# Random dense bipartite graphs and directed graphs with specified degrees* 

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#### Abstract

Let $\boldsymbol{s}$ and $\boldsymbol{t}$ be vectors of positive integers with the same sum. We study the uniform distribution on the space of simple bipartite graphs with degree sequence $\boldsymbol{s}$ in one part and $\boldsymbol{t}$ in the other; equivalently, binary matrices with row sums $s$ and column sums $\boldsymbol{t}$. In particular, we find precise formulae for the probabilities that a given bipartite graph is edge-disjoint from, a subgraph of, or an induced subgraph of a random graph in the class. We also give similar formulae for the uniform distribution on the set of simple directed graphs with out-degrees $s$ and indegrees $\boldsymbol{t}$. In each case, the graphs or digraphs are required to be sufficiently dense, with the degrees varying within certain limits, and the subgraphs are required to be sufficiently sparse. Previous results were restricted to spaces of sparse graphs. Our theorems are based on an enumeration of bipartite graphs avoiding a given set of edges, proved by multidimensional complex integration. As a sample application, we determine the expected permanent of a random binary matrix with row sums $s$ and column sums $\boldsymbol{t}$.


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## 1 Introduction

Let $\boldsymbol{s}=\left(s_{1}, \ldots, s_{m}\right)$ and $\boldsymbol{t}=\left(t_{1}, \ldots, t_{n}\right)$ be vectors of positive integers with $\sum_{j=1}^{m} s_{j}=$ $\sum_{k=1}^{n} t_{k}$. Define $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ to be the set of simple bipartite graphs with vertices $\left\{u_{1}, \ldots, u_{m}\right\} \cup$ $\left\{v_{1}, \ldots, v_{n}\right\}$, such that vertex $u_{j}$ has degree $s_{j}$ for $j=1, \ldots, m$ and vertex $v_{k}$ has degree $t_{k}$ for $k=1, \ldots, n$. Equivalently, we may think of $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ as the set of all $m \times n$ matrices over $\{0,1\}$ with $j$ th row sum equal to $s_{j}$ for $j=1, \ldots, m$ and $k$ th column sum equal to $t_{k}$ for $k=1, \ldots, n$.

In addition, let $H$ be a fixed bipartite graph on the same vertex set. In this paper we find precise formulae for the probabilities that $H$ is edge-disjoint from $G \in \mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$, that $H$ is a subgraph of $G$, and that $H$ is an induced subgraph of $G$. These probabilities are defined for the uniform distribution on $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$. In general, whenever we refer to a random element of a set, we always mean an element chosen uniformly at random.

These formulae are obtained when the graphs in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ are sufficiently dense, the graph $H$ is sufficiently sparse and the entries of $\boldsymbol{s}$ and $\boldsymbol{t}$ only vary within certain limits. The exact conditions are stated in Section 2. The starting point of the calculations is an enumeration of the set $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)$ of graphs in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ which are edge-disjoint from $H$; see Theorem 2.1.

In the case $m=n$, the $n \times n$ binary matrix associated with the bipartite graph can also be interpreted as the adjacency matrix of a digraph which has no multiple edges but may have loops. By excluding the diagonal we obtain a parallel series of results for simple digraphs (digraphs without multiple edges or loops). These are presented in Section 3,

These subgraph probabilities enable the development of a theory of random graphs and digraphs in these classes. As examples of computations made possible by this theory, we calculate the expected number of subgraphs isomorphic to a given regular subgraph. A particular case of interest is the permanent of a random 0-1 matrix with row sums $\boldsymbol{s}$ and column sums $\boldsymbol{t}$.

Now we briefly review the history of this problem. All previous precise asymptotics were restricted to sparse graphs. Define $g=\max \left\{s_{1}, \ldots, s_{m}, t_{1}, \ldots, t_{n}\right\}, x=$ $\max \left\{x_{1}, \ldots, x_{m}, y_{1}, \ldots, y_{n}\right\}$ and $N=\sum_{j} s_{j}$. Asymptotic estimates for bounded $g$ were found by Bender [2] and Wormald [22]. This was extended by Bollobás and McKay [3] to the case $g, x=O\left(\min \{\log m, \log n\}^{1 / 3}\right)$ and by McKay [12] to the case $g^{2}+x g=o\left(N^{1 / 2}\right)$. Estimates which are sometimes more widely applicable were given by McKay [11]. The best enumerative results for $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ in the sparse domain appear in [8, 17].

Although results about sparse digraphs with specified in-degree and out-degree sequences can be deduced from the above, we are not aware of this having been done. Some
results using the pairings model have appeared [6]. For digraphs in the dense regime, some related work includes enumeration of tournaments by score sequence with possible forbidden subgraph [13, 16, 15, 7], Eulerian digraphs [13, 19], Eulerian oriented graphs [13, 21], and digraphs with a given excess sequence [20].

For the case of dense bipartite graphs with specified degrees, an asymptotic formula for the case of empty $H$ was given by Canfield and McKay [5] for semiregular graphs and by Canfield, Greenhill and McKay [4] for irregular graphs. The latter study is the inspiration for the present one. A similar study for graphs which are not necessarily bipartite is in preparation [14].

In related work using different methods, Barvinok [1] gives upper and lower bounds for $|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|$ which hold very generally (from sparse to dense graphs) but which can differ by a factor of $(m n)^{O(m+n)}$. Barvinok's results also give insight into the structure of a "typical" element of $\mathcal{B}(s, \boldsymbol{t}, H)$, which he proves is close to a certain "maximum entropy" matrix.

The paper is structured as follows. The results for bipartite graphs are presented in Section 2 and the corresponding results for digraphs can be found in Section 3, Then Section 4 presents a proof of the fundamental enumeration result, Theorem 2.1, from which everything else follows.

Throughout the paper, the asymptotic notation $O(f(m, n))$ refers to the passage of $m$ and $n$ to $\infty$. We also use a modified notation $\widetilde{O}(f(m, n))$, which is to be taken as a shorthand for $O\left(f(m, n) n^{O(1) \varepsilon}\right)$, where the $O(1)$ factor is uniform over $\varepsilon$ provided $\varepsilon$ is small enough.

## 2 Subgraphs of random bipartite graphs

In this section we state our results for bipartite graphs.
The starting point of the investigation is the enumeration formula given in the following theorem. Define $m, n, s, \boldsymbol{t}$ as in the Introduction and further define

$$
s=m^{-1} \sum_{j=1}^{m} s_{j}, \quad t=n^{-1} \sum_{k=1}^{n} t_{k}, \quad \lambda=s / n=t / m, \quad A=\frac{1}{2} \lambda(1-\lambda) .
$$

Note that $s$ is the average degree on one side of the vertex bipartition, $t$ is the average degree on the other side, and $\lambda$ is the edge density (the number of edges divided by $m n$ ).

Let $H$ be a fixed bipartite graph on the same vertex set that defines $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$, namely $\left\{u_{1}, \ldots, u_{m}\right\} \cup\left\{v_{1}, \ldots, v_{n}\right\}$. For $j=1, \ldots, m, k=1, \ldots, n$, let $x_{j}$ and $y_{k}$ be the degrees
of vertices $u_{j}$ and $v_{k}$ of $H$, respectively, and further define

$$
\delta_{j}=s_{j}-s+\lambda x_{j}, \quad \quad \eta_{k}=t_{k}-t+\lambda y_{k}
$$

Also define

$$
\begin{array}{ll}
X=\sum_{j=1}^{m} x_{j}=\sum_{k=1}^{n} y_{k}, & Y=\sum_{j k \in H} \delta_{j} \eta_{k}, \\
R=\sum_{j=1}^{m}\left(s_{j}-s\right)^{2}, & C=\sum_{k=1}^{n}\left(t_{k}-t\right)^{2} .
\end{array}
$$

In the case of $Y$ and similar notation used in this section, the summation is over all $j \in\left\{u_{1}, \ldots, u_{m}\right\}$ and $k \in\left\{v_{1}, \ldots, v_{n}\right\}$ such that $u_{j} v_{k}$ is an edge of $H$.

Theorem 2.1. For some $\varepsilon>0$, suppose that $s_{j}-s, x_{j}, t_{k}-t$ and $y_{k}$ are uniformly $O\left(n^{1 / 2+\varepsilon}\right)$ for $1 \leq j \leq m$ and $1 \leq k \leq n$, and $X=O\left(n^{1+2 \varepsilon}\right)$, for $m, n \rightarrow \infty$. Let $a, b>0$ be constants such that $a+b<\frac{1}{2}$. Suppose that $m, n \rightarrow \infty$ with $n=o\left(m^{1+\varepsilon}\right), m=o\left(n^{1+\varepsilon}\right)$ and

$$
\frac{(1-2 \lambda)^{2}}{8 A}\left(1+\frac{5 m}{6 n}+\frac{5 n}{6 m}\right) \leq a \log n
$$

Then, provided $\varepsilon>0$ is small enough, we have

$$
\begin{aligned}
|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|= & \binom{m n-X}{\lambda m n}^{-1} \prod_{j=1}^{m}\binom{n-x_{j}}{s_{j}} \prod_{k=1}^{n}\binom{m-y_{k}}{t_{k}} \\
& \times \exp \left(-\frac{1}{2}\left(1-\frac{R}{2 A m n}\right)\left(1-\frac{C}{2 A m n}\right)-\frac{Y}{2 A m n}+O\left(n^{-b}\right)\right) .
\end{aligned}
$$

The proof of Theorem 2.1 will be presented in Section 4. As in the special case of empty $H$ proved in [4], the formula for $|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|$ has an intuitive interpretation. The first binomial and the two products of binomials are, respectively, the number of graphs with $\lambda m n$ edges that avoid $H$, the number of such graphs with row sums $\boldsymbol{s}$, and the number of such graphs with column sums $\boldsymbol{t}$. Therefore, the exponential factor measures the nonindependence of the events of having row sums $\boldsymbol{s}$ and having column sums $\boldsymbol{t}$. Another expression for the product of binomials in the theorem is given below in equation (18).

We can now employ Theorem 2.1 to explore the uniform probability space over $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$. First we need a little more notation. For all nonnegative integers $h, \ell$ define

$$
R_{h, \ell}=\sum_{j=1}^{m} \delta_{j}^{h} x_{j}^{\ell}, \quad C_{h, \ell}=\sum_{k=1}^{n} \eta_{k}^{h} y_{k}^{\ell}
$$

We will abbreviate $R_{h, 0}=R_{h}$ and $C_{h, 0}=C_{h}$. Also note that $R_{1}=C_{1}=\lambda X$ and $R_{0,1}=C_{0,1}=X$. Finally, let

$$
Y_{1,1}=\sum_{j k \in H} x_{j} y_{k}, \quad Y_{0,1}=\sum_{j k \in H} \delta_{j} y_{k}, \quad Y_{1,0}=\sum_{j k \in H} x_{j} \eta_{k}
$$

Theorem 2.2. Under the conditions of Theorem 2.1, the following are true for a random graph $G \in \mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ provided $\varepsilon>0$ is small enough:
(i) the probability that $G$ is edge-disjoint from $H$ is $(1-\lambda)^{X} \operatorname{miss}(m, n)$;
(ii) the probability that $G$ contains $H$ as a subgraph is $\lambda^{X} \operatorname{hit}(m, n)$,
where

$$
\begin{aligned}
\operatorname{miss}(m, n)=\exp ( & \frac{\lambda X}{2(1-\lambda)}\left(\frac{1}{n}+\frac{1}{m}\right)+\frac{\lambda X^{2}}{2(1-\lambda) m n}-\frac{1}{1-\lambda}\left(\frac{R_{1,1}}{n}+\frac{C_{1,1}}{m}\right) \\
& -\frac{Y}{\lambda(1-\lambda) m n}+\frac{\lambda}{2(1-\lambda)}\left(\frac{R_{0,2}}{n}+\frac{C_{0,2}}{m}\right)+\frac{\lambda(1-2 \lambda)}{6(1-\lambda)^{2}}\left(\frac{R_{0,3}}{n^{2}}+\frac{C_{0,3}}{m^{2}}\right) \\
& \left.-\frac{1-2 \lambda}{2(1-\lambda)^{2}}\left(\frac{R_{1,2}}{n^{2}}+\frac{C_{1,2}}{m^{2}}\right)-\frac{1}{2(1-\lambda)^{2}}\left(\frac{R_{2,1}}{n^{2}}+\frac{C_{2,1}}{m^{2}}\right)+O\left(n^{-b}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{hit}(m, n)=\exp & \left(\frac{(1-\lambda) X}{2 \lambda}\left(\frac{1}{n}+\frac{1}{m}\right)+\frac{(1-\lambda) X^{2}}{2 \lambda m n}+\frac{1}{\lambda}\left(\frac{R_{1,1}}{n}+\frac{C_{1,1}}{m}\right)\right. \\
& -\frac{1}{2 \lambda^{2}}\left(\frac{R_{2,1}}{n^{2}}+\frac{C_{2,1}}{m^{2}}\right)-\frac{1+\lambda}{2 \lambda}\left(\frac{R_{0,2}}{n}+\frac{C_{0,2}}{m}\right)+\frac{1+2 \lambda}{2 \lambda^{2}}\left(\frac{R_{1,2}}{n^{2}}+\frac{C_{1,2}}{m^{2}}\right) \\
& \left.-\frac{(1+\lambda)(1+2 \lambda)}{6 \lambda^{2}}\left(\frac{R_{0,3}}{n^{2}}+\frac{C_{0,3}}{m^{2}}\right)-\frac{Y-Y_{0,1}-Y_{1,0}+Y_{1,1}}{\lambda(1-\lambda) m n}+O\left(n^{-b}\right)\right) .
\end{aligned}
$$

Proof. The first probability in the statement of Theorem 2.2 is

$$
\frac{|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|}{|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})|}
$$

which can be expanded using Theorem [2.1. (One method is to apply (18) below.) The second probability can be derived in similar fashion, or can be deduced from the first on noting that the probability that $G$ includes $H$ is the probability that the complement of $G$ avoids $H$.

In the standard model of random bipartite graphs on $m+n$ vertices with expected edge density $\lambda$, each of the $m n$ possible edges is present independently with probability $\lambda$. The probability that a random bipartite graph taken from the standard model is disjoint
from or contains a given set of $X$ edges is $(1-\lambda)^{X}$ or $\lambda^{X}$, respectively. Therefore, the quantities $\operatorname{miss}(m, n)$ and $\operatorname{hit}(m, n)$ given in Theorem 2.2 can be interpreted as a measure of how far these probabilities differ in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ compared to the standard model. Suppose that in addition to the conditions of Theorem 2.2, we also have

$$
\begin{gather*}
X \max _{j}\left|s_{j}-s\right|+\lambda R_{0,2}=o((1-\lambda) n)  \tag{1}\\
X \max _{k}\left|t_{k}-t\right|+\lambda C_{0,2}=o((1-\lambda) m) .
\end{gather*}
$$

Then $\operatorname{miss}(m, n)=1+o(1)$. Similarly, if we have

$$
\begin{align*}
X \max _{j}\left|s_{j}-s\right|+(1-\lambda) R_{0,2} & =o(\lambda n) \\
X \max _{k}\left|t_{k}-t\right|+(1-\lambda) C_{0,2} & =o(\lambda m) \tag{2}
\end{align*}
$$

then $\operatorname{hit}(m, n)=1+o(1)$. Requirements (11) and (2) are both met, for example, if $X=O\left(n^{1 / 2-2 \varepsilon}\right)$. Another interesting case is when $s_{j}-s, x_{j}, t_{k}-t$ and $y_{k}$ are uniformly $O\left(n^{\varepsilon}\right)$ and $X=O\left(n^{1-2 \varepsilon}\right)$.

To assist with the application of Theorem 2.2, we will give the simplifications that result when the graphs in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ are semiregular or when the graph $H$ is semiregular.

Corollary 2.1. In addition to the conditions of Theorem 2.2, assume that $s_{j}=s$ and $t_{k}=t$ for all $j, k$. Then

$$
\begin{aligned}
\operatorname{miss}(m, n)=\exp & \left(\frac{\lambda X}{2(1-\lambda)}\left(\frac{1}{n}+\frac{1}{m}\right)+\frac{\lambda X^{2}}{2(1-\lambda) m n}-\frac{\lambda Y_{1,1}}{(1-\lambda) m n}\right. \\
& \left.-\frac{\lambda}{2(1-\lambda)}\left(\frac{R_{0,2}}{n}+\frac{C_{0,2}}{m}\right)-\frac{\lambda(2-\lambda)}{6(1-\lambda)^{2}}\left(\frac{R_{0,3}}{n^{2}}+\frac{C_{0,3}}{m^{2}}\right)+O\left(n^{-b}\right)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{hit}(m, n)=\exp & \left(\frac{(1-\lambda) X}{2 \lambda}\left(\frac{1}{n}+\frac{1}{m}\right)+\frac{(1-\lambda) X^{2}}{2 \lambda m n}-\frac{(1-\lambda) Y_{1,1}}{\lambda m n}\right. \\
& \left.-\frac{1-\lambda}{2 \lambda}\left(\frac{R_{0,2}}{n}+\frac{C_{0,2}}{m}\right)-\frac{1-\lambda^{2}}{6 \lambda^{2}}\left(\frac{R_{0,3}}{n^{2}}+\frac{C_{0,3}}{m^{2}}\right)+O\left(n^{-b}\right)\right)
\end{aligned}
$$

Corollary 2.2. In addition to the conditions of Theorem 2.2, assume that $x_{j}=x$ and $y_{k}=y$ for all $j, k$. (Note that Theorem 2.2 requires $x, y=O\left(n^{2 \varepsilon}\right)$ in that case.) Then

$$
\operatorname{miss}(m, n)=\exp \left(-\frac{\lambda(x y-x-y)}{2(1-\lambda)}-\frac{y R+x C}{2(1-\lambda)^{2} m n}-\frac{\widehat{Y}}{\lambda(1-\lambda) m n}+O\left(n^{-b}\right)\right)
$$

and

$$
\operatorname{hit}(m, n)=\exp \left(-\frac{(1-\lambda)(x y-x-y)}{2 \lambda}-\frac{y R+x C}{2 \lambda^{2} m n}-\frac{\widehat{Y}}{\lambda(1-\lambda) m n}+O\left(n^{-b}\right)\right)
$$

where $\widehat{Y}=\sum_{j k \in H}\left(s_{j}-s\right)\left(t_{k}-t\right)$.

The next question we will address is the probability of $H$ appearing as an induced subgraph. To be precise, suppose that $H$ has no edges outside $\left\{u_{1}, \ldots, u_{J}\right\} \times\left\{v_{1}, \ldots, v_{K}\right\}$ and let $H_{J, K}$ denote the subgraph of $H$ induced by those vertices. We will only consider the situation when the graphs in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ are semiregular. The corresponding result for irregular graphs can also be obtained using the same approach.

The probability that $H_{J, K}$ is an induced subgraph of $G \in \mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ is simpler to state in terms of some new variables. For $\ell=1,2,3$, define

$$
\omega_{\ell}=\sum_{j=1}^{J}\left(x_{j}-\lambda K\right)^{\ell}, \quad \omega_{\ell}^{\prime}=\sum_{k=1}^{K}\left(y_{k}-\lambda J\right)^{\ell} .
$$

Note that $\omega_{1}=\omega_{1}^{\prime}=X-\lambda J K$.
Theorem 2.3. Adopt the assumptions of Theorem 2.1 with $s_{j}=s$ and $t_{k}=t$ for all $j, k$, and assume that $J, K=O\left(n^{1 / 2+\varepsilon}\right)$. Then the probability that a random graph in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ has $H_{J, K}$ as an induced subgraph is

$$
\begin{aligned}
\lambda^{X}(1-\lambda)^{J K-X} \exp ( & \left(\frac{J K}{2}+\frac{(1-2 \lambda) \omega_{1}}{4 A}\right)\left(\frac{1}{m}+\frac{1}{n}\right)-\frac{\omega_{1}^{2}}{4 A m n} \\
& \left.-\frac{(n+K) \omega_{2}}{4 A n^{2}}-\frac{(m+J) \omega_{2}^{\prime}}{4 A m^{2}}-\frac{1-2 \lambda}{24 A^{2}}\left(\frac{\omega_{3}}{n^{2}}+\frac{\omega_{3}^{\prime}}{m^{2}}\right)+O\left(n^{-b}\right)\right) .
\end{aligned}
$$

Proof. Let $H^{*}$ be the complete bipartite graph on the parts $\left\{u_{1}, \ldots, u_{J}\right\}$ and $\left\{v_{1}, \ldots, v_{K}\right\}$. Then the probability that a random graph in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ has $H_{J, K}$ as an induced subgraph is

$$
\frac{\left|\mathcal{B}\left(\boldsymbol{s}-\boldsymbol{x}, \boldsymbol{t}-\boldsymbol{y}, H^{*}\right)\right|}{|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})|}
$$

This ratio can be estimated using Theorem 2.1 (or by combining Theorems 2.1 and 2.2).

The argument of the exponential in Theorem [2.3 is $o(1)$ if $J K^{2}=o(A n)$ and $J^{2} K=$ $o(A m)$. So, in those circumstances, the probabilities of induced subgraphs asymptotically match the standard bipartite random graph model for edge probability $\lambda$.

A related question asks for the distribution of the number of subgraphs of given type in a random graph in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$. This deserves a serious study, which we will only just initiate here. A colour-preserving isomorphism of two bipartite graphs on $\left\{u_{1}, \ldots, u_{m}\right\} \cup$ $\left\{v_{1}, \ldots, v_{n}\right\}$ is an isomorphism that preserves the sets $\left\{u_{1}, \ldots, u_{m}\right\}$ and $\left\{v_{1}, \ldots, v_{n}\right\}$. Let $\mathcal{I}(H)$ be the set of all graphs isomorphic to $H$ by a colour-preserving isomorphism. We know that

$$
|\mathcal{I}(H)|=\frac{m!n!}{\operatorname{aut}(H)}
$$

where $\operatorname{aut}(H)$ is the number of colour-preserving automorphisms of $H$.
When the graphs in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ are semiregular, the expected number of elements of $\mathcal{I}(H)$ that are contained in or edge-disjoint from a random graph in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ is clearly just $|\mathcal{I}(H)|$ times the probability given by Theorem 2.2 and Corollary 2.1.

If this regularity condition does not hold, the calculation is more complex. Here we consider the case that the graph $H$ is semiregular and leave the most general case for a future paper.

We will need the following averaging lemma.
Lemma 2.1. Let $\boldsymbol{z}^{(0)}=\left(z_{1}^{(0)}, z_{2}^{(0)}, \ldots, z_{n}^{(0)}\right)$ be a vector in $[-1,1]^{n}$ such that $\sum_{j=1}^{n} z_{j}^{(0)}=0$. Form $\boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}, \ldots$ as follows: for each $r \geq 0$, if $z_{i}^{(r)}$ is the first of the smallest elements of $\boldsymbol{z}^{(r)}$ and $z_{\ell}^{(r)}$ is the first of the largest elements of $\boldsymbol{z}^{(r)}$, then $\boldsymbol{z}^{(r+1)}$ is the same as $\boldsymbol{z}^{(r)}$ except that $z_{i}^{(r+1)}$ and $z_{\ell}^{(r+1)}$ are both equal to $\left(z_{i}^{(r)}+z_{\ell}^{(r)}\right) / 2$. Then $\boldsymbol{z}^{(n)} \in\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}$.

Proof. If $\boldsymbol{z}^{(r)} \notin\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}$, then the fact that $\sum_{j=1}^{n} z_{j}^{(r)}=0$ implies that $z_{i}^{(r)}<0$ and $z_{\ell}^{(r)}>0$. Therefore $\boldsymbol{z}^{(r+1)}$ has at least one fewer element outside $\left[-\frac{1}{2}, \frac{1}{2}\right]$ than $\boldsymbol{z}^{(r)}$ does. The lemma follows. (In fact, $\boldsymbol{z}^{(\lfloor(2 n-1) / 3\rfloor)} \in\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}$, but this improvement is not necessary for our application.)

Theorem 2.4. Suppose that the conditions of Theorem 2.1 apply with $x_{j}=x$ and $y_{k}=y$ for all $j, k$. Then the following is true of a random graph $G$ in $\mathcal{B}(\boldsymbol{s}, \boldsymbol{t})$ :
(i) the expected number of graphs in $\mathcal{I}(H)$ that are subgraphs of $G$ is

$$
\lambda^{X}|\mathcal{I}(H)| \exp \left(-\frac{(1-\lambda)(x y-x-y)}{2 \lambda}-\frac{y R+x C}{2 \lambda^{2} m n}+O\left(n^{-b}\right)\right)
$$

(ii) the expected number of graphs in $\mathcal{I}(H)$ that are edge-disjoint from $G$ is

$$
(1-\lambda)^{X}|\mathcal{I}(H)| \exp \left(-\frac{\lambda(x y-x-y)}{2(1-\lambda)}-\frac{y R+x C}{2(1-\lambda)^{2} m n}+O\left(n^{-b}\right)\right)
$$

Proof. Define $\boldsymbol{z}^{(0)}, \boldsymbol{z}^{(1)}, \ldots$ as in Lemma 2.1, with $\boldsymbol{z}_{j}^{(0)}=s_{j}-s$ for $1 \leq j \leq n$. For $r \geq 0$, define

$$
Y^{(r)}(g, h)=\sum_{j=1}^{m} z_{j^{g}}^{(r)} T_{j, h}, \text { where } T_{j, h}=\sum_{k: u_{j} v_{k} \in E(H)}\left(t_{k^{h}}-t\right),
$$

and

$$
F^{(r)}=\sum_{(g, h) \in S_{m} \times S_{n}} \exp \left(-\frac{Y^{(r)}(g, h)}{2 A m n}\right) .
$$

For a permutation pair $(g, h) \in S_{m} \times S_{n}$, define $H^{g, h}$ to be the isomorph of $H$ with edge set $\left\{u_{j}{ }^{g} v_{k^{h}} \mid u_{j} v_{k} \in E(H)\right\}$. As $(g, h)$ runs over $S_{m} \times S_{n}$, each isomorph of $H$ appears as $H^{g, h}$ exactly $\operatorname{aut}(H)$ times. Therefore, by Corollary 2.2, the expectation required in part (i) of the theorem is

$$
\frac{\lambda^{X} F^{(0)}}{\operatorname{aut}(H)} \exp \left(-\frac{(1-\lambda)(x y-x-y)}{2 \lambda}-\frac{y R+x C}{2 \lambda^{2} m n}+O\left(n^{-b}\right)\right)
$$

For some $r \geq 0$, suppose that $\boldsymbol{z}^{(r+1)}$ is formed from $\boldsymbol{z}^{(r)}$ by averaging $z_{i}^{(r)}$ and $z_{\ell}^{(r)}$ as in Lemma 2.1. Then $\left\{(i \ell) g \mid g \in S_{n}\right\}=S_{n}$, so

$$
\begin{aligned}
F^{(r)}= & \frac{1}{2} \sum_{(g, h)}\left(\exp \left(-\frac{Y^{(r)}(g, h)}{2 A m n}\right)+\exp \left(-\frac{Y^{(r)}((i \ell) g, h)}{2 A m n}\right)\right) \\
= & \frac{1}{2} \sum_{(g, h)} \exp \left(-\frac{\sum_{j \notin\{i, \ell\}} z_{j}^{(r)} T_{j, h}}{2 A m n}\right)\left(\exp \left(-\frac{z_{i}^{(r)} T_{i, h}+z_{\ell}^{(r)} T_{\ell, h}}{2 A m n}\right)\right. \\
& \left.\quad+\exp \left(-\frac{z_{i}^{(r)} T_{\ell, h}+z_{\ell}^{(r)} T_{i, h}}{2 A m n}\right)\right) \\
= & \sum_{(g, h)} \exp \left(-\frac{\sum_{j \notin\{i, \ell\}} z_{j}^{(r)} T_{j, h}}{2 A m n}-\frac{z_{i}^{(r+1)} T_{i, h}+z_{\ell}^{(r+1)} T_{\ell, h}}{2 A m n}+\widetilde{O}\left(n^{-2}\right)\right) \\
= & \sum_{(g, h)} \exp \left(-\frac{Y^{(r+1)}(g, h)}{2 A m n}+\widetilde{O}\left(n^{-2}\right)\right) \\
= & F^{(r+1)} \exp \left(\widetilde{O}\left(n^{-2}\right)\right) .
\end{aligned}
$$

By Lemma 2.1 there is some $r_{0}=O(n \log n)$ such that $\boldsymbol{z}^{\left(r_{0}\right)} \in\left[-n^{-1 / 2}, n^{-1 / 2}\right]^{n}$. By the definition of $F^{\left(r_{0}\right)}$, we have $F^{\left(r_{0}\right)}=m!n!\exp \left(\widetilde{O}\left(n^{-1}\right)\right)$, so $F^{(0)}=m!n!\exp \left(\widetilde{O}\left(n^{-1}\right)\right)$ by induction. Part (i) of the theorem follows. Part (ii) is proved in identical fashion.

A simple example of Theorem 2.4 at work is the enumeration of perfect matchings in the case $m=n$. Equivalently, this is the permanent of the corresponding $n \times n$ binary matrix. Most previous research has focussed on the case that the matrix has constant row and column sums. For $s=t=o\left(n^{1 / 3}\right)$, the asymptotic expectation and variance are known, while for $s=t=n-O\left(n^{1-\epsilon}\right)$, the asymptotic expectation is known [3]. In the intermediate range of densities covered by the current paper, it appears that only bounds are known. The van der Waerden lower bound $n!\lambda^{n}$ (proved independently by Egorychev and Falikman) was improved by Gurvits [9] to $s!\left((s-1)^{s-1} / s^{s-2}\right)^{n-s}$. The best upper bound is $s!^{1 / \lambda} \sim n!\lambda^{n+1 /(2 \lambda)}(2 \pi n)^{(1-\lambda) /(2 \lambda)}$ conjectured by Minc and proved by Bregman. See Timashëv [18] for references and discussion.

Applying Theorem 2.4(i) with $x=y=1$ gives the following.
Theorem 2.5. Suppose that $m=n$ and $\boldsymbol{s}, \boldsymbol{t}, \lambda$ satisfy the requirements of Theorem 2.1. Then the expected permanent of a random $n \times n$ matrix over $\{0,1\}$ with row sums $\boldsymbol{s}$ and column sums $\boldsymbol{t}$ is

$$
n!\lambda^{n} \exp \left(\frac{1-\lambda}{2 \lambda}-\frac{R+C}{2 \lambda^{2} n^{2}}+O\left(n^{-b}\right)\right)
$$

It is interesting to note that in the regular case $R=C=0$, the average given in Theorem 2.5 is only higher than Gurvits' lower bound [9] by a factor of $\lambda^{-1 / 2}(1+o(1))$.

## 3 Subdigraphs of random digraphs

The adjacency matrix of a simple digraph is a square $\{0,1\}$-matrix with zero diagonal. Therefore, Theorem 2.1 can be applied to enumerate digraphs with specified degrees, and the result can then be used to explore the corresponding uniform probability space.

In this section, $H$ denotes a fixed simple digraph on the vertices $\left\{w_{1}, \ldots, w_{n}\right\}$. Let $\mathcal{D}(\boldsymbol{s}, \boldsymbol{t})$ be the set of all simple digraphs on vertices $\left\{w_{1}, \ldots, w_{n}\right\}$ with out-degrees $\boldsymbol{s}$ and in-degrees $\boldsymbol{t}$, and let $\mathcal{D}(\boldsymbol{s}, \boldsymbol{t}, H)$ be the subset of $\mathcal{D}(\boldsymbol{s}, \boldsymbol{t})$ containing those digraphs which are arc-disjoint from $H$.

For $1 \leq j \leq n$ let $x_{j}, y_{j}$ denote the out-degree and in-degree of vertex $w_{j}$ in $H$, respectively. The quantities $s=t, \lambda, \delta_{j}, \eta_{j}, X, Y$, and so forth are all defined by the same formulae as in Section 2 with $m=n$. In the definition of $Y$, the summation over $j k \in H$ should now be interpreted as summation over $j, k \in\{1, \ldots, n\}$ such that $w_{j} w_{k}$ is an arc of $H$. But note that $\lambda$ does not represent the arc-density of a digraph in $\mathcal{D}(\boldsymbol{s}, \boldsymbol{t})$. Instead the arc-density of a digraph in $\mathcal{D}(\boldsymbol{s}, \boldsymbol{t})$ is given by

$$
p=s /(n-1) .
$$

We begin with the basic enumeration result for digraphs.
Theorem 3.1. For some $\varepsilon>0$, suppose that $s_{j}-s, x_{j}, t_{j}-s$ and $y_{j}$ are uniformly $O\left(n^{1 / 2+\varepsilon}\right)$ for $1 \leq j, k \leq n$, and $X=O\left(n^{1+2 \varepsilon}\right)$, for $n \rightarrow \infty$. Let $a, b>0$ be constants such that $a+b<\frac{1}{2}$. Suppose that $n \rightarrow \infty$ with

$$
\frac{(1-2 \lambda)^{2}}{3 A} \leq a \log n
$$

Then, provided $\varepsilon>0$ is small enough, we have

$$
\begin{aligned}
& |\mathcal{D}(\boldsymbol{s}, \boldsymbol{t}, H)| \\
& \quad=\binom{n^{2}-X-n}{\lambda n^{2}}^{-1} \prod_{j=1}^{n}\binom{n-x_{j}-1}{s_{j}}\binom{n-y_{j}-1}{t_{j}} \\
& \quad \times \exp \left(-\frac{1}{2}\left(1-\frac{R}{2 A n^{2}}\right)\left(1-\frac{C}{2 A n^{2}}\right)-\frac{Y+\sum_{j=1}^{n}\left(s_{j}-s\right)\left(t_{j}-s\right)}{2 A n^{2}}+O\left(n^{-b}\right)\right) .
\end{aligned}
$$

Proof. Let $\widetilde{H}$ be the bipartite graph obtained from $H$ by replacing each vertex $w_{j}$ by two vertices $u_{j}, v_{j}$, replacing each arc $w_{j} w_{k}$ of $H$ by the edge $u_{j} v_{k}$ of $\widetilde{H}$, and finally adding the perfect matching $\left\{u_{j} v_{j} \mid j=1, \ldots, n\right\}$ to the edge set of $\widetilde{H}$. Then the degree sequences on the left and right of $\widetilde{H}$ and the total number of edges in $\widetilde{H}$ are given by

$$
\widetilde{x}_{j}=x_{j}+1, \quad \widetilde{y}_{j}=y_{j}+1, \quad \widetilde{X}=X+n
$$

respectively $(1 \leq j \leq n)$. The quantity $\widetilde{Y}$ for $\widetilde{H}$ satisfies

$$
\widetilde{Y}=Y+\sum_{j=1}^{n}\left(s_{j}-s\right)\left(t_{j}-s\right)+O\left(n^{2-b}\right)
$$

for any positive constant $b<1 / 2$. Using this fact while applying Theorem 2.1 to $\widetilde{H}$ completes the proof.

This formula for $|\mathcal{D}(\boldsymbol{s}, \boldsymbol{t}, H)|$ has an intuitive interpretation which is analogous to that given after Theorem 2.1 for the bipartite graph case.

Using this enumeration theorem, we can explore the uniform probability space over $\mathcal{D}(\boldsymbol{s}, \boldsymbol{t})$. In each case, the proof is analogous to that of the corresponding theorem for bipartite graphs in Section 2,

Theorem 3.2. Under the conditions of Theorem 3.1, the following are true for a random digraph $G \in \mathcal{D}(\boldsymbol{s}, \boldsymbol{t})$ if $\varepsilon>0$ is small enough:
(i) the probability that $G$ is arc-disjoint from $H$ is $(1-p)^{X} \operatorname{miss}(n, n)$;
(ii) the probability $G$ contains $H$ as a subdigraph is $p^{X} \operatorname{hit}(n, n)$, where $\operatorname{miss}(m, n)$ and $\operatorname{hit}(m, n)$ are defined in Theorem 2.2.

The special cases of $\operatorname{miss}(m, n)$ and $\operatorname{hit}(m, n)$ provided by Corollaries 2.1 and 2.2 apply here as well, as do the sufficient conditions (1) and (2) for the probabilities in Theorem 3.2 to asymptotically match those in the standard random digraph model with arc probability $p$.

Next suppose that each arc of $H$ has both ends in $\left\{w_{1}, \ldots, w_{J}\right\}$. Let $H_{J}$ be the subdigraph of $H$ induced by those vertices. For $\ell=1,2,3$, define

$$
\chi_{\ell}=\sum_{j=1}^{J}\left(x_{j}-p(J-1)\right)^{\ell}, \quad \chi_{\ell}^{\prime}=\sum_{k=1}^{J}\left(y_{k}-p(J-1)\right)^{\ell} .
$$

Note that $\chi_{1}=\chi_{1}^{\prime}=X-p J(J-1)$.
Theorem 3.3. Adopt the assumptions of Theorem 3.2 with $s_{j}=s$ and $t_{k}=t$ for all $j, k$, and assume that $J=O\left(n^{1 / 2+\varepsilon}\right)$. The probability that a random digraph in $\mathcal{D}(s, \boldsymbol{t})$ has $H_{J}$ as an induced subdigraph is

$$
\begin{aligned}
& p^{X}(1-p)^{J(J-1)-X} \exp \left(\frac{J^{2}}{n}+\frac{(1-2 \lambda) \chi_{1}}{2 A n}-\frac{\chi_{1}^{2}}{4 A n^{2}}\right. \\
&\left.-\frac{(n+J)\left(\chi_{2}+\chi_{2}^{\prime}\right)}{4 A n^{2}}-\frac{(1-2 \lambda)\left(\chi_{3}+\chi_{3}^{\prime}\right)}{24 A^{2} n^{2}}+O\left(n^{-b}\right)\right)
\end{aligned}
$$

The argument of the exponential in Theorem 3.3 is $o(1)$ if $J^{3}=o(A n)$. So in that case, the probabilities of induced subdigraphs asymptotically matches the standard random digraph model for arc probability $p$.

Let $\mathcal{I}(H)$ be the isomorphism class of $H$ and note that $|\mathcal{I}(H)|=n!/ \operatorname{aut}(H)$, where $\operatorname{aut}(H)$ is the number of automorphisms of $H$. By the same averaging technique as used to prove Theorem 2.4, we obtain the following.

Theorem 3.4. Suppose that the conditions of Theorem 3.1 apply with $x_{j}=y_{j}=x$ for all $j$. Then the following is true of a random digraph $G$ in $\mathcal{D}(\boldsymbol{s}, \boldsymbol{t})$ :
(i) the expected number of digraphs in $\mathcal{I}(H)$ that are subgraphs of $G$ is

$$
p^{X}|\mathcal{I}(H)| \exp \left(-\frac{(1-\lambda) x(x-2)}{2 \lambda}-\frac{(R+C) x}{2 \lambda^{2} n^{2}}+O\left(n^{-b}\right)\right) ;
$$

(ii) the expected number of digraphs in $\mathcal{I}(H)$ that are arc-disjoint from $G$ is

$$
(1-p)^{X}|\mathcal{I}(H)| \exp \left(-\frac{\lambda x(x-2)}{2(1-\lambda)}-\frac{(R+C) x}{2(1-\lambda)^{2} n^{2}}+O\left(n^{-b}\right)\right)
$$

## 4 Proof of Theorem 2.1

In the remainder of the paper we give the proof of Theorem 2.1. The overall method and many of the calculations will parallel [4], albeit with extra twists at each step, so we acknowledge our considerable debt to Rod Canfield.

Outline of proof of Theorem 2.1. The basic idea is to identify $|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|$ as a coefficient in a multivariable generating function and to extract that coefficient using the saddlepoint method. In Subsection 4.1, we write $|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|=P(\boldsymbol{s}, \boldsymbol{t}, H) I(\boldsymbol{s}, \boldsymbol{t}, H)$, where $P(\boldsymbol{s}, \boldsymbol{t}, H)$ is a rational expression and $I(\boldsymbol{s}, \boldsymbol{t}, H)$ is an integral in $m+n$ complex dimensions. Both depend on the location of the saddle point, which is the solution of some nonlinear equations. Those equations are solved in Subsection 4.2, and this leads to the value of $P(\boldsymbol{s}, \boldsymbol{t}, H)$ in (19). In Subsections 4.3-4.6, the integral $I(\boldsymbol{s}, \boldsymbol{t}, H)$ is estimated in a small region $\mathcal{R}^{\prime}$ defined in (30). The result is given by Lemma 4.3 together with (22). Finally, in Subsection 4.7, it is shown that the integral $I(s, \boldsymbol{t}, H)$ restricted to the exterior of $\mathcal{R}^{\prime}$ is negligible. Theorem 2.1 then follows from (4), (19), Lemmas 4.3 4.7 and (22).

We will use a shorthand notation for summation over doubly subscripted variables. If $z_{j k}$ is a variable for $1 \leq j \leq m$ and $1 \leq k \leq n$, then

$$
\begin{array}{lll}
z_{\bullet \bullet}=\sum_{k=1}^{n} z_{j k}, & z_{\bullet k}=\sum_{j=1}^{m} z_{j k}, & z_{\bullet \bullet}=\sum_{j=1}^{m} \sum_{k=1}^{n} z_{j k}, \\
z_{j *}=\sum_{k=1}^{n-1} z_{j k}, & z_{* k}=\sum_{j=1}^{m-1} z_{j k}, & z_{* *}=\sum_{j=1}^{m-1} \sum_{k=1}^{n-1} z_{j k},
\end{array}
$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$.
For $1 \leq j \leq m$ and $1 \leq k \leq n$ define $h_{j k}=1$ if $u_{j} v_{k}$ is an edge of $H$ and $h_{j k}=0$ otherwise. Then define the sets

$$
\begin{aligned}
X_{j} & =\left\{k \mid 1 \leq k \leq n, h_{j k}=1\right\}, & & \bar{X}_{j}=\left\{k \mid 1 \leq k \leq n, h_{j k}=0\right\} \\
Y_{k} & =\left\{j \mid 1 \leq j \leq n, h_{j k}=1\right\}, & & \bar{Y}_{k}=\left\{j \mid 1 \leq j \leq n, h_{j k}=0\right\}
\end{aligned}
$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$.
The notations $\sum_{j k \in H}$ and $\sum_{j k \in \bar{H}}$ indicate sums over the sets $\{(j, k) \mid 1 \leq j \leq m$, $\left.1 \leq k \leq n, h_{j k}=1\right\}$ and $\left\{(j, k) \mid 1 \leq j \leq m, 1 \leq k \leq n, h_{j k}=0\right\}$, respectively, and
similarly for products. We also define summations whose domain is limited by $H$.

$$
\begin{array}{lll}
z_{j \bullet \mid H}=\sum_{k \in X_{j}} z_{j k}, & z_{\bullet k \mid H}=\sum_{j \in Y_{k}} z_{j k}, & z_{\bullet \bullet \mid H}=\sum_{j k \in H} z_{j k}, \\
z_{j \bullet \mid \bar{H}}=\sum_{k \in \bar{X}_{j}} z_{j k}, & z_{\bullet k \mid \bar{H}}=\sum_{j \in \bar{Y}_{k}} z_{j k}, & z_{\bullet \bullet \mid \bar{H}}=\sum_{j k \in \bar{H}} z_{j k} .
\end{array}
$$

Under the assumptions of Theorem 2.1, we have $m=\widetilde{O}(n)$ and $n=\widetilde{O}(m)$. We also have that $8 \leq A^{-1} \leq O(\log n)$, so $A^{-1}=\widetilde{O}(1)$. More generally, $A^{c_{1}} m^{c_{2}+c_{3} \varepsilon} n^{c_{4}+c_{5} \varepsilon}=$ $\widetilde{O}\left(n^{c_{2}+c_{4}}\right)$ if $c_{1}, c_{2}, c_{3}, c_{4}, c_{5}$ are constants.

We now show that the assumptions of Theorem 2.1 imply that

$$
\begin{equation*}
m=o\left(A^{2} n^{1+\varepsilon}\right), \quad n=o\left(A^{2} m^{1+\varepsilon}\right) \tag{3}
\end{equation*}
$$

If $A \geq \frac{3}{32}$ then (3) follows immediately. If $A<\frac{3}{32}$ then $(1-2 \lambda)^{2}>\frac{1}{4}$ and so the assumptions of Theorem 2.1 imply that $1 / A=O(\log n /(m / n+n / m))$. This implies (3).

### 4.1 Expressing the desired quantity as an integral

In this section we express $|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|$ as a contour integral in $(m+n)$-dimensional complex space, then begin to estimate its value using the saddle-point method.

Firstly, notice that $|\mathcal{B}(s, \boldsymbol{t}, H)|$ is the coefficient of $u_{1}^{s_{1}} \cdots u_{m}^{s_{m}} w_{1}^{t_{1}} \cdots w_{n}^{t_{n}}$ in the function

$$
\prod_{j k \in \bar{H}}\left(1+u_{j} w_{k}\right)
$$

By Cauchy's coefficient theorem this equals

$$
|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|=\frac{1}{(2 \pi i)^{m+n}} \oint \cdots \oint \frac{\prod_{j k \in \bar{H}}\left(1+u_{j} w_{k}\right)}{u_{1}^{s_{1}+1} \cdots u_{m}^{s_{m}+1} w_{1}^{t_{1}+1} \cdots w_{n}^{t_{n}+1}} d u_{1} \cdots d u_{m} d w_{1} \cdots d w_{n}
$$

where each integral is along a simple closed contour enclosing the origin anticlockwise. It will suffice to take each contour to be a circle; specifically, we will write

$$
u_{j}=q_{j} e^{i \theta_{j}} \quad \text { and } \quad w_{k}=r_{k} e^{i \phi_{k}}
$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$. Also define

$$
\lambda_{j k}=\frac{q_{j} r_{k}}{1+q_{j} r_{k}}
$$

for $1 \leq j \leq m$ and $1 \leq k \leq n$. Then $|\mathcal{B}(\boldsymbol{s}, \boldsymbol{t}, H)|=P(\boldsymbol{s}, \boldsymbol{t}, H) I(\boldsymbol{s}, \boldsymbol{t}, H)$ where

$$
\begin{align*}
P(\boldsymbol{s}, \boldsymbol{t}, H) & =\frac{\prod_{j k \in \bar{H}}\left(1+q_{j} r_{k}\right)}{(2 \pi)^{m+n} \prod_{j=1}^{m} q_{j}^{s_{j}} \prod_{k=1}^{n} r_{k}^{t_{k}}} \\
I(\boldsymbol{s}, \boldsymbol{t}, H) & =\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{\prod_{j k \in \bar{H}}\left(1+\lambda_{j k}\left(e^{i\left(\theta_{j}+\phi_{k}\right)}-1\right)\right)}{\exp \left(i \sum_{j=1}^{m} s_{j} \theta_{j}+i \sum_{k=1}^{n} t_{k} \phi_{k}\right)} d \boldsymbol{\theta} d \boldsymbol{\phi}, \tag{4}
\end{align*}
$$

$\boldsymbol{\theta}=\left(\theta_{1}, \ldots, \theta_{m}\right)$ and $\boldsymbol{\phi}=\left(\phi_{1}, \ldots, \phi_{n}\right)$.
We will choose the radii $q_{j}, r_{k}$ so that there is no linear term in the logarithm of the integrand of $I(\boldsymbol{s}, \boldsymbol{t}, H)$ when expanded for small $\boldsymbol{\theta}, \boldsymbol{\phi}$. This gives the equation

$$
\sum_{j k \in \bar{H}} \lambda_{j k}\left(\theta_{j}+\phi_{k}\right)-\sum_{j=1}^{m} s_{j} \theta_{j}-\sum_{k=1}^{n} t_{k} \phi_{k}=0 .
$$

For this to hold for all $\boldsymbol{\theta}, \boldsymbol{\phi}$, we require

$$
\begin{array}{ll}
\lambda_{j \bullet \mid \bar{H}}=s_{j} & (1 \leq j \leq m),  \tag{5}\\
\lambda_{\bullet k \mid \bar{H}}=t_{k} & (1 \leq k \leq n) .
\end{array}
$$

The quantities $\lambda_{j k}$ have an interesting interpretation. If edge $u_{j} v_{k}$ is chosen with probability $\lambda_{j k}$ independently for all $j, k \in \bar{H}$, then the expected degrees are $\boldsymbol{s}, \boldsymbol{t}$.

In addition to the quantities defined before the statement of Theorem [2.2 we define for $j=1, \ldots, m, k=1, \ldots, n$,

$$
J_{j}=\sum_{k \in X_{j}} \eta_{k}, \quad K_{k}=\sum_{j \in Y_{k}} \delta_{j} .
$$

### 4.2 Locating the saddle-point

In this subsection we solve (5) and derive some of the consequences of the solution. As with the whole paper, we work under the assumptions of Theorem 2.1.

Change variables to $\left\{a_{j}\right\}_{j=1}^{m},\left\{b_{k}\right\}_{k=1}^{n}$ as follows:

$$
\begin{equation*}
q_{j}=r \frac{1+a_{j}}{1-r^{2} a_{j}}, \quad r_{k}=r \frac{1+b_{k}}{1-r^{2} b_{k}} \tag{6}
\end{equation*}
$$

where

$$
r=\sqrt{\frac{\lambda}{1-\lambda}}
$$

Equation (5) is slightly underdetermined, which we will exploit to impose an additional condition. If $\left\{q_{j}\right\},\left\{r_{k}\right\}$ satisfy (5) and $c>0$ is a constant, then $\left\{c q_{j}\right\},\left\{r_{k} / c\right\}$ also
satisfy (5). From this we can see that, if there is a solution to (5) at all, there is one for which $\sum_{j=1}^{m}\left(n-x_{j}\right) a_{j}<0$ and $\sum_{k=1}^{n}\left(m-y_{k}\right) b_{k}>0$, and also a solution for which $\sum_{j=1}^{m}\left(n-x_{j}\right) a_{j}>0$ and $\sum_{k=1}^{n}\left(m-y_{k}\right) b_{k}<0$. It follows from the Intermediate Value Theorem that there is a solution for which

$$
\begin{equation*}
\sum_{j=1}^{m}\left(n-x_{j}\right) a_{j}=\sum_{k=1}^{n}\left(m-y_{k}\right) b_{k}, \tag{7}
\end{equation*}
$$

so we will seek a common solution to (5) and (7).
From (6) we find that

$$
\begin{equation*}
\lambda_{j k} / \lambda=1+a_{j}+b_{k}+Z_{j k} \tag{8}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{j k}=\frac{a_{j} b_{k}\left(1-r^{2}-r^{2} a_{j}-r^{2} b_{k}\right)}{1+r^{2} a_{j} b_{k}} \tag{9}
\end{equation*}
$$

and that equations (5) can be rewritten as

$$
\begin{align*}
& \frac{\delta_{j}}{\lambda}=\left(n-x_{j}\right) a_{j}+\sum_{k \in \bar{X}_{j}} b_{k}+Z_{j \bullet \mid \bar{H}}  \tag{10}\\
& \frac{\eta_{k}}{\lambda}=\left(m-y_{k}\right) b_{k}+\sum_{j \in \bar{Y}_{k}} a_{j}+Z_{\bullet k \mid \bar{H}}
\end{align*}
$$

Summing (10) over all $j, k$, respectively, we find in both cases that that

$$
\begin{equation*}
X=\sum_{j=1}^{m}\left(n-x_{j}\right) a_{j}+\sum_{k=1}^{n}\left(m-y_{k}\right) b_{k}+Z_{\bullet \bullet \mid \bar{H}} \tag{11}
\end{equation*}
$$

Equations (7) and (11) together imply that

$$
\sum_{j=1}^{m}\left(n-x_{j}\right) a_{j}=\sum_{k=1}^{n}\left(m-y_{k}\right) b_{k}=\frac{1}{2}\left(X-Z_{\bullet \bullet \mid \bar{H}}\right)
$$

Substituting back into (10), we obtain

$$
\begin{align*}
a_{j} & =\mathbb{A}_{j}\left(a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}\right)  \tag{12}\\
b_{k} & =\mathbb{B}_{k}\left(a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}\right)
\end{align*}
$$

for $1 \leq j \leq m, 1 \leq k \leq n$, where

$$
\begin{aligned}
& \mathbb{A}_{j}\left(a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}\right)= \frac{\delta_{j}}{\lambda n}- \\
&-\frac{X}{2 m n}+\frac{a_{j} x_{j}}{n} \\
&-\frac{\sum_{k=1}^{n} y_{k} b_{k}}{m n}+\frac{\sum_{k \in X_{j}} b_{k}}{n}-\frac{Z_{j \bullet \mid \bar{H}}}{n}+\frac{Z_{\bullet \bullet \mid \bar{H}}}{2 m n}, \\
& \mathbb{B}_{k}\left(a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}\right)= \frac{\eta_{k}}{\lambda m}-\frac{X}{2 m n}+\frac{b_{k} y_{k}}{m} \\
&-\frac{\sum_{j=1}^{m} x_{j} a_{j}}{m n}+\frac{\sum_{j \in Y_{k}} a_{j}}{m}-\frac{Z_{\bullet k \mid \bar{H}}}{m}+\frac{Z_{\bullet \bullet \mid \bar{H}}}{2 m n} .
\end{aligned}
$$

By the same argument as in [4], equation (12) defines a convergent iteration starting with $a_{j}=b_{k}=0$ for all $j, k$. Four iterations give the following estimate of $a_{j}$. The value of $b_{k}$ follows by symmetry, while $Z_{j k}$ follows from (9).

$$
\begin{aligned}
a_{j}= & \frac{\delta_{j}}{\lambda n}+\frac{\delta_{j} x_{j}}{\lambda n^{2}}+\frac{\delta_{j} x_{j}^{2}}{\lambda n^{3}}+\frac{\delta_{j} x_{j}^{3}}{\lambda n^{4}}-\frac{X}{2 m n}-\frac{(1-2 \lambda) \delta_{j} X}{4 A m n^{2}}+\frac{\delta_{j}^{2} X}{4 A m n^{3}} \\
& -\frac{x_{j} X}{m n^{2}}-\frac{x_{j}^{2} X}{m n^{3}}+\frac{\lambda(7-10 \lambda) X^{2}}{16 A m^{2} n^{2}}-\frac{3(1-2 \lambda) \delta_{j} x_{j} X}{4 A m n^{3}}-\frac{(1-2 \lambda) Y}{4 \lambda A m^{2} n^{2}} \\
& +\frac{\delta_{j} C_{2}}{2 \lambda A m^{2} n^{2}}+\frac{(1-2 \lambda) \delta_{j}^{2} C_{2}}{4 \lambda A^{2} m^{2} n^{3}}+\frac{\delta_{j} x_{j} C_{2}}{\lambda A m^{2} n^{3}}-\frac{3 X C_{2}}{8 A m^{3} n^{2}}-\frac{X R_{2}}{8 A m^{2} n^{3}} \\
& -\frac{(1-2 \lambda) R_{2} C_{2}}{8 \lambda A^{2} m^{3} n^{3}}-\frac{x_{j} R_{1,1}}{\lambda m n^{3}}-\frac{C_{1,1}}{\lambda m^{2} n}-\frac{(1-2 \lambda) \delta_{j} C_{1,1}}{2 \lambda A m^{2} n^{2}}-\frac{x_{j} C_{1,1}}{\lambda m^{2} n^{2}} \\
& -\frac{Y_{0,1}}{\lambda m^{2} n^{2}}-\frac{C_{1,2}}{\lambda m^{3} n}+\frac{J_{j}}{\lambda m n}+\frac{(1-2 \lambda) \delta_{j} J_{j}}{2 \lambda A m n^{2}}-\frac{\delta_{j}^{2} J_{j}}{2 \lambda A m n^{3}}+\frac{x_{j} J_{j}}{\lambda m n^{2}} \\
& +\frac{x_{j}^{2} J_{j}}{\lambda m n^{3}}+\frac{(1-2 \lambda) \delta_{j} x_{j} J_{j}}{\lambda A m n^{3}}-\frac{3(1-2 \lambda) X J_{j}}{4 A m^{2} n^{2}}+\frac{(1-2 \lambda) J_{j}^{2}}{2 \lambda A m^{2} n^{2}} \\
& +\left(\frac{R_{2}}{n}+\frac{C_{2}}{m}\right) \frac{J_{j}}{2 \lambda A m^{2} n^{2}}+\frac{1}{\lambda m^{2} n^{2}} \sum_{\left(j^{\prime}, k^{\prime}\right)} \delta_{j^{\prime} y_{k^{\prime}}+\frac{1}{\lambda m^{3} n} \sum_{k \in X_{j}} \eta_{k} y_{k}^{2}}^{\left(j^{\prime}, k^{\prime}\right)} \\
& -\frac{X}{m^{2} n^{2}} \sum_{k \in X_{j}} y_{k}-\frac{\delta_{j}}{2 \lambda A m^{2} n^{2}} \sum_{k \in X_{j}} \eta_{k}^{2}+\frac{(1-2 \lambda)}{2 \lambda A m^{2} n^{2}} \sum_{\left.j^{\prime}, \eta_{k^{\prime}}\right)} \\
& +\left(\frac{1}{\lambda m^{2} n^{2}}+\frac{(1-2 \lambda) \delta_{j}}{2 \lambda A m^{2} n^{3}}+\frac{x_{j}}{\lambda m^{2} n^{3}}\right)\left(n \sum_{k \in X_{j}} \eta_{k} y_{k}+m \sum_{\left(j^{\prime}, k^{\prime}\right)} \delta_{j^{\prime}}\right) \\
& +\frac{1}{\lambda m n^{3}} \sum_{k^{\prime}} \delta_{j^{\prime}} x_{j^{\prime}}+\frac{1}{\lambda m^{2} n^{2}} \sum_{k^{\prime}} \sum_{k^{\prime \prime}}+\widetilde{O}\left(n^{-5 / 2}\right),
\end{aligned}
$$

where the notation $\sum_{\left(j^{\prime}, k^{\prime}\right)}$ means $\sum_{k^{\prime} \in X_{j}} \sum_{j^{\prime} \in Y_{k^{\prime}}}$.
A sufficient approximation of $\lambda_{j k}$ is given by substituting this estimate into (8). In evaluating the integral $I(\boldsymbol{s}, \boldsymbol{t}, H)$, the following approximations will be required:

$$
\begin{align*}
\lambda_{j k}\left(1-\lambda_{j k}\right)= & \lambda(1-\lambda)+\frac{(1-2 \lambda) \delta_{j}}{n}+\frac{(1-2 \lambda) \eta_{k}}{m}-\frac{\delta_{j}^{2}}{n^{2}}-\frac{\eta_{k}^{2}}{m^{2}} \\
& +\frac{(1-12 A) \delta_{j} \eta_{k}}{2 A m n}+\frac{(1-2 \lambda) \delta_{j} x_{j}}{n^{2}}+\frac{(1-2 \lambda) \eta_{k} y_{k}}{m^{2}}  \tag{13}\\
& +\frac{(1-2 \lambda)\left(J_{j}+K_{k}-\lambda X\right)}{m n}+\widetilde{O}\left(n^{-3 / 2}\right),
\end{align*}
$$

$$
\begin{gather*}
\lambda_{j k}\left(1-\lambda_{j k}\right)\left(1-2 \lambda_{j k}\right)=\lambda(1-\lambda)(1-2 \lambda)+\frac{(1-12 A) \delta_{j}}{n}  \tag{14}\\
+ \\
\lambda_{j k}\left(1-\lambda_{j k}\right)\left(1-6 \lambda_{j k}+6 \lambda_{j k}^{2}\right)=  \tag{15}\\
m(1-\lambda)(1-12 A)+\widetilde{O}\left(n^{-1 / 2}\right) .
\end{gather*}
$$

We now estimate the factor $P(\boldsymbol{s}, \boldsymbol{t}, H)$. If

$$
\Lambda=\prod_{j k \in \bar{H}} \lambda_{j k}^{\lambda_{j k}}\left(1-\lambda_{j k}\right)^{1-\lambda_{j k}}
$$

then

$$
\begin{aligned}
\Lambda^{-1} & =\prod_{j k \in \bar{H}}\left(\frac{1+q_{j} r_{k}}{q_{j} r_{k}}\right)^{\lambda_{j k}}\left(1+q_{j} r_{k}\right)^{1-\lambda_{j k}} \\
& =\prod_{j k \in \bar{H}}\left(1+q_{j} r_{k}\right)\left(\prod_{j=1}^{m} q_{j}^{\lambda_{j \bullet \mid \bar{H}}} \prod_{k=1}^{n} r_{k}^{\lambda_{\bullet k \mid \bar{H}}}\right)^{-1} \\
& =\prod_{j k \in \bar{H}}\left(1+q_{j} r_{k}\right) \prod_{j=1}^{m} q_{j}^{-s_{j}} \prod_{k=1}^{n} r_{k}^{-t_{k}}
\end{aligned}
$$

using (5). Therefore, the factor $P(\boldsymbol{s}, \boldsymbol{t}, H)$ in front of the integral in (4) is given by

$$
P(s, \boldsymbol{t}, H)=(2 \pi)^{-(m+n)} \Lambda^{-1}
$$

We proceed to estimate $\Lambda$. Writing $\lambda_{j k}=\lambda\left(1+z_{j k}\right)$, we have

$$
\begin{align*}
& \log \left(\frac{\lambda_{j k}^{\lambda_{j k}}\left(1-\lambda_{j k}\right)^{1-\lambda_{j k}}}{\lambda^{\lambda}(1-\lambda)^{1-\lambda}}\right)=\lambda z_{j k} \log \left(\frac{\lambda}{1-\lambda}\right)  \tag{16}\\
& \quad+\frac{\lambda}{2(1-\lambda)^{2}} z_{j k}^{2}-\frac{\lambda(1-2 \lambda)}{6(1-\lambda)^{2}} z_{j k}^{3}+\frac{\lambda\left(1-3 \lambda+3 \lambda^{2}\right)}{12(1-\lambda)^{3}} z_{j k}^{4}+O\left(\frac{z_{j k}^{5}}{(1-\lambda)^{4}}\right) .
\end{align*}
$$

We know from (5) that $\lambda_{\bullet \bullet \mid \bar{H}}=\lambda m n$, which implies that $z_{\bullet \bullet \mid \bar{H}}=X$, hence the first term on the right side of (16) contributes $\lambda^{\lambda X}(1-\lambda)^{-\lambda X}$ to $\Lambda$. Now using (8) we can write $z_{j k}=a_{j}+b_{k}+Z_{j k}$ and apply the above estimates to obtain

$$
\begin{align*}
& \Lambda=\left(\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right)^{m n}(1-\lambda)^{-X} \\
& \quad \times \exp ( \frac{R_{2}}{4 A n}+\frac{C_{2}}{4 A m}+\frac{R_{2} C_{2}}{8 A^{2} m^{2} n^{2}}-\frac{\lambda^{2} X^{2}}{4 A m n}-\frac{(1-2 \lambda)}{24 A^{2}}\left(\frac{R_{3}}{n^{2}}+\frac{C_{3}}{m^{2}}\right)  \tag{17}\\
&\left.\quad+\frac{(1-6 A)}{96 A^{3}}\left(\frac{R_{4}}{n^{3}}+\frac{C_{4}}{m^{3}}\right)+\frac{Y}{2 A m n}+\frac{R_{2,1}}{4 A n^{2}}+\frac{C_{2,1}}{4 A m^{2}}+\widetilde{O}\left(n^{-1 / 2}\right)\right) .
\end{align*}
$$

As in [4], our answer will be simpler when written in terms of binomial coefficients. Using an accurate approximation of the binomial coefficients (such as [4, Equation 18]), we obtain that

$$
\begin{gather*}
\binom{m n-X}{\lambda m n}^{-1} \prod_{j=1}^{m}\binom{n-x_{j}}{s_{j}} \prod_{k=1}^{n}\binom{m-y_{k}}{t_{k}}=\frac{\left(\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right)^{-m n}(1-\lambda)^{X}}{(4 \pi A)^{(m+n-1) / 2} m^{(n-1) / 2} n^{(m-1) / 2}} \\
\times \exp \left(-\frac{R_{2}}{4 A n}-\frac{C_{2}}{4 A m}-\frac{1-2 A}{24 A}\left(\frac{m}{n}+\frac{n}{m}\right)+\frac{1-4 A}{16 A^{2}}\left(\frac{R_{2}}{n^{2}}+\frac{C_{2}}{m^{2}}\right)\right.  \tag{18}\\
\quad+\frac{1-2 \lambda}{24 A^{2}}\left(\frac{R_{3}}{n^{2}}+\frac{C_{3}}{m^{2}}\right)-\frac{1-6 A}{96 A^{3}}\left(\frac{R_{4}}{n^{3}}+\frac{C_{4}}{m^{3}}\right) \\
\left.\quad+\frac{\lambda^{2} X(m+n+X)}{4 A m n}-\frac{R_{2,1}}{4 A n^{2}}-\frac{C_{2,1}}{4 A m^{2}}+\widetilde{O}\left(n^{-1 / 2}\right)\right)
\end{gather*}
$$

Putting (17) and (18) together, we find that

$$
\begin{align*}
P(s, \boldsymbol{t}, H)= & \Lambda^{-1}(2 \pi)^{-(m+n)} \\
= & \frac{A^{(m+n-1) / 2} m^{(n-1) / 2} n^{(m-1) / 2}}{2 \pi^{(m+n+1) / 2}}\binom{m n-X}{\lambda m n}^{-1} \prod_{j=1}^{m}\binom{n-x_{j}}{s_{j}} \prod_{k=1}^{n}\binom{m-y_{k}}{t_{k}} \\
& \times \exp \left(\frac{1-2 A}{24 A}\left(\frac{m}{n}+\frac{n}{m}\right)-\frac{R_{2} C_{2}}{8 A^{2} m^{2} n^{2}}-\frac{1-4 A}{16 A^{2}}\left(\frac{R_{2}}{n^{2}}+\frac{C_{2}}{m^{2}}\right)\right.  \tag{19}\\
& \left.\quad-\frac{\lambda^{2} X}{4 A}\left(\frac{1}{m}+\frac{1}{n}\right)-\frac{Y}{2 A m n}+\widetilde{O}\left(n^{-1 / 2}\right)\right) .
\end{align*}
$$

### 4.3 Evaluating the integral

Our next task is to evaluate the integral $I(\boldsymbol{s}, \boldsymbol{t}, H)$ given (4).
Let $C$ be the ring of real numbers modulo $2 \pi$, which we can interpret as points on a circle, and let $z$ be the canonical mapping from $C$ to the real interval $(-\pi, \pi]$. An open half-circle is $C_{t}=(t-\pi / 2, t+\pi / 2) \subseteq C$ for some $t$. Now define

$$
\widehat{C}^{N}=\left\{\boldsymbol{v}=\left(v_{1}, \ldots, v_{N}\right) \in C^{N} \mid v_{1}, \ldots, v_{N} \in C_{t} \text { for some } t \in \mathbb{R}\right\}
$$

If $\boldsymbol{v}=\left(v_{1}, \ldots, v_{N}\right) \in C_{0}^{N}$ then define

$$
\overline{\boldsymbol{v}}=z^{-1}\left(\frac{1}{N} \sum_{j=1}^{N} z\left(v_{j}\right)\right) .
$$

More generally, if $\boldsymbol{v} \in C_{t}^{N}$ then define $\overline{\boldsymbol{v}}=t+\overline{\left(v_{1}-t, \ldots, v_{N}-t\right)}$. The function $\boldsymbol{v} \rightarrow \overline{\boldsymbol{v}}$ is well-defined and continuous for $\boldsymbol{v} \in \widehat{C}^{N}$.

Let $\mathcal{R}$ denote the set of vector pairs $(\boldsymbol{\theta}, \boldsymbol{\phi}) \in \widehat{C}^{m} \times \widehat{C}^{n}$ such that

$$
\begin{align*}
|\overline{\boldsymbol{\theta}}+\overline{\boldsymbol{\phi}}| & \leq(m n)^{-1 / 2+2 \varepsilon} \\
\left|\hat{\theta}_{j}\right| & \leq n^{-1 / 2+\varepsilon} \quad(1 \leq j \leq m),  \tag{20}\\
\left|\hat{\phi}_{k}\right| & \leq m^{-1 / 2+\varepsilon} \quad(1 \leq k \leq n),
\end{align*}
$$

where $\hat{\theta}_{j}=\theta_{j}-\overline{\boldsymbol{\theta}}$ and $\hat{\phi}_{k}=\phi_{k}-\overline{\boldsymbol{\phi}}$. In this definition, values are considered in $C$. The constant $\varepsilon$ is the sufficiently-small value required by Theorem [2.1.

Let $I_{\mathcal{R}^{\prime \prime}}(\boldsymbol{s}, \boldsymbol{t}, H)$ denote the integral $I(\boldsymbol{s}, \boldsymbol{t}, H)$ restricted to any region $\mathcal{R}^{\prime \prime}$. In this subsection, we estimate $I_{\mathcal{R}^{\prime}}(\boldsymbol{s}, \boldsymbol{t}, H)$ in a certain region $\mathcal{R}^{\prime} \supseteq \mathcal{R}$. In Subsection 4.7 we will show that the remaining parts of $I(\boldsymbol{s}, \boldsymbol{t}, H)$ are negligible. We begin by analysing the integrand in $\mathcal{R}$, but for future use when we expand the region to $\mathcal{R}^{\prime}$ (to be defined in (30)), note that all the approximations we establish for the integrand in $\mathcal{R}$ also hold in the superset of $\mathcal{R}^{\prime}$ defined by

$$
\begin{align*}
|\overline{\boldsymbol{\theta}}+\overline{\boldsymbol{\phi}}| & \leq 3(m n)^{-1 / 2+2 \varepsilon} \\
\left|\hat{\theta}_{j}\right| & \leq 3 n^{-1 / 2+\varepsilon} \quad(1 \leq j \leq m-1) \\
\left|\hat{\theta}_{m}\right| & \leq 2 n^{-1 / 2+3 \varepsilon}  \tag{21}\\
\left|\hat{\phi}_{k}\right| & \leq 3 m^{-1 / 2+\varepsilon} \quad(1 \leq k \leq n-1) \\
\left|\hat{\phi}_{n}\right| & \leq 2 m^{-1 / 2+3 \varepsilon}
\end{align*}
$$

Define $\hat{\boldsymbol{\theta}}=\left(\hat{\theta}_{1}, \ldots, \hat{\theta}_{m-1}\right)$ and $\hat{\boldsymbol{\phi}}=\left(\hat{\phi}_{1}, \ldots, \hat{\phi}_{n-1}\right)$. Let $T_{1}$ be the transformation $T_{1}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, \psi)=(\boldsymbol{\theta}, \boldsymbol{\phi})$ defined by

$$
\nu=\overline{\boldsymbol{\theta}}+\overline{\boldsymbol{\phi}}, \quad \psi=\overline{\boldsymbol{\theta}}-\overline{\boldsymbol{\phi}},
$$

together with $\hat{\theta}_{j}=\theta_{j}-\overline{\boldsymbol{\theta}}(1 \leq j \leq m-1)$ and $\hat{\phi}_{k}=\phi_{k}-\overline{\boldsymbol{\phi}}(1 \leq k \leq n-1)$. We also define the 1-many transformation $T_{1}^{*}$ by

$$
T_{1}^{*}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu)=\bigcup_{\psi} T_{1}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, \psi)
$$

After applying the transformation $T_{1}$ to $I_{\mathcal{R}}(\boldsymbol{s}, \boldsymbol{t}, H)$, the new integrand is easily seen to be independent of $\psi$, so we can multiply by the range of $\psi$ and remove it as an independent variable. Therefore, we can continue with an $(m+n-1)$-dimensional integral over $\mathcal{S}$ such that $\mathcal{R}=T_{1}^{*}(\mathcal{S})$. More generally, if $\mathcal{S}^{\prime \prime} \subseteq\left(-\frac{1}{2} \pi, \frac{1}{2} \pi\right)^{m+n-2} \times(-2 \pi, 2 \pi]$ and $\mathcal{R}^{\prime \prime}=T_{1}^{*}\left(\mathcal{S}^{\prime \prime}\right)$, we have

$$
\begin{equation*}
I_{\mathcal{R}^{\prime \prime}}(\boldsymbol{s}, \boldsymbol{t}, H)=2 \pi m n \int_{\mathcal{S}^{\prime \prime}} G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu) d \hat{\boldsymbol{\theta}} d \hat{\boldsymbol{\phi}} d \nu \tag{22}
\end{equation*}
$$

where $G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu)=F\left(T_{1}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, 0)\right)$ with $F(\boldsymbol{\theta}, \boldsymbol{\phi})$ being the integrand of $I(\boldsymbol{s}, \boldsymbol{t}, H)$. The factor $2 \pi m n$ combines the range of $\psi$, which is $4 \pi$, and the Jacobian of $T_{1}$, which is $m n / 2$.

Note that $\mathcal{S}$ is defined by the same inequalities (20) as define $\mathcal{R}$. The first inequality is now $|\nu| \leq(m n)^{-1 / 2+2 \varepsilon}$ and the bounds on

$$
\hat{\theta}_{m}=-\sum_{j=1}^{m-1} \hat{\theta}_{j} \text { and } \hat{\phi}_{m}=-\sum_{k=1}^{n-1} \hat{\phi}_{k}
$$

still apply even though these are no longer variables of integration.
In the region $\mathcal{S}$, the integrand of (22) can be expanded as

$$
\begin{aligned}
& G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu)= \exp \left(-\sum_{j k \in \bar{H}}\left(A+\alpha_{j k}\right)\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{2}-i \sum_{j k \in \bar{H}}\left(A_{3}+\beta_{j k}\right)\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{3}\right. \\
&\left.+\sum_{j k \in \bar{H}}\left(A_{4}+\gamma_{j k}\right)\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{4}+O\left(A \sum_{j k \in \bar{H}}\left|\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right|^{5}\right)\right) \\
&=\exp \left(-\sum_{j=1}^{m} \sum_{k=1}^{n}\left(A+\alpha_{j k}\right)\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{2}-i \sum_{j=1}^{m} \sum_{k=1}^{n}\left(A_{3}+\beta_{j k}\right)\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{3}\right. \\
&\left.+\sum_{j=1}^{m} \sum_{k=1}^{n}\left(A_{4}+\gamma_{j k}\right)\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{4}+\sum_{j k \in H} A\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{2}+\widetilde{O}\left(n^{-1 / 2}\right)\right) .
\end{aligned}
$$

Here $\alpha_{j k}, \beta_{j k}$, and $\gamma_{j k}$ are defined by

$$
\begin{align*}
\frac{1}{2} \lambda_{j k}\left(1-\lambda_{j k}\right) & =A+\alpha_{j k}, \\
\frac{1}{6} \lambda_{j k}\left(1-\lambda_{j k}\right)\left(1-2 \lambda_{j k}\right) & =A_{3}+\beta_{j k},  \tag{23}\\
\frac{1}{24} \lambda_{j k}\left(1-\lambda_{j k}\right)\left(1-6 \lambda_{j k}+6 \lambda_{j k}^{2}\right) & =A_{4}+\gamma_{j k},
\end{align*}
$$

where

$$
A=\frac{1}{2} \lambda(1-\lambda), A_{3}=\frac{1}{6} \lambda(1-\lambda)(1-2 \lambda), \text { and } A_{4}=\frac{1}{24} \lambda(1-\lambda)\left(1-6 \lambda+6 \lambda^{2}\right) .
$$

Approximations for $\alpha_{j k}, \beta_{j k}, \gamma_{j k}$ were given in (13)-(15). Note that $\alpha_{j k}$ in this paper is slightly different from in [4], but it is still true that $\alpha_{j k}, \beta_{j k}, \gamma_{j k}=\widetilde{O}\left(n^{-1 / 2}\right)$ uniformly over $j, k$.

### 4.4 Another change of variables

We now make a second change of variables $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu)=T_{2}(\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu)$, where $\boldsymbol{\zeta}=\left(\zeta_{1}, \ldots, \zeta_{m-1}\right)$ and $\boldsymbol{\xi}=\left(\xi_{1}, \ldots, \xi_{n-1}\right)$, whose purpose is to almost diagonalize the quadratic part of $G$. The diagonalization will be completed in the next subsection. The transformation $T_{2}$ is defined as follows. For $1 \leq j \leq m-1$ and $1 \leq k \leq n-1$ let

$$
\hat{\theta}_{j}=\zeta_{j}+c \pi_{1}, \quad \hat{\phi}_{k}=\xi_{k}+d \rho_{1}
$$

where

$$
c=-\frac{1}{m+m^{1 / 2}} \quad \text { and } \quad d=-\frac{1}{n+n^{1 / 2}}
$$

and, for $1 \leq h \leq 4$,

$$
\pi_{h}=\sum_{j=1}^{m-1} \zeta_{j}^{h}, \quad \rho_{h}=\sum_{k=1}^{n-1} \xi_{k}^{h} .
$$

The Jacobian of the transformation is $(m n)^{-1 / 2}$. In [5], this transformation was seen to exactly diagonalize the quadratic part of the integrand in the semiregular case. In the present irregular case, the diagonalization is no longer exact but still provides useful progress.

By summing the equations $\hat{\theta}_{j}=\zeta_{j}+c \pi_{1}$ and $\hat{\phi}_{k}=\xi_{k}+d \rho_{1}$, we find that

$$
\begin{gather*}
\pi_{1}=m^{1 / 2} \sum_{j=1}^{m-1} \hat{\theta}_{j}, \quad\left|\pi_{1}\right| \leq m^{1 / 2} n^{-1 / 2+\varepsilon}, \\
\rho_{1}=n^{1 / 2} \sum_{k=1}^{n-1} \hat{\phi}_{k}, \quad\left|\rho_{1}\right| \leq n^{1 / 2} m^{-1 / 2+\varepsilon}, \tag{24}
\end{gather*}
$$

where the inequalities come from the bounds on $\hat{\theta}_{m}$ and $\hat{\phi}_{n}$. This implies that

$$
\begin{aligned}
\zeta_{j} & =\hat{\theta}_{j}+\widetilde{O}\left(n^{-1}\right) \quad(1 \leq j \leq m-1) \\
\xi_{k} & =\hat{\phi}_{k}+\widetilde{O}\left(n^{-1}\right) \quad(1 \leq k \leq n-1)
\end{aligned}
$$

The transformed region of integration is $T_{2}^{-1}(\mathcal{S})$, but for convenience we will expand it a little to be the region defined by the inequalities

$$
\begin{align*}
\left|\zeta_{j}\right| & \leq \frac{3}{2} n^{-1 / 2+\varepsilon} \quad(1 \leq j \leq m-1) \\
\left|\xi_{k}\right| & \leq \frac{3}{2} m^{-1 / 2+\varepsilon} \quad(1 \leq k \leq n-1) \\
\left|\pi_{1}\right| & \leq m^{1 / 2} n^{-1 / 2+\varepsilon}  \tag{25}\\
\left|\rho_{1}\right| & \leq n^{1 / 2} m^{-1 / 2+\varepsilon} \\
|\nu| & \leq(m n)^{-1 / 2+2 \varepsilon}
\end{align*}
$$

We now consider the new integrand $E_{1}=\exp \left(L_{1}\right)=G \circ T_{2}$. As in [5], the semiregular parts of $L_{1}$ (those not involving $\alpha_{j k}, \beta_{j k}, \gamma_{j k}$ or $H$ ) transform to

$$
\begin{align*}
& -A m n \nu^{2}-A n \pi_{2}-A m \rho_{2}-3 i A_{3} n \nu \pi_{2}-3 i A_{3} m \nu \rho_{2}+6 A_{4} \pi_{2} \rho_{2}  \tag{26}\\
& \quad-i A_{3} n \pi_{3}-i A_{3} n \rho_{3}-3 i A_{3} c n \pi_{1} \pi_{2}-3 i A_{3} d m \rho_{1} \rho_{2}+A_{4} n \pi_{4}+A_{4} m \rho_{4}+\widetilde{O}\left(n^{-1 / 2}\right)
\end{align*}
$$

To see the effect of the transformation on the irregular parts of the integrand, write $\zeta_{m}=\hat{\theta}_{m}-c \pi_{1}$ and $\xi_{n}=\hat{\theta}_{n}-d \rho_{1}$. From (24) we can see that $\zeta_{m}=\widetilde{O}\left(n^{-1 / 2}\right)$ and $\xi_{n}=\widetilde{O}\left(n^{-1 / 2}\right)$. Thus we have, for all $1 \leq j \leq m$ and $1 \leq k \leq n, \zeta_{j}+\xi_{k}=\widetilde{O}\left(n^{-1 / 2}\right)$ and $c \pi_{1}+d \rho_{1}+\nu=\widetilde{O}\left(n^{-1}\right)$. Recalling also that $\alpha_{j k}, \beta_{j k}, \gamma_{j k}=\widetilde{O}\left(n^{-1 / 2}\right)$, we have

$$
\begin{aligned}
& \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_{j k}\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{2} \\
& \quad=\sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_{j k}\left(\left(\zeta_{j}+\xi_{k}\right)^{2}+2\left(\zeta_{j}+\xi_{k}\right)\left(\nu+c \pi_{1}+d \rho_{1}\right)\right)+\widetilde{O}\left(n^{-1 / 2}\right), \\
& \sum_{j=1}^{m} \sum_{k=1}^{n} \beta_{j k}\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{3}=\sum_{j=1}^{m} \sum_{k=1}^{n} \beta_{j k}\left(\zeta_{j}+\xi_{k}\right)^{3}+\widetilde{O}\left(n^{-1 / 2}\right), \\
& \sum_{j=1}^{m} \sum_{k=1}^{n} \gamma_{j k}\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{4}=\widetilde{O}\left(n^{-1 / 2}\right), \\
& \sum_{j k \in H}\left(\nu+\hat{\theta}_{j}+\hat{\phi}_{k}\right)^{2}=\sum_{j k \in H}\left(\zeta_{j}+\xi_{k}\right)^{2}+\widetilde{O}\left(n^{-1 / 2}\right)
\end{aligned}
$$

Moreover, the terms on the right sides of the above that involve $\zeta_{m}$ or $\xi_{n}$ contribute only $\widetilde{O}\left(n^{-1 / 2}\right)$ in total, so we can drop them. Combining this with (26), we have

$$
\begin{align*}
L_{1}= & -A m n \nu^{2}-A n \pi_{2}-A m \rho_{2}-3 i A_{3} n \nu \pi_{2}-3 i A_{3} m \nu \rho_{2}+6 A_{4} \pi_{2} \rho_{2} \\
& -i A_{3} n \pi_{3}-i A_{3} n \rho_{3}-3 i A_{3} c n \pi_{1} \pi_{2}-3 i A_{3} d m \rho_{1} \rho_{2}+A_{4} n \pi_{4}+A_{4} m \rho_{4} \\
& -\sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \alpha_{j k}\left(\left(\zeta_{j}+\xi_{k}\right)^{2}+2\left(\zeta_{j}+\xi_{k}\right)\left(\nu+c \pi_{1}+d \rho_{1}\right)\right)  \tag{27}\\
& -i \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \beta_{j k}\left(\zeta_{j}+\xi_{k}\right)^{3}+A \sum_{j k \in H}\left(\zeta_{j}+\xi_{k}\right)^{2}+\widetilde{O}\left(n^{-1 / 2}\right) .
\end{align*}
$$

### 4.5 Completing the diagonalization

The quadratic form in $L_{1}$ is the following function of the $m+n-1$ variables $\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu$ :

$$
\begin{align*}
Q= & -A m n \nu^{2}-A n \pi_{2}-A m \rho_{2}+A \sum_{j k \in H}\left(\zeta_{j}+\xi_{k}\right)^{2} \\
& -\sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \alpha_{j k}\left(\left(\zeta_{j}+\xi_{k}\right)^{2}+2\left(\zeta_{j}+\xi_{k}\right)\left(\nu+c \pi_{1}+d \rho_{1}\right)\right) . \tag{28}
\end{align*}
$$

We will make a third change of variables, $(\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu)=T_{3}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)$, that diagonalizes this quadratic form, where $\boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{m-1}\right)$ and $\boldsymbol{\tau}=\left(\tau_{1}, \ldots, \tau_{n-1}\right)$. This is achieved using a slight extension of [16, Lemma 3.2].

Lemma 4.1. Let $\boldsymbol{U}$ and $\boldsymbol{Y}$ be square matrices of the same order, such that $\boldsymbol{U}^{-1}$ exists and all the eigenvalues of $\boldsymbol{U}^{-1} \boldsymbol{Y}$ are less than 1 in absolute value. Then

$$
\left(\boldsymbol{I}+\boldsymbol{Y} \boldsymbol{U}^{-1}\right)^{-1 / 2}(\boldsymbol{U}+\boldsymbol{Y})\left(\boldsymbol{I}+\boldsymbol{U}^{-1} \boldsymbol{Y}\right)^{-1 / 2}=\boldsymbol{U}
$$

where the fractional powers are defined by the binomial expansion.
Note that $\boldsymbol{U}^{-1} \boldsymbol{Y}$ and $\boldsymbol{Y} \boldsymbol{U}^{-1}$ have the same eigenvalues, so the eigenvalue condition on $\boldsymbol{U}^{-1} \boldsymbol{Y}$ applies equally to $\boldsymbol{Y} \boldsymbol{U}^{-1}$. If we also have that both $\boldsymbol{U}$ and $\boldsymbol{Y}$ are symmetric, then $\left(\boldsymbol{I}+\boldsymbol{Y} \boldsymbol{U}^{-1}\right)^{-1 / 2}$ is the transpose of $\left(\boldsymbol{I}+\boldsymbol{U}^{-1} \boldsymbol{Y}\right)^{-1 / 2}$, as proved in [4]. Let $\boldsymbol{V}$ be the symmetric matrix associated with the quadratic form $Q$. Write $\boldsymbol{V}=\boldsymbol{V}_{\mathrm{d}}+\boldsymbol{V}_{\text {nd }}$ where $\boldsymbol{V}_{\mathrm{d}}$ has all off-diagonal entries equal to zero and matches $V$ on the diagonal entries, and $\boldsymbol{V}_{\text {nd }}$ has all diagonal entries zero and matches $\boldsymbol{V}$ on the off-diagonal entries. We will apply Lemma 4.1 with $\boldsymbol{U}=\boldsymbol{V}_{\mathrm{d}}$ and $\boldsymbol{Y}=\boldsymbol{V}_{\mathrm{nd}}$. Note that $\boldsymbol{V}_{\mathrm{d}}$ is invertible and that both $\boldsymbol{V}_{\mathrm{d}}$ and $\boldsymbol{V}_{\text {nd }}$ are symmetric. Let $T_{3}$ be the transformation given by $T_{3}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)^{T}=(\boldsymbol{\zeta}, \boldsymbol{\xi}, \nu)^{T}=$ $\left(\boldsymbol{I}+\boldsymbol{V}_{\mathrm{d}}^{-1} \boldsymbol{V}_{\mathrm{nd}}\right)^{-1 / 2}(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)^{T}$. If the eigenvalue condition of Lemma 4.1 is satisfied then this transformation diagonalizes the quadratic form $Q$, keeping the diagonal entries unchanged.

From the formula for $Q$ we extract the following coefficients, which tell us the diagonal and off-diagonal entries of $\boldsymbol{V}$. Define $x_{j}^{\prime}=x_{j}-h_{j n}$ for $1 \leq j \leq m-1$, and $y_{k}^{\prime}=y_{k}-h_{m k}$ for $1 \leq k \leq n-1$. Then:

$$
\begin{aligned}
{\left[\zeta_{j}^{2}\right] Q } & =-A n-(1+2 c) \alpha_{j *}+A x_{j}^{\prime} \\
{\left[\xi_{k}^{2}\right] Q } & =-A m-(1+2 d) \alpha_{* k}+A y_{k}^{\prime} \\
{\left[\nu^{2}\right] Q } & =-A m n
\end{aligned}
$$

$$
\begin{aligned}
{\left[\zeta_{j_{1}} \zeta_{j_{2}}\right] Q } & =-2 c\left(\alpha_{j_{1} *}+\alpha_{j_{2} *}\right) \\
{\left[\zeta_{j} \xi_{k}\right] Q } & =-2 \alpha_{j k}-2 d \alpha_{j *}-2 c \alpha_{* k}+2 A h_{j k}, \\
{\left[\xi_{k_{1}} \xi_{k_{2}}\right] Q } & =-2 d\left(\alpha_{* k_{1}}+\alpha_{* k_{2}}\right) \\
{\left[\zeta_{j} \nu\right] Q } & =-2 \alpha_{j *}, \\
{\left[\xi_{k} \nu\right] Q } & =-2 \alpha_{* k}
\end{aligned}
$$

Using these equations we find that all off-diagonal entries of $\boldsymbol{V}_{\mathrm{d}}^{-1} \boldsymbol{V}_{\text {nd }}$ are $\widetilde{O}\left(n^{-3 / 2}\right)$, except for the column corresponding to $\nu$, which has off-diagonal entries of size $\widetilde{O}\left(n^{-1 / 2}\right)$, and the entries corresponding to $\zeta_{j} \xi_{k}$ for $h_{j k}=1$, which have size $\widetilde{O}\left(n^{-1}\right)$. Similarly, the off-diagonal entries of $\boldsymbol{V}_{\mathrm{nd}} \boldsymbol{V}_{\mathrm{d}}^{-1}$ are all $\widetilde{O}\left(n^{-3 / 2}\right)$, except for the row corresponding to $\nu$, which has off-diagonal entries of size $\widetilde{O}\left(n^{-1 / 2}\right)$, and the entries corresponding to $\zeta_{j} \xi_{k}$ for $h_{j k}=1$, which have size $\widetilde{O}\left(n^{-1}\right)$. To see that these conditions imply that the eigenvalues of $\boldsymbol{V}_{\mathrm{d}}^{-1} \boldsymbol{V}_{\mathrm{nd}}$ are less than one, recall that the value of any matrix norm is greater than or equal to the greatest absolute value of an eigenvalue. The $\infty$-norm (maximum row sum of absolute values) of $\boldsymbol{V}_{\mathrm{d}}^{-1} \boldsymbol{V}_{\mathrm{nd}}$ is $\widetilde{O}\left(n^{-1 / 2}\right)$, so the eigenvalues are all $\widetilde{O}\left(n^{-1 / 2}\right)$.

We also need to know the Jacobian of the transformation $T_{3}$.
Lemma 4.2 ([4]). Let $\boldsymbol{M}$ be a matrix of order $O(m+n)$ with all eigenvalues uniformly $\widetilde{O}\left(n^{-1 / 2}\right)$. Then

$$
\operatorname{det}(\boldsymbol{I}+\boldsymbol{M})=\exp \left(\operatorname{tr} \boldsymbol{M}-\frac{1}{2} \operatorname{tr} \boldsymbol{M}^{2}+\widetilde{O}\left(n^{-1 / 2}\right)\right)
$$

Let $\boldsymbol{M}=\boldsymbol{V}_{\mathrm{d}}^{-1} \boldsymbol{V}_{\mathrm{nd}}$. As noted before, the eigenvalues of $\boldsymbol{M}$ are all $\widetilde{O}\left(n^{-1 / 2}\right)$ so Lemma 4.2 applies. Noting that $\operatorname{tr}(\boldsymbol{M})=0$ and calculating that $\operatorname{tr}\left(\boldsymbol{M}^{2}\right)=\widetilde{O}\left(n^{-1}\right)$, we conclude that the Jacobian of $T_{3}$ is

$$
\operatorname{det}\left((\boldsymbol{I}+\boldsymbol{M})^{-1 / 2}\right)=(\operatorname{det}(\boldsymbol{I}+\boldsymbol{M}))^{-1 / 2}=1+\widetilde{O}\left(n^{-1 / 2}\right)
$$

To derive $T_{3}$ explicitly, we can expand $\left(\boldsymbol{I}+\boldsymbol{V}_{\mathrm{d}}^{-1} \boldsymbol{V}_{\mathrm{nd}}\right)^{-1 / 2}$ while noting that $\alpha_{j *}=$ $O\left(n^{1 / 2+\varepsilon}\right)$ for all $j, \alpha_{* k}=O\left(m^{1 / 2+\varepsilon}\right)$ for all $k, \alpha_{* *}=O\left(m n^{2 \varepsilon}+n m^{2 \varepsilon}\right), R \leq m n^{1+2 \varepsilon}$ and $C \leq n m^{1+2 \varepsilon}$.

This gives

$$
\begin{aligned}
\sigma_{j}=\zeta_{j} & +\sum_{j^{\prime}=1}^{m-1}\left(\frac{c\left(\alpha_{j *}+\alpha_{j^{\prime} *}\right)}{2 A n}+\widetilde{O}\left(n^{-2}\right)\right) \zeta_{j^{\prime}} \\
& +\sum_{k=1}^{n-1}\left(\frac{\alpha_{j k}+d \alpha_{j *}+c \alpha_{* k}}{2 A n}+\widetilde{O}\left(n^{-2}\right)\right) \xi_{k}+\left(\frac{\alpha_{j *}}{2 A n}+\widetilde{O}\left(n^{-1}\right)\right) \nu+\widetilde{O}\left(n^{-3 / 2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\tau_{k}= & \xi_{k}+\sum_{j=1}^{m-1}\left(\frac{\alpha_{j k}+d \alpha_{j *}+c \alpha_{* k}}{2 A m}+\widetilde{O}\left(n^{-2}\right)\right) \zeta_{j} \\
& +\sum_{k^{\prime}=1}^{n-1}\left(\frac{d\left(\alpha_{* k}+\alpha_{* k^{\prime}}\right)}{2 A m}+\widetilde{O}\left(n^{-2}\right)\right) \xi_{k^{\prime}}+\left(\frac{\alpha_{* k}}{2 A m}+\widetilde{O}\left(n^{-1}\right)\right) \nu+\widetilde{O}\left(n^{-3 / 2}\right), \\
\mu= & +\sum_{j=1}^{m-1}\left(\frac{\alpha_{j *}}{2 A m n}+\widetilde{O}\left(n^{-2}\right)\right) \zeta_{j}+\sum_{k=1}^{n-1}\left(\frac{\alpha_{* k}}{2 A m n}+\widetilde{O}\left(n^{-2}\right)\right) \xi_{k}+\widetilde{O}\left(n^{-1}\right) \nu
\end{aligned}
$$

for $1 \leq j \leq m-1,1 \leq k \leq n-1$.
The transformation $T_{3}^{-1}$ perturbs the region of integration in an irregular fashion that we must bound. From the explicit form of $T_{3}$ above, we have

$$
\begin{aligned}
& \sigma_{j}=\zeta_{j}+\sum_{j^{\prime}=1}^{m-1} \widetilde{O}\left(n^{-3 / 2}\right) \zeta_{j^{\prime}}+\sum_{k=1}^{n-1} \widetilde{O}\left(n^{-3 / 2}\right) \xi_{k}+\widetilde{O}\left(n^{-1 / 2}\right) \nu+\widetilde{O}\left(n^{-3 / 2}\right)=\zeta_{j}+\widetilde{O}\left(n^{-1}\right) \\
& \tau_{k}=\xi_{k}+\sum_{j=1}^{m-1} \widetilde{O}\left(n^{-3 / 2}\right) \zeta_{j}+\sum_{k^{\prime}=1}^{n-1} \widetilde{O}\left(n^{-3 / 2}\right) \xi_{k^{\prime}}+\widetilde{O}\left(n^{-1 / 2}\right) \nu+\widetilde{O}\left(n^{-3 / 2}\right)=\xi_{k}+\widetilde{O}\left(n^{-1}\right)
\end{aligned}
$$

for $1 \leq j \leq m-1,1 \leq k \leq n-1$, so $\boldsymbol{\sigma}, \boldsymbol{\tau}$ are only slightly different from $\boldsymbol{\zeta}, \boldsymbol{\xi}$.
For $\mu$ versus $\nu$ we have

$$
\begin{aligned}
\mu & =\nu+O\left(n^{-1+2 \varepsilon} / A\right)+O\left(m^{-1+2 \varepsilon} / A\right) \\
& =\nu+o\left((m n)^{-1 / 2+2 \varepsilon}\right)
\end{aligned}
$$

where the second step requires (3). This shows that the bound $|\nu| \leq(m n)^{-1 / 2+2 \varepsilon}$ is adequately covered by $|\mu| \leq 2(m n)^{-1 / 2+2 \varepsilon}$.

For $1 \leq h \leq 4$, define

$$
\mu_{h}=\sum_{j=1}^{m-1} \sigma_{j}^{h}, \quad \nu_{h}=\sum_{k=1}^{n-1} \tau_{k}^{h} .
$$

From (25), we see that $\left|\pi_{1}\right| \leq m^{1 / 2} n^{-1 / 2+\varepsilon}$ and $\left|\rho_{1}\right| \leq m^{-1 / 2+\varepsilon} n^{1 / 2}$ are the remaining constraints that define the region of integration. We next apply these constraints to bound $\mu_{1}$ and $\nu_{1}$. From the explicit form of $T_{3}$, we have

$$
\begin{aligned}
\mu_{1}=\pi_{1} & +\sum_{j=1}^{m-1} \sum_{j^{\prime}=1}^{m-1}\left(\frac{c\left(\alpha_{j *}+\alpha_{j^{\prime} *}\right)}{2 A n}+\widetilde{O}\left(n^{-2}\right)\right) \zeta_{j^{\prime}} \\
& +\sum_{j=1}^{m-1} \sum_{k=1}^{n-1}\left(\frac{\alpha_{j k}+d \alpha_{j *}+c \alpha_{* k}}{2 A n}+\widetilde{O}\left(n^{-2}\right)\right) \xi_{k}+\sum_{j=1}^{m-1}\left(\frac{\alpha_{j *}}{2 A n}+\widetilde{O}\left(n^{-1}\right)\right) \nu+\widetilde{O}\left(n^{-1 / 2}\right)
\end{aligned}
$$

$$
\begin{align*}
= & \pi_{1}+\frac{c \alpha_{* *}}{2 A n} m^{1 / 2} n^{-1 / 2+\varepsilon}+\frac{d \alpha_{* *}}{2 A n} m^{-1 / 2+\varepsilon} n^{1 / 2}+\frac{\alpha_{* *}}{2 A n} \nu \\
& +(1+c(m-1)) \sum_{k=1}^{n-1} \frac{\alpha_{* k}}{2 A n} \xi_{k}+\frac{c(m-1)}{2 A n} \sum_{j^{\prime}=1}^{m-1} \alpha_{j^{\prime} *} \zeta_{j^{\prime}}+\widetilde{O}\left(n^{-1 / 2}\right) \\
= & \pi_{1}+\frac{c(m-1)}{2 A n} \sum_{j^{\prime}=1}^{m-1} \alpha_{j^{\prime} *} \zeta_{j^{\prime}}+\widetilde{O}\left(n^{-1 / 2}\right)  \tag{29}\\
= & \pi_{1}+O\left(A^{-1} m n^{-1+2 \varepsilon}\right) \\
= & \pi_{1}+o\left(m^{1 / 2} n^{-1 / 2+5 \varepsilon / 2}\right) .
\end{align*}
$$

To derive the above we have used $1+c(m-1)=m^{1 / 2}$ and the bounds we have established on the various variables. For the last step, we need (3), which implies that $A^{-1} m n^{-1+2 \varepsilon}=$ $o\left(m^{1 / 2} n^{-1 / 2+5 \varepsilon / 2}\right)$.

Since our region of integration has $\left|\pi_{1}\right| \leq m^{1 / 2} n^{-1 / 2+\varepsilon}$, we see that this implies the bound $\left|\mu_{1}\right| \leq m^{1 / 2} n^{-1 / 2+3 \varepsilon}$. By a parallel argument, we have

$$
\nu_{1}=\rho_{1}+o\left(m^{-1 / 2+5 \varepsilon / 2} n^{1 / 2}\right),
$$

which implies $\left|\nu_{1}\right| \leq n^{1 / 2} m^{-1 / 2+3 \varepsilon}$. Putting together all the bounds we have derived, we see that

$$
T_{3}^{-1}\left(T_{2}^{-1}(\mathcal{S})\right) \subseteq \mathcal{Q} \cap \mathcal{M}
$$

where

$$
\begin{aligned}
& \mathcal{Q}=\left\{\left|\sigma_{j}\right| \leq 2 n^{-1 / 2+\varepsilon}, j=1, \ldots, m-1\right\} \cap\left\{\left|\tau_{k}\right| \leq 2 m^{-1 / 2+\varepsilon}, k=1, \ldots, n-1\right\} \\
& \quad \cap\left\{|\mu| \leq 2(m n)^{-1 / 2+2 \varepsilon}\right\}, \\
& \mathcal{M}=\left\{\left|\mu_{1}\right| \leq m^{1 / 2} n^{-1 / 2+3 \varepsilon}\right\} \cap\left\{\left|\nu_{1}\right| \leq n^{1 / 2} m^{-1 / 2+3 \varepsilon}\right\} .
\end{aligned}
$$

Now define

$$
\begin{align*}
\mathcal{S}^{\prime} & =T_{2}\left(T_{3}(\mathcal{Q} \cap \mathcal{M})\right), \\
\mathcal{R}^{\prime} & =T_{1}^{*}\left(\mathcal{S}^{\prime}\right) \tag{30}
\end{align*}
$$

We have proved that $\mathcal{S}^{\prime} \supseteq \mathcal{S}$. Also notice that $\mathcal{R}^{\prime}$ is contained in the region defined by the inequalities (21). As we forecast at that time, our estimates of the integrand have been valid inside this expanded region. It remains to apply the transformation $T_{3}^{-1}$ to the integrand (27) so that we have it in terms of $(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu)$. The explicit form of $T_{3}^{-1}$ is similar to the explicit form for $T_{3}$, namely:

$$
\begin{aligned}
\zeta_{j}=\sigma_{j}-\sum_{j^{\prime}=1}^{m-1} & \left(\frac{c\left(\alpha_{j *}+\alpha_{j^{\prime} *}\right)}{2 A n}+\widetilde{O}\left(n^{-2}\right)\right) \sigma_{j^{\prime}}-\sum_{k=1}^{n-1}\left(\frac{\alpha_{j k}+d \alpha_{j *}+c \alpha_{* k}}{2 A n}+\widetilde{O}\left(n^{-2}\right)\right) \tau_{k} \\
& -\left(\frac{\alpha_{j *}}{2 A n}+\widetilde{O}\left(n^{-1}\right)\right) \mu+\widetilde{O}\left(n^{-3 / 2}\right)
\end{aligned}
$$

$$
\begin{gathered}
\xi_{k}=\tau_{k}-\sum_{j=1}^{m-1}\left(\frac{\alpha_{j k}-d \alpha_{j *}+c \alpha_{* k}}{2 A m}+\widetilde{O}\left(n^{-2}\right)\right) \sigma_{j}-\sum_{k^{\prime}=1}^{n-1}\left(\frac{d\left(\alpha_{* k}+\alpha_{* k^{\prime}}\right)}{2 A m}+\widetilde{O}\left(n^{-2}\right)\right) \tau_{k^{\prime}} \\
\quad-\left(\frac{\alpha_{* k}}{2 A m}+\widetilde{O}\left(n^{-1}\right)\right) \mu+\widetilde{O}\left(n^{-3 / 2}\right), \\
\nu=\mu-\sum_{j=1}^{m-1}\left(\frac{\alpha_{j *}}{2 A m n}+\widetilde{O}\left(n^{-2}\right)\right) \sigma_{j}-\sum_{k=1}^{n-1}\left(\frac{\alpha_{* k}}{2 A m n}+\widetilde{O}\left(n^{-2}\right)\right) \tau_{k}+\widetilde{O}\left(n^{-1}\right) \mu
\end{gathered}
$$

for $1 \leq j \leq m-1,1 \leq k \leq n-1$. In addition to the relationships between the old and new variables that we proved before, we can note that $\pi_{2}=\mu_{2}+\widetilde{O}\left(n^{-1 / 2}\right), \rho_{2}=\nu_{2}+\widetilde{O}\left(n^{-1 / 2}\right)$, $\pi_{3}=\mu_{3}+\widetilde{O}\left(n^{-1}\right), \rho_{3}=\nu_{3}+\widetilde{O}\left(n^{-1}\right), \pi_{4}=\mu_{4}+\widetilde{O}\left(n^{-3 / 2}\right)$, and $\rho_{4}=\nu_{4}+\widetilde{O}\left(n^{-3 / 2}\right)$.

The quadratic part of $L_{1}$, which we called $Q$ in (28), loses its off-diagonal parts according to our design of $T_{3}$. Thus, what remains is

$$
\begin{aligned}
-A m n \mu^{2}- & \sum_{j=1}^{m-1}\left(A n+(1+2 c) \alpha_{j *}-A x_{j}^{\prime}\right) \sigma_{j}^{2}-\sum_{k=1}^{n-1}\left(A m+(1+2 d) \alpha_{* k}-A y_{k}^{\prime}\right) \tau_{k}^{2} \\
= & -A m n \mu^{2}-A n \mu_{2}-A m \nu_{2} \\
& \quad-\sum_{j=1}^{m-1}\left(\alpha_{j *}-A x_{j}^{\prime}\right) \sigma_{j}^{2}-\sum_{k=1}^{n-1}\left(\alpha_{* k}-A y_{k}^{\prime}\right) \tau_{k}^{2}+\widetilde{O}\left(n^{-1 / 2}\right) .
\end{aligned}
$$

Next consider the cubic terms of $L_{1}$. These are

$$
\begin{aligned}
& -3 i A_{3} n \nu \pi_{2}-3 i A_{3} m \nu \rho_{2}-i A_{3} n \pi_{3}-i A_{3} n \rho_{3} \\
& \quad-3 i A_{3} c n \pi_{1} \pi_{2}-3 i A_{3} d n \rho_{1} \rho_{2}-i \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \beta_{j k}\left(\zeta_{j}+\xi_{k}\right)^{3}
\end{aligned}
$$

We calculate the following in $\mathcal{Q} \cap \mathcal{M}$ :

$$
\begin{align*}
& -3 i A_{3} n \nu \pi_{2}=-3 i A_{3} n \mu \mu_{2}+\frac{3 i A_{3} \mu_{2}}{2 A m}\left(\sum_{j=1}^{m-1} \alpha_{j *} \sigma_{j}+\sum_{k=1}^{n-1} \alpha_{* k} \tau_{k}\right)+\widetilde{O}\left(n^{-1 / 2}\right), \\
& -i A_{3} n \pi_{3}=-i A_{3} n \mu_{3}+\frac{3 i A_{3}}{2 A}\left(\sum_{j, j^{\prime}=1}^{m-1} c\left(\alpha_{j *}+\alpha_{j^{\prime} *}\right) \sigma_{j}^{2} \sigma_{j^{\prime}},\right. \\
& \\
& \left.\quad+\sum_{j=1}^{m-1} \sum_{k=1}^{n-1}\left(\alpha_{j k}+d \alpha_{j *}+c \alpha_{* k}\right) \sigma_{j}^{2} \tau_{k}\right)+\widetilde{O}\left(n^{-1 / 2}\right),  \tag{31}\\
& -3 i A_{3} c n \pi_{1} \pi_{2}=-3 i A_{3} c n \mu_{1} \mu_{2}+\frac{3 i A_{3} c^{2} m \mu_{2}}{2 A} \sum_{j=1}^{m-1} \alpha_{j *} \sigma_{j}+\widetilde{O}\left(n^{-1 / 2}\right), \\
& -i \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \beta_{j k}\left(\zeta_{j}+\xi_{k}\right)^{3}=-i \sum_{j=1}^{m-1} \sum_{k=1}^{n-1} \beta_{j k}\left(\sigma_{j}+\tau_{k}\right)^{3}+\widetilde{O}\left(n^{-1 / 2}\right),
\end{align*}
$$

and the remaining cubic terms are each parallel to one of those. The proof of (31) is similar to the proof of (29).

Finally we come to the quartic part of $L_{1}$, which is

$$
6 A_{4} \pi_{2} \rho_{2}+A_{4} n \pi_{4}+A_{4} m \rho_{4}=6 A_{4} \mu_{2} \nu_{2}+A_{4} n \mu_{4}+A_{4} m \nu_{4}+\widetilde{O}\left(n^{-1 / 2}\right)
$$

In summary, the value of the integrand for $(\boldsymbol{\sigma}, \boldsymbol{\tau}, \mu) \in \mathcal{Q} \cap \mathcal{M}$ is $\exp \left(L_{2}+\widetilde{O}\left(n^{-1 / 2}\right)\right)$, where

$$
\begin{aligned}
L_{2}= & -A m n \mu^{2}-A n \mu_{2}-A m \nu_{2}-\sum_{j=1}^{m-1}\left(\alpha_{j *}-A x_{j}^{\prime}\right) \sigma_{j}^{2}-\sum_{k=1}^{n-1}\left(\alpha_{* k}-A y_{k}^{\prime}\right) \tau_{k}^{2}+6 A_{4} \mu_{2} \nu_{2} \\
& +A_{4} n \mu_{4}+A_{4} m \nu_{4}-i A_{3} n \mu_{3}-i A_{3} m \nu_{3}-3 i A_{3} c n \mu_{1} \mu_{2}-3 i A_{3} d m \nu_{1} \nu_{2} \\
& -3 i A_{3} n \mu \mu_{2}-3 i A_{3} m \mu \nu_{2}-i \sum_{j=1}^{m-1} \beta_{j *} \sigma_{j}^{3}-i \sum_{k=1}^{n-1} \beta_{* k} \tau_{k}^{3} \\
& +i \sum_{j, j^{\prime}=1}^{m-1} g_{j j^{\prime}} \sigma_{j} \sigma_{j^{\prime}}^{2}+i \sum_{k, k^{\prime}=1}^{n-1} h_{k k^{\prime}} \tau_{k} \tau_{k^{\prime}}^{2}+i \sum_{j=1}^{m-1} \sum_{k=1}^{n-1}\left(u_{j k} \sigma_{j} \tau_{k}^{2}+v_{j k} \sigma_{j}^{2} \tau_{k}\right),
\end{aligned}
$$

with

$$
\begin{aligned}
g_{j j^{\prime}} & =\frac{3 A_{3}}{2 A m}\left(\left(1+c m+c^{2} m^{2}\right) \alpha_{j *}+c m \alpha_{j^{\prime} *}\right)=O\left(n^{-1 / 2+\varepsilon}\right) \\
h_{k k^{\prime}} & =\frac{3 A_{3}}{2 A n}\left(\left(1+d n+d^{2} n^{2}\right) \alpha_{* k}+d n \alpha_{* k^{\prime}}\right)=O\left(m^{-1 / 2+\varepsilon}\right) \\
u_{j k} & =\frac{3 A_{3}}{2 A n}\left(n \alpha_{j k}+(1+d n) \alpha_{j *}+c n \alpha_{* k}\right)-3 \beta_{j k}=O\left(m^{-1 / 2+2 \varepsilon}+n^{-1 / 2+2 \varepsilon}\right) \\
v_{j k} & =\frac{3 A_{3}}{2 A m}\left(m \alpha_{j k}+(1+c m) \alpha_{* k}+d m \alpha_{j *}\right)-3 \beta_{j k}=O\left(m^{-1 / 2+2 \varepsilon}+n^{-1 / 2+2 \varepsilon}\right) .
\end{aligned}
$$

Note that the $O(\cdot)$ estimates in the last four lines are uniform over $j, j^{\prime}, k, k^{\prime}$.

### 4.6 Estimating the main part of the integral

Define $E_{2}=\exp \left(L_{2}\right)$. We have shown that the value of the integrand in $\mathcal{Q} \cap \mathcal{M}$ is $E_{1}=E_{2}\left(1+\widetilde{O}\left(n^{-1 / 2}\right)\right)$. Denote the complement of the region $\mathcal{M}$ by $\mathcal{M}^{c}$. We can approximate our integral as follows:

$$
\begin{align*}
\int_{\mathcal{Q} \cap \mathcal{M}} E_{1} & =\int_{\mathcal{Q} \cap \mathcal{M}} E_{2}+\widetilde{O}\left(n^{-1 / 2}\right) \int_{\mathcal{Q} \cap \mathcal{M}}\left|E_{2}\right| \\
& =\int_{\mathcal{Q} \cap \mathcal{M}} E_{2}+\widetilde{O}\left(n^{-1 / 2}\right) \int_{\mathcal{Q}}\left|E_{2}\right| \\
& =\int_{\mathcal{Q}} E_{2}+O(1) \int_{\mathcal{Q} \cap \mathcal{M}^{c}}\left|E_{2}\right|+\widetilde{O}\left(n^{-1 / 2}\right) \int_{\mathcal{Q}}\left|E_{2}\right| . \tag{32}
\end{align*}
$$

It suffices to estimate the value of each integral in (32). This can be done using the same calculation as in Section 4.3 of [4], using $\hat{\alpha}_{j k}=\alpha_{j k}-A h_{j k}$ in place of the variable $\alpha_{j k}$ used in that paper. A potential problem with this analogy is that the variable $\alpha_{j k}$ used in [4] has the property $\alpha_{j k}=\widetilde{O}\left(n^{-1 / 2}\right)$, whereas it is not true that $\hat{\alpha}_{j k}=\widetilde{O}\left(n^{-1 / 2}\right)$. However, a careful look at Section 4.3 of [4] confirms that only the properties $\hat{\alpha}_{j *}=\alpha_{j *}-A x_{j}^{\prime}=$ $\widetilde{O}\left(n^{1 / 2}\right), \hat{\alpha}_{* k}=\alpha_{* k}-A y_{k}^{\prime}=\widetilde{O}\left(n^{1 / 2}\right)$, and the bounds on $g_{j j^{\prime}}, h_{k k^{\prime}}, u_{j k}, v_{j k}$, are required.

The result is that

$$
\begin{align*}
& \int_{\mathcal{Q}} E_{2}=\left(\frac{\pi}{A m n}\right)^{1 / 2}\left(\frac{\pi}{A n}\right)^{(m-1) / 2}\left(\frac{\pi}{A m}\right)^{(n-1) / 2} \\
& \times \exp \left(-\frac{9 A_{3}^{2}}{8 A^{3}}+\frac{3 A_{4}}{2 A^{2}}+\left(\frac{m}{n}+\frac{n}{m}\right)\left(\frac{3 A_{4}}{4 A^{2}}-\frac{15 A_{3}^{2}}{16 A^{3}}\right)\right. \\
&-\left(\frac{1}{2 A m}+\frac{1}{2 A n}\right) \hat{\alpha}_{* *}+\frac{1}{4 A^{2} m^{2}} \sum_{k=1}^{n-1}\left(\hat{\alpha}_{* k}\right)^{2}  \tag{33}\\
&\left.+\frac{1}{4 A^{2} n^{2}} \sum_{j=1}^{m-1}\left(\hat{\alpha}_{j *}\right)^{2}+\widetilde{O}\left(n^{-b}\right)\right),
\end{align*}
$$

where $b$ is specified in Theorem 2.1.
Using (13) and the conditions of Theorem [2.1, we calculate that

$$
\begin{aligned}
\hat{\alpha}_{* *} & =-\frac{1}{2}\left(\frac{R_{2}}{n}+\frac{C_{2}}{m}\right)-\frac{1}{2} \lambda^{2} X+\widetilde{O}\left(n^{1 / 2}\right), \\
\sum_{j=1}^{m-1}\left(\hat{\alpha}_{j *}\right)^{2} & =\frac{1}{4}(1-2 \lambda)^{2} R_{2}+\widetilde{O}\left(n^{3 / 2}\right), \\
\sum_{k=1}^{n-1}\left(\hat{\alpha}_{* k}\right)^{2} & =\frac{1}{4}(1-2 \lambda)^{2} C_{2}+\widetilde{O}\left(n^{3 / 2}\right) .
\end{aligned}
$$

Substituting these values into (33) together with the actual values of $A, A_{3}, A_{4}$, we conclude that

$$
\begin{align*}
\int_{\mathcal{Q}} E_{2}=( & \left.\frac{\pi}{A m n}\right)^{1 / 2}\left(\frac{\pi}{A n}\right)^{(m-1) / 2}\left(\frac{\pi}{A m}\right)^{(n-1) / 2} \\
& \times \exp \left(-\frac{1}{2}-\frac{1-2 A}{24 A}\left(\frac{m}{n}+\frac{n}{m}\right)+\frac{1-4 A}{16 A^{2}}\left(\frac{R_{2}}{n^{2}}+\frac{C_{2}}{m^{2}}\right)\right.  \tag{34}\\
& \left.+\frac{R_{2}+C_{2}}{4 A m n}+\frac{\lambda^{2} X}{4 A}\left(\frac{1}{m}+\frac{1}{n}\right)+O\left(n^{-b}\right)\right) .
\end{align*}
$$

By the same argument as in [4], the other two terms in (32) have value $O\left(n^{-b}\right) \int_{Q} E_{2}$. Multiplying (34) by the Jacobians of the transformations $T_{2}$ and $T_{3}$, we have proved the following.

Lemma 4.3. The region $\mathcal{S}^{\prime}$ given by (30) contains $\mathcal{S}$ and

$$
\begin{aligned}
& \int_{\mathcal{S}^{\prime}} G(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu) d \hat{\boldsymbol{\theta}} d \hat{\boldsymbol{\phi}} d \nu=(m n)^{-1 / 2}\left(\frac{\pi}{A m n}\right)^{1 / 2}\left(\frac{\pi}{A n}\right)^{(m-1) / 2}\left(\frac{\pi}{A m}\right)^{(n-1) / 2} \\
& \times \exp \left(-\frac{1}{2}-\frac{1-2 A}{24 A}\left(\frac{m}{n}+\frac{n}{m}\right)+\frac{1-4 A}{16 A^{2}}\left(\frac{R_{2}}{n^{2}}+\frac{C_{2}}{m^{2}}\right)\right. \\
&\left.+\frac{R_{2}+C_{2}}{4 A m n}+\frac{\lambda^{2} X}{4 A}\left(\frac{1}{m}+\frac{1}{n}\right)+O\left(n^{-b}\right)\right) .
\end{aligned}
$$

### 4.7 Bounding the remainder of the integral

In the previous subsection, we estimated the value of the integral $I_{\mathcal{R}^{\prime}}(\boldsymbol{s}, \boldsymbol{t}, H)$, which is the same as $I(\boldsymbol{s}, \boldsymbol{t}, H)$ except that it is restricted to a certain region $\mathcal{R}^{\prime} \supseteq \mathcal{R}$. In this subsection, we extend this to an estimate of $I(s, t, H)$ by showing that the remainder of the region of integration contributes negligibly.

For $1 \leq j \leq m, 1 \leq k \leq n$, let $A_{j k}=A+\alpha_{j k}=\frac{1}{2} \lambda_{j k}\left(1-\lambda_{j k}\right)$ (recall (23)), and define $A_{\text {min }}=\min _{j k} A_{j k}=A+\widetilde{O}\left(n^{-1 / 2}\right)$. We begin with two technical lemmas whose proofs are omitted, and a well-known bound of Hoeffding.

## Lemma 4.4.

$$
|F(\boldsymbol{\theta}, \boldsymbol{\phi})|=\prod_{j k \in \bar{H}} f_{j k}\left(\theta_{j}+\phi_{k}\right),
$$

where

$$
f_{j k}(z)=\sqrt{1-4 A_{j k}(1-\cos z)} .
$$

Moreover, for all real z,

$$
0 \leq f_{j k}(z) \leq \exp \left(-A_{j k} z^{2}+\frac{1}{12} A_{j k} z^{4}\right)
$$

Lemma 4.5. For all $c>0$,

$$
\int_{-8 \pi / 75}^{8 \pi / 75} \exp \left(c\left(-x^{2}+\frac{7}{3} x^{4}\right)\right) d x \leq \sqrt{\pi / c} \exp (3 / c)
$$

Lemma 4.6 ([10]). Let $X_{1}, \ldots, X_{N}$ be independent random variables such that $\mathbb{E} X_{i}=0$ and $\left|X_{i}\right| \leq M$ for all $i$. Then, for any $t \geq 0$,

$$
\operatorname{Prob}\left(\sum_{i=1}^{N} X_{i} \geq t\right) \leq \exp \left(-\frac{t^{2}}{2 N M^{2}}\right)
$$

Lemma 4.7. Let $F(\boldsymbol{\theta}, \boldsymbol{\phi})$ be the integrand of $I(\boldsymbol{s}, \boldsymbol{t}, H)$ as defined in (4). Then, under the conditions of Theorem 2.1,

$$
\int_{\mathcal{R}^{c}}|F(\boldsymbol{\theta}, \boldsymbol{\phi})| d \boldsymbol{\theta} d \boldsymbol{\phi}=O\left(n^{-1}\right) \int_{\mathcal{R}^{\prime}} F(\boldsymbol{\theta}, \boldsymbol{\phi}) d \boldsymbol{\theta} d \boldsymbol{\phi}
$$

where $\mathcal{R}^{c}$ denotes the complement of $\mathcal{R}$.
Proof. Our approach will be to bound $\int|F(\boldsymbol{\theta}, \boldsymbol{\phi})|$ over a variety of regions whose union covers $\mathcal{R}^{c}$. To make the comparison of these bounds with $\int_{\mathcal{R}^{\prime}} F(\boldsymbol{\theta}, \boldsymbol{\phi})$ easier, we note that

$$
\begin{equation*}
\int_{\mathcal{R}^{\prime}} F(\boldsymbol{\theta}, \boldsymbol{\phi}) d \boldsymbol{\theta} d \boldsymbol{\phi}=\exp \left(A^{-1} O\left(m^{\varepsilon}+n^{\varepsilon}\right)\right) I_{0}=\exp \left(O\left(m^{3 \varepsilon}+n^{3 \varepsilon}\right)\right) I_{1} \tag{35}
\end{equation*}
$$

where

$$
\begin{aligned}
& I_{0}=\left(\frac{\pi}{A_{\bullet \bullet}}\right)^{1 / 2} \prod_{j=1}^{m}\left(\frac{\pi}{A_{\bullet \bullet}}\right)^{1 / 2} \prod_{k=1}^{n}\left(\frac{\pi}{A_{\bullet k}}\right)^{1 / 2} \\
& I_{1}=\left(\frac{\pi}{A n}\right)^{m / 2}\left(\frac{\pi}{A m}\right)^{n / 2}
\end{aligned}
$$

To see this, expand

$$
A_{j \bullet}=A n+\alpha_{j \bullet}=A n \exp \left(\frac{\alpha_{j \bullet}}{A n}-\frac{\alpha_{j \bullet}^{2}}{2 A^{2} n^{2}}+\cdots\right),
$$

and similarly for $A_{\bullet \bullet}$, and compare the result to Lemma 4.3 using the assumptions of Theorem 2.1, It may help to recall the calculation following (33).

Take $\kappa=\pi / 300$ and define $w_{0}, w_{1}, \ldots, w_{299}$ by $w_{\ell}=2 \ell \kappa$. For any $\ell$, let $\mathcal{S}_{1}(\ell)$ be the set of $(\boldsymbol{\theta}, \boldsymbol{\phi})$ such that $\theta_{j} \in\left[w_{\ell}-\kappa, w_{\ell}+\kappa\right]$ for at least $\kappa m / \pi$ values of $j$ and $\phi_{k} \notin\left[-w_{\ell}-2 \kappa,-w_{\ell}+2 \kappa\right]$ for at least $n^{\varepsilon}$ values of $k$. For $(\boldsymbol{\theta}, \boldsymbol{\phi}) \in \mathcal{S}_{1}(\ell), \theta_{j}+\phi_{k} \notin[-\kappa, \kappa]$ for at least $\kappa\left(m-O\left(m^{\varepsilon}\right)\right) n^{\varepsilon} / \pi$ pairs $(j, k)$ with $h_{j k}=0$ so, by Lemma 4.4, $|F(\boldsymbol{\theta}, \boldsymbol{\phi})| \leq$ $\exp \left(-c_{1} A_{\min } m n^{\varepsilon}\right)$ for some $c_{1}>0$ which is independent of $\ell$.

Next define $\mathcal{S}_{2}(\ell)$ to be the set of $(\boldsymbol{\theta}, \boldsymbol{\phi})$ such that $\theta_{j} \in\left[w_{\ell}-\kappa, w_{\ell}+\kappa\right]$ for at least $\kappa m / \pi$ values of $j, \phi_{k} \in\left[-w_{\ell}-2 \kappa,-w_{\ell}+2 \kappa\right]$ for at least $n-n^{\varepsilon}$ values of $k$ and $\theta_{j} \notin\left[w_{\ell}-3 \kappa, w_{\ell}+3 \kappa\right]$ for at least $m^{\varepsilon}$ values of $j$. By the same argument with the roles of $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ reversed, $|F(\boldsymbol{\theta}, \boldsymbol{\phi})| \leq \exp \left(-c_{2} A_{\min } m^{\varepsilon} n\right)$ for some $c_{2}>0$ independent of $\ell$ when $(\boldsymbol{\theta}, \boldsymbol{\phi}) \in \mathcal{S}_{2}(\ell)$.

Now define $\mathcal{R}_{1}(\ell)$ to be the set of pairs $(\boldsymbol{\theta}, \boldsymbol{\phi})$ such that $\theta_{j} \in\left[w_{\ell}-3 \kappa, w_{\ell}+3 \kappa\right]$ for at least $m-m^{\varepsilon}$ values of $j$, and $\phi_{k} \in\left[-w_{\ell}-3 \kappa,-w_{\ell}+3 \kappa\right]$ for at least $n-n^{\varepsilon}$ values of $k$. By the pigeonhole principle, for any $\boldsymbol{\theta}$ there is some $\ell$ such that $\left[w_{\ell}-\kappa, w_{\ell}+\kappa\right]$ contains at least $\kappa m / \pi$ values of $\theta_{j}$. Therefore,

$$
\left(\bigcup_{\ell=0}^{299} \mathcal{R}_{1}(\ell)\right)^{c} \subseteq \bigcup_{\ell=0}^{299}\left(\mathcal{S}_{1}(\ell) \cup \mathcal{S}_{2}(\ell)\right)
$$

Since the total volume of $\left(\bigcup_{\ell} \mathcal{R}_{1}(\ell)\right)^{c}$ is at most $(2 m)^{m+n}$, we find that for some $c_{3}>0$,

$$
\begin{align*}
\int_{\left(\bigcup_{\ell} \mathcal{R}_{1}(\ell)\right)^{c}} & |F(\boldsymbol{\theta}, \boldsymbol{\phi})| d \boldsymbol{\theta} d \boldsymbol{\phi} \\
& \leq(2 \pi)^{m+n}\left(\exp \left(-c_{3} A_{\min } m n^{\varepsilon}\right)+\exp \left(-c_{3} A_{\min } m^{\varepsilon} n\right)\right) \\
& \leq e^{-n} I_{1} \tag{36}
\end{align*}
$$

We are left with $(\boldsymbol{\theta}, \boldsymbol{\phi}) \in \bigcup_{\ell} \mathcal{R}_{1}(\ell)$. If we subtract $w_{\ell}$ from each $\theta_{j}$ and add $w_{\ell}$ to each $\phi_{k}$ the integrand $F(\boldsymbol{\theta}, \boldsymbol{\phi})$ is unchanged, so we can assume for convenience that $\ell=0$ and that $(\boldsymbol{\theta}, \boldsymbol{\phi}) \in \mathcal{R}_{1}=\mathcal{R}_{1}(0)$. The bounds we obtain on parts of the integral we seek to reject will be at least $1 / 300$ of the total and thus be of the right order of magnitude. We will not mention this point again.

For a given $\boldsymbol{\theta}$, partition $\{1,2, \ldots, m\}$ into sets $J_{0}=J_{0}(\boldsymbol{\theta}), J_{1}=J_{1}(\boldsymbol{\theta})$ and $J_{2}=J_{2}(\boldsymbol{\theta})$, containing the indices $j$ such that $\left|\theta_{j}\right| \leq 3 \kappa, 3 \kappa<\left|\theta_{j}\right| \leq 15 \kappa$ and $\left|\theta_{j}\right|>15 \kappa$, respectively. Similarly partition $\{1,2, \ldots, n\}$ into $K_{0}=K_{0}(\boldsymbol{\phi}), K_{1}=K_{1}(\boldsymbol{\phi})$ and $K_{2}=K_{2}(\boldsymbol{\phi})$. The value of $|F(\boldsymbol{\theta}, \boldsymbol{\phi})|$ can now be bounded using

$$
\begin{aligned}
& f_{j k}\left(\theta_{j}+\phi_{k}\right) \\
& \quad \leq \begin{cases}\exp \left(-A_{\min }\left(\theta_{j}+\phi_{k}\right)^{2}+\frac{1}{12} A_{\min }\left(\theta_{j}+\phi_{k}\right)^{4}\right) & \text { if }(j, k) \in\left(J_{0} \cup J_{1}\right) \times\left(K_{0} \cup K_{1}\right), \\
\sqrt{1-4 A_{\min }(1-\cos (12 \kappa))} \leq e^{-A_{\min } / 64} & \text { if }(j, k) \in\left(J_{0} \times K_{2}\right) \cup\left(J_{2} \times K_{0}\right), \\
1 & \text { otherwise }\end{cases}
\end{aligned}
$$

Let $I_{2}\left(m_{2}, n_{2}\right)$ be the contribution to $\int_{\mathcal{R}_{1}}|F(\boldsymbol{\theta}, \boldsymbol{\phi})|$ of those $(\boldsymbol{\theta}, \boldsymbol{\phi})$ with $\left|J_{2}\right|=m_{2}$ and $\left|K_{2}\right|=n_{2}$. Recall that $\left|J_{0}\right|>m-m^{\varepsilon}$ and $\left|K_{0}\right|>n-n^{\varepsilon}$. We have

$$
\begin{align*}
I_{2}\left(m_{2}, n_{2}\right) \leq & \binom{m}{m_{2}}\binom{n}{n_{2}}(2 \pi)^{m_{2}+n_{2}}  \tag{37}\\
& \times \exp \left(-\frac{1}{64} A_{\min }\left(n-O\left(n^{\varepsilon}\right)\right) m_{2}-\frac{1}{64} A_{\min }\left(m-O\left(m^{\varepsilon}\right)\right) n_{2}\right) I_{2}^{\prime}\left(m_{2}, n_{2}\right),
\end{align*}
$$

where

$$
I_{2}^{\prime}\left(m_{2}, n_{2}\right)=\int_{-15 \kappa}^{15 \kappa} \cdots \int_{-15 \kappa}^{15 \kappa} \exp \left(-A_{\min } \sum_{j k \in \bar{H}}^{\prime}\left(\theta_{j}+\phi_{k}\right)^{2}+\frac{1}{12} A_{\min } \sum_{j k \in \bar{H}}^{\prime}\left(\theta_{j}+\phi_{k}\right)^{4}\right) d \boldsymbol{\theta}^{\prime} d \boldsymbol{\phi}^{\prime}
$$

and the primes denote restriction to $j \in J_{0} \cup J_{1}$ and $k \in K_{0} \cup K_{1}$, in the case of the summations in addition to the restriction given by the summation limits. Write $m^{\prime}=$ $m-m_{2}$ and $n^{\prime}=n-n_{2}$ and define $\overline{\boldsymbol{\theta}}^{\prime}=\left(m^{\prime}\right)^{-1} \sum_{j}^{\prime} \theta_{j}, \breve{\theta}_{j}=\theta_{j}-\overline{\boldsymbol{\theta}}^{\prime}$ for $j \in J_{0} \cup J_{1}$, $\overline{\boldsymbol{\phi}}^{\prime}=\left(n^{\prime}\right)^{-1} \sum_{k}^{\prime} \phi_{k}, \breve{\phi}_{k}=\phi_{k}-\overline{\boldsymbol{\phi}}^{\prime}$ for $k \in K_{0} \cup K_{1}, \nu^{\prime}=\overline{\boldsymbol{\phi}}^{\prime}+\overline{\boldsymbol{\theta}}^{\prime}$ and $\psi^{\prime}=\overline{\boldsymbol{\theta}}^{\prime}-\overline{\boldsymbol{\phi}}^{\prime}$. Change variables from $\left(\boldsymbol{\theta}^{\prime}, \boldsymbol{\phi}^{\prime}\right)$ to $\left\{\breve{\theta}_{j} \mid j \in J_{3}\right\} \cup\left\{\breve{\phi}_{k} \mid k \in K_{3}\right\} \cup\left\{\nu^{\prime}, \psi^{\prime}\right\}$, where $J_{3}$ is some subset
of $m^{\prime}-1$ elements of $J_{0} \cup J_{1}$ and $K_{3}$ is some subset of $n^{\prime}-1$ elements of $K_{0} \cup K_{1}$. From Subsection 4.3 we know that the Jacobian of this transformation is $m^{\prime} n^{\prime} / 2$. The integrand of $I_{2}^{\prime}$ can now be bounded using

$$
\sum_{j k \in \bar{H}}^{\prime}\left(\theta_{j}+\phi_{k}\right)^{2}=\left(n^{\prime}-O\left(n^{\varepsilon}\right)\right) \sum_{j}^{\prime} \breve{\theta}_{j}^{2}+\left(m^{\prime}-O\left(m^{\varepsilon}\right)\right) \sum_{k}^{\prime} \breve{\phi}_{k}^{2}+\left(m^{\prime} n^{\prime}-O(X)\right) \nu^{\prime 2}
$$

and

$$
\sum_{j k \in \bar{H}}^{\prime}\left(\theta_{j}+\phi_{k}\right)^{4} \leq 27 n^{\prime} \sum_{j}^{\prime} \breve{\theta}_{j}^{4}+27 m^{\prime} \sum_{k}^{\prime} \breve{\phi}_{k}^{4}+27 m^{\prime} n^{\prime} \nu^{\prime 4} .
$$

The latter follows from the inequality $(x+y+z)^{4} \leq 27\left(x^{4}+y^{4}+z^{4}\right)$ valid for all $x, y, z$. Therefore,

$$
\begin{aligned}
I_{2}^{\prime}\left(m_{2}, n_{2}\right) \leq \frac{O(1)}{m^{\prime} n^{\prime}} \int_{-30 \kappa}^{30 \kappa} \int_{-30 \kappa}^{30 \kappa} \cdots \int_{-30 \kappa}^{30 \kappa} \exp ( & A_{\min }\left(n^{\prime}-O\left(n^{\varepsilon}\right)\right) \sum_{j}^{\prime} g\left(\breve{\theta}_{j}\right) \\
& +A_{\min }\left(m^{\prime}-O\left(m^{\varepsilon}\right)\right) \sum_{k}^{\prime} g\left(\breve{\phi}_{k}\right) \\
& \left.+A_{\min }\left(m^{\prime} n^{\prime}-O(X)\right) g\left(\nu^{\prime}\right)\right) d \breve{\theta}_{j \in J_{3}} d \breve{\phi}_{k \in K_{3}} d \nu^{\prime}
\end{aligned}
$$

where $g(z)=-z^{2}+\frac{7}{3} z^{4}$. Since $g(z) \leq 0$ for $|z| \leq 30 \kappa$, and we only need an upper bound, we can restrict the summations in the integrand to $j \in J_{3}$ and $k \in K_{3}$. The integral now separates into $m^{\prime}+n^{\prime}-1$ one-dimensional integrals and Lemma 4.5 (by monotonicity) gives that

$$
\begin{aligned}
I_{2}^{\prime}\left(m_{2}, n_{2}\right)= & O(1) \frac{\pi^{\left(m^{\prime}+n^{\prime}\right) / 2}}{A_{\min }^{\left(m^{\prime}+n^{\prime}-1\right) / 2}\left(m^{\prime}-O\left(m^{\varepsilon}\right)\right)^{n^{\prime} / 2-1}\left(n^{\prime}-O\left(n^{\varepsilon}\right)\right)^{m^{\prime} / 2-1}} \\
& \times \exp \left(O\left(m^{\prime} /\left(A_{\min } n^{\prime}\right)+n^{\prime} /\left(A_{\min } m^{\prime}\right)\right)\right) .
\end{aligned}
$$

Applying (35) and (37), we find that

$$
\begin{equation*}
\sum_{\substack{m_{2}=0 \\ m_{2}+n_{2} \geq 1}}^{m^{\varepsilon}} \sum_{n_{2}=0}^{n^{\varepsilon}} I_{2}\left(m_{2}, n_{2}\right)=O\left(e^{-c_{4} A m}+e^{-c_{4} A n}\right) I_{1} \tag{38}
\end{equation*}
$$

for some $c_{4}>0$.
We have now bounded contributions to the integral of $|F(\boldsymbol{\theta}, \boldsymbol{\phi})|$ from everywhere outside the union of 300 equivalent translates of $\mathcal{X}-\mathcal{R}$, where

$$
\mathcal{X}=\left\{(\boldsymbol{\theta}, \phi)| | \theta_{j}\left|,\left|\phi_{k}\right| \leq 15 \kappa \text { for } 1 \leq j \leq m, 1 \leq k \leq n\right\}\right.
$$

By Lemma 4.4, we have for $(\boldsymbol{\theta}, \boldsymbol{\phi}) \in \widehat{C}^{m+n}$ (which includes $\mathcal{X}$ ) that

$$
|F(\boldsymbol{\theta}, \phi)| \leq \exp \left(-\sum_{j k \in \bar{H}} A_{j k}\left(\hat{\theta}_{j}+\hat{\phi}_{k}+\nu\right)^{2}+\frac{1}{12} \sum_{j=1}^{m} \sum_{k=1}^{n} A_{j k}\left(\hat{\theta}_{j}+\hat{\phi}_{k}+\nu\right)^{4}\right),
$$

where $\hat{\theta}_{j}=\theta_{j}-\overline{\boldsymbol{\theta}}, \hat{\phi}_{k}=\phi_{k}-\overline{\boldsymbol{\phi}}$ and $\nu=\overline{\boldsymbol{\theta}}+\overline{\boldsymbol{\phi}}$. As before, the integrand is independent of $\psi=\overline{\boldsymbol{\theta}}-\overline{\boldsymbol{\phi}}$ and our notation will tend to ignore $\psi$ for that reason; for our bounds it will suffice to remember that $\psi$ has a bounded range.

We proceed by exactly diagonalizing the ( $m+n+1$ )-dimensional quadratic form. Since $\sum_{j=1}^{m} \hat{\theta}_{j}=\sum_{k=1}^{n} \hat{\phi}_{k}=0$, we have

$$
\begin{aligned}
\sum_{j k \in \bar{H}} A_{j k}\left(\hat{\theta}_{j}+\hat{\phi}_{k}+\nu\right)^{2}= & \sum_{j=1}^{m} A_{j \bullet \mid \bar{H}} \hat{\theta}_{j}^{2}+\sum_{k=1}^{n} A_{\bullet k \mid \bar{H}} \hat{\phi}_{k}^{2}+A_{\bullet \bullet \mid \bar{H}} \nu^{2} \\
& +2 \sum_{j=1}^{m} \sum_{k=1}^{n}\left(\alpha_{j k}-A_{j k} h_{j k}\right) \hat{\theta}_{j} \hat{\phi}_{k} \\
& +2 \nu \sum_{j=1}^{m}\left(\alpha_{j \bullet}-A_{j \bullet \mid H}\right) \hat{\theta}_{j}+2 \nu \sum_{k=1}^{n}\left(\alpha_{\bullet k}-A_{\bullet k \mid H}\right) \hat{\phi}_{k}
\end{aligned}
$$

This is almost diagonal, because $\alpha_{j k}=\widetilde{O}\left(n^{-1 / 2}\right), A_{j \bullet \mid H}=\widetilde{O}(1), A_{\bullet k \mid H}=\widetilde{O}(1)$. The coefficients $-2 A_{j k} h_{j k}$ can be larger but only in the $\widetilde{O}(n)$ places where $h_{j k}=1$. We can make the quadratic form exactly diagonal using the slight additional transformation $\left(\boldsymbol{I}+\boldsymbol{U}^{-1} \boldsymbol{Y}\right)^{-1 / 2}$ described by Lemma 4.1, where $\boldsymbol{U}$ is a diagonal matrix with diagonal entries $A_{\bullet \bullet \mid \bar{H}}, A_{\bullet k \mid \bar{H}}$ and $A_{\bullet \bullet \mid \bar{H}}$. The matrix $\boldsymbol{Y}$ has zero diagonal and other entries of magnitude $\widetilde{O}\left(n^{-1 / 2}\right)$ apart from the row and column indexed by $\nu$, which have entries of magnitude $\widetilde{O}\left(n^{1 / 2}\right)$, and the $\widetilde{O}(n)$ just-mentioned entries of order $\widetilde{O}(1)$. By the same argument as used in Subsection 4.5, all eigenvalues of $\boldsymbol{U}^{-1} \boldsymbol{Y}$ have magnitude $\widetilde{O}\left(n^{-1 / 2}\right)$, so the transformation is well-defined. The new variables $\left\{\hat{\vartheta}_{j}\right\},\left\{\hat{\varphi}_{k}\right\}$ and $\dot{\nu}$ are related to the old by

$$
\left(\hat{\theta}_{1}, \ldots, \hat{\theta}_{m}, \hat{\phi}_{1}, \ldots, \hat{\phi}_{n}, \nu\right)^{T}=\left(\boldsymbol{I}+\boldsymbol{U}^{-1} \boldsymbol{Y}\right)^{-1 / 2}\left(\hat{\vartheta}_{1}, \ldots, \hat{\vartheta}_{m}, \hat{\varphi}_{1}, \ldots, \hat{\varphi}_{n}, \dot{\nu}\right)^{T} .
$$

We will keep the variable $\psi$ as a variable of integration but, as noted before, our notation will generally ignore it.

More explicitly, for some $d_{1}, \ldots, d_{m}, d_{1}^{\prime}, \ldots, d_{n}^{\prime}=\widetilde{O}\left(n^{-3 / 2}\right)$, we have uniformly over
$j=1, \ldots, m, k=1, \ldots, n$ that

$$
\begin{align*}
\hat{\theta}_{j} & =\hat{\vartheta}_{j}+\sum_{q=1}^{m} \widetilde{O}\left(n^{-2}\right) \hat{\vartheta}_{q}+\sum_{k=1}^{n} \widetilde{O}\left(n^{-3 / 2}+n^{-1} h_{j k}\right) \hat{\varphi}_{k}+\widetilde{O}\left(n^{-1 / 2}\right) \dot{\nu} \\
\hat{\phi}_{k} & =\hat{\varphi}_{k}+\sum_{j=1}^{m} \widetilde{O}\left(n^{-3 / 2}+n^{-1} h_{j k}\right) \hat{\vartheta}_{j}+\sum_{q=1}^{n} \widetilde{O}\left(n^{-2}\right) \hat{\varphi}_{q}+\widetilde{O}\left(n^{-1 / 2}\right) \dot{\nu}  \tag{39}\\
\nu & =\dot{\nu}+\sum_{j=1}^{m} d_{j} \hat{\vartheta}_{j}+\sum_{k=1}^{n} d_{k}^{\prime} \hat{\varphi}_{k}+\widetilde{O}\left(n^{-1}\right) \dot{\nu}
\end{align*}
$$

Note that the expressions $\widetilde{O}(\cdot)$ in (39) represent values that depend on $m, n, \boldsymbol{s}, \boldsymbol{t}$ but not on $\left\{\hat{\vartheta}_{j}\right\},\left\{\hat{\varphi}_{k}\right\}, \dot{\nu}$.

The region of integration $\mathcal{X}$ is $(m+n)$-dimensional. In place of the variables $(\boldsymbol{\theta}, \boldsymbol{\phi})$ we can use $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, \psi)$ by applying the identities $\hat{\theta}_{m}=-\sum_{j=1}^{m-1} \hat{\theta}_{j}$ and $\hat{\phi}_{n}=-\sum_{k=1}^{n-1} \hat{\phi}_{k}$. (Recall that $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ don't include $\hat{\theta}_{m}$ and $\hat{\phi}_{n}$.) The additional transformation (39) maps the two just-mentioned identities into identities that define $\hat{\vartheta}_{m}$ and $\hat{\varphi}_{n}$ in terms of $(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu})$, where $\hat{\boldsymbol{\vartheta}}=\left(\hat{\vartheta}_{1}, \ldots, \hat{\vartheta}_{m-1}\right)$ and $\hat{\boldsymbol{\varphi}}=\left(\hat{\varphi}_{1}, \ldots, \hat{\varphi}_{n-1}\right)$. These have the form

$$
\begin{align*}
& \hat{\vartheta}_{m}=-\sum_{j=1}^{m-1}\left(1+\widetilde{O}\left(n^{-1}\right)\right) \hat{\vartheta}_{j}+\sum_{k=1}^{n-1} \widetilde{O}\left(n^{-1 / 2}\right) \hat{\varphi}_{k}+\widetilde{O}\left(n^{1 / 2}\right) \dot{\nu} \\
& \hat{\varphi}_{n}=\sum_{j=1}^{m-1} \widetilde{O}\left(n^{-1 / 2}\right) \hat{\vartheta}_{j}-\sum_{k=1}^{n-1}\left(1+\widetilde{O}\left(n^{-1}\right)\right) \hat{\varphi}_{k}+\widetilde{O}\left(n^{1 / 2}\right) \dot{\nu} \tag{40}
\end{align*}
$$

Therefore, we can now integrate over $(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}, \psi)$. The Jacobian of the transformation from $(\boldsymbol{\theta}, \boldsymbol{\phi})$ to $(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu, \psi)$ is $m n / 2$.

Next consider the transformation $T_{4}(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu})=(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \nu)$ defined by (39). The matrix of partial derivatives can be obtained by substituting (40) into (39). Without loss of generality, we can suppose that $x_{m}, y_{n}=\widetilde{O}(1)$. Recall that the Frobenius norm of a matrix is the square root of the sum of squares of absolute values of the entries. After multiplying by $n^{1 / 2}$ the row indexed by $\nu$ and dividing by $n^{1 / 2}$ the column indexed by $\dot{\nu}$ (these two operations together not changing the determinant), the Frobenius norm of the matrix is $\widetilde{O}\left(n^{-1 / 2}\right)$. Since the Frobenius norm bounds the eigenvalues, we can apply Lemma 4.2 to find that the Jacobian of this transformation is $1+\widetilde{O}\left(n^{-1 / 2}\right)$.

The transformation $T_{4}$ changes the region of integration only by a factor $1+\widetilde{O}\left(n^{-1 / 2}\right)$ in each direction, since the inverse of (39) has exactly the same form except that the constants $\left\{d_{j}\right\},\left\{d_{k}^{\prime}\right\}$, while still of magnitude $\widetilde{O}\left(n^{-3 / 2}\right)$, may be different. Therefore, the image of region $\mathcal{X}$ lies inside the region

$$
\mathcal{Y}=\left\{(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu})| | \hat{\vartheta}_{j}\left|,\left|\hat{\varphi}_{k}\right| \leq 31 \kappa(1 \leq j \leq m, 1 \leq k \leq n),|\dot{\nu}| \leq 31 \kappa\right\} .\right.
$$

We next bound the value of the integrand in $\mathcal{Y}$. By repeated application of the inequality $x y \leq \frac{1}{2} x^{2}+\frac{1}{2} y^{2}$, we find that

$$
\frac{1}{12} \sum_{j=1}^{m} \sum_{k=1}^{n} A_{j k}\left(\hat{\theta}_{j}+\hat{\phi}_{k}+\nu\right)^{4} \leq \frac{23}{10}\left(\sum_{j=1}^{m} A_{j \bullet} \hat{\vartheta}_{j}^{4}+\sum_{k=1}^{n} A_{\bullet k} \hat{\varphi}_{k}^{4}+A_{\bullet \bullet} \dot{\nu}^{4}\right)
$$

where we have chosen $\frac{23}{10}$ as a convenient value greater than $\frac{9}{4}$ (to cover the small variations in the coefficients) and less than $\frac{7}{3}$ (to allow us to use Lemma 4.5). Now define $h(z)=$ $-z^{2}+\frac{23}{10} z^{4}$. Then, for $(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) \in \mathcal{Y}$,

$$
\begin{align*}
|F(\boldsymbol{\theta}, \boldsymbol{\phi})| & \leq \exp \left(\sum_{j=1}^{m} A_{j \bullet \mid \bar{H}} h\left(\hat{\vartheta}_{j}\right)+\sum_{k=1}^{n} A_{\bullet k \mid \bar{H}} h\left(\hat{\varphi}_{k}\right)+A_{\bullet \bullet \mid \bar{H}} h(\dot{\nu})\right) \\
& \leq \exp \left(\sum_{j=1}^{m-1} A_{j \bullet \mid \bar{H}} h\left(\hat{\vartheta}_{j}\right)+\sum_{k=1}^{n-1} A_{\bullet k \mid \bar{H}} h\left(\hat{\varphi}_{k}\right)+A_{\bullet \bullet \mid \bar{H}} h(\dot{\nu})\right)  \tag{41}\\
& =\exp \left(A_{\bullet \bullet \mid \bar{H}} h(\dot{\nu})\right) \prod_{j=1}^{m-1} \exp \left(A_{j \bullet \mid \bar{H}} h\left(\hat{\vartheta}_{j}\right)\right) \prod_{k=1}^{n-1} \exp \left(A_{\bullet k \mid \bar{H}} h\left(\hat{\varphi}_{k}\right)\right), \tag{42}
\end{align*}
$$

where the second line holds because $h(z) \leq 0$ for $|z| \leq 31 \kappa$.
Define

$$
\begin{aligned}
& \mathcal{W}_{0}=\left\{(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) \in \mathcal{Y} \left\lvert\, \begin{array}{rl}
\left|\hat{\vartheta}_{j}\right| & \leq \frac{1}{2} n^{-1 / 2+\varepsilon} \\
\left\lvert\, \begin{array}{rl}
\hat{\varphi}_{k} \mid & \leq \frac{1}{2} m^{-1 / 2+\varepsilon} \quad(1 \leq j \leq m-1), \\
& \left.|\dot{\nu}| \leq \frac{1}{2}(m n)^{-1 / 2+2 \varepsilon}\right\},
\end{array}\right. \\
\mathcal{W}_{1}=\mathcal{Y}-\mathcal{W}_{0}, & \\
\mathcal{W}_{2}=\left\{(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) \in \mathcal{Y}| | \sum_{j=1}^{m-1} d_{j} \hat{\vartheta}_{j}+\sum_{k=1}^{n-1} d_{k}^{\prime} \hat{\varphi}_{k} \mid \leq n^{-5 / 4}\right\} .
\end{array}\right.\right.
\end{aligned}
$$

Also define similar regions $\mathcal{W}_{0}^{\prime}, \mathcal{W}_{1}^{\prime}, \mathcal{W}_{2}^{\prime}$ by omitting the variables $\hat{\vartheta}_{1}, \hat{\varphi}_{1}$ instead of $\hat{\vartheta}_{m}, \hat{\varphi}_{n}$ starting at (41). (Note that without loss of generality we can also assume that $x_{1}, y_{1}=$ $\widetilde{O}(1)$.$) Using (39), we see that T_{4}$, and the corresponding transformation that omits $\hat{\vartheta}_{1}$ and $\hat{\varphi}_{1}$, map $\mathcal{R}$ to a superset of $\mathcal{W}_{0} \cap \mathcal{W}_{2} \cap \mathcal{W}_{0}^{\prime} \cap \mathcal{W}_{2}^{\prime}$. Therefore, $\mathcal{X}-\mathcal{R}$ is mapped to a subset of $\mathcal{W}_{1} \cup\left(\mathcal{W}_{0}-\mathcal{W}_{2}\right) \cup \mathcal{W}_{1}^{\prime} \cup\left(\mathcal{W}_{0}^{\prime}-\mathcal{W}_{2}^{\prime}\right)$ and it will suffice to find a tight bound on the integral in each of the four latter regions.

Denoting the right side of (42) by $F_{0}(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu})$, Lemma 4.5 gives

$$
\begin{equation*}
\int_{\mathcal{Y}} F_{0}(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) d \hat{\boldsymbol{\vartheta}} d \hat{\boldsymbol{\varphi}} d \dot{\nu}=\exp \left(O\left(m^{\varepsilon}+n^{\varepsilon}\right)\right) I_{0} \tag{43}
\end{equation*}
$$

Also note that

$$
\begin{equation*}
\int_{z_{0}}^{31 \kappa} \exp \left(c_{0} h(z)\right)=O(1) \exp \left(c_{0} h\left(z_{0}\right)\right) \tag{44}
\end{equation*}
$$

for $c_{0}, z_{0}>0$ and $z_{0}=o(1)$, since $h(z) \leq h\left(z_{0}\right)$ for $z_{0} \leq z \leq 31 \kappa$. By applying (44) to each of the factors of (42) in turn,

$$
\begin{equation*}
\int_{\mathcal{W}_{1}} F_{0}(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) d \hat{\boldsymbol{\vartheta}} d \hat{\boldsymbol{\varphi}} d \dot{\nu}=O\left(e^{-c_{6} A m^{2 \varepsilon}}+e^{-c_{6} A n^{2 \varepsilon}}\right) I_{0} \tag{45}
\end{equation*}
$$

for some $c_{6}>0$ and so, by (43) and (45),

$$
\int_{\mathcal{W}_{0}} F_{0}(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) d \hat{\boldsymbol{\vartheta}} d \hat{\boldsymbol{\varphi}} d \dot{\nu}=\exp \left(O\left(m^{\varepsilon}+n^{\varepsilon}\right)\right) I_{0}
$$

Applying Lemma 4.6 twice, once to the variables $d_{1} \hat{\vartheta}_{1}, \ldots, d_{m-1} \hat{\vartheta}_{m-1}, d_{1}^{\prime} \hat{\varphi}_{1}, \ldots, d_{n-1}^{\prime} \hat{\varphi}_{n-1}$ and once to their negatives, using $M=\widetilde{O}\left(n^{-2}\right), N=m+n-2$ and $t=n^{-5 / 4}$, we find that

$$
\begin{align*}
\int_{\mathcal{W}_{0}-\mathcal{W}_{2}} F_{0}(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) d \hat{\boldsymbol{\vartheta}} d \hat{\boldsymbol{\varphi}} d \dot{\nu} & =O\left(e^{-n^{1 / 4}}\right) \int_{\mathcal{W}_{0}} F_{0}(\hat{\boldsymbol{\vartheta}}, \hat{\boldsymbol{\varphi}}, \dot{\nu}) d \hat{\boldsymbol{\vartheta}} d \hat{\boldsymbol{\varphi}} d \dot{\nu} \\
& =O\left(e^{-n^{1 / 5}}\right) I_{0} . \tag{46}
\end{align*}
$$

Finally, parallel computations give the same bounds on the integrals over $\mathcal{W}_{1}^{\prime}$ and $\mathcal{W}_{0}^{\prime}-\mathcal{W}_{2}^{\prime}$.

We have now bounded $\int|F(\boldsymbol{\theta}, \boldsymbol{\phi})|$ in regions that together cover the complement of $\mathcal{R}$. Collecting these bounds from (36), (38), (45), (46), and the above-mentioned analogues of (45) and (46), we conclude that

$$
\int_{\mathcal{R}^{c}}|F(\boldsymbol{\theta}, \boldsymbol{\phi})| d \boldsymbol{\theta} d \boldsymbol{\phi}=O\left(e^{-c_{7} A m^{2 \varepsilon}}+e^{-c_{7} A n^{2 \varepsilon}}\right) I_{0}
$$

for some $c_{7}>0$, which Lemma 4.7 by (35).

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[^0]:    *This is a revised expanded version of "Asymptotic enumeration of dense 0-1 matrices with specified line sums and forbidden positions", submitted to arXiv on 22 Jan 2007. A concise version will appear in Random Structures and Algorithms.
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