# Homological Connectivity of Random $k$-dimensional Complexes 

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#### Abstract

Let $\Delta_{n-1}$ denote the ( $n-1$ )-dimensional simplex. Let $Y$ be a random $k$-dimensional subcomplex of $\Delta_{n-1}$ obtained by starting with the full $(k-1)$-dimensional skeleton of $\Delta_{n-1}$ and then adding each $k$-simplex independently with probability $p$. Let $H_{k-1}(Y ; R)$ denote the ( $k-1$ )-dimensional reduced homology group of $Y$ with coefficients in a finite abelian group $R$. It is shown that for any fixed $R$ and $k \geq 1$ and for any function $\omega(n)$ that tends to infinity


$$
\lim _{n \rightarrow \infty} \operatorname{Pr}\left[H_{k-1}(Y ; R)=0\right]= \begin{cases}0 & p=\frac{k \log n-\omega(n)}{n} \\ 1 & p=\frac{k \log n+\omega(n)}{n}\end{cases}
$$

## 1 Introduction

Let $G(n, p)$ denote the probability space of graphs on the vertex set $[n]=$ $\{1, \ldots, n\}$ with independent edge probabilities $p$. Let log denote the natural logarithm. A classical result of Erdős and Rényi [2] asserts that the threshold probability for connectivity of $G \in G(n, p)$ coincides with the threshold for

[^0]the non-existence of isolated vertices in $G$. In particular, for any function $\omega(n)$ that tends to infinity
\[

\lim _{n \rightarrow \infty} \operatorname{Pr}[G \in G(n, p): G connected]= $$
\begin{cases}0 & p=\frac{\log n-\omega(n)}{n} \\ 1 & p=\frac{\log n+\omega(n)}{n}\end{cases}
$$
\]

A 2-dimensional analogue of the Erdős-Rényi result was considered in [3], where the threshold for homological 1-connectivity of random 2-dimensional complexes was determined (see below). In this paper we study the homological $(k-1)$-connectivity of random $k$-dimensional complexes for a general fixed $k$.

We recall some topological terminology (see e.g. [4]) . Let $X$ be a finite simplicial complex on the vertex set $V$. let $X^{(k)}=\{\sigma \in X: \operatorname{dim} \sigma \leq k\}$ denote the $k$-dimensional skeleton of $X$, and let $X(k)$ denote the set of $k$ dimensional simplices in $X$, each taken with an arbitrary but fixed orientation. Denote by $f_{k}(X)=|X(k)|$ the number of $k$-dimensional simplices in $X$. Let $R$ be a fixed finite abelian group of cardinality $r$. A simplicial $k$-cochain is an $R$-valued skew-symmetric function on all ordered $k$-simplices of $X$. For $k \geq 0$ let $C^{k}(X)$ denote the group of $k$-cochains on $X$. The $i$ face of an ordered $(k+1)$-simplex $\sigma=\left[v_{0}, \ldots, v_{k+1}\right]$ is the ordered $k$-simplex $\sigma_{i}=\left[v_{0}, \ldots, \widehat{v_{i}}, \ldots, v_{k+1}\right]$. The coboundary operator $d_{k}: C^{k}(X) \rightarrow C^{k+1}(X)$ is given by

$$
d_{k} \phi(\sigma)=\sum_{i=0}^{k+1}(-1)^{i} \phi\left(\sigma_{i}\right)
$$

It is convenient to augment the cochain complex $\left\{C^{i}(X)\right\}_{i=0}^{\infty}$ with the $(-1)$ degree term $C^{-1}(X)=R$ with the coboundary map $d_{-1}: C^{-1}(X) \rightarrow C^{0}(X)$ given by $d_{-1} a(v)=a$ for $a \in R, v \in V$. Let $Z^{k}(X)=\operatorname{ker}\left(d_{k}\right)$ denote the space of $k$-cocycles and let $B^{k}(X)=\operatorname{Im}\left(d_{k-1}\right)$ denote the space of $k$ coboundaries. For $k \geq 0$ let $H^{k}(X ; R)=Z^{k}(X) / B^{k}(X)$ denote the $k$ th reduced cohomology group of $X$ with coefficients in $R$. We abbreviate $H^{k}(X)=H^{k}(X ; R)$.

Let $\Delta_{n-1}$ denote the ( $n-1$ )-dimensional simplex on the vertex set $V=[n]$. Let $Y_{k}(n, p)$ denote the probability space of complexes $\Delta_{n-1}^{(k-1)} \subset Y \subset \Delta_{n-1}^{(k)}$ with probability measure

$$
\operatorname{Pr}(Y)=p^{f_{k}(Y)}(1-p)^{\binom{n}{k+1}-f_{k}(Y)}
$$

A $(k-1)$-simplex $\sigma \in \Delta_{n-1}(k-1)$ is isolated in $Y$ if it is not contained in any of the $k$-simplices of $Y$. If $\sigma$ is isolated then the indicator function of $\sigma$ is a non-trivial $(k-1)$-cocycle of $Y$, hence $H^{k-1}(Y) \neq 0$. Our main result is that the threshold probability for the vanishing of $H^{k-1}(Y)$ coincides with the threshold for the non-existence of isolated $(k-1)$-simplices in $Y$.

Theorem 1.1. Let $k \geq 1$ and $R$ be fixed, and let $\omega(n)$ be any function which satisfies $\omega(n) \rightarrow \infty$ then

$$
\lim _{n \rightarrow \infty} \operatorname{Pr}\left[Y \in Y_{k}(n, p): H^{k-1}(Y ; R)=0\right]=\left\{\begin{array}{cl}
0 & p=\frac{k \log n-\omega(n)}{n}  \tag{1}\\
1 & p=\frac{k \log n+\omega(n)}{n}
\end{array}\right.
$$

## Remarks:

1. Theorem 1.1 remains true when $H^{k-1}(Y)$ is replaced by the $(k-1)$-th reduced homology group $H_{k-1}(Y)=H_{k-1}(Y ; R)$. This follows from the universal coefficient theorem since $H_{k-2}(Y)=0$ for $Y \in Y_{k}(n, p)$.
2. The $k=1$ case of Theorem 1.1 is the Erdős-Rényi result. For $k=2$ and $R=\mathbb{Z}_{2}$ the theorem was proved in [3]. Our approach to the general case combines the method of [3] with some additional new ideas.

The case $p=\frac{k \log n-\omega(n)}{n}$ of Theorem 1.1 is straightforward: Let $g(Y)$ denote the number of isolated $(k-1)$-simplices of $Y$. Then

$$
E[g]=\binom{n}{k}(1-p)^{n-k}=\Omega(\exp (\omega(n))) .
$$

A standard second moment argument then shows that

$$
\operatorname{Pr}\left[H^{k-1}(Y)=0\right] \leq \operatorname{Pr}[g=0]=o(1)
$$

The case $p=\frac{k \log n+\omega(n)}{n}$ is more involved. For a $\phi \in C^{k-1}\left(\Delta_{n-1}\right)$ denote by [ $\phi$ ] the image of $\phi$ in $H^{k-1}\left(\Delta_{n-1}^{(k-1)}\right)$. Let

$$
b(\phi)=\left|\left\{\tau \in \Delta_{n-1}(k): d_{k-1} \phi(\tau) \neq 0\right\}\right| .
$$

For any complex $Y \supset \Delta_{n-1}^{(k-1)}$ we identify $H^{k-1}(Y)$ with its image under the natural injection $H^{k-1}(Y) \rightarrow H^{k-1}\left(\Delta_{n-1}^{(k-1)}\right)$. It follows that for $\phi \in$ $C^{k-1}\left(\Delta_{n-1}\right)$

$$
\operatorname{Pr}\left[[\phi] \in H^{k-1}(Y)\right]=(1-p)^{b(\phi)}
$$

For $\phi \in C^{k-1}\left(\Delta_{n-1}\right)$ let $\operatorname{supp}(\phi)=\left\{\sigma \in \Delta_{n-1}(k-1): \phi(\sigma) \neq 0\right\}$. The weight of such $\phi$ is defined by

$$
\begin{aligned}
w(\phi)= & \min \left\{\left|\operatorname{supp}\left(\phi^{\prime}\right)\right|: \phi^{\prime} \in C^{k-1}\left(\Delta_{n-1}\right),\left[\phi^{\prime}\right]=[\phi]\right\}= \\
& \min \left\{\left|\operatorname{Supp}\left(\phi+d_{k-2} \psi\right)\right|: \psi \in C^{k-2}\left(\Delta_{n-1}\right)\right\} .
\end{aligned}
$$

A $k$-uniform hypergraph $\mathcal{F} \subset\binom{[n]}{k}$ is connected if for any $\sigma, \tau \in \mathcal{F}$ there exists a sequence $\sigma=\sigma_{0}, \ldots, \sigma_{t}=\tau \in \mathcal{F}$ such that $\left|\sigma_{i} \cap \sigma_{i-1}\right|=k-1$ for all $1 \leq i \leq t$. Let

$$
\mathcal{G}_{n}=\left\{0 \neq \phi \in C^{k-1}\left(\Delta_{n-1}\right): \operatorname{supp}(\phi) \text { is connected }, w(\phi)=|\operatorname{supp}(\phi)|\right\}
$$

If $H^{k-1}(Y) \neq 0$ and $\phi \in C^{k-1}\left(\Delta_{n-1}\right)$ is a cochain of minimum support size such that $0 \neq[\phi] \in H^{k-1}(Y)$, then $\phi \in \mathcal{G}_{n}$. Therefore

$$
\operatorname{Pr}\left[H^{k-1}(Y) \neq 0\right] \leq \sum_{\phi \in \mathcal{G}_{n}} \operatorname{Pr}\left[[\phi] \in H^{k-1}(Y)\right]=\sum_{\phi \in \mathcal{G}_{n}}(1-p)^{b(\phi)}
$$

Theorem 1.1 will thus follow from
Theorem 1.2. For $p=\frac{k \log n+\omega(n)}{n}$

$$
\begin{equation*}
\sum_{\phi \in \mathcal{G}_{n}}(1-p)^{b(\phi)}=o(1) \tag{2}
\end{equation*}
$$

The main ingredients in the proof of Theorem 1.2 are a lower bound on $b(\phi)$ given in Section 2, and an estimate for the number of $\phi \in \mathcal{G}_{n}$ with prescribed values of $b(\phi)$ given in Section 3. In Section 4 we combine these results to derive Theorem 1.2. The group $R$ and the dimension $k$ are fixed throughout the paper. We use $c_{i}=c_{i}(r, k)$ to denote constants depending on $r$ and $k$ alone.

## 2 A lower bound on $b(\phi)$

We bound $b(\phi)$ in terms of the weight $w(\phi)$.
Proposition 2.1. For $\phi \in C^{k-1}\left(\Delta_{n-1}\right)$

$$
\begin{equation*}
b(\phi) \geq \frac{n w(\phi)}{k+1} \tag{3}
\end{equation*}
$$

Proof: For an ordered simplex $\tau=\left[v_{0}, \ldots, v_{\ell}\right]$ and a vertex $v \notin \tau$, let $v \tau=\left[v, v_{0}, \ldots, v_{\ell}\right]$. For $u \in V$ define $\phi_{u} \in C^{k-2}\left(\Delta_{n-1}\right)$ by

$$
\phi_{u}(\tau)= \begin{cases}\phi(u \tau) & u \notin \tau  \tag{4}\\ 0 & u \in \tau .\end{cases}
$$

Let $\sigma \in \Delta_{n-1}(k-1)$ and $u \in V$. Then

$$
\phi(\sigma)-d_{k-2} \phi_{u}(\sigma)= \begin{cases}d_{k-1} \phi(u \sigma) & u \notin \sigma \\ 0 & u \in \sigma\end{cases}
$$

It follows that

$$
\begin{gathered}
(k+1)\left|\operatorname{supp}\left(d_{k-1} \phi\right)\right|=\left|\left\{(\tau, u): u \in \tau \in \operatorname{supp}\left(d_{k-1} \phi\right)\right\}\right|= \\
\left|\left\{(\sigma, u) \in \Delta_{n-1}(k-1) \times V: \sigma \in \operatorname{supp}\left(\phi-d_{k-2} \phi_{u}\right)\right\}\right|= \\
\sum_{u \in V}\left|\operatorname{supp}\left(\phi-d_{k-2} \phi_{u}\right)\right| \geq n w(\phi)
\end{gathered}
$$

Remark: The following example shows that equality can be attained in (3). Let $n$ be divisible by $k+1$, and let $[n]=\cup_{i=0}^{k} V_{i}$ be a partition of $[n]$ with $\left|V_{i}\right|=\frac{n}{k+1}$. Consider the unique cochain $\phi \in C^{k-1}\left(\Delta_{n-1}\right)$ that satisfies

$$
\phi\left(\left[v_{0}, \ldots, v_{k-1}\right]\right)= \begin{cases}1 & v_{i} \in V_{i} \text { for all } 0 \leq i \leq k-1 \\ 0 & \left|\left\{v_{0}, \ldots, v_{k-1}\right\} \cap V_{i}\right| \neq 1 \text { for some } 0 \leq i \leq k-1\end{cases}
$$

Then $b(\phi)=\left(\frac{n}{k+1}\right)^{k+1}$, and it can be shown that $w(\phi)=|\operatorname{supp}(\phi)|=\left(\frac{n}{k+1}\right)^{k}$.

## 3 The number of $\phi$ with prescribed $b(\phi)$

Let

$$
\mathcal{G}_{n}(m)=\left\{\phi \in \mathcal{G}_{n}:|\operatorname{supp}(\phi)|=m\right\}
$$

and for $0 \leq \theta \leq 1$ let

$$
\mathcal{G}_{n}(m, \theta)=\left\{\phi \in \mathcal{G}_{n}(m): b(\phi)=(1-\theta) m n\right\}
$$

Write $g_{n}(m)=\left|\mathcal{G}_{n}(m)\right|$ and $g_{n}(m, \theta)=\left|\mathcal{G}_{n}(m, \theta)\right|$. Proposition 2.1 implies that $g_{n}(m, \theta)=0$ for $\theta>\frac{k}{k+1}$. Our main estimate is the following

Proposition 3.1. There exists a constant $c_{1}=c_{1}(r, k)$ such that for any $n \geq 10 k^{2}, m \geq \frac{n}{2 k}$, and $\theta \geq \frac{1}{2 k}$

$$
\begin{equation*}
g_{n}(m, \theta) \leq\left(c_{1} \cdot n^{(k-1)\left(1-\theta\left(1-\frac{1}{2 k^{2}}\right)\right)}\right)^{m} \tag{5}
\end{equation*}
$$

The proof of Proposition 3.1 depends on a certain partial domination property of hypergraphs. Let $\mathcal{F} \subset\binom{[n]}{k}$ be a $k$-uniform hypergraph of cardinality $|\mathcal{F}|=m$. For $\sigma \in \mathcal{F}$ let

$$
\beta_{\mathcal{F}}(\sigma)=\left|\left\{\tau \in\binom{[n]}{k+1}:\binom{\tau}{k} \cap \mathcal{F}=\{\sigma\}\right\}\right|
$$

and let $\beta(\mathcal{F})=\sum_{\sigma \in \mathcal{F}} \beta_{\mathcal{F}}(\sigma)$. Clearly $\beta_{\mathcal{F}}(\sigma) \leq n-k$ and $\beta(\mathcal{F}) \leq m(n-k)$. For $S \subset \mathcal{F}$ let

$$
\Gamma(S)=\{\eta \in \mathcal{F}:|\eta \cap \sigma|=k-1 \text { for some } \sigma \in S\} .
$$

Claim 3.2. Let $0<\epsilon \leq \frac{1}{2}$ and $n>2 \log \frac{1}{\epsilon}+k$. Suppose that

$$
\beta(\mathcal{F}) \leq(1-\theta) m(n-k)
$$

for some $0<\theta \leq 1$. Then there exists a subfamily $S \subset \mathcal{F}$ such that

$$
|\Gamma(S)| \geq(1-\epsilon) \theta m
$$

and

$$
|S|<\left(20 \log \frac{1}{\epsilon}\right) \cdot \frac{m}{n-k}+2 \log \frac{1}{\epsilon \theta} .
$$

proof: Let $c(\epsilon)=2 \log \frac{1}{\epsilon}$. Choose a random subfamily $S \subset \mathcal{F}$ by picking each $\sigma \in \mathcal{F}$ independently with probability $\frac{c(\epsilon)}{n-k}$. For any $\sigma \in \mathcal{F}$ there exist distinct $v_{1}, \ldots, v_{n-k-\beta_{\mathcal{F}}(\sigma)} \in[n]-\sigma$ and $\tau_{1}, \ldots, \tau_{n-k-\beta_{\mathcal{F}}(\sigma)} \in\binom{\sigma}{k-1}$ such that $\tau_{i} \cup\left\{v_{i}\right\} \in \mathcal{F}$ for all $i$. In particular

$$
\operatorname{Pr}[\sigma \notin \Gamma(S)] \leq\left(1-\frac{c(\epsilon)}{n-k}\right)^{n-k-\beta_{\mathcal{F}}(\sigma)}
$$

hence

$$
\begin{equation*}
E[|\mathcal{F}-\Gamma(S)|] \leq \sum_{\sigma \in \mathcal{F}}\left(1-\frac{c(\epsilon)}{n-k}\right)^{n-k-\beta_{\mathcal{F}}(\sigma)} \tag{6}
\end{equation*}
$$

Since

$$
\sum_{\sigma \in \mathcal{F}}\left(n-k-\beta_{\mathcal{F}}(\sigma)\right)=m(n-k)-\beta(\mathcal{F}) \geq \theta m(n-k)
$$

it follows by convexity from (6) that

$$
\begin{gathered}
E[|\mathcal{F}-\Gamma(S)|] \leq(1-\theta) m+\theta m\left(1-\frac{c(\epsilon)}{n-k}\right)^{n-k} \leq \\
(1-\theta) m+\theta m e^{-c(\epsilon)}=(1-\theta) m+\theta m \epsilon^{2}
\end{gathered}
$$

Therefore

$$
E[|\Gamma(S)|] \geq\left(1-\epsilon^{2}\right) \theta m
$$

Hence, since $|\Gamma(S)| \leq|\mathcal{F}|=m$, it follows that

$$
\begin{equation*}
\operatorname{Pr}[|\Gamma(S)| \geq(1-\epsilon) \theta m]>\epsilon(1-\epsilon) \theta \tag{7}
\end{equation*}
$$

On the other hand

$$
E[|S|]=\frac{c(\epsilon) m}{n-k}
$$

and by the large deviation inequality (see e.g. Theorem A.1.12 in [1])

$$
\begin{equation*}
\operatorname{Pr}\left[|S|>\lambda \frac{c(\epsilon) m}{n-k}\right]<\left(\frac{e}{\lambda}\right)^{\lambda \frac{c(\epsilon) m}{n-k}} \tag{8}
\end{equation*}
$$

for all $\lambda \geq 1$. Let

$$
\lambda=10+\frac{n-k}{m}\left(\frac{\log \frac{1}{\theta}}{\log \frac{1}{\epsilon}}+1\right)
$$

then

$$
\epsilon(1-\epsilon) \theta>\left(\frac{e}{\lambda}\right)^{\lambda \frac{c(\epsilon) m}{n-k}} .
$$

Hence by (7) and (8) there exists an $S \subset \mathcal{F}$ such that $|\Gamma(S)| \geq(1-\epsilon) \theta m$ and

$$
|S| \leq \lambda \frac{c(\epsilon) m}{n-k}=\left(20 \log \frac{1}{\epsilon}\right) \cdot \frac{m}{n-k}+2 \log \frac{1}{\epsilon \theta}
$$

Proof of Proposition 3.1: Define

$$
\mathcal{F}_{n}(m, \theta)=\left\{\mathcal{F} \subset\binom{[n]}{k}:|\mathcal{F}|=m, \beta(\mathcal{F}) \leq(1-\theta) m n\right\}
$$

and let $f_{n}(m, \theta)=\left|\mathcal{F}_{n}(m, \theta)\right|$. If $\phi \in \mathcal{G}_{n}(m, \theta)$, then $\mathcal{F}=\operatorname{Supp}(\phi) \in$ $\mathcal{F}_{n}(m, \theta)$. Indeed, if $\tau \in\binom{[n]}{k+1}$ satisfies $\binom{\tau}{k} \cap \mathcal{F}=\{\sigma\}$, then $d_{k-1} \phi(\tau)=$ $\pm \phi(\sigma) \neq 0$, hence $\beta(\mathcal{F}) \leq b(\phi)=(1-\theta) m n$. Therefore

$$
g_{n}(m, \theta) \leq(r-1)^{m} f_{n}(m, \theta) .
$$

We next estimate $f_{n}(m, \theta)$. Let $\mathcal{F} \in \mathcal{F}_{n}(m, \theta)$, then

$$
\beta(\mathcal{F}) \leq(1-\theta) m n=\left(1-\frac{\theta n-k}{n-k}\right) m(n-k)
$$

Applying Claim 3.2 with $\theta^{\prime}=\frac{\theta n-k}{n-k}$ and $\epsilon=\frac{1}{2 k^{2}}$, it follows that there exists an $S \subset \mathcal{F}$ of cardinality $|S| \leq \frac{c_{2} m}{n}$ with $c_{2}=c_{2}(k)$, such that $|\Gamma(S)| \geq$ $\left(1-\frac{1}{2 k^{2}}\right) \theta^{\prime} m$. The injectivity of the mapping

$$
\mathcal{F} \rightarrow(S, \Gamma(S), \mathcal{F}-\Gamma(S))
$$

implies that

$$
\begin{gathered}
f_{n}(m, \theta) \leq \sum_{i=0}^{c_{2} m / n}\binom{n}{k} \cdot 2^{\left(\frac{c_{2} m}{n}\right) k n} \cdot \sum_{j=0}^{\left(1-\theta^{\prime}\left(1-\frac{1}{2 k^{2}}\right)\right) m}\left(\begin{array}{c}
n \\
k \\
k
\end{array}\right) \leq \\
c_{3}^{m}\binom{\binom{n}{k}}{\left(1-\theta^{\prime}\left(1-\frac{1}{2 k^{2}}\right)\right) m} \leq \\
c_{4}^{m}\left(\frac{n^{k}}{m}\right)^{\left(1-\theta^{\prime}\left(1-\frac{1}{2 k^{2}}\right)\right) m} .
\end{gathered}
$$

Therefore

$$
\begin{gathered}
g_{n}(m, \theta) \leq(r-1)^{m} f_{n}(m, \theta) \leq \\
(r-1)^{m} c_{4}^{m}\left(\frac{n^{k}}{m}\right)^{\left(1-\theta^{\prime}\left(1-\frac{1}{2 k^{2}}\right)\right) m} \leq \\
\left(c_{1} \cdot n^{(k-1)\left(1-\theta\left(1-\frac{1}{2 k^{2}}\right)\right)}\right)^{m}
\end{gathered}
$$

## 4 Proof of Theorem 1.2

Proof of Theorem 1.2: Let $\omega(n) \rightarrow \infty$ and let $p=\frac{k \log n+\omega(n)}{n}$. We have to show that

$$
\begin{equation*}
\sum_{m \geq 1} \sum_{\phi \in \mathcal{G}_{n}(m)}(1-p)^{b(\phi)}=o(1) \tag{9}
\end{equation*}
$$

We deal separately with two intervals of $m$ :
(i) $1 \leq m \leq \frac{n}{2 k}$. If $\phi \in \mathcal{G}_{n}(m)$ then $\operatorname{supp}(\phi) \subset\binom{[n]}{k}$ is a connected $k$-uniform hypergraph, hence there exists a subset $S \subset[n]$ of cardinality $|S| \leq m+k-1$ such that $\operatorname{supp}(\phi) \subset\binom{S}{k}$. Since $d_{k-1} \phi(u \sigma)=\phi(\sigma) \neq 0$ for any $\sigma \in \operatorname{supp}(\phi)$ and $u \notin S$, it follows that $b(\phi) \geq m(n-m-k+1)$. The trivial estimate

$$
g_{n}(m) \leq(r-1)^{m}\binom{n}{k} ~ \leq c_{5}^{m}\left(\frac{n^{k}}{m}\right)^{m}
$$

implies that for $n \geq 6 k$

$$
\begin{gathered}
g_{n}(m)(1-p)^{m(n-m-k+1)} \leq \\
c_{5}^{m} \frac{n^{k m}}{m^{m}}\left(1-\frac{k \log n+w(n)}{n}\right)^{m(n-m-k+1)} \leq \\
c_{5}^{m} \frac{n^{k m}}{m^{m}} n^{\frac{-k(n-m-k+1) m}{n}} e^{\frac{-w(n)(n-m-k+1) m}{n}} \leq \\
c_{6}^{m}\left(\frac{n^{k}}{m} n^{\frac{-k(n-m)}{n}}\right)^{m} e^{-\frac{w(n)}{3} m}= \\
\left(c_{6} \frac{n^{\frac{k m}{n}}}{m} e^{-\frac{w(n)}{3}}\right)^{m}
\end{gathered}
$$

Since

$$
\frac{n^{\frac{k m}{n}}}{m} \leq \begin{cases}n^{k n^{-1 / 3}} & m \leq n^{2 / 3} \\ n^{-1 / 6} & n^{2 / 3} \leq m \leq \frac{n}{2 k}\end{cases}
$$

it follows that there exists a $c_{7}=c_{7}(r, k)$ such that for $m \leq \frac{n}{2 k}$ and $n \geq 6 k$

$$
g_{n}(m)(1-p)^{m(n-m-k+1)} \leq\left(c_{7} e^{-\frac{w(n)}{3}}\right)^{m}
$$

Therefore

$$
\begin{gather*}
\sum_{m=1}^{n / 2 k} \sum_{\phi \in \mathcal{G}_{n}(m)}(1-p)^{b(\phi)} \leq \sum_{m=1}^{n / 2 k} g_{n}(m)(1-p)^{m(n-m-k+1)} \leq \\
\sum_{m=1}^{n / 2 k}\left(c_{7} e^{-\frac{w(n)}{3}}\right)^{m}=O\left(e^{-\frac{w(n)}{3}}\right)=o(1) . \tag{10}
\end{gather*}
$$

(ii) $m \geq \frac{n}{2 k}$. Then

$$
\begin{gather*}
\sum_{m \geq n / 2 k} \sum_{\theta \leq 1 / 2 k} \sum_{\phi \in \mathcal{G}_{n}(m, \theta)}(1-p)^{b(\phi)}= \\
\sum_{m \geq n / 2 k} \sum_{\theta \leq 1 / 2 k} g_{n}(m, \theta)(1-p)^{(1-\theta) m n} \leq \\
\sum_{m \geq n / 2 k} g_{n}(m)(1-p)^{\left(1-\frac{1}{2 k}\right) m n} \leq \\
\sum_{m \geq n / 2 k}\left(\frac{c_{5} n^{k}}{m}\right)^{m} n^{-\left(1-\frac{1}{2 k}\right) k m} \leq \\
\sum_{m \geq n / 2 k}\left(2 k c_{5} n^{k-1}\right)^{m} n^{-\left(1-\frac{1}{2 k}\right) k m}= \\
\sum_{m \geq n / 2 k}\left(2 k c_{5} n^{-1 / 2}\right)^{m}=n^{-\Omega(n)} \tag{11}
\end{gather*}
$$

Next note that by Proposition 2.1, $g_{n}(m, \theta)=0$ for $\theta>\frac{k}{k+1}$. Hence, by Proposition 3.1

$$
\begin{gathered}
\sum_{m \geq n / 2 k} \sum_{\theta \geq 1 / 2 k} \sum_{\phi \in \mathcal{G}_{n}(m, \theta)}(1-p)^{b(\phi)}= \\
\sum_{m \geq n / 2 k} \sum_{\theta \geq 1 / 2 k} g_{n}(m, \theta)(1-p)^{(1-\theta) m n} \leq \\
\sum_{m \geq n / 2 k} \sum_{\substack{\theta \geq 1 / 2 k \\
g_{n}(m, \theta) \neq 0}}\left(c_{1} \cdot n^{(k-1)\left(1-\theta\left(1-\frac{1}{2 k^{2}}\right)\right)}\right)^{m} \cdot n^{-(1-\theta) k m}=
\end{gathered}
$$

$$
\begin{gather*}
\sum_{m \geq n / 2 k} \sum_{\substack{\theta \geq 1 / 2 k \\
g_{n}(m, \theta) \neq 0}}\left(c_{1} \cdot n^{\theta\left(1+\frac{k-1}{2 k^{2}}\right)-1}\right)^{m} \leq n^{k+1} \sum_{m \geq n / 2 k}\left(c_{1} \cdot n^{\frac{k}{k+1}\left(1+\frac{k-1}{2 k^{2}}\right)-1}\right)^{m}= \\
n^{k+1} \sum_{m \geq n / 2 k}\left(c_{1} n^{-1 / 2 k}\right)^{m}=n^{-\Omega(n)} . \tag{12}
\end{gather*}
$$

Finally (9) follows from (10), (11) and (12).

## 5 Concluding Remarks

We have shown that in the model $Y_{k}(n, p)$ of random $k$-complexes on $n$ vertices, the threshold for the vanishing of $H^{k-1}(Y ; R)$ occurs at $p=\frac{k \log n}{n}$, provided that both $k$ and the finite coefficient group $R$ are fixed. One natural concrete question is whether $p=\frac{k \log n}{n}$ is also the threshold for the vanishing of $H^{k-1}(Y ; \mathbb{Z})$.

More generally, in view of the detailed understanding of the evolution of random graphs (see e.g. [1]), it would be interesting to formulate and prove analogous statements concerning the topology of random complexes. For example, what is the higher dimensional counterpart of the remarkable double-jump phenomenon that occurs in random graphs?

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