constant factor and obtaining upper and lower bounds differing by a factor of less than 2. We have moved the model to continuous time and made n infinite, since from our view as probabilists this is the most natural way to frame such a model. Nevertheless, our results apply to the setting of [GHZ98] as well (Corollary 3). Although the model seems simple, we remark that we were unable to prove many things about the model, including whether certain limits exist.

1.2 Statement of the model

The one-dimensional integer lattice is decorated with 0's and 1's, arbitrarily except that there must be some point to the left of which lie only 0's. Each site has a clock that goes off at times distributed as independent mean-one exponentials. When a clock rings, if there is a 0 there nothing happens, but if there is 1 there, then it and all (infinitely many) of the sites to the right flip as well -0's become 1's and 1's become 0's. Later arguments will use random variables involved in the construction of this continuous time Markov chain, so we give a formal construction as follows.

Let S be the subset of $\{0,1\}^{\mathbb{Z}}$ consisting of sequences of 0's and 1's that have a leftmost 1 (equivalently, have finitely many 1's to the left of the origin). Let $\{N(j,t) : t \geq 0\}_{j \in \mathbb{Z}}$ be a collection of IID Poisson counting processes, that is, step functions increasing by 1 at random times, which we denote $\xi_{j,i}$, having independent increments distributed as exponentials of mean 1. As usual, the filtration is defined by letting \mathcal{F}_t denote the σ -field generated by $\{N(j,s) : s \leq t\}$. For any $\omega \in S$, we define a Markov chain starting from ω as follows. If ω is the zero string, the chain remains at ω . Otherwise, let *i* be the position of the leftmost 1 in ω . First, fix $j > i \in \mathbb{Z}$ and define a Markov chain on configurations on the index set $\{i, i + 1, \ldots, j\}$ in the finitary model: each site *s* attempts to flip at times $\xi_{s,r}$ for all $r \geq 0$. The flip is successful if and only if there is a 1 in position *s* at time $\xi_{s,r}^-$, in which case sites $s + 1, \ldots j$ flip as well. This is a well defined process because for each *T* there are only finitely many jump times $\xi_{s,r}$ with $i \leq s \leq j$ and $r \leq T$. It is clear that for j' > j the Markov chain thus defined on configurations on [i, j'] projects down to the Markov chain on configurations on [i, j] by ignoring the bits to the right of *j*. By Kolmogorov's Extension Theorem there is an inverse limit as $j \to \infty$. This is a Markov chain, which we denote $\{\omega_t\}$, taking values in trajectories $\omega : [0, \infty) \to S$, and having jumps at a countable set of times $\xi_{s,r}$, at which a 1 flips to a 0 and all bits to its right flip as well.

We are interested in the speed at which the leftmost 1 drifts to the right. Because this chain converges weakly to the zero state, it is convenient to renormalize by shifting to the leftmost 1. Consequently, we define the space Ξ to be the space of those sequences of 0's and 1's indexed by the nonnegative integers that begin with a 1. We define two functionals fibzeros and fiones on Ξ by letting fiones(\mathbf{x}) ≥ 1 be the number of leading 1's:

$$\operatorname{Bones}(\mathbf{x}) := \inf\{j \ge 1 : \mathbf{x}(j) = 0\}$$

and letting $\beta zeros(\mathbf{x}) \ge 0$ be the number of successive 0's after the first 1:

$$\beta zeros(\mathbf{x}) := -1 + \inf\{j \ge 1 : \mathbf{x}(j) = 1\}$$

Let $\beta ones_j$ denote the set $\{\mathbf{x} : \beta ones(\mathbf{x}) = j\}$ which form a partition of Ξ , and let $\{\beta zeros_j\}$ denote the analogous partition with respect to the values of $\beta zeros$.

We now define a Markov chain on the space Ξ whose law starting from $\mathbf{x} \in \Xi$ is denoted $\mathbf{Q}_{\mathbf{x}}$. Pick $\omega \in S$ such that the leftmost 1 of ω is in some position i and $\omega(i + j) = \mathbf{x}(j)$ for all $j \ge 0$. We construct the Markov chain $\{X_t\}$ on Ξ as a function of $\{M_t\}$ as follows. First, define σ_0 to be 0, and i_0 to be the position of the leading 1 in ω . Now recursively we let σ_n be the first time after σ_{n-1} for which $N(i_{n-1}, \cdot)$ increases. For $\sigma_{n-1} \le t < \sigma_n$ we let X_t be M_t shifted so that i_{n-1} is at the origin (and ignoring negative indices). We let β_{jump_n} denote $\beta_{ones}(\sigma_n^-)$ and we let $i_n = i_{n-1} + \beta_{jump_n}$.

Sometimes, it is convenient to look at the chain sampled at times σ_n . Thus we let

$$Y_n := X_{\sigma_n}$$

which is now a discrete-time Markov chain. We let $\mathbb{P}_{\mathbf{x}}$ denote the law of this chain starting from \mathbf{x} .

Since the conditional distribution of $\sigma_n - \sigma_{n-1}$ given $\mathcal{F}_{\sigma_{n-1}}$ is exponential of mean 1, it follows that $\sigma_n/n \to 1$. The distance the leading 1 has moved to the right by time t is the sum $\sum_{n:\sigma_n \leq t} \beta_{jump_n}$, and therefore the average speed $\beta_{spd}(n)$ up to time σ_n is the random quantity

$$\operatorname{fspd}(n) := \sigma_n^{-1} \sum_{j=1}^n \operatorname{fjump}_j \sim n^{-1} \sum_{j=1}^n \operatorname{fjump}_j.$$
(1)

It is not a priori clear, nor in fact can we prove that Bspd(n) has a limit as $n \to \infty$. Consequently we define

$$\begin{split} & \text{ßinf-spd} := \text{liminf}_n \text{\$spd}(n) ; \\ & \text{\$sup-spd} := \text{limsup}_n \text{\$spd}(n) . \end{split}$$

Problem 1: Show that the limiting speed exists.

The Markov chain $\{Y_n\}$ is a time-homogeneous process on a compact space, so taking cesaro averages of the marginals, we see these must have at least one weak limit. Any weak limit is a stationary distribution.

Problem 2: Show that there is a unique stationary distribution for the chain $\{Y_n\}$.

This would imply a positive solution to Problem 1. In particular, it would imply that

$$n^{-1}\sum_{j=1}^n \operatorname{Bjump}_j \to \int \operatorname{Bjump}_1 d\pi$$

where π is the unique stationary measure; hence the speed would not only exist but would be almost surely constant independent of the starting state. Although we do not have a solution to Problem 2, we believe something stronger may hold. **Problem 3a:** Let $T^{j}\pi$ denote the composition of the measure π with a translation by j bits, e.g., if A is the event that there is a 1 in position r, then $T^{j}\pi(A) = \pi(A^{j})$, where A_{j} is the event that there is a 1 in position r+j. Prove that $T^{j}\pi \to M$ where π is the stationary measure and M is IID fair coin flipping.

Problem 3b: Prove or disprove that the unique stationary measure π is equivalent (mutually absolutely continuous) to M.

To see why we believe (3a) to be true, consider a window of positions of any size k, located far to the right of the initial 1. Between each time a clock rings in this window there are many times a 1 turns to a 0 to the left of the window. Each time this happens, the bits in the window all flip. Projecting configuration space onto what is visible in the window, and again onto a space of half the size by identifying each configuration with its complement, it seems reasonable to approximate the projection by a Markov chain. Specifically, from a state $\{x, x^c\}$, for each position $i \in [1, \ldots, k]$, exactly one of x or x^c will be able to flip the bit in position i, so one may imagine the pair $\{x, x^c\}$ as flipping at rate 1/2 in every position. This chain has the uniform distribution as the unique stationary distribution.

2 Statement of main result and lemmas

In this section we state the results that we do know how to prove, namely bounds on the lim inf and lim sup speeds. The following theorem is to be interpreted as referring to the lim inf and lim sup speeds (until we have proved the speed exists).

Theorem 2 The speed of the drift of the leftmost 1 satisfies

$$1.646 < \beta inf-spd < \beta sup-spd < 2.92$$
.

Relating back to the performance of the random edge rule on Klee-Minty cubes, we have:

Corollary 3 For sufficiently large n, starting from a uniform random vertex of the Klee-Minty cube,

$$0.086n^2 \le E_n \le 0.152n^2$$
.

PROOF: Gärtner, Henk and Ziegler consider another way of counting steps, where instead of choosing an edge at random among all those decreasing the objective function, they choose an edge at random from among all edges, but suppress the move if the edge increases the objective function. For a vector \mathbf{x} of 0's and 1's of length n, let $L^*(\mathbf{x}) = L^*(\mathbf{x}, n)$ denote the expectation of the number $N^{(r)}$ of moves starting from \mathbf{x}^r , including the suppressed moves, before the minimum is reached. They prove the following identity [GHZ98, Lemma 4].

$$E_n = \frac{1}{2n} \sum_{r=1}^n L^*(\mathbf{x}^r, n)$$
 (2)

where \mathbf{x}^r is the vector of length r consisting of all 1's. Including suppressed moves in the count corresponds in our infinite, continuous-time model to counting the number of clock events (only among the first r vertices). Let $T^{(r)}$ be the time it takes starting from \mathbf{x}^r to reach the minimum. By the strong law of large numbers, $N^{(r)}/T^{(r)} \sim r$ as $r \to \infty$. Also by the strong law, if the limit and limsup speed are known to be in the interval (a, b), then

$$\frac{r}{b} < T^{(r)} < \frac{r}{a}$$

for sufficiently large r. Hence, for sufficiently large r,

$$\frac{nr}{b} < N^{(r)} < \frac{nr}{a}$$

and plugging into (2) and summing from r = 1 to n gives

$$\frac{n^2}{4b} < E_n < \frac{n^2}{4a}$$

for sufficiently large n. Plugging in a = 2.92 and b = 1.646 proves the corollary.

The lower bound of Theorem 2 is proved in Section 4. The lower bound may in principle be improved so as to be arbitrarily near the actual speed. For the upper bound, we state some lemmas. Let

$$H_k := \sum_{j=1}^k \frac{1}{j}$$

denote the k^{th} harmonic number. Let $\{S_n\}$ be a random walk whose increments are equal k with probability 2/((k+1)(k+2)) for each integer $k \ge 1$. Let $S := S_{G-1}$ where G is an independent geometric random variable with mean 2. Let Θ be a random variable satisfying

$$\mathbb{P}(\Theta \ge j) = 1 - F_{\Theta}(j-1) = \sum_{k=1}^{\infty} \frac{1}{k} \frac{1}{k+j} = \frac{H_j}{j}.$$

Assume $\{S_n\}$ and Θ are independent of each other and of $\{\mathcal{F}_t\}$; denote expectation with respect to $\mathbb{P}_{\mathbf{x}}$ by $\mathbb{E}_{\mathbf{x}}$ and let \mathbb{E} denote expectation with respect to the laws of S and Θ . Analogously with $\beta \operatorname{zeros}(\mathbf{x})$ we define the quantity

$$\beta zeros^*(\mathbf{x}) := -1 + \inf\{j > \beta ones(\mathbf{x}) : \mathbf{x}(\beta ones(\mathbf{x}) - 1 + j) = 1\}$$

to be the number of zeros after the first block of ones; thus $\beta z = 1$, $\beta z = 1$, $\beta z = 1$, $\beta z = 0$ if and only if $\beta z = 0$ if and only if $\beta z = 0$.

Lemma 4 For any $\mathbf{x} \in \beta ones_i$,

$$\mathbb{E}_{\mathbf{x}}\beta jump_1 = \sum_{k=1}^j \frac{1}{k} \,.$$

Equivalently, for any $\mathbf{x} \in \Xi$,

$$\mathbb{E}_{\mathbf{x}}\beta jump_1 = H_{\beta ones(\mathbf{x})}.$$

Lemma 5 For any $j \ge 1$, any $\mathbf{x} \in \beta zeros_j$, and any integer $L \ge 1$,

$$\mathbb{P}_{\mathbf{x}}(\beta ones(Y_1) \ge L) \le \mathbb{P}(S+j \ge L)$$

When $\beta zeros(\mathbf{x}) = 0$ then

$$\mathbb{P}_{\mathbf{x}}(\beta ones(Y_1) \ge L) \le \mathbb{P}(\Theta + \beta zeros^*(\mathbf{x}) \cdot B + S \ge L)$$

where B is a Bernoulli with mean 1/2, and Θ , B and S are all independent.

Since $S + \Theta + \beta \text{zeros}^*(\mathbf{x})$ is an upper bound for both quantities $S + \beta \text{zeros}^*(\mathbf{x})$ and $S + \Theta + B\beta \text{zeros}^*(\mathbf{x})$ appearing as stochastic upper bounds in Lemma 5, and since H_n increases in n, we may put this together with Lemma 4 to obtain

Corollary 6 For any x,

$$\mathbb{E}_{\mathbf{x}}\beta jump_2 \leq \mathbb{E}H_{\Theta + \beta zeros^*(\mathbf{x}) + S} \,.$$

Lemma 7 The conditional distribution of $\beta zeros^*(Y_n)$ given $\mathcal{F}_{\sigma_{n-1}}$ is always bounded above stochastically by the law of Θ . In other words, for all j,

$$\mathbb{P}(\beta zeros^*(Y_n) > j \mid \mathcal{F}_{\sigma_{n-1}}) \le 1 - F_{\Theta}(j).$$

PROOF OF THEOREM 2 FROM THE LEMMAS: It suffices to show that for any $\mathbf{x} \in \Xi$,

$$\mathbb{E}_{\mathbf{x}}\beta jump_3 \leq 2.92$$
.

We simply iterate conditional expectations and compute. By the Markov property, and Corollary 6,

$$\mathbb{E}_{\mathbf{x}} \beta_{jump_{3}} = \mathbb{E}_{\mathbf{x}} \mathbb{E}_{Y_{1}} \beta_{jump_{2}}$$

$$\leq \mathbb{E}_{\mathbf{x}} \left(\mathbb{E}_{y} H_{\mathcal{S}+\Theta+\beta zeros^{*}(y)} \right) \Big|_{y=Y_{1}} .$$

Since H_n is increasing in n we may use the stochastic upper bound in Lemma 7 for any \mathbf{x} to see that this is at most $\mathbb{E}H_{S+\Theta^{(2)}}$ where $\Theta^{(2)}$ is the sum of two independent copies of Θ . The upper bound in Theorem 2 is completed by computing an upper bound for this.

The function H is concave and $\Theta^{(2)} \ge 1$, so

$$H(\Theta^{(2)} + j) - H(\Theta^{(2)}) \le H(j+1) - 1$$

and we may therefore conclude that

$$\mathbb{E}H_{\Theta^{(2)}+S} \le \mathbb{E}H_{\Theta^{(2)}} + \mathbb{E}H_{S+1} - 1.$$
(3)

To compute the quantity $\mathbb{E}H_{\Theta^{(2)}}$, let Θ_1 and Θ_2 independently have the distribution of Θ and write

$$\begin{split} \mathbb{E}H_{\Theta_1+\Theta_2} &= 1 + \sum_{1 \le j,k} \frac{1}{j+k} \mathbb{P}(\Theta_1 \ge j) \mathbb{P}(\Theta_2 = k) \\ &= 1 + \sum_{1 \le j,k} \frac{1}{j+k} \frac{H_j}{j} \left(\frac{H_k}{k} - \frac{H_{k+1}}{k+1} \right) \\ &= 1 + \sum_{1 \le j,k} \frac{1}{j+k} \frac{H_j}{j} \left(\frac{H_k}{k} - \frac{H_{k+1}}{k} + \frac{H_{k+1}}{k} - \frac{H_{k+1}}{k+1} \right) \\ &= 1 + \sum_{1 \le j,k} \frac{1}{j+k} \frac{H_j}{j} \frac{H_{k+1}-1}{k(k+1)} \,. \end{split}$$

We may evaluate this numerically as just under 2.

For the last term on the RHS of (3), let $\phi(z) = \mathbb{E}z^S = \sum_{n=0}^{\infty} z^n \mathbb{P}(S = n)$ be the generating function for S, so $z\phi$ is the generating function for S + 1. For any positive integer j there is an identity

$$\int_0^1 \frac{1-z^j}{1-z} \, dz = \int_0^1 (1+\cdots+z^{j-1}) \, dz = H_j \, .$$

Consequently, we may write

$$\mathbb{E}H_{S+1} - 1 = \int_0^1 \frac{z - \mathbb{E}z^{S+1}}{1 - z} \, dz = \int_0^1 z \frac{1 - \phi(z)}{1 - z} \, dz \,. \tag{4}$$

To compute the generating function ϕ , first compute the generating function f for the increments of $\{S_n\}$:

$$f(z) = \sum_{k=1}^{\infty} \frac{2}{(k+1)(k+2)} z^k = \frac{2z - z^2 - 2(1-z)\log\frac{1}{1-z}}{z^2}$$

Then $\phi = 1/(2 - f)$ and the integral in (4) becomes

$$\mathbb{E}H_{S+1} = \int_0^1 \frac{2z(\log\frac{1}{1-z} - z)}{3z^2 - 2z + 2(1-z)\log\frac{1}{1-z}} \, dz \, .$$

One may evaluate this numerically to approximately 0.918797. Adding this to the value for $\mathbb{E}H_{\Theta^{(2)}}$ gives a little under 2.92. Rigorous bounds may be obtained with more care.

3 Proofs of Lemmas

PROOF OF LEMMA 4: By definition, each $\mathbf{x} \in \text{gones}_j$ begins with j 1's in positions $0, \ldots, j-1$ followed by a zero. The evolution of $\{M_t\}$ decreases the binary representation $\sum_k 2^{-k} \mathbf{x}(k)$, whence $M_{\text{Bjump}_1^-} \in \text{gones}_k$ for some $k \leq j$, that is, there is always a zero in some position in [0, j]. Furthermore, once there is a zero in position k for some k < j, then there is always a zero at or to the left of position k. It follows that Bjump_1 is equal to the least k < jfor which $\xi_{j,1} < \xi_{0,1}$, that is, for which the clock in position k goes off before the clock at position 0. The minimum is taken to be j if there is no such k. It follows that $\mathbb{P}_{\mathbf{x}}(\text{fjump}_1 = j) = 1/j$ and that for 0 < k < j,

$$\mathbb{P}_{\mathbf{x}}(\text{fjump}_1 = j) = \frac{1}{k(k+1)}.$$
(5)

To see this, note that $\text{fjump}_1 = k$ if and only if $\xi_{0,1}$ is the minimum of the exchangeable variables $\{\xi_{k,1} : 0 \leq k < j\}$. Similarly, for 0 < k < j, $\text{fjump}_1 = k$ if and only if $\xi_{k,1}$ is the minimum of the variables $\{\xi_{r,1} : 0 \leq r \leq k\}$ and $\xi_{0,1}$ is the next least of the values. Computing expectations via (5) proves the lemma.

PROOF OF LEMMA 7: Again let us denote $q = \text{fones}(\mathbf{x})$. We recall that $\mathbb{P}(\text{fijump}_1 = k) = 1/q$ for k = q and 1/(k(k+1)) for $1 \le k \le q-1$. We claim that for any $k \le q$,

$$\mathbb{P}(\beta \operatorname{zeros}^*(Y_1) \ge j \,|\, \beta \operatorname{jump}_1 = k) \le \frac{k+1}{k+j} \,. \tag{6}$$

If we can show this, then we will have

$$\mathbb{P}(\text{fizeros}^*(Y_1) \ge j) \le \frac{1}{q} \frac{q+1}{q+j} + \sum_{k=1}^{q-1} \frac{1}{k(k+1)} \frac{k+1}{k+j}.$$

Changing q to q+1 increases this by

$$\frac{j-1}{(q+1)(q+j)(q+j+1)}$$

which is nonnegative. Setting $q = \infty$ then yields the upper bound in the lemma, and it remains to show (6).

Observe first that it suffices to show this for k = q. This is because when k < q, the event $\{\beta_{jump_1} = k\}$ necessitates $\xi_{k,1} = \min\{\xi_{r,1} : 0 \le r \le k\}$. Thus to evaluate $\mathbb{P}(\beta_{zeros}^*(Y_1) \ge j \mid \beta_{jump_1} = k)$ we may wait until time $\xi_{k,1}$, at which point if no bit to the left of k has flipped yet, the new conditional probability $\mathbb{P}(\beta_{zeros}^*(Y_1) \ge j \mid \mathcal{F}_{\xi_{k,1}}, \beta_{jump_1} = k)$ is always at most (k+1)/(k+j) by applying the claim for q = k.

Assuming now that k = q, we note that the event { $\beta_{jump_1} = k$ } that we are conditioning on is just the event

$$A := \{\xi_{0,1} = \min_{0 \le i \le k-1} \xi_{i,1}\}$$

is the event that the clock at 0 goes off before any clock in positions $1, \ldots, k-1$. Conditioning on A then makes the law of $\xi_{0,1}$ an exponential of mean 1/k without affecting the joint distribution of $\{\xi_{r,s} : r > k\}$. Now let m_1 be the position at time $\xi_{k,1}$ of the first 1 to the right of k, and let t_1 be the time this 1 flips. Inductively, define m_{r+1} to be the position of the first 1 to the right of k after time t_r and let t_{r+1} be the first time after t_r that this 1 flips.

If the positions $m_r, \ldots, m_r + j - 1$ are not filled with ones at time t_{r-1} (define $t_0 = 0$) then it is not possible to have $\beta \operatorname{zeros}^*(Y_1) \ge j$ and A and $t_{r-1} < \xi_{0,1} < t_r$. That is, one cannot get from fewer than j ones in the first block of ones to the right of k to at least jones at the time of the flip at 0 without having the leftmost one in this block flip. On the other hand, if these j positions are filled with ones at time t_{r-1} , then

$$\mathbb{P}(\beta \text{zeros}^*(Y_1) \ge j, \xi_{0,1} < t_r \mid A, \mathcal{F}_{t_{r-1}}, \xi_{0,1} > t_{r-1}) \le \frac{k}{k+j}$$

since the event { $\beta zeros^*(Y_1) \ge j, \xi_{0,1} < t_r$ } requires that the clock at 0 go off before the clocks in positions $m_r, \ldots, m_r + j - 1$ (recall that conditioning on A has elevated the rate of the clock at 0 to rate k). Similarly, $\mathbb{P}(\xi_{0,1} < t_r \mid A, \mathcal{F}_{t_{r-1}}, \xi_{0,1} > t_{r-1}) = k/(k+1)$. Therefore,

$$\frac{\mathbb{P}(\text{fizeros}^*(Y_1) \ge j, \xi_{0,1} < t_r \,|\, A, \mathcal{F}_{t_{r-1}}, \xi_{0,1} > t_{r-1})}{\mathbb{P}(\xi_{0,1} < t_r \,|\, A, \mathcal{F}_{t_{r-1}}, \xi_{0,1} > t_{r-1})} \le \frac{k+1}{k+j}$$

By the craps principle, the RHS is then an upper bound for the probability of $\beta zeros^*(Y_1) \ge j$ conditioned only on $\beta jump_1 = k$.

PROOF OF LEMMA 5: Let $\mathbf{x} \in \beta \text{zeros}_j$ and first assume $j \geq 1$. We prove the statement for n = 1, the proof for greater n being identical, conditioned on $\mathcal{F}_{\sigma_{n-1}}$. It is simple to check whether $\beta \text{ones}(Y_1) = j$. The bits in positions $1, \ldots, j$ will remain 0's until the leading 1 flips at time σ_1 , so the only thing to check is whether $\sigma_1 = \xi_{0,1}$ is less than or greater than $\xi_{j+1,1}$. With probability 1/2, $\xi_{0,1} < \xi_{j+1,1}$ and in exactly this case $\beta \text{ones}(Y_1) = j$.

Now condition on this inequality going the other way: $\xi_{01} > \xi_{j+1,1}$. Let $t_1 := \xi_{j,1}$. Let $j + 1 + k_1$ be the position of the first 0 of **x** to the right of position j + 1. Then at time t_1 , the position of the first one to the right of j + 1 is $j + Z_1$, where Z_1 is the least $l \in [1, k_1 - 1]$ for which $\xi_{j+1+l,1} < \xi_{j+1,1}$. If no such l exists, then $Z_1 = k_1$. We compute $\mathbb{P}_{\mathbf{x}}(Z_1 = l)$ as follows.

The variables $\{\xi_{r,1} : r \in \{0\} \cup [j+1, j+k_1]\}$ are independent exponentials. For $1 \leq l < k_1$, the event that $Bones(Y_1) \neq j$ and $Z_1 = l$ is the intersection of the event A that $\xi_{j+1,1}$ is less than $\xi_{0,1}$ and $\xi_{r,1}$ for all $j+2 \leq r \leq j+l$ with the event *B* that $\xi_{j+1+l,1} < \xi_{j+1,1}$. In other words, among l+2 independent exponentials, the index of the least and second least must be j+2 and j+1 respectively. The unconditional probability of this is 1/((l+1)(l+2)). Having conditioned on the larger event $\{\xi_{j+1,1} < \xi_{0,1}\}$, the conditional probability is therefore equal to 2/((l+1)(l+2)). This holds for $l < k_1$, where $Z_1 = k_1$ with the complementary probability. To sum up, Z_1 is distributed as $S_1 \wedge k_1$ where S_1 has the distribution of the random walk increments described in the lemma.

The last step is to invoke the Markov property. Condition on \mathcal{F}_{t_1} . The chain from here evolves under the law $\mathbb{P}_{X(t_1)}$. Iterating the previous argument, there are two cases. The first case, which happens with probability 1/2 is that the clock at the origin goes off before the next alarm at location $j + 1 + Z_1$. In this case, $\text{flones}(Y_1) = j + Z_1$. In the alternative case, we let t_2 be the time at which the clock at location $j + 1 + Z_1$ next goes off. We let Z_2 be the number of consecutive 1's at time t_2^- starting from position $j + 1 + Z_1$. Then conditional on \mathcal{F}_{t_1} , Z_2 is distributed as $S_1 \wedge k_2$ where k_2 is the number of consecutive 1's at time t_1 starting at position $j + 1 + Z_1$.

Iterating in this way, we have the following inductive definitions. Let $t_0 = 0$. Let τ be the least r for which the clock at the origin goes off after time t_r but before the first alarm at location $j + 1 + \sum_{i=1}^{r} Z_i$. For each $r \leq \tau$, we may define k_r to be the number of consecutive 1's at time t_{r-1} starting at location $j + 1 + \sum_{i=1}^{r-1} Z_i$. We may then define t_r to be the first time after t_{r-1} that the alarm at location $j + 1 + \sum_{i=1}^{r-1} Z_i$ goes off, and we may define Z_r so that $j + 1 + \sum_{i=1}^{r} Z_i$ is the location of the first zero to the right of $j + 1 + \sum_{i=1}^{r-1} Z_i$ at time t_r^- .

The upshot of all of this is that

$$\operatorname{Bones}(Y_1) = j + \sum_{i=1}^{\tau} Z_i$$

and that the joint distributions of τ and $\{Z_i : 1 \leq i \leq \tau\}$ are easily described. Conditioned on $\tau \geq r$ and on \mathcal{F}_{t_r} , the probability of $\tau = r + 1$ is always 1/2; as well, Z_{r+1} given $\tau \geq r + 1$ and \mathcal{F}_{t_r} is always distributed as a truncation of S_1 . We conclude that $\text{Bones}(Y_1)$ is stochastically dominated by the sum of τ independent copies of S_1 , hence as S_{G-1} .

Finally, we consider the case where $\beta \operatorname{zeros}^*(\mathbf{x}) = l > \beta \operatorname{zeros}(\mathbf{x}) = 0$. Let $q = \beta \operatorname{ones}(\mathbf{x})$, so that \mathbf{x} begins with q ones, followed by l zeros, followed by a one in position q + l. A

preliminary observation is that if we begin with a one at the origin, the position W(t) of the leading one at a later time t is an increasing function of t; hence, if T_{μ} is an independent exponential with mean μ , the distribution of $W(T_{\mu})$ is stochastically increasing in μ .

Begin by writing

$$\mathbb{P}_{\mathbf{x}}(\mathrm{\betaones}(Y_1) \ge j) = \sum_{k=1}^{q} \mathbb{P}(\mathrm{\betaones}(Y_1) \ge j, \mathrm{\betajump}_1 = k)$$

Let l^* denote the number of zeros consecutively starting from position k at time $\xi_{k,1}$ if $\beta_{jump_1} = k < q$, and $l^* = l$ if k = q. In other words, $l^* = \beta_{zeros}(\mathbf{x}')$ where \mathbf{x}' is the word at the last time t that β_{ones} changes before the leading bit flips ($t = \xi_{\beta_{jump_1,1}}$ if $\beta_{jump_1} < q$ and t = 0 otherwise). We may then describe $\beta_{ones}(Y_1)$ as $l^* + W$, where W is the number of consecutive positions starting from position $\beta_{jump_1} + l^*$ that turn to zeros between time t and $\xi_{0,1}$. Now we break into two cases.

Condition first on $\{\beta_{jump_1} = q\}$. The time $\xi_{0,1}$ is now an exponential of mean 1/q, and before this time, the bits from position q + l onward evolve independently. We may describe $\beta_{0,1}$ as $l + W(\xi_{0,1})$, where W is the number of consecutive positions starting at q + l which have become zeros in the time from 0 to $\xi_{0,1}$. The first part of this lemma established that when ξ has rate 1, then $W(\xi) \leq S$. Our preliminary observation now shows, conditional on $\{\beta_{jump_1} = q\}$, that $W(\xi_{0,1}) \leq S$.

Next, let us condition on $\beta \text{jump}_1 = k < q$, observing that then $l^* \leq q - k$. In order to have $l^* \geq r$, it is necessary that $\xi_{k,1} = \min\{\xi_{s,1} : k \leq s \leq k + r - 1\}$. Having conditioned on $\beta \text{jump}_1 = k$, the distribution of $\xi_{k,1}$ becomes an exponential of rate k + 1, so that the conditional probability of this clock going off before r - 1 other conditionally independent clocks of rate 1 is just (k + 1)/(k + r). Since the event $\{\beta \text{jump}_1 = k < q\}$ has probability 1/(k(k + 1)) if k < q and zero otherwise, we may remove the conditioning and sum to get

$$\mathbb{P}(l^* \ge r, \operatorname{\betajump}_1 < q) \le \sum_{k=1}^{q-1} \frac{1}{k(k+1)} \frac{k+1}{k+r} \le \mathbb{P}(\theta^* \ge r)$$

We also still have in this case $W \leq S$.

Putting together the cases $\beta_{jump_1} = q$ and $\beta_{jump_1} < q$, we see that $l^* = l$ with probability 1/q and otherwise $l^* \leq \Theta$. The crude bound $1/q \leq 1/2$ gives $l^* \leq \Theta + \beta_{zeros}^*(\mathbf{x}) \cdot B$. Since $l^* \in \sigma(\mathcal{F}_t)$ and the bound $W \leq S$ holds conditionally on \mathcal{F}_t , we arrive at $\beta_{zeros}(Y_1) \leq \Theta + \beta_{zeros}(\mathbf{x}) \cdot B + S$. \Box

4 Lower bound

To arrive at a lower bound, consider a tree whose vertices are positive integers, identified with their binary expansions. The root is 1, and the children of x are 2x and 2x + 1. Associated with each node x are a set of n(x) possible transitions, where n(x) is the number of 1's in the binary expansion of x. The transitions are to numbers gotten by flipping a 1 and simultaneously flipping all bits to its right. Note that all transitions from x are to numbers less than x. Let r(x) denote the *reward* if the leading bit of x is flipped, namely one less than the number of leading 1's in the binary expansion of x. Recursively, we assign to each x a mean reward and maturation time, (a(x), b(x)) as follows. Fix a set B of *bad nodes*, to be determined later. On first reading, take B to be empty. For any x, let y_1, \ldots, y_n be the possible transitions from x. Let (a(x), b(x)) := (0, 0) if $x \in B$ and otherwise, let

$$(a(x), b(x)) := \frac{1}{n} \left((r(x), 1) + \sum_{i=1}^{n} (a(y_i), b(y_i)) \right) \,.$$

Let \mathcal{T} be any finite binary rooted subtree, meaning that any vertex in the subtree has either zero or two children in the subtree.

Lemma 8 Suppose no leaf of \mathcal{T} is in B. Then an almost sure lower bound for the lim inf speed from any starting configuration is given by the minimum over leaves x of T of 1 + a(x)/b(x).

The lower bound in Theorem 2 will follow from Lemma 8 together with an implementation of the recursion. Below is some code written in C that implements the recursion for a complete binary tree of depth 15, with the set B chosen to give a good bound without much trouble.

```
#include <stdio.h>
#define N 18
#define M
              524288
main() {
        float a[M][2];
        int i , j , k, n, min_index, locmin, power[N+2], bitcount=0, lastcount=0;
float min_ratio;
       float locrat;
       float average;
a[0][0] = 0; a[0][1]=0; a[1][0] = 0; a[1][1] = 0; power[0] = 1; power[1]=2; average = 0;
/* the n loop stratifies the recursion by levels of the tree */
for (n=1; n <= N; n++)</pre>
{
 power[n+1] = 2*power[n];
  /* the i loop indexes vertices inside a level */
  for (i=power[n]; i < power[n+1]; i++)</pre>
  {
    bitcount = 0; lastcount=0; a[i][0] = 0; a[i][1] = 0;
    for (j=0; j <= n; j++)</pre>
    {
      k = i^(power[j]);
      if (k < i)
      {
        /* k < i iff there is a 1 in position k */
        k = power[j+1]*(i / (power[j+1])) + power[j+1] - 1 - i % power[j+1];
        a[i][0] += a[k][0];
        a[i][1] += a[k][1];
        bitcount++;
}
      else lastcount=bitcount;
      /* lastcount will eventually say how many 1's after the leftmost zero */
    }
    a[i][1]++;
    a[i][0] += bitcount - lastcount - 1;
    a[i][0] = a[i][0] / bitcount;
    a[i][1] = a[i][1] / bitcount;
    /* a particular choice of the set B is given in literal form */
```

```
if ( i==2 || i==4 || i==5 || i==8 ||
          i==9 || i==10 || i==11 ||
          i==18 || i==19 || i==20 || i==21 || i==22 || i==23 ||
          i==36 || i==37 || i==38 ||
          i==40 || i==41 || i==42 || i==43 || i==44 || i==45 || i==46 || i==47 ||
          i==73 || i==74 || i==75 ||
          i==80 || i==81 || i==82 || i==83 || i==84 || i==85 || i==86 || i==87 ||
          i==88 || i==89 || i==90 || i==91 ||
          i==160 || i==161 || i==162 || i==163 || i==164 || i==165 || i==166 || i==167 ||
  i==168 || i==169 || i==170 || i==171 || i==172 || i==173 || i==174 || i==175 || i==178 ||
  i==180 || i==181 || i==182 || i==183 ||
          i==324 || i==325 || i==326 || i==328 || i==329 ||
          i==330 || i==331 || i==332 || i==333 || i==334 || i==335 ||
          i==336 || i==337 || i==338 || i==339 || i==340 || i==341 || i==342 || i==343 ||
          i==344 || i==345 || i==346 || i==347 || i==348 || i==349 ||
          i==350 || i==351 ||
          i==361 || i==362 || i==363 ||
          i==656 || i==657 || i==658 || i==659 || i==660 || i==661 || i==662 || i==663 ||
  i==672 || i==673 || i==674 || i==675 || i==676 || i==677 || i==678 || i==679 ||
          i==680 || i==681 || i==682 || i==683 ||
          i==684 || i==685 || i==686 || i==687 || i==688 || i==689 ||
          i==690 || i==691 || i==692 || i==693 || i==694 || i==695 || i==697 || i==698 || i==699||
          i==1322 || i==1323 ||
          i==1346 || i==1347 || i==1348 || i==1349 ||
          i==1350 || i==1351 || i==1352 || i==1353 || i==1354|| i==1355 ||
          i==1356 || i==1357 || i==1358 || i==1359 ||
          i==1360 || i==1361 || i==1362 || i==1363 || i==1364 || i==1365 ||
          i==1366 || i==1367 || i==1368 || i==1369 || i==1370 || i==1371 ||
          i==1372 || i==1373 || i==1374 || i==1375 || i==1380 || i==1381 ||
          i==1384 || i==1385 || i==1386 || i==1387 || i==1388 || i==1389 || i==1390 || i==1391 ||
          i==2704 || i==2705 || i==2706 || i==2707 || i==2708 || i==2709 || i==2710 || i==2711 ||
          i==2720 || i==2721 || i==2722 || i==2723 || i==2724 || i==2725 || i==2726 || i==2727 ||
          i==2728 || i==2729 || i==2730 || i==2731 || i==2732 || i==2733 || i==2734 || i==2735 ||
          i==2736 || i==2737 || i==2738 || i==2739 || i==2740 || i==2741 || i==2742 || i==2743 ||
          i==2745 || i==2746 || i==2747 ||
          i==5456 || i==5457 || i==5458 || i==5459 || i==5467 ||
          i==5460 || i==5461 || i==5462 || i==5463 || i==5465 || i==5466 )
  ſ
a[i][0] = 0; a[i][1] = 0;
```

```
}
    if(i == (M/2))
     {
min_index = i;
min_ratio = a[i][0]/a[i][1];
      }
     if(i > (M/2))
      {
if((a[i][0]/a[i][1]) < min_ratio)
  {
   min_index = i;
   min_ratio = a[i][0]/a[i][1];
  }
      }
     if (i > (M/2))
   {
      average = average + (a[i][0]/a[i][1]);
          }
      if(i == 3)
     {
locmin = i;
locrat = a[i][0]/a[i][1];
     }
     if(i < (M/2))
{ if (i > 71)
         {
    if(a[i][1] > 0)
            { if ((a[i][0]/a[i][1]) < locrat)
       {
       locmin = i;
       locrat = (a[i][0])/(a[i][1]);
      }
     }
          }
        }
  }
  printf("\n");
  }
```

A look at the data shows the minimum value of a(x)/b(x) on each level to be obtained when the binary expansion of x alternates. In particular, the global minimum is at x =349525 and has value 0.646..., which proves the lower bound.

PROOF OF LEMMA 8: Any finite rooted binary subtree induces a prefix rule, that is, a map η from infinite sequences beginning with a 1 to leaves of \mathcal{T} , defined by $\eta(\mathbf{x}) = w$ for the unique leaf of \mathcal{T} that is a prefix of η .

Given a trajectory of the Markov chain $\{X_t\}$, define a sequence of elements of \mathcal{T} as follows. Let $x_0 := \eta(X_0)$ be the prefix of the initial state of the trajectory. Let $\tau_0 := 0$. As the definition proceeds, verify inductively that for $\tau_k \leq t < \tau_{k+1}$, the string x_k will be an initial segment of X_t . The recursion is as follows. Let τ_{k+1} be the first time after τ_k that a 1 flips in the initial segment x_k of X_t . Let x'_{k+1} be the string gotten from x_k by flipping this bit and all bits to its right. If $x'_{k+1} \notin B$ and x'_{k+1} is not the zero string then let x_{k+1} be x'_{k+1} , stripped of any leading zeros. If $x'_{k+1} \in B$ or x'_{k+1} is the zero string, then let $x_{k+1} = \eta(X_{\tau_{k+1}})$.

Let ρ_0, ρ_1, \ldots be the successive times that this latter transition occurs, that is, successive times t in the recursion that $\eta(X_t)$ is computed. Given $x \in \mathcal{T}$, suppose we begin counting every time the sequence $\{x_k\}$ hits x and stop counting at every time ρ_k . When we are counting, we count how many times the leading 1 flips, and how many 1's flip together when this happens, or more precisely, we keep a cumulative count, each time adding one less than the number of 1's that have flipped together. Let this cumulative count be denoted (A(x,t), B(x,t)), where A counts flips of the leading 1. If $x \in B$, then by convention we do not count anything.

Claim: For each $x \in \mathcal{T} \setminus B$, if x is visited infinitely often then the ratio of A(x,t)/B(x,t)converges as $t \to \infty$ to a(x)/b(x). Proof: Conditioned on the past, each time we start counting we are equally likely to transition to each of the n(x) possible transitions. The claim then follows from induction on x. Each x defines a set of time where x is on, that is, where we are counting leading 1 flips and rewards for that x. As x ranges over the leaves of \mathcal{T} , the on-timesets for x partition $[0, \infty)$. In each such time set, we have shown that the mean number of recorded simultaneous 1-flips per leading 1 flip is 1 + a(x)/b(x) in the limit, as long as the time set is unbounded. Since the partition is finite, it follows that the lim inf speed is at least the minimum of the values.

We remark that the only place this computation is not sharp is when the number of 1's flipping together exceeds the number recorded, because the present knowledge of the prefix was a string of all 1's and there were more 1's after this that also flipped. Thus by making the tree \mathcal{T} big enough, even without increasing B, we can get arbitrarily close to the true value.

5 Further observations

The following argument almost solves Problem (3a), and perhaps may be strengthened to a proof. Lemma 4 of [GHZ98] is proved by means of a duality result. The result is that the probability, starting from a uniform random state, of finding a 1 in position r after t steps (counting suppressed transitions), is equal to half the probability that \mathbf{x}^r has not reached the minimum yet after t steps (again counting suppressed transitions). The argument that proves this may be generalized by introducing a simultaneous coupling of the process from all starting states. The probability, from a uniform starting state, of finding a 1 in every position in a set A after t steps, is then the expectation of the function that is zero if the column vector of all 1's is not in the span of the columns of the matrix whose rows are the states reached at time t starting at \mathbf{x}^r , as r varies over A, and is 2^{-u} if the column vector of all 1's is in the span and the matrix has rank u. The kernel of the matrix is the set of starting configurations that reach the minimum by time t (the simultaneous coupling is linear). Hence, as long as A and t are such that the probability of reaching the minimum from any \mathbf{x}^r by time t goes to zero, the rank of the matrix will be |A| and the probability of finding all 1's in positions in A at time t will go to $2^{-|A|}$. In particular, if a window of fixed size moves rightward faster than the limsup speed, then what one sees in this window approaches uniformity. This is not good enough to imply uniformity of a window a fixed distance to the right of the leftmost 1.

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