Parameter testing in bounded degree graphs of subexponential growth

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Abstract. Parameter testing algorithms are using constant number of queries to estimate the value of a certain parameter of a very large finite graph. It is well-known that graph parameters such as the independence ratio or the edit-distance from 3-colorability are not testable in bounded degree graphs. We prove, however, that these and several other interesting graph parameters are testable in bounded degree graphs of subexponential growth.

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Contents

1	Introduction		3
	1.1	Dense graph sequences	3
	1.2	Bounded degree graphs	3
	1.3	Hyperfinite graph classes	4
	1.4	Union-closed monotone properties	5
	1.5	Continuous graph parameters in Graph^f_d	6
	1.6	The main theorem	6
2	The canonical limit object		7
	2.1	Hyperfinite graphings	7
	2.2	Graphings as graph limits	9
	2.3	B-graphs	10
	2.4	The canonical colouring of a B -graph	12
	2.5	Random <i>B</i> -colourings of convergent graph sequences $\ldots \ldots \ldots \ldots \ldots \ldots$	13
	2.6	Generic elements	14
3	Graph sequences and ultraproducts		14
	3.1	Basic notions	14
	3.2	The ultraproduct of B -valued functions $\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$	15
	3.3	The canonical action on the ultraproduct space $\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$	15
	3.4	The canonical map preserves the measure	16
	3.5	The ultraproduct of finite graphs	17
4	The	e proof of Theorem 1	19
5	Testing union-closed monotone graph properties		21
	5.1	Edit-distance from a union-closed monotone graph property	21
	5.2	Testability versus continuity	22
6	Continuous parameters in Graph^f_d		23
	6.1	Integrated density of states	23
	62	Independence ratio entropy and log-partitions	27

1 Introduction

1.1 Dense graph sequences

The main motivation for our paper is to develop a theory analogous to that recently developed for dense graph sequences [9],[10],[22]. First let us recall some basic notions. A sequence of finite simple graphs $\mathbf{G} = \{G_n\}_{n=1}^{\infty}, |V(G_n)| \to \infty$ is called **convergent** if for any finite simple graph F, $\lim_{n\to\infty} t(F, G_n)$ exists where

$$t(F,G) = \frac{|\mathrm{hom}\,(F,G)|}{|V(G)|^{|V(F)|}}$$

is the probability that a random map from V(F) into V(G) is a graph homomorphism. The convergence structure above defines a metrizable compactification of the sets of finite graphs. The **limit objects** of the graph sequences were first introduced in [22]. They are measurable symmetric functions

$$W: [0,1] \times [0,1] \to [0,1]$$

A graph sequence $\{G_n\}_{n=1}^{\infty}$ converges to W if for every finite simple graph F,

$$\lim_{n \to \infty} t(F, G_n) = \int_{[0,1]^{V(F)}} \prod_{(i,j) \in E(F)} W(x_i, x_j) dx_1 dx_2 \dots dx_{|V(F)|}$$

For any such function W one can find a graph sequence $\{G_n\}_{n=1}^{\infty}$ converging to W and conversely for any graph sequence $\{G_n\}_{n=1}^{\infty}$ there exists a measurable function W such that the sequence converges to W. Consequently, the boundary points of the compactification can be identified with equivalence classes of such measurable functions [22]. Note that if $\{G_n\}_{n=1}^{\infty}$ is a **sparse** sequence with $\lim_{n\to\infty} \frac{|E(G_n)|}{|V(G_n)|^2} = 0$, then $\{G_n\}_{n=1}^{\infty}$ in fact converges to the zero function.

A graph parameter is a real function on the sets of finite simple graphs that is invariant under graph isomorphims. A parameter ϕ is continuous if $\lim_{n\to\infty} \phi(G_n)$ exists for any convergent sequence $\{G_n\}_{n=1}^{\infty}$. It was shown by Fischer and Newman [17] that continuous graph parameters are exactly the ones that are **testable** by random samplings. It has been proved first in [3] and then later in [22] that the edit-distance from a hereditary graph property is a continuous graph parameter.

1.2 Bounded degree graphs

Let $d \ge 2$ be a positive integer and let Graph_d be the set of finite graphs G (up to isomorphisms) such that $\operatorname{deg}(x) \le d$ for any $x \in V(G)$. The notion of **weak convergence** for the class Graph_d was introduced by Benjamini and Schramm [6]. Let us start with some definitions. A rooted (r, d)-ball is a finite, simple, connected graph H such that

- $\deg(y) \le d$ if $y \in V(H)$.
- H has a distinguished vertex x (the root).
- $d_G(x, y) \leq r$ for any $y \in V(H)$.

For $r \ge 1$, we denote by $U^{r,d}$ the finite set of rooted isomorphism classes of rooted (r, d)-balls. Let G(V, E) be a finite graph with vertex degree bound d. For $\alpha \in U^{r,d}$, $T(G, \alpha)$ denotes the set of vertices $x \in V(G)$ such that there exists a rooted isomorphism between α and the rooted r-ball $B_r(x)$ around x. Set $p_G(\alpha) := \frac{|T(G,\alpha)|}{|V(G)|}$. Thus we associated to G a probability distribution on $U^{r,d}$ for any $r \ge 1$. Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \text{Graph}_d$ be a sequence of finite simple graphs such that $\lim_{n\to\infty} |V(G_n)| = \infty$. Then \mathbf{G} is called **weakly convergent** if for any $r \ge 1$ and $\alpha \in U^{r,d}$, $\lim_{n\to\infty} p_{G_n}(\alpha)$ exists. The convergence structure above defines a metrizable compactification of Graph_d in the following way. Let $\alpha_1, \alpha_2, \ldots$ be an enumeration of the elements of $\cup_{r=1}^{\infty} U^{r,d}$. For a graph G we associate a sequence

$$s(G) = \{\frac{1}{|V(G)|}, p_G(\alpha_1), p_G(\alpha_2), \ldots\} \in [0, 1]^{\mathbb{N}}.$$

By definition, $\{G_n\}_{n=1}^{\infty}$ is weakly convergent if and only if $\{s(G_n)\}_{n=1}^{\infty}$ converge pointwise. We consider the closure of $s(\operatorname{Graph}_d)$ in the compact space $[0,1]^{\mathbb{N}}$. This set can be viewed as the compactification of Graph_d . Again, a graph parameter ϕ : $\operatorname{Graph}_d \to \mathbb{R}$ is called continuous if $\lim_{n\to\infty} \phi(G_n)$ exists for any weakly convergent sequence. Equivalently, ϕ is continuous if it extends continuously to the compactification above.

1.3 Hyperfinite graph classes

Hyperfinite graph classes were introduced in [14] and studied in depth in [23],[7]. Also, under the name of non-expanding bounded degree graph classes they were studied in [11] as well. A class $\mathcal{H} \subset \operatorname{Graph}_d$ is called **hyperfinite** if for any $\epsilon > 0$ there exists K > 0 such that if $G \in \mathcal{H}$ then one can delete $\epsilon |E(G)|$ edges from G in such a way that all the components in the remaining graph G' have size at most K. Planar graphs, graphs with bounded treewidth or, in general, all the minor-closed graph classes are hyperfinite [7]. Let $f : \mathbb{N} \to \mathbb{N}$ be a function of subexponential growth. That is, for any $\delta > 0$ there exists $C_{\delta} > 0$ such that for all $n \geq 1$: $f(n) \leq C_{\delta}(1+\delta)^n$. The class Graph_d^f consists of graphs $G \in \operatorname{Graph}_d$ such that $|B_r(x)| \leq f(r)$ for each $x \in V(G)$. The classes Graph_d^f are also hyperfinite [14]. We will call a graph parameter ϕ continuous on Graph_d^f if $\lim_{n\to\infty} \phi(G_n)$ exists whenever $\{G_n\}_{n=1}^{\infty} \subset \operatorname{Graph}_d^f$ is a weakly convergent sequence. Equivalently, ϕ is continuous on Graph_d^f if it extends continuously to the closure of Graph_d^f in $[0,1]^{\mathbb{N}}$.

1.4 Union-closed monotone properties

Let $\mathcal{P} \subset \text{Graph}_d$. We say that \mathcal{P} is union-closed monotone graph class (or being in \mathcal{P} is a union-closed monotone property) if the following conditions are satisfied:

- if |E(A)| = 0 then $A \in \mathcal{P}$
- if $A \in \mathcal{P}$ and $B \subset A$ is a subgraph, then $B \in \mathcal{P}$ (we consider spanning subgraphs, that is if $B \subset A$ then V(B) = V(A))
- if $A \in \mathcal{P}$ and $B \in \mathcal{P}$ then the disjoint union of A and B is also in \mathcal{P} .

Let us list some union-closed monotone graph classes :

- planar graphs
- bipartite graphs
- *k*-colorable graphs
- graphs that are not containing some fixed graph H.

If G and H are finite graphs with the same vertex set V then their **edge-distance** is defined as

$$d_e(G,H) := \frac{|E(G) \triangle E(H)|}{|V|} \,.$$

The **edit-distance** from a class \mathcal{P} is defined as

$$d_e(G, \mathcal{P}) = \inf_{V(H)=V(G), H \in \mathcal{P}} d_e(G, H).$$

It is important to note that $d_e(*, \mathcal{P})$ is not continuous on Graph_d even for such a simple class as the set of bipartite graphs Bip. Indeed, Bollobás [8] constructed a large girth sequence of cubic graphs such that $d_e(G_n, Bip) > \epsilon > 0$ for any $n \ge 1$. On the other hand there are bipartite large girth sequences of cubic graphs. Since by the definition of weak convergence all sequences of cubic graphs with large girth converge to the same elements of the compactification of Graph_d , $d_e(*, \mathcal{P})$ is not continuous. We shall see, however, that in the class Graph_d^f the graph parameter $d_e(*, \mathcal{P})$ is continuous if \mathcal{P} is a union-closed monotone property (Theorem 2). We also prove that continuous graph parameters are effectively testable via random samplings (Theorem 3).

1.5 Continuous graph parameters in Graph_d^f

In Section 6 we prove that the independence ratio as well as the matching ratio are continuous parameters for the class Graph_d^f (Theorem 5). We also prove that the log-partition functions associated to independent subsets resp. to matchings are continuous graph parameters in Graph_d^f . This shows that for certain aperiodic graphs such as the Penrose tilings, in which all neighbourhood patterns can be seen in a given frequency, the thermodynamical limit of the log-partition functions exists. Such results are well-known for lattices. We also show a similar convergence result for the integrated density of states for discrete Schrödinger operators with random potentials extending some recent results in [20] and [21](Theorem 4).

1.6 The main theorem

It is known [2] that there exists $\delta > 0$ such that to construct an independent set that approximates the size of a maximum independent set within an error of $\delta |V(G)|$ in a 3-regular graph G is NP-hard. The situation is dramatically different in the case of planar graphs. For any fixed $\delta > 0$ there exists a polynomial time algorithm to construct an independent set that approximates the size of the maximum independent set within an error of $\delta |V(G)|$ for cubic planar graphs G [4] (note that finding a maximum independent set in a planar cubic graph is still NP-hard). First, using a polynomial time algorithm one can delete $\frac{\delta |E(G)|}{3}$ edges from G to obtain a graph G' with components of size at most $K(\delta)$. For each component of G' one can find a maximum independent set in $L(\delta)$ steps. Obviously, the union of these sets can not be smaller in size that the maximum independent set in G. If we delete all the vertices from the union that are on some previously deleted edges, then we get an independent subset of the original graph G. Since the number of deleted vertices is at most $\delta |V(G)|$ we obtained an approximation of the maximum independent set within an error of $\delta |V(G)|$.

How can we use this idea for constant-time algorithms ? Let f be a function of subexponential growth and $G \in \operatorname{Graph}_d^f$. Fix $\epsilon > 0$. Since Graph_d^f is a hyperfinite class one can delete $\epsilon |E(G)|$

edges from G to obtain a graph G' with components of size at most $K(\epsilon)$. Let $A(d, K(\epsilon))$ be the finite set of all finite connected graphs of size at most $K(\epsilon)$. If for each $H \in A(d, K(\epsilon))$ someone tells us how many components of G' are isomorphic to H we can calculate the size of the maximum independent set in G'. What we need is to test the following value : the number of components in G' isomorphic to H divided by |V(G)|. Unfortunately, this is not a well-defined graph parameter since there are many ways to delete edges from G to obtain graphs with small components. Informally speaking, what we need to show is that if two graphs $G_1, G_2 \in \operatorname{Graph}_d^f$ are close to each other in terms of local neighborhood statistics, then one can delete edges from G_1 resp. G_2 in such a way that in the remaining graphs G'_1 resp. G'_2 the ratios of H-components are close to each other for any fixed $H \in A(d, K(\epsilon))$. That is exactly what we prove in our main theorem, which is the main tool of our paper.

Theorem 1 Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \operatorname{Graph}_d^f$ be a weakly convergent sequence of finite graphs. Then for any $\epsilon > 0$ there exists a constant K > 0 and also, for all connected simple graphs $H \in \operatorname{Graph}_d^f$ with $|V(H)| \leq K$ a real constant c_H such that for any $n \geq 1$ one can remove $\epsilon |E(G_n)|$ edges from G_n satisfying the following conditions:

- The number of vertices in each component of the remaining graph G'_n is not greater than K.
- If Vⁿ_H ⊆ V(G_n) is the set of vertices that are contained in a component of G'_n isomorphic to H then

$$\lim_{n \to \infty} \frac{|V_H^n|}{|V(G_n)|} = c_H$$

Note that the second condition is equivalent to saying that $\{G'_n\}$ is a convergent sequence. In order to prove the theorem we combine the limit object method of Benjamini and Schramm [6] and the non-standard analytic technique developed in [16].

2 The canonical limit object

2.1 Hyperfinite graphings

In this subsection we briefly recall the basic properties of graphings (graphed equivalence relations) [18]. Let \mathbf{F}_2^{∞} be the free product of countably many copies of the cycle group of order two. Thus

$$\mathbf{F}_2^{\infty} = \langle \{s_i\}_{i \in \mathbb{N}} | s_i^2 = 1 \rangle$$

is a presentation of the group \mathbf{F}_2^{∞} , where the s_i 's are generators of order two. Suppose that the edges of a simple graph H (finite or infinite) are coloured by natural numbers **properly**, that is, any two edges having a common vertex are coloured differently. Then, the colouring induces an action of \mathbf{F}_2^{∞} on the vertex set V(H) in the following way:

- $s_i(x) = y$ if $e = (x, y) \in E(H)$ and e is coloured by i.
- $s_i(x) = x$ if no edge incident to x is coloured by i.

We regard graphings as the measure theoretical analogues of N-coloured graphs. Let (X, μ) be a probability measure space with a measure-preserving action of \mathbf{F}_2^{∞} that is not necessarily free such that if $s_i(p) = q \neq p$ and $s_j(p) = q$ then i = j. Let $E \subset X \times X$ be the set of pairs (p,q) such that $\gamma(p) = q$ for some $\gamma \in \mathbf{F}_2^{\infty}$. Thus E is the **measurable equivalence relation** induced by the \mathbf{F}_2^{∞} -action. Connect the points $p \in X, q \in X, p \neq q$ by an edge of colour i if $s_i(p) = q$ for some generator element s_i . Thus we obtain a properly N-coloured graph with a measurable structure, the **graphing** \mathcal{G} . If $p \in X$ then \mathcal{G}_p denotes the component of \mathcal{G} containing p. In this paper we consider only bounded degree graphings, that is, graphings for which all the degrees of the vertices are bounded by a certain constant d. Thus, for any $p \in X$ the number of generators $\{s_i\}$ which do not fix p is at most d. The edge-set of the graphing $\mathcal{G}, E(\mathcal{G})$ has a natural measure space structure as well. Let $i \geq 1$ and $A \subseteq X$ be a measurable subset of vertices such that

• If $a \in A$ then $s_i(a) \in A$, $s_i(a) \neq a$.

Let $\hat{A}_i \subseteq E(\mathcal{G})$ be the set of edges such that their endpoints belong to A. Then we call \hat{A}_i a measurable edge-set of colour i. These measurable edge-sets form a σ -algebra with a measure μ_{E_i} ,

$$\mu_{E_i}(\hat{A}_i) = \frac{1}{2}\mu(A) \,.$$

Clearly, the σ -algebra above contains the set E_i consisting of all edges coloured by i. Then $E(\mathcal{G}) = \bigcup_{i=1}^{\infty} E_i$. The set $M \subset E(\mathcal{G})$ is measurable if for all $i \ M \cap E_i$ is measurable and

$$\mu_E(M) = \sum_{i=1}^{\infty} \mu_{E_i}(M \cap E_i) \,.$$

A measurable subgraphing $\mathcal{H} \subseteq \mathcal{G}$ is a measurable subset of $E(\mathcal{G})$ such that the components of \mathcal{H} are induced subgraphs of \mathcal{G} . A subgraphing \mathcal{H} is called component-finite if all of its components are finite graphs. It is easy to see that if $\mathcal{H} \subset \mathcal{G}$ is a component-finite subgraphing and F is a finite connected simple graph, then

$$\mathcal{H}_F = \{ p \in X \mid \mathcal{H}_p \cong F \}$$

is measurable and the span of \mathcal{H}_F is a component-finite subgraphing having components isomorphic to F.

The graphing \mathcal{G} is called **hyperfinite** if there exist component-finite subgraphings $\mathcal{H}_1 \subset \mathcal{H}_2 \subset \ldots$ such that

$$\lim_{n \to \infty} \mu_E(E(\mathcal{G}) \setminus E(\mathcal{H}_n)) = 0.$$

Suppose that $f : \mathbb{N} \to \mathbb{R}$ is a function of subexponential growth and $|B_r(x)| \leq f(r)$ for all the balls of radius r in \mathcal{G} . Then we call \mathcal{G} a graphing of subexponential growth. By the result of Adams and Lyons [1] graphings of subexponential growth are always hyperfinite.

2.2 Graphings as graph limits

Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \operatorname{Graph}_d$ be weakly convergent graph sequence as in the Introduction. Let (\mathcal{G}, X, μ) be a graphing. If $\alpha \in U^{r,d}$ then let $T(\mathcal{G}, \alpha)$ be the set of points $p \in X$ such that the ball $B_r(p) \subset \mathcal{G}_p$ is rooted isomorphic to α . Clearly, $T(\mathcal{G}, \alpha)$ is a measurable set. We say that \mathbf{G} converges to \mathcal{G} if for any $r \geq 1$ and $\alpha \in U^{r,d}$

$$p_{\mathbf{G}}(\alpha) := \lim_{n \to \infty} p_{G_n}(\alpha) = \mu(T(\mathcal{G}, \alpha)).$$

In [13] we proved that any weakly convergent graph sequence admits such **limit graphings**. There is however an other even more natural limit object for weakly convergent graph sequences constructed by Benjamini and Schramm [6]. Let \mathbf{Gr}_d be the set of all countable connected rooted graphs (up to rooted isomorphism) with uniform vertex degree bound d. For each $\alpha \in U^{r,d}$ we associate a closed-open set $R(\alpha)$, the set of elements $G \in \mathbf{Gr}_d$ such that $B_r(x) \cong \alpha$, where x is the root of G. Then \mathbf{Gr}_d is a metrizable, compact space. Now let $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ be a weakly convergent sequence in Graph_d . Then

$$\hat{\mu}_{\mathbf{G}}(R(\alpha)) := \lim_{n \to \infty} p_{G_n}(\alpha) = p_{\mathbf{G}}(\alpha)$$

defines a measure $\hat{\mu}_{\mathbf{G}}$ on \mathbf{Gr}_d . This measure space can be considered as the primary limit object for weakly convergent graph sequences.

Note that if $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ is a Følner-sequence in the Cayley-graph of a finitely generated amenable group Γ then the limit measure $\hat{\mu}_{\mathbf{G}}$ is concentrated on one single point in \mathbf{Gr}_d namely on the point representing the Cayley-graph itself. In order to avoid this technical difficulty, in the following subsections we introduce a combination of the limit graphing and the Benjamini-Schramm construction.

2.3 B-graphs

Let $B = \{0,1\}^{\mathbb{N}}$ be the Bernoulli space of 0 - 1-sequences with the standard product measure ν . A rooted *B*-graph is a rooted connected graph *G* equipped with a function $\tau_G : V(G) \to B$. We say that the rooted *B*-graphs *G* and *H* are isomorphic if there exists a rooted graph isomorphism $\psi : G \to H$ such that $\tau_H(\psi(x)) = \tau_G(x)$ for any $x \in V(G)$. Let **BGr**_d be the set of such isomorphism classes of countable rooted *B*-graphs with vertex degree bound *d*. Let $\alpha \in U^{r,d}$ and consider a rooted *r*-ball *T* representing the class α . Consider the product space $B(T) = B^{V(T)}$ with the product measure $\nu^{|V(T)|} = \nu_T$. Note that the finite group of rooted automorphisms Aut(*T*) acts continuously on B(T) preserving the measure ν_T . Let us consider the quotient space $Q(T) = B(T)/\operatorname{Aut}(T)$ and the natural projection $\pi_T : B(T) \to Q(T)$. For a Borel-set $W \subseteq Q(T)$ let us define the measure λ by $\lambda(W) = \nu_T(\pi_T^{-1}(W))$. Obviously if *T* and *S* are rooted isomorphic balls then Q(T) and Q(S) are naturally isomorphic. Hence we shall denote the quotient space by $Q(\alpha)$. Let $\beta \in U^{r+1,d}, \alpha \in U^{r,d}$ such that the *r*-ball around the root in β is isomorphic to α . Then we have a natural projection $\pi_{\beta,\alpha} : Q(\beta) \to Q(\alpha)$. Indeed if $f \in B(T)$ for some rooted ball *T* representing β and the restriction of *f* on the *r*-ball around the root is *g*, then the class of *f* is mapped to the class of *g*.

Lemma 2.1 If $W \subseteq Q(\alpha)$ is a Borel-set then

$$\lambda(\pi_{\beta,\alpha}^{-1}(W)) = \lambda(W) \,.$$

Proof. Let $\pi_{T,S} : B(T) \to B(S)$ be the natural projection, where S is the r-ball around the root. Then

$$\pi_S \circ \pi_{T,S} = \pi_{\beta,\alpha} \circ \pi_T.$$

Also, since $\nu_T(\pi_{T,S}^{-1}(A)) = \nu_S(A)$ for any Borel-set $A \subseteq B(S)$,

$$\lambda(\pi_{\beta,\alpha}^{-1}(W)) = \nu_T(\pi_T^{-1} \circ \pi_{\beta,\alpha}^{-1}(W)) \text{ and } \lambda(W) = \nu_S(\pi_S^{-1}(W)) = \nu_T(\pi_{T,S}^{-1} \circ \pi_S^{-1}(W)).$$

Now the lemma follows.

Hence we have the compact spaces $Q^r_d := \bigcup_{\alpha \in U^{r,d}} Q(\alpha)$ and the projections

$$Q_d^1 \stackrel{\pi^1}{\leftarrow} Q_d^2 \stackrel{\pi^2}{\leftarrow} \dots$$

where π^r is defined as $\pi_{\beta,\alpha}$ on $Q(\beta)$. It is easy to see that the elements of $\lim_{\leftarrow} Q_d^r$ are in a oneto-one correspondence with the rooted isomorphism classes of the countable rooted *B*-graphs with vertex degree bound *d*. Hence from now on we regard \mathbf{BGr}_d as a compact metrizable space. Note that the forgetting functor provides us a continuous map $\mathcal{F} : \mathbf{BGr}_d \to \mathbf{Gr}_d$. Note that the forgetting functor maps a *B*-graph to its underlying graph in \mathbf{Gr}_d .

Now let $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ be a weakly convergent graph sequence and $\hat{\mu}_{\mathbf{G}}$ be the limit measure on \mathbf{Gr}_d .

Proposition 2.1 Let $\alpha \in U^{r,d}$ and $W \subseteq Q(\alpha)$ be a Borel-set. We define the measure $\widetilde{\mu}_{\mathbf{G}}$ by

$$\widetilde{\mu}_{\mathbf{G}}(W) := \lambda(W) p_{\mathbf{G}}(\alpha) \,.$$

Then $\tilde{\mu}_{\mathbf{G}}$ is a Borel-measure on \mathbf{BGr}_d and $\mathcal{F}_*(\tilde{\mu}_{\mathbf{G}}) = \hat{\mu}_{\mathbf{G}}$.

Proof. Clearly, we define a measure $\widetilde{\mu}^r_{\mathbf{G}}$ on Q^r_d by

$$\widetilde{\mu}^{r}_{\mathbf{G}}(\cup_{\alpha\in U^{r,d}}W_{\alpha}):=\sum_{\alpha\in U^{r,d}}p_{\mathbf{G}}(\alpha)\lambda(W_{\alpha}).$$

We only need to prove that

$$\pi^r_*(\widetilde{\mu}^{r+1}_{\mathbf{G}}) = \widetilde{\mu}^r_{\mathbf{G}} \,.$$

Let $U_{\alpha}^{r+1,d}$ be the set of classes such that the rooted *r*-ball around the root is just α . Then

• $\bigcup_{\alpha \in U^{r,d}} U_{\alpha}^{r+1,d} = U^{r+1,d}$.

•
$$(\pi^r)^{-1}(Q(\alpha)) = \bigcup_{\beta \in U_\alpha^{r+1,d}} Q(\beta)$$

•
$$p_{\mathbf{G}}(\alpha) = \sum_{\beta \in U_{\alpha}^{r+1,d}} p_{\mathbf{G}}(\beta)$$
.

If $W \subseteq Q(\alpha)$ then

$$(\pi^r)^{-1}(W) = \bigcup_{\beta \in U^{r+1,d}_{\alpha}} \pi^{-1}_{\beta,\alpha}(W).$$

Thus by Lemma 2.1, $\widetilde{\mu}^{r+1}_{\bf G}(\pi^{-1}_{\beta,\alpha}(W))=p_{\bf G}(\beta)\lambda(W)$. Therefore

$$\widetilde{\mu}_{\mathbf{G}}^{r+1}((\pi^r)^{-1}(W)) = \widetilde{\mu}_{\mathbf{G}}^r(W) \,.$$

Consequently, $\tilde{\mu}_{\mathbf{G}}$ is a well-defined Borel-measure on \mathbf{BGr}_d . Since $\hat{\mu}_{\mathbf{G}}(R(\alpha)) = \tilde{\mu}_{\mathbf{G}}(Q(\alpha))$, $\mathcal{F}_*(\tilde{\mu}_{\mathbf{G}}) = \hat{\mu}_{\mathbf{G}}$.

2.4 The canonical colouring of a *B*-graph

Let us consider the triples (p, q, n), where $1 \le p \le d$, $1 \le q \le d$, $n \ge 1$. Let $G(V, E, \tau_G)$ be a countable *B*-graph such that $\tau_G(x) \ne \tau_G(y)$ if $x \ne y$. These *B*-graphs are called **separated**. Now colour the edge $e = (x, y) \in E$ by (p, q, n) if

- $\tau_G(x) < \tau_G(y)$ (in the lexicographic ordering of $\{0,1\}^{\mathbb{N}}$) and $l_1 < l_2 < \ldots < l_{\deg(x)}$ are the values of τ_G at the neighbours of x and $\tau_G(y) = l_p$.
- $m_1 < m_2 < \ldots < m_{\deg(y)}$ are the values of τ_G at the neighbours of y and $\tau_G(x) = m_q$.
- $\tau_G(x) = \{a_1, a_2, \ldots\} \in B, \ \tau_G(y) = \{b_1, b_2, \ldots\} \in B, \ a_1 = b_1, a_2 = b_2, \ldots, a_{n-1} = b_{n-1}, a_n \neq b_n.$

Lemma 2.2 If $(a, b) \in E$, $(a, c) \in E$ then the colours of (a, b) and (a, c) are different.

Proof. If the colour of (a, b) and (a, c) are the same, then either $\tau_G(b) > \tau_G(a), \tau_G(c) > \tau_G(a)$ or $\tau_G(b) < \tau_G(a), \tau_G(c) < \tau_G(a)$. Hence by the definition of the colouring $\tau_G(b) = \tau_G(c)$ leading to a contradiction.

Now consider $\mathbf{O}_d \subset \mathbf{BGr}_d$, the Borel-set of separated *B*-graphs. Clearly $\tilde{\mu}_{\mathbf{G}}(\mathbf{O}_d) = 1$. The colouring construction above defines a canonical Borel \mathbf{F}_2^{∞} -action on \mathbf{O}_d as follows. Suppose that $\underline{z} \in \mathbf{O}_d$ represents the rooted *B*-graph *G* with root $a \in V(G)$. Consider the free generators of order two $\{s_{\delta}\}_{\delta \in I}$, where

$$I = \{1, 2, \dots, d\} \times \{1, 2, \dots, d\} \times \mathbb{N}.$$

Let $\alpha \in I$, $\alpha = (p, q, n)$. Then

- If there exists an edge $(a, b) \in E(G)$ coloured by (p, q, n) then define $s_{\alpha}(\underline{z}) = \underline{w}$, where \underline{w} represents the same *B*-graph as \underline{z} , but with root *b*.
- If there exists no edge $(a,b) \in E(G)$ coloured by (p,q,n) then let $s_{\alpha}(\underline{z}) = \underline{z}$.

Observe that we constructed an \mathbf{F}_2^{∞} -action on \mathbf{O}_d such a way that if $\underline{z} \in \mathbf{O}_d$ represents a graph G then the orbit graph of \underline{z} is isomorphic to G, We call this action the canonical \mathbf{F}_2^{∞} -action on the **canonical limit object** $(\mathbf{O}_d, \tilde{\mu}_{\mathbf{G}})$. In Corollary 3.1 we shall prove that the measure $\tilde{\mu}_{\mathbf{G}}$ is invariant under the canonical action.

2.5 Random *B*-colourings of convergent graph sequences

Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \operatorname{Graph}_d$ be a weakly convergent sequence of graphs. Let $\Omega = B^{\bigcup_{n=1}^{\infty} V(G_n)}$ be the space of *B*-valued functions κ on the vertices of the graph sequence. We equip Ω with the standard product measure ν_{Ω} .

Now let $\alpha \in U^{r,d}$ and let $Q(\alpha)$ be the quotient space as in Subsection 2.3. Let $U \subseteq Q(\alpha)$ be a Borel-subset, $\kappa \in \Omega$ and $T(G_n, \kappa, U)$ be the set of vertices $p \in V(G_n)$ such that

- $p \in T(G_n, \alpha)$.
- $\kappa_{|B_r(p)} \in U$.

Proposition 2.2 For any Borel-set $U \in Q(\alpha)$,

$$\lim_{n \to \infty} \frac{|T(G_n, \kappa, U)|}{|V(G_n)|} = \lambda(U) p_{\mathbf{G}}(\alpha) = \widetilde{\mu}_{\mathbf{G}}(U)$$
(1)

holds for almost all $\kappa \in \Omega$.

Proof. We may suppose that $p_{\mathbf{G}}(\alpha) \neq 0$, since if $p_{\mathbf{G}}(\alpha) = 0$ then both sides of the equation (1) vanish. Let $x \in T(G_n, \alpha)$. Then we define $A_x^U \subset \Omega$ by

$$A_x^U := \left\{ \kappa \in \Omega \ | \ x \in T(G_n, \kappa, U) \right\}.$$

Clearly, $\nu_{\Omega}(A_x^U) = \lambda(U)$. Note however that if $x \neq y \in T(G_n, \alpha)$ then A_x^U and A_y^U might not be independent subsets. On the other hand, if $x \in T(G_n, \alpha)$, $y \in T(G_m, \alpha)$ and $n \neq m$ then A_x^U and A_y^U are independent. Also, if $S \subset T(G_n, \alpha)$, $S = \{x_1, x_2, \ldots, x_k\}$ and $d_{G_n}(x_i, x_j) > 2r$ if $i \neq j$ then $A_{x_1}^U, A_{x_2}^U, \ldots, A_{x_k}^U$ are jointly independent.

Lemma 2.3 There exists a natural number l > 0 (depending on r and d) and a partition $\cup_{i=1}^{l} B_{i}^{n} = T(G_{n}, \alpha)$ for any $n \ge 1$ such that if $x \ne y \in B_{i}^{n}$ then $d_{G_{n}}(x, y) > 2r$.

Proof. Let H_n be a graph with vertex set $V(G_n)$. Let $(x, y) \in E(H_n)$ if and only if $d_{G_n}(x, y) \leq 2r$. Then $\deg(x) \leq d^{r+1}$ for any $x \in V(H_n)$. Let $l = d^{r+1} + 1$ then H_n is vertex-colorable by the colours c_1, c_2, \ldots, c_l . Let B_i^n be the set of vertices coloured by c_i .

To conclude the proof of Proposition 2.2, let us fix $q \ge 2$. Let $B_{i_1}^n, B_{i_2}^n, \ldots, B_{i_{n,q}}^n$ be those elements of the partition of the previous lemma such that

$$\frac{|B_{i_j}^n|}{|V(G_n)|} > \frac{2^{-q}}{l} \,.$$

Then by the law of large numbers, for almost all $\kappa \in \Omega$

$$\lim_{n \to \infty} \frac{|T(G_n, \kappa, U) \cap B_{i_j}^n|}{|B_{i_j}^n|} = \lambda(U) \,,$$

for any choice of i_j . An easy calculation shows that for the same κ

$$\lim_{n \to \infty} \frac{|T(G_n, \kappa, U) \cap (\bigcup_{j=1}^{i_{n,q}} B_{i_j}^n)|}{|\bigcup_{j=1}^{i_{n,q}} B_{i_j}^n|} = \lambda(U).$$
(2)

On the other hand,

$$\frac{|T(G_n,\alpha) \setminus \bigcup_{j=1}^{i_{n,q}} B_{i_j}^n|}{|V(G_n)|} \le 2^{-q}$$

and $\lim_{n\to\infty} \frac{|T(G_n,\alpha)|}{|V(G_n)|} = p_{\mathbf{G}}(\alpha)$. Hence letting $q \to \infty$, (1) follows.

2.6 Generic elements

For any $\alpha \in U^{r,d}$ let us choose closed-open sets $\{U_{\alpha}^k\}_{k=1}^{\infty}$ such that they form a Boolean-algebra and generate all the Borel-sets in $Q(\alpha)$. We call $\kappa \in \Omega$ generic if for any $k \ge 1$ and $\alpha \in U^{r,d}$

$$\lim_{n \to \infty} \frac{|T(G_n \kappa, U_{\alpha}^k)|}{|V(G_n)|} = \lambda(U_{\alpha}^k) p_{\mathbf{G}}(\alpha)$$

and for any $n \ge 1$, $\kappa(p) \ne \kappa(q)$ if $p \ne q \in V(G_n)$. By Proposition 2.2, almost all $\kappa \in \Omega$ are generic.

3 Graph sequences and ultraproducts

3.1 Basic notions

In this section we briefly recall some of the basic notions on the ultraproducts of finite sets [16]. Let $\{X_i\}_{i=1}^{\infty}$ be finite sets, $|X_i| \to \infty$. Let ω be a non-principal ultrafilter and $\lim_{\omega} : l^{\infty}(\mathbb{N}) \to \mathbb{R}$ be the corresponding ultralimit. The ultraproduct of the sets X_i is defined as follows. Let $\widetilde{X} = \prod_{i=1}^{\infty} X_i$. We say that $\widetilde{p} = \{p_i\}_{i=1}^{\infty}, \widetilde{q} = \{q_i\}_{i=1}^{\infty} \in \widetilde{X}$ are equivalent, $\widetilde{p} \sim \widetilde{q}$, if

$$\{i \in \mathbb{N} \mid p_i = q_i\} \in \omega$$
.

We shall denote the equivalence class of $\{p_i\}_{i=1}^{\infty}$ by $[\{p_i\}_{i=1}^{\infty}]$. Define $\mathbf{X} := \widetilde{X} / \sim$. Now let $\mathcal{R}(X_i)$ denote the Boolean algebra of subsets of X_i , with the normalised measure $\mu_i(A) = \frac{|A|}{|X_i|}$. Then let $\widetilde{\mathcal{R}} = \prod_{i=1}^{\infty} \mathcal{R}(X_i)$ and $\mathcal{R} = \widetilde{\mathcal{R}}/I$, where I is the ideal of elements $\{A_i\}_{i=1}^{\infty}$ such that $\{i \in \mathbb{N} \mid A_i = \emptyset\} \in \omega$. It is important to note that the elements of \mathcal{R} can be identified with certain subsets of **X**: If

$$\mathbf{p} = [\{p_i\}_{i=1}^{\infty}] \in \mathbf{X} \text{ and } \mathbf{A} = [\{A_i\}_{i=1}^{\infty}] \in \mathcal{R}$$

then $\mathbf{p} \in \mathbf{A}$ if $\{i \in \mathbb{N} \mid p_i \in A_i\} \in \omega$. One can easily see that \mathcal{R} is a Boolean-algebra on \mathbf{X} . Now let $\mu_{\mathbf{G}}(\mathbf{A}) = \lim_{\omega} \mu_i(A_i)$. Then $\mu_{\mathbf{G}} : \mathcal{R} \to \mathbb{R}$ is a finitely additive probability measure. We call $\mathbf{N} \subseteq \mathbf{X}$ a nullset if for any $\epsilon > 0$ there exists $\mathbf{A}_{\epsilon} \in \mathcal{R}$ such that $\mathbf{N} \subset \mathbf{A}_{\epsilon}$ and $\mu(\mathbf{A}_{\epsilon}) \leq \epsilon$. We call $\mathbf{B} \subset \mathbf{X}$ measurable if there exists $\hat{\mathbf{B}} \subset \mathcal{R}$ such that $\mathbf{B} \triangle \hat{\mathbf{B}}$ is a nullset. The measurable sets form a σ -algebra \mathcal{B} and $\mu_{\mathbf{G}}(\mathbf{B}) = \mu_{\mathbf{G}}(\hat{\mathbf{B}})$ defines a probability measure on \mathcal{B} .

3.2 The ultraproduct of *B*-valued functions

Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \operatorname{Graph}_d$ be a weakly convergent sequence of graphs. We shall denote by $\mathbf{X}_{\mathbf{G}}$ the ultraproduct of the vertex sets $\{V(G_n)\}_{n=1}^{\infty}$. Now consider an element $\kappa \in \Omega = B^{\bigcup_{n=1}^{\infty} V(G_n)}$. We define the *B*-valued function F_{κ} on $\mathbf{X}_{\mathbf{G}}$ the following way. Let $\mathbf{p} = [\{p_n\}_{n=1}^{\infty}]$ then $F_{\kappa}(\mathbf{p}) := \lim_{\omega} \kappa(p_i)$. Note that if $\{b_n\}_{n=1}^{\infty} \subset B$ is a sequence of elements of the Bernoulli product space then $\lim_{\omega} b_n = b$ is the unique element of *B* such that for any neighbourhood $b \in U \subseteq B$

$$\{n \in \mathbb{N} \mid b_n \in U\} \in \omega$$
.

Lemma 3.1 F_{κ} is a measurable *B*-valued function on $\mathbf{X}_{\mathbf{G}}$.

Proof. Let $O_{x_1,x_2,...,x_n}$ be the basic closed-open set in B, where $x_i \in \{0,1\}$ and $b \in O_{x_1,x_2,...,x_n}$ if $b(i) = x_i$. It is enough to prove that $F_{\kappa}^{-1}(O_{x_1,x_2,...,x_n}) \in \mathcal{R}$. Let

$$O_{x_1,x_2,...,x_n}^i := \{ p_n \in V(G_n) \mid \kappa(p_i) \in O_{x_1,x_2,...,x_n} \}.$$

Since O_{x_1,x_2,\ldots,x_n}^i is an closed-open set $[\{O_{x_1,x_2,\ldots,x_n}^i\}_{i=1}^\infty] = F_{\kappa}^{-1}(O_{x_1,x_2,\ldots,x_n})$. Thus our lemma follows.

3.3 The canonical action on the ultraproduct space

Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \operatorname{Graph}_d$ and $\mathbf{X}_{\mathbf{G}}$ be as in the previous subsections and let $\kappa \in \Omega$ be a fixed generic element. Then κ determines a separated *B*-function on each vertex space $V(G_n)$.

Now let $\beta \in \{1, 2, ..., d\} \times \{1, 2, ..., d\} \times \mathbb{N}$ and let $S^i_{\beta} : V(G_i) \to V(G_i)$ be the bijection $S^i_{\beta}(p) = s_{\beta}(p)$ (as in Subsection 2.4). The ultraproduct of $\{S^i_{\beta}\}_{i=1}^{\infty}$ is defined the following way:

$$\mathbf{S}_{\beta}([\{p_i\}_{i=1}^{\infty}]) = [\{S_{\beta}^i(p_i)\}_{i=1}^{\infty}].$$

Then \mathbf{S}_{β} is a measure-preserving bijection on the ultraproduct space $\mathbf{X}_{\mathbf{G}}$. Indeed if $\mathbf{A} = [\{A_i\}_{i=1}^{\infty}] \in \mathcal{R}$ then $\mathbf{S}_{\beta}(\mathbf{A}) = [\{S_{\beta}^i(A_i)\}_{i=1}^{\infty}]$. Clearly $\mathbf{S}_{\beta}^2 = \mathrm{Id}$, hence we defined a measure-preserving action of \mathbf{F}_2^{∞} on $\mathbf{X}_{\mathbf{G}}$.

Lemma 3.2 Each component $\mathcal{G}_{\mathbf{p}}$ of the graphing \mathcal{G} induced by the action above has vertex degree bound d.

Proof. Let $\mathbf{p} = [\{p_i\}_{i=1}^{\infty}] \in \mathbf{X}_{\mathbf{G}}$. If $\mathbf{S}_{\beta}(\mathbf{p}) \neq \mathbf{p}$ then $s_{\beta}(p_i) \neq p_i$ for ω -almost all $i \in \mathbb{N}$. Therefore if $\mathbf{S}_{\beta_1}, \mathbf{S}_{\beta_2}, \dots, \mathbf{S}_{\beta_{d+1}}$ are bijections such that $\mathbf{S}_{\beta_j}(\mathbf{p}) \neq \mathbf{p}$ then $s_{\beta_j}(p_i) \neq p_i$ for ω -almost all $i \in \mathbb{N}$ and $1 \leq j \leq d+1$. This leads to a contradiction.

By the previous lemma each graph $\mathcal{G}_{\mathbf{p}}$ is a rooted *B*-graph of vertex degree bound *d*, where \mathbf{p} is the root and the *B*-colouring on the vertices of $\mathcal{G}_{\mathbf{p}}$ is induced by F_{κ} . Consequently, we have a canonical map $\rho : \mathbf{X}_{\mathbf{G}} \to \mathbf{B}\mathbf{Gr}_d$ (depending on the fixed generic element κ of course) such that for each $\mathbf{p}, \rho(\mathbf{p})$ is the rooted *B*-graph representing the component $\mathcal{G}_{\mathbf{p}}$.

3.4 The canonical map preserves the measure

The goal of this subsection is to prove the main technical tool of our paper.

Proposition 3.1 $\rho : (\mathbf{X}_{\mathbf{G}}, \mu_{\mathbf{G}}) \to (\mathbf{B}\mathbf{G}\mathbf{r}_d, \widetilde{\mu}_{\mathbf{G}})$ is a measure-preserving map.

Proof. We need to prove that for any Borel-set $W \subseteq \mathbf{BGr}_d$, $\rho^{-1}(W)$ is a measurable set in $\mathbf{X}_{\mathbf{G}}$ and $\mu_{\mathbf{G}}(\rho^{-1}(W)) = \widetilde{\mu}_{\mathbf{G}}(W)$.

Lemma 3.3 Suppose that the r-neighborhood of $\mathbf{p} = [\{p_i\}_{i=1}^{\infty}] \in \mathbf{X}_{\mathbf{G}}$ represents $\alpha \in U^{r,d}$. Then for ω -almost all $i \in \mathbb{N}$ the r-neighbourhood of $p_i \in V(G_i)$ represents α as well.

Proof. Let $\mathbf{q} \in B_r(\mathbf{p})$. Then there exists a path $\mathbf{p}^0, \mathbf{p}^1, \dots, \mathbf{p}^r$ in the graph $\mathcal{G}_{\mathbf{p}}$ such that $\mathbf{p}^0 = \mathbf{p}, \mathbf{p}^r = \mathbf{q}$. Therefore for ω -almost all $i \in \mathbb{N}, (p_i^k, p_i^{k+1}) \in E(G_i)$ thus if $\mathbf{q} = [\{q_i\}_{i=1}^{\infty}]$ then $q_i \in B_r(p_i)$ for ω -almost $i \in \mathbb{N}$. Obviously if \mathbf{q} and \mathbf{q}' are vertices in $B_r(\mathbf{p})$ then $(\mathbf{q}, \mathbf{q}') \in E(\mathcal{G}_{\mathbf{p}})$ if and only if $(q_i, q_i') \in E(G_i)$ for ω -almost all $i \in \mathbb{N}$. Also, if $\deg(\mathbf{q}) = k$ then $\deg(q_i) = k$ for ω -almost all $i \in \mathbb{N}$. This shows that $B_r(p_i) \cong B_r(\mathbf{p})$ for ω -almost all $i \in \mathbb{N}$. **Lemma 3.4** Let $U \subseteq Q(\alpha)$ be a closed-open subset. Then $F_{\kappa|B_r(\mathbf{p})} \in U$ if and only if $\kappa_{|B_r(p_i)} \in U$ for ω -almost all $i \in \mathbb{N}$, where $\kappa_{|B_r(p_i)}$ denotes the restriction of κ onto the set $B_r(p_i)$.

Observe that $F_{\kappa|B_r(\mathbf{p})} = \lim_{\omega} \kappa_{|B_r(p_i)}$. Note that the ultralimit of a sequence in a compact metric space is in the closure of the sequence. Therefore the lemma easily follows.

By Lemma 3.4, if $U \subseteq Q(\alpha)$ be a closed-open subset

$$\mu_{\mathbf{G}}(\{\mathbf{p} \in \mathbf{X}_{\mathbf{G}} \mid B_{r}(\mathbf{p}) \cong \alpha \text{ and } F_{|B_{r}(\mathbf{p})}^{\kappa} \in U\}) = \lim_{\omega} \frac{|T(G_{i}, \kappa, U)|}{|V(G_{i})|}$$

Since κ is a generic element in Ω , $\mu_{\mathbf{G}}(\rho^{-1}(U_{\alpha}^{k})) = \widetilde{\mu}_{\mathbf{G}}(U_{\alpha}^{k})$ for any $\alpha \in U^{r,d}$ and $k \geq 1$. Since $\{U_{\alpha}^{k}\}_{k=1}^{\infty}$ is a generating Boolean-algebra $\mu_{\mathbf{G}}(\rho^{-1}(W)) = \widetilde{\mu}_{\mathbf{G}}(W)$ holds for any Borel-set $W \subseteq \mathbf{BGr}_{d}$.

Corollary 3.1

- (a) For almost all $\mathbf{p} \in \mathbf{X}_{\mathbf{G}}$, $\mathcal{G}_{\mathbf{p}}$ is a separated B-graph.
- (b) The \mathbf{F}_2^{∞} -action on $\mathbf{O}_d \subseteq \mathbf{BGr}_d$ preserves the measure.

Proof. (a) follows from the fact that $\mu_{\mathbf{G}}(\rho^{-1}(\mathbf{O}_d)) = 1$. On the other hand ρ commutes with the \mathbf{F}_2^{∞} -action, that implies (b).

3.5 The ultraproduct of finite graphs

The goal of this subsection is to prove some auxiliary lemmas that shall be used in the proof of our main theorem. Let $\mathbf{G} = \{G_n\}_{n=1}^{\infty}$ be a weakly convergent sequence of finite graphs with vertex degree bound d. Let $\mathbf{X}_{\mathbf{G}}$ be the ultraproduct of their vertex sets and $\mathcal{G}_{\mathbf{X}}$ be the graphing constructed in the previous subsection. Then we have two notions for measure space of edge-sets. The first is the one constructed in Subsection 2.1. On the other hand, similarly to the ultraproduct of the vertex sets we can also define the ultraproducts of edge sets with normalised measure $\mu_{\mathbf{E}}$,

$$\mu_{\mathbf{E}}(L) = \lim_{\omega} \frac{|E(L_n)|}{|V(G_n)|},$$

where $L = [\{L_n\}_{n=1}^{\infty}], L_n \subseteq E(G_n)$. Again the ultraproduct sets $L = [\{L_n\}_{n=1}^{\infty}]$ form a Boolean-algebra $\mathcal{R}_{\mathbf{E}}$ and we can define the σ -algebra of measurable edge-sets by $\mathcal{M}_{\mathbf{E}}$ as well. It is easy to see that the two measure spaces above coincide. If $\mathbf{A} \in \mathcal{M}_{\mathbf{E}}$, then let $V(\mathbf{A})$ be the set of points \mathbf{p} in $\mathbf{X}_{\mathbf{G}}$ for which there exists $\mathbf{q} \in \mathbf{X}_{\mathbf{G}}$, $(\mathbf{p}, \mathbf{q}) \in \mathbf{A}$. Clearly if $\mathbf{A} \in \mathcal{R}_{\mathbf{E}}$ then $V(\mathbf{A}) \in \mathcal{R}$ and if **N** is a nullset of edges then $V(\mathbf{N})$ is a nullset of vertices. Consequently if $A \in \mathcal{M}_{\mathbf{E}}$ then $V(\mathbf{A}) \in \mathcal{M}$. Note that we can regard the elements of $\mathcal{M}_{\mathbf{E}}$ as measurable subgraphs of $\mathcal{G}_{\mathbf{X}}$.

Lemma 3.5 Let $\mathbf{H}_F \in \mathcal{M}_{\mathbf{E}}$ be a subgraph such that all of its components are isomorphic to a finite simple graph F. Then for any $\gamma > 0$ there exists $\mathbf{S}_F \subset \mathbf{H}_F$ such that

- $\mathbf{S}_F \in \mathcal{R}_{\mathbf{E}}$
- All the components of \mathbf{S}_F are isomorphic to F.
- $\mu_{\mathbf{G}}(V(\mathbf{H}_F \setminus \mathbf{S}_F)) < \gamma$.

Proof. Let $\mathbf{H}'_F \in \mathcal{R}_{\mathbf{E}}$ be a subgraph such that $\mu_{\mathbf{E}}(\mathbf{H}_F \triangle \mathbf{H}'_F) = 0$. Then $V(\mathbf{H}_F \triangle \mathbf{H}'_F)$ is a nullset in $\mathbf{X}_{\mathbf{G}}$. Consequently, $\mathbf{Q} = \operatorname{Orb}(V(\mathbf{H}_F \triangle \mathbf{H}'_F))$ is still a nullset in $\mathbf{X}_{\mathbf{G}}$, where

$$\mathbf{Q} = \bigcup_{\mathbf{p} \in V(\mathbf{H}_F \triangle \mathbf{H}'_F)} V(\mathcal{G}_{\mathbf{p}})$$

is the union of the orbits of the vertices in $\mathbf{H}_F \triangle \mathbf{H}'_F$. Let $\mathbf{Q} \subset \mathbf{K} \in \mathcal{R}$, $\mu_{\mathbf{G}}(\mathbf{K}) \leq \gamma$. Consider the subset $V(\mathbf{H}') \setminus \mathbf{K} \in \mathcal{R}$. Then the *r*-neighbourhood of $V(\mathbf{H}') \setminus \mathbf{K}$, $B_r(V(\mathbf{H}') \setminus \mathbf{K})$ is also an element of \mathcal{R} for any $r \geq 1$. Indeed, if $V(\mathbf{H}') \setminus \mathbf{K} = [\{A_i\}_{i=1}^{\infty}]$ then

$$B_r(V(\mathbf{H}')\backslash \mathbf{K}) = [\{B_r(A_i)\}_{i=1}^{\infty}]$$

Let $r > \operatorname{diam}(F)$ and \mathbf{S}_F be the spanned subgraph of $B_r(V(\mathbf{H}')\backslash \mathbf{K})$ in \mathbf{H}' . Then clearly $\mathbf{S}_F \in \mathcal{R}_{\mathbf{E}}$. Also, \mathbf{S}_F does not contain any vertex of $V(\mathbf{H}' \bigtriangleup \mathbf{H})$. Thus $V(\mathbf{S}_F) \subseteq V(\mathbf{H})$ and if $\mathbf{p} \in V(\mathbf{S}_F)$ then the component of \mathbf{p} in \mathbf{S}_F is just the component of \mathbf{p} in \mathbf{H} . Clearly, $\mu_{\mathbf{G}}(V(\mathbf{H}\backslash \mathbf{S}_F)) < \gamma$ thus our lemma follows.

Lemma 3.6 Let $\mathbf{S} = [\{S_n\}_{n=1}^{\infty}] \in \mathcal{R}_{\mathbf{E}}$ be a subgraph such that all of its components are isomorphic to the finite simple graph F. Then for ω -almost all n each component of S_n is isomorphic to F.

Proof. We prove the lemma by contradiction. Suppose that there exists $T \in \omega$ such that for any $n \in T$ there exists $p_n \in V(S_n)$ such that the component of S_n containing p_n is not isomorphic to F.

<u>Case 1</u>: If for ω -almost all elements of T there exists $q_n \in V(S_n)$ such that $d(p_n, q_n) = r > \operatorname{diam}(F)$ then there exists $(\mathbf{p}, \mathbf{q}) \in \mathbf{S}$ such that $d_{\mathbf{S}}(\mathbf{p}, \mathbf{q}) > \operatorname{diam}(F)$, leading to a contradiction.

<u>Case 2</u>: If for ω -almost all elements of T the component containing p_n has diameter less than 3r, then for ω -almost all elements of T the component containing p_n is isomorphic to the same finite graph G, where G is not isomorphic to F. Then there exists $\mathbf{p} \in V(\mathbf{S})$ such that the component of \mathbf{S} containing p is isomorphic to G. This also leads to a contradiction.

4 The proof of Theorem 1

Let $\epsilon > 0$ and $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \subset \operatorname{Graph}_d^f$ be a weakly convergent sequence of graphs. Also let $(\mathbf{BGr}_d, \tilde{\mu}_{\mathbf{G}})$ be the canonical limit object as in Subsection 2.4. Let \mathbf{BGr}_f be the \mathbf{F}_2^{∞} invariant subspace of graphs G in \mathbf{BGr}_d satisfying $|B_r(x)| \leq f(r)$ for all $x \in V(G)$. Let $\mathbf{O}_f = \mathbf{BGr}_f \cap \mathbf{O}_d$. Then \mathbf{O}_f is also \mathbf{F}_2^{∞} -invariant and $\tilde{\mu}_{\mathbf{G}}(\mathbf{BGr}_d \setminus \mathbf{O}_f) = 0$. Consider the induced graphing $(\mathcal{G}, \mathbf{O}_f, \tilde{\mu}_{\mathbf{G}})$. Since all the component graphs are of subexponential growth, by the theorem of Adams and Lyons [1], this graphing is hyperfinite. Therefore there exists a K > 0 and a component-finite subgraphing $\mathcal{H} \subset \mathcal{G}$ such that

- $\mu_E(E(\mathcal{G})\backslash E(\mathcal{H})) \leq \frac{\epsilon}{2}$.
- $\mathbf{O}_f = \bigcup_{H, |V(H)| \leq K} \mathcal{H}_H$, where \mathcal{H}_H is the set of points in \mathbf{O}_f contained in a component of \mathcal{H} isomorphic to H.
- $E(\mathcal{H}) = \bigcup_{H, |V(H)| \le K} E(\mathcal{H}_H)$

Let $c_H = \tilde{\mu}_{\mathbf{G}}(\mathcal{H}_H)$. Now suppose that our theorem does not hold. Therefore there exists a subsequence $\{G_{n_i}\}_{i=1}^{\infty}$ such that one can not remove $\epsilon E(G_{n_i})$ edges from any G_{n_i} to satisfy condition (1) of our Theorem with the extra condition that

$$\left|\frac{|V_H^{n_i}|}{|V(G_{n_i})|} - c_H\right| < \delta,$$

for any finite simple graph H, $|V(H)| \leq K$. Let $\mathbf{X}_{\mathbf{G}}$ be the ultraproduct of the graphs $\{G_{n_i}\}_{n=1}^{\infty}$. Note that the canonical limit objects of the sequence $\{G_{n_i}\}_{i=1}^{\infty}$ and of $\{G_n\}_{n=1}^{\infty}$ are the same. Therefore we have a measure-preserving map

$$\rho: (\mathbf{O}_{\mathbf{G}}, \mu_{\mathbf{G}}) \to (\mathbf{O}_{f}, \widetilde{\mu}_{\mathbf{G}}),$$

where O_G is the set of elements $\mathbf{p} \in \mathbf{X}_G$ such that $\mathcal{G}_{\mathbf{p}}$ is separated. Note that

• $\mu_{\mathbf{G}}(\mathbf{X}_{\mathbf{G}} \setminus \mathbf{O}_{\mathbf{G}}) = 0.$

- ρ commutes with the canonical \mathbf{F}_2^{∞} -actions.
- ρ preserves the isomorphism type of the orbit graphs.

Clearly, ρ extends to a measure-preserving map

$$\hat{\rho}: (E(\mathbf{O}_{\mathbf{G}}), \mu_{\mathbf{E}}) \to (E(\mathbf{O}_{f}), \tilde{\mu}_{E}),$$

where $\mu_{\mathbf{E}}$ and $\tilde{\mu}_{E}$ denote the induced measures on the edge-sets. Now fix a constant $\gamma > 0$. Let $\mathbf{A}_{H} = \hat{\rho}^{-1}(\mathcal{H}_{H})$. Then $\{\mathbf{A}_{H}\}_{H,|V(H)|\leq K}$ are component-finite subgraphings and all the components of \mathbf{A}_{H} are isomorphic to H. Observe that $\mu_{\mathbf{G}}(V(\mathbf{A}_{H})) = c_{H}$. Now first apply Lemma 3.5 to obtain subgraphings $\mathbf{S}_{H} \subset \mathbf{A}_{H}$ such that $\mu_{\mathbf{G}}(V(\mathbf{A}_{H} \setminus \mathbf{S}_{H})) < \gamma$ for each H. Then we apply Lemma 3.6 to obtain the graphs $\{S_{n_{i}}^{H}\}_{i=1}^{\infty}$ for each H such that

- $S_{n_i}^H \subset G_{n_i}$
- All the components of $S_{n_i}^H$ are isomorphic to H.
- $\lim_{\omega} \left| \frac{|V(S_{n_i}^H)|}{|V(G_{n_i})|} c_H \right| < \gamma$.

Thus for ω -almost all $i \in \mathbb{N}$

- $\bullet \left| \frac{|V(S_{n_i}^H)|}{|V(G_{n_i})|} c_H \right| < 2\gamma$
- $\left|\frac{E(G_{n_i}) \setminus \bigcup_{H, |V(H)| \le K} E(S_{n_i}^H)}{|V(G_{n_i})|}\right| < 2d\gamma g_K + \frac{\epsilon}{2}$ where g_K is the number of graphs having vertices not greater than K.

Since γ can be chosen arbitrarily we are in contradiction with our assumption on the graphs $\{G_{n_i}\}_{i=1}^{\infty}$.

Remark : In [23], Schramm proved that a graph sequence $\{G_n\}_{n=1}^{\infty}$ is hyperfinite if and only if its unimodular limit measure is hyperfinite. This idea was used in [7] to show that planarity is a testable property for bounded degree graphs. If we prove that the canonical limit of a hyperfinite graph sequence is always hyperfinite, then we can extend the results of our paper to arbitrary hyperfinite classes. This is subject of ongoing research [15]. In [11], the authors studied hereditary hyperfinite classes (see Corollary 3.2 of their paper). A graph class is hereditary if it is closed under vertex removal. Thus planar graphs of bounded degree d and Graph^f are both hereditary hyperfinite classes. The main result of [11] is that hereditary properties are testable in hyperfinite classes. It means that a tester accepts the graph if it has the property and rejects the graph with probability at least $(1 - \epsilon)$ if the graph is ϵ -far from the property in edit-distance. It would be interesting to see whether the edit-distance from a hereditary property is testable in a hereditary hyperfinite graph class.

5 Testing union-closed monotone graph properties

5.1 Edit-distance from a union-closed monotone graph property

Theorem 2 Let \mathcal{P} be a union-closed monotone graph property as in the Introduction. Then $\zeta(G) = d_e(G, \mathcal{P})$ is a continuous graph parameter on Graph_d^f .

Proof. We define the **normal distance** from a union-closed monotone class by

$$d_n(G, \mathcal{P}) = \inf_{H \subset G, H \in \mathcal{P}} d_e(G, H).$$

Lemma 5.1 $d_n(G, \mathcal{P}) = d_e(G, \mathcal{P})$

Proof. Clearly, $d_e(G, \mathcal{P}) \leq d_n(G, \mathcal{P})$. Now let $J \in \mathcal{P}$, V(J) = V(G). Then the spanning graph $J \cap G$ has also property \mathcal{P} and $d_e(G, J \cap G) \leq d_e(G, J)$. Therefore $d_e(G, \mathcal{P}) \geq d_n(G, \mathcal{P})$.

Now we prove a simple continuity lemma.

Lemma 5.2 If $G' \subseteq G$, $d_e(G, G') \leq \delta$ then $|d_n(G', \mathcal{P}) - d_n(G, \mathcal{P})| \leq \delta$.

Proof. Let $H' \subseteq G', H' \in \mathcal{P}$. Then $d_e(G, H') \leq d_e(G', H') + \delta$. Consequently, $d_n(G, \mathcal{P}) \leq d_n(G', \mathcal{P}) + \delta$. Now let $H \subseteq G, H \in \mathcal{P}$. Then $H \cap G' \in \mathcal{P}$. Since $d_e(G', H \cap G') \leq d_e(G, H)$, we obtain that $d_n(G', \mathcal{P}) \leq d_n(G, \mathcal{P})$.

Lemma 5.3 Let $A_1, A_2, A_3, \ldots, A_l$ be finite simple graphs. Suppose that the graph A consists of m_1 disjoint copies of A_1 and m_2 disjoint copies of A_2 ... and m_l disjoint copies of A_l . That is $|V(A)| = \sum_{i=1}^l m_i |V(A_i)|$. Then

$$d_n(A, \mathcal{P}) = \sum_{i=1}^l w_i d_n(A_i, \mathcal{P}),$$

where $w_i = \frac{m_i |V(A_i)|}{\sum_{i=1}^{l} m_i |V(A_i)|}$.

Proof. Let $B \subset A$ be the closest subgraph in \mathcal{P} . Then $B \cap A_i^j$ is the closest subgraph in \mathcal{P} in each copy of A_i . Hence

$$d_n(A, \mathcal{P}) = \frac{|E(A \setminus B)|}{\sum_{i=1}^l m_i |V(A_i)|} = \frac{\sum_{i=1}^l m_i E(A_i \setminus B)}{\sum_{i=1}^l m_i |V(A_i)|} =$$
$$= \frac{\sum_{i=1}^l m_i d_n(A_i, \mathcal{P}) |V(A_i)|}{\sum_{i=1}^l m_i |V(A_i)|} \blacksquare$$

Now let $\mathbf{G} = \{G_n\}_{n=1}^{\infty} \in \operatorname{Graph}_d^f$ be a weakly convergent graph sequence and $\epsilon > 0$. Consider the graphs G'_n in Theorem 1. Then by Lemma 5.2, $|d_n(G_n, \mathcal{P}) - d(G'_n, \mathcal{P})| < \epsilon$. Let s_H^n be the number of components in G'_n isomorphic to H. By Lemma 5.3,

$$d_n(G'_n, \mathcal{P}) = \sum_{H, |H| \le K} \frac{s_H^n |V(H)| d_n(H, \mathcal{P})}{|V(G_n)|} \,.$$

By Theorem 1,

$$\lim_{n \to \infty} \frac{s_H^n |V(H)|}{|V(G_n)|} = c_H$$

Therefore $\lim_{n\to\infty} d_n(G'_n, \mathcal{P}) = \sum_{H,|H|\leq K} c_H d_n(H, \mathcal{P})$. Hence if n, m are large enough then $|d_n(G_n, \mathcal{P}) - d_n(G_m, \mathcal{P})| < 3\epsilon$. Consequently, $\lim_{n\to\infty} d_n(G_n, \mathcal{P})$ exists.

5.2 Testability versus continuity

Let ζ : $\operatorname{Graph}_d^f \to \mathbb{R}$ be a continuous graph parameter. Let $\epsilon > 0$ be a real constant and $N(\zeta, \epsilon) > 0, r(\zeta, \epsilon) > 0, k(\zeta, \epsilon) > 0$ be integer numbers. An (ϵ, N, r, k) -random sampling is the following process. For a graph $G \in \operatorname{Graph}_d^f$, $|V(G)| \ge N(\zeta, \epsilon)$ we randomly pick $k(\zeta, \epsilon)$ vertices of G. Then by examining the $r(\zeta, \epsilon)$ -neighbourhood of the chosen vertices we obtain an empirical distribution

$$Y: \bigcup_{s \le r} U^{s,d} \to \mathbb{R} \,.$$

A (ζ, ϵ) -tester is an algorithm T which takes the empirical distribution Y as an input and calculates the real number T(Y). We say that ζ is **testable** if for any $\epsilon > 0$ there exist constants $N(\zeta, \epsilon) > 0$, $r(\zeta, \epsilon) > 0$, $k(\zeta, \epsilon) > 0$ and a (ζ, ϵ) -tester such that

$$\operatorname{Prob}(|T(Y) - \zeta(G)| > \epsilon) < \epsilon.$$

In other words, the tester estimates the value of ζ on G using a random sampling and guarantees that the error shall be less than ϵ with probability $1 - \epsilon$.

Theorem 3 Any continuous graph parameter ζ on $Graph_d^f$ is testable.

Proof. Since ζ is continuous on the compactification of Graph_d^f , for any $\epsilon > 0$ there exist constants $r(\zeta, \epsilon) > 0$ and $\delta(\epsilon, \zeta) > 0$ such that

If
$$|p_G(\alpha) - p_{G'}(\alpha)| < \delta$$
 for all $\alpha \in U^{s,d}, s \le r$ then $|\zeta(G) - \zeta(G')| < \epsilon$. (3)

Also, by the total boundedness of compact metric spaces, there exists a finite family of graphs (depending on ζ and ϵ) $\{G_1, G_2, \ldots, G_t\} \subset \operatorname{Graph}_d^f$ such that for any $G \in \operatorname{Graph}_d^f$ there exists at least one G_k , $1 \leq k \leq t$ such that

$$|p_{G_k}(\alpha) - p_G(\alpha)| < \frac{\delta}{2}$$

for each $\alpha \in U^{s,d}$, $s \leq r$. By the law of large numbers there exist constants $N(\zeta, \epsilon) > 0$ and $k(\zeta, \epsilon) > 0$ such that if Y is the empirical distribution of an (ϵ, N, r, k) -random sampling then the probability that there exists an $\alpha \in U^{s,d}$ for some $s \leq r$ satisfying

$$|p_G(\alpha) - Y(\alpha)| > \frac{\delta}{2}$$

is less than ϵ . Note that the sampling is taking place on the vertices of G and $|V(G)| > N(\zeta, \epsilon)$. The tester works as follows. First the sampler measures Y. Then the algorithm compares the vector $\{Y(\alpha)\}_{\alpha \in U^{s,d}, s \leq r}$ to a finite database containing the vectors

$$\{p_{G_1}(\alpha)\}_{\alpha \in U^{s,d}, s \leq r}, \{p_{G_2}(\alpha)\}_{\alpha \in U^{s,d}, s \leq r}, \dots, \{p_{G_t}(\alpha)\}_{\alpha \in U^{s,d}, s \leq r}.$$

Now with probability at least $(1-\epsilon)$ the algorithm finds $1 \le k \le t$ such that $|p_G(\alpha) - p_{G_k}(\alpha)| \le \delta$ for any $\alpha \in U^{s,d}, s \le r$. Then the output T(Y) shall be $\zeta(G_k)$. By (3) the probability that $|\zeta(G) - T(Y)| > \epsilon$ is less than ϵ .

6 Continuous parameters in Graph_d^f

6.1 Integrated density of states

Integrated density of states is a fundamental concept in mathematical physics. Let us explain, how this notion is related to graph parameters. Recall that the Laplacian on the finite graphs $G, \Delta_G : l^2(V(G)) \to l^2(V(G))$ is a positive, self-adjoint operator defined by

$$\Delta_G(f)(x) := \deg(x)f(x) - \sum_{(x,y)\in E(G)} f(y).$$

For a finite dimensional self-adjoint linear operator $A: \mathbb{R}^n \to \mathbb{R}^n$, the normalised spectral distribution of A is given by

$$N_A(\lambda) := \frac{s_A(\lambda)}{n},$$

where $s_A(\lambda)$ is the number of eigenvalues of A not greater than λ counted with multiplicities. Therefore $N_{\Delta_G}(\lambda)$ is a graph parameter for every $\lambda \geq 0$. Now consider the 3-dimensional lattice graph \mathbb{Z}^3 . The finite cubes C_n are the graphs induced on the sets $\{-n, -n+1, \ldots, n-1, n\}^3$. It is known for decades that for any $\lambda \geq 0 \lim_{n\to\infty} N_{\Delta_{C_n}}(\lambda)$ exists and in fact the convergence is uniform in λ . In other words, the integrated density of states exists in the uniform sense. The discovery of quasicrystals led to the study of certain infinite graphs that are not periodic as the lattice graph. What sort of graphs are we talking about ?

Let G be an infinite connected graph such that $|B_r(x)| \leq f(r)$, for any $x \in V(G)$. We say that a sequence of finite induced subgraphs $\{F_n\}_{n=1}^{\infty}, \cup_{n=1}^{\infty} F_n = G$ form a Følner-sequence if $\lim_{n\to\infty} \frac{|\partial F_n|}{|V(F_n)|} = 0$, where

$$\partial F_n := \{ p \in V(G) \ | \ p \in V(F_n) \text{ and there exists } q \notin V(F_n) \text{ such that } (p,q) \in E(G) \}$$

Note that subexponential growth implies that for any $x \in V(G)$, $\{B_n(x)\}_{n=1}^{\infty}$ contains a Følnersubsequence.

We say that an infinite graph G of subexponential growth has **uniform patch frequency** if all of its Følner-subgraph sequences are weakly convergent. Obviously, the lattices \mathbb{Z}^n are of uniform patch frequency, but there are plenty of aperiodic UPF graphs as well, among them the graph of a Penrose tiling, or other Delone-systems [20].

Using ergodic theory, Lenz and Stollmann proved the existence of the integrated density of states in the uniform sense for such Delone-systems [20] and later we extended their results for all UPF graphs of subexponential growth [14]. This last result can be interpreted the following way : $N_{\Delta_G}(\lambda)$ are continuous graph parameters in Graph_d^f for any $\lambda \geq 1$.

In this subsection we apply our Theorem 1 to extend the theorem in [14] for discrete Schrödinger operators with random potentials (as a general reference see the lecture notes of Kirsch [19]). Let X be a random variable taking finitely many real values $\{r_1, r_2, \ldots, r_m\}$. Let

$$\operatorname{Prob}(X=r_i)=p_i\,.$$

For the vertices p of G we consider independent random variables X_p with the same distribution

as X. Let Ω_G^X be the space of $\{r_1, r_2, \ldots, r_m\}$ -valued functions on V(G) with the product measure ν_G . That is

$$\mu_G(\{\omega \mid \omega(x_1) = r_{i_1}, \omega(x_2) = r_{i_2}, \dots, \omega(x_k) = r_{i_k}\}) = \prod_{j=1}^k p_{i_j}$$

for any k-tuple $(x_1, x_2, \ldots, x_k) \subset V(G)$. Thus for each $\omega \in \Omega_G^X$ we have a self-adjoint operator $\Delta_G^{\omega} : l^2(V(G)) \to l^2(V(G))$, given by

$$\Delta_G^{\omega}(f)(x) = \Delta_G(f)(x) + \omega(x)f(x) \,.$$

This operator is a discrete Schrödinger operator with random potential. The following theorem is the extension of the main theorem of [14] for such operators. Note that in the case of Euclidean lattices a similar result was proved by Delyon and Souillard [12].

Theorem 4 Let f be a function of subexponential growth. Let G be an infinite connected graph such that $|B_r(x)| \leq f(r)$, for any $x \in V(G)$ with UPF and let $\{G_n\}_{n=1}^{\infty}$ be a Følnersequence. Then for almost all $\omega \in \Omega_G^X \{N_{\Delta_{G_n}}\}_{n=1}^{\infty}$ uniformly converges to an integrated density of states N_G^X that does not depend on ω That is the integrated density of state for such discrete Schröedinger operator with random potential is non-random. (see also [21] and the references therein)

Proof. Let $\epsilon > 0$ and $G'_n \subset G_n$ be the spanning graphs as in Theorem 1. First we prove the analog of Lemma 5.2.

Lemma 6.1 For any $\omega \in \Omega_G^X$,

$$|N_{\Delta_{G_n}^{\omega}}(\lambda) - N_{\Delta_{G_n}^{\omega}}(\lambda)| \le \epsilon d \,,$$

for any $-\infty < \lambda < \infty$, where d is the uniform bound on the degrees of $\{G_n\}_{n=1}^{\infty}$.

Proof. Observe that

$$\operatorname{Rank}(\Delta_{G_n}^{\omega} - \Delta_{G'_n}^{\omega}) < 2\epsilon |E(G_n)|.$$
(4)

Indeed, let $1_p \in l^2(V(G_n))$ be the function, where $1_p(q) = 0$ if $p \neq q$ and $1_p(p) = 1$. Then $(\Delta_{G_n}^{\omega} - \Delta_{G'_n}^{\omega})(1_p) = 0$ if the edges incident to p are the same in G_n as in G'_n . Therefore

$$\dim_{\mathbb{R}} \operatorname{Ker}(\Delta_{G_n}^{\omega} - \Delta_{G'_n}^{\omega}) \ge |V(G_n)| - 2\epsilon |E(G_n)|.$$

Consequently (4) holds. Hence the lemma follows immediately from Lemma 3.5 [14]. Now we prove the analog of Lemma 5.3.

Lemma 6.2 Let the finite graph A be the disjoint union of m_1 copies of A_1 , m_2 copies of A_2, \ldots, m_l copies of A_l as in Lemma 5.3. Then

$$N_{\Delta_A^{\omega}}(\lambda) = \sum_{i=1}^l w_i N_{\Delta_{A_i}^{\omega}}(\lambda) \,,$$

where $w_i = \frac{m_i |V(A_i)|}{\sum_{i=1}^l m_i |V(A_i)|}$.

Proof. Clearly, $s_{\Delta_{H}^{\omega}}(\lambda) = \sum_{i=1}^{l} m_{i} s_{\Delta_{A_{i}}^{\omega}}(\lambda)$. Therefore

$$N_{\Delta_{H}^{\omega}}(\lambda) = \frac{\sum_{i=1}^{l} m_{i} s_{\Delta_{A_{i}}^{\omega}}(\lambda)}{\sum_{i=1}^{l} m_{i} |V(A_{i})|} = \sum_{i=1}^{l} w_{i} N_{\Delta_{A_{i}}^{\omega}}(\lambda).$$

For each $\omega \in \Omega_G^X$ we have a natural vertex-labeling of G (and of its subgraphs), where $\omega(p)$ is the label of the vertex p. Clearly, if ω and ω' coincide on the finite graph F then $\Delta_F^{\omega} = \Delta_F^{\omega'}$. Now let H be a finite simple graph, $|V(H)| \leq K$, where K is the constant in Theorem 1. Let $\{H_\alpha\}_{\alpha \in I_H}$ be the set of all vertex-labellings of H by $\{r_1, r_2, \ldots, r_m\}$ up to labelled-isomorphisms. By the law of large numbers, for almost all $\omega \in \Omega_G^X$ the number of labelled vertices in G'_n belonging to a vertex-labelled component labelled-isomorphic to H_α divided by $|V(G_n)|$ converges to a constant $q(H_\alpha)$. Notice that $q(H_\alpha) = c_H p(H_\alpha)$, where $p(H_\alpha)$ is the probability that a X-random labelling of the vertices of H is labelled-isomorphic to H_α .

Hence by Lemma 6.2, for almost all $\omega \in \Omega_G^X$ the functions $\{N_{\Delta_{G'_n}}\}_{n=1}^{\infty}$ converge uniformly to a function N_{ϵ} that does not depend on ω . Let $W \subset \Omega_G^X$ be the set of elements such that $\{N_{\Delta_{G'_n}}\}_{n=1}^{\infty}$ converge uniformly to $N_{\frac{1}{k}}$ for any $k \geq 1$. Clearly, $\nu_G(W) = 1$. The following lemma finishes the proof of our Theorem.

Lemma 6.3 $\{N_{\frac{1}{k}}\}_{k=1}^{\infty}$ converge uniformly to a function N_G^X . Also, for any $\omega \in W$ $\{N_{\Delta_{G_n}}\}_{n=1}^{\infty}$ converge uniformly to N_G^X .

Proof. By Lemma 6.1 if $\omega \in W$ then for large enough n

$$|N_{\Delta_{G_n}^{\omega}}(\lambda) - N_{\frac{1}{k}}(\lambda)| \le 2d\frac{1}{k} \text{ for } -\infty < \lambda < \infty.$$

Therefore $\{N_{\frac{1}{k}}\}_{k=1}^{\infty}$ form a Cauchy-sequence and consequently $\{N_{\Delta_{G_n}}\}_{n=1}^{\infty}$ converge uniformly to N_G^X .

6.2 Independence ratio, entropy and log-partitions

First recall the notion of some graph parameters associated to independent sets and matchings. Let H be a finite graphs.

- Let I(H) be the maximal size of an independent subset in V(H). The number $\frac{I(H)}{|V(H)|}$ is called the independence ratio of H.
- Let M(H) be the maximal size of a matching in E(H). The number $\frac{M(H)}{|V(H)|}$ is called the matching ratio of H.
- Let $\pi_H^I(\lambda) = \sum_{\{S \subset V(H), S \text{ is independent}\}} \lambda^{|S|}$ be the partition function corresponding to the system of independent subsets.
- Let $\pi_H^M(\lambda) = \sum_{\{T \subset E(H), T \text{ is a matching}\}} \lambda^{|T|}$ be the partition function corresponding to the system of matchings.

Now we prove that all the graph parameters above are continuous in Graph_d^f . That is we show the analog of the existence of the integrated density of states for the quantities above.

Theorem 5 Let G be an infinite connected graph such that $|B_r(x)| \leq f(r)$, for any $x \in V(G)$ (where f is of subexponential growth) with UPF and $\{G_n\}_{n=1}^{\infty}$ be a Følner-sequence. Then

$$\begin{array}{l} (a) \ \lim_{n \to \infty} \frac{I(G_n)}{|V(G_n)|} \ exists. \\ (b) \ \lim_{n \to \infty} \frac{M(G_n)}{|V(G_n)|} \ exists. \\ (c) \ \lim_{n \to \infty} \frac{\log \pi_{G_n}^I(\lambda)}{|V(G_n)|} \ exists \ for \ all \ 0 < \lambda < \infty \,. \\ (d) \ \lim_{n \to \infty} \frac{\log \pi_{G_n}^M(\lambda)}{|V(G_n)|} \ exists \ for \ all \ 0 < \lambda < \infty \,. \end{array}$$

Note that if $\lambda = 1$ then the limit value is the associated entropy.

Proof. We prove only (a) and (c), since the proofs of (b) and (d) are completely similar. Let $\epsilon > 0$ and $G'_n \subset G_n$ be the spanning graphs as in Theorem 1. Again, we prove the continuity lemma.

Lemma 6.4

$$\begin{split} \left| \frac{I(G'_n)}{|V(G_n)|} - \frac{I(G_n)}{|V(G_n)|} \right| &< \epsilon d \\ \left| \frac{\log \pi^I_{G'_n}(\lambda)}{|V(G_n)|} - \frac{\log \pi^I_{G_n}(\lambda)}{|V(G_n)|} \right| &< (\log(\max(1,\lambda)) + 2)\epsilon d \,. \end{split}$$

Proof. Clearly, $I(G'_n) \ge I(G_n)$. Let A be a maximal independent subset of G'_n . Then if we delete the vertices from A that are incident to an edge of $E(G_n) \setminus E(G'_n)$, the remaining set is an independent subset of the graph G_n . Hence

$$\left|\frac{I(G'_n)}{|V(G_n)|} - \frac{I(G_n)}{|V(G_n)|}\right| < \epsilon d.$$

Now let S be an independent subset of the graph G_n . Denote by Q(S) the set of independent subsets S' of G'_n such that $S \subseteq S'$ and if $p \in S' \setminus S$ then p is incident to an edge of $E(G_n) \setminus E(G'_n)$. Observe that

$$\sum_{\{S' \subset V(G_n) \mid S' \text{ is independent in } G'_n\}} \lambda^{|S'|} \leq \sum_{\{S \subset V(G_n) \mid S \text{ is independent in } G_n\}} \sum_{S' \in Q(S)} \lambda^{|S'|} \leq \sum_{\{S \subset V(G_n) \mid S \text{ is independent in } G_n\}} \lambda^{|S|} \max(1, \lambda)^{\epsilon d \mid V(G) \mid} 2^{\epsilon d \mid V(G) \mid}.$$

That is

$$\log \pi_{G'_n}^I(\lambda) \le \log \pi_{G_n}^I(\lambda) + \left(\log(\max(1,\lambda)) + 2\right)\epsilon d|V(G_n)|.$$

Lemma 6.5 Let $A_1, A_2, A_3, \ldots, A_l$ be finite simple graphs. Suppose that the graph H consists of m_1 disjoint copies of A_1 and m_2 disjoint copies of A_2, \ldots and m_l disjoint copies of A_l . Then

$$\frac{I(H)}{|V(H)|} = \sum_{i=1}^{l} w_i \frac{I(A_i)}{|V(A_i)|}$$
(5)

$$\frac{\log \pi_H^I(\lambda)}{|V(H)|} = \sum_{i=1}^l w_i \frac{\log \pi_{A_i}^I(\lambda)}{|V(A_i)|},\tag{6}$$

where $w_i = \frac{m_i |V(A_i)|}{\sum_{i=1}^k m_i |V(A_i)|}$.

Proof. Note that $I(H) = \sum_{i=1}^{l} m_i I(A_i)$ and $\pi_H^I(\lambda) = \prod_{i=1}^{l} (\pi_{A_i}^I)^{m_i}$. That is $\log(\pi_H^I(\lambda)) = \sum_{i=1}^{l} m_i \log(\pi_{A_i}^I)$. Now we proceed as in Lemma 6.1 \blacksquare . By the two preceding lemmas Theorem 5 easily follows.

Remark: In Theorem 3. [5] the authors proved that $\lim_{n\to\infty} \frac{I(G_n)}{|V(G_n)|}$ exists if $\{G_n\}_{n=1}^{\infty}$ is a *r*-regular large girth sequence, where $2 \le r \le 5$.

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