# DISTANCES BETWEEN PAIRS OF VERTICES AND VERTICAL PROFILE IN CONDITIONED GALTON-WATSON TREES 

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#### Abstract

We consider a conditioned Galton-Watson tree and prove an estimate of the number of pairs of vertices with a given distance, or, equivalently, the number of paths of a given length.

We give two proofs of this result, one probabilistic and the other using generating functions and singularity analysis.

Moreover, the second proof yields a more general estimate for generating functions, which is used to prove a conjecture by Bousquet-Mélou and Janson [5], saying that the vertical profile of a randomly labelled conditioned Galton-Watson tree converges in distribution, after suitable normalization, to the density of ISE (Integrated Superbrownian Excursion).


## 1. Introduction and results

Let $T_{n}$ be a conditioned Galton-Watson tree, i.e., the random rooted tree $\mathcal{T}$ obtained as the family tree of a Galton-Watson process with some given offspring distribution $\xi$, conditioned on the number of vertices $|\mathcal{T}|=n$. We will always assume that

$$
\begin{equation*}
\mathbb{E} \xi=1 \quad \text { and } \quad 0<\sigma^{2}:=\operatorname{Var} \xi<\infty \tag{1.1}
\end{equation*}
$$

In other words, the Galton-Watson process is critical and with finite variance, and $\mathbb{P}(\xi=1)<1$. (Note that this entails $0<\mathbb{P}(\xi=0)<1$.) It is well-known that this assumption is without essential loss of generality, and that the resulting random trees are essentially the same as the simply generated families of trees introduced by Meir and Moon [13]. The importance of this construction lies in that many combinatorially interesting random trees are of this type, for example random plane (= ordered) trees, random unordered labelled trees (Cayley trees), random binary trees, and (more generally) random $d$-ary trees. For further examples see e.g. Aldous [1] and Devroye [6].

[^0]We consider only $n$ such that $T_{n}$ exists, i.e., such that $\mathbb{P}(|\mathcal{T}|=n)>0$. The span of $\xi$ is defined to be the largest integer $d$ such that $\xi \in d \mathbb{Z}$ a.s. If the span of $\xi$ is $d$, then $T_{n}$ exists only for $n \equiv 1(\bmod d)$, and it exists for all large such $n$.

We consider in this paper two types of properties of $T_{n}$ that turn out to have proofs using a common argument. First, for an arbitrary rooted tree $\tau$, let $P_{k}(\tau), k \geq 1$, be the number of (unordered) pairs of vertices $\{v, w\}$ in $\tau$ such that the distance $d(v, w)=k$; equivalently, $P_{k}(\tau)$ is the number of paths of length $k$ in $\tau$. Our first result is an estimate, uniform in all $k$ and $n$, of the expectation of this number $P_{k}\left(T_{n}\right)$ for a conditioned Galton-Watson tree $T_{n}$.

We let in this paper $C_{1}, C_{2}$ and $c_{1}, c_{2}$ denote various positive constants that may depend on (the distribution of) $\xi$, and sometimes later $\eta$ introduced below, but not on $n, k$ and other variables unless explicitly stated. Recall that we tacitly assume (1.1).

Theorem 1.1. There exists a constant $C_{1}$ such that for all $k \geq 1$ and $n \geq 1$, $\mathbb{E} P_{k}\left(T_{n}\right) \leq C_{1} n k$.

One way to interpret this result is that the expected number of vertices of distance $k$ from a randomly chosen vertex in $T_{n}$ is $O(k)$. In other words, if $T_{n}^{*}$ is $T_{n}$ randomly rerooted, and $Z_{k}(\tau)$ is the number of vertices of distance $k$ from the root in a rooted tree $\tau$, then the following holds.
Corollary 1.2. $\mathbb{E} Z_{k}\left(T_{n}^{*}\right)=O(k)$, uniformly in all $k \geq 1$ and $n \geq 1$.
This can be compared to [10, Theorem 1.13], which shows that

$$
\begin{equation*}
\mathbb{E} Z_{k}\left(T_{n}\right)=O(k), \tag{1.2}
\end{equation*}
$$

again uniformly in $k$ and $n$. Note that in the special case when $T_{n}$ is a random (unordered) labelled tree, $T_{n}^{*}$ has the same distribution, so Corollary 1.2 reduces to (1.2). However, in general, a randomly rerooted conditioned Galton-Watson tree is not a conditioned Galton-Watson tree.

Remark. The emphasis is on uniformity in both $k$ and $n$. If we, on the contrary, fix $k$ and consider limits as $n \rightarrow \infty$, we have $\mathbb{E} Z_{k}\left(T_{n}\right) \rightarrow 1+k \sigma^{2}$, see Meir and Moon [13] and Janson [10; 11]. It is shown in [11] that the sequence $\mathbb{E} Z_{k}\left(T_{n}\right)$ is not always monotone in $n$.

We give a probabilistic proof of Theorem 1.1, and thus of Corollary 1.2 too, in Section 2.

We also give a second proof by first proving a corresponding estimate for the generating function. (We present two different proofs, since we find both methods interesting, and both methods yield as intermediary steps in the proofs other results that we find interesting.) Let $f_{n}(z)$ be the generating function defined by

$$
f_{n}(z):=\sum_{k=1}^{\infty} \mathbb{E} P_{k}\left(T_{n}\right) z^{k}
$$

We will use standard singularity analysis, see e.g. Flajolet and Sedgewick [9], and define the domain, for $0<\beta<\pi / 2$ and $\delta>0$,

$$
\Delta(\beta, \delta):=\{z \in \mathbb{C}:|z|<1+\delta, z \neq 1,|\arg (z-1)|>\pi / 2-\beta\} .
$$

Note that $|\arg (z-1)|>\pi / 2-\beta$ is equivalent to $|\arg (1-z)|<\pi / 2+\beta$.
Theorem 1.3. For every $\xi$ there exist positive constants $C_{2}, \beta, \delta$ such that for all $n \geq 1, f_{n}$ extends to an analytic function in $\Delta(\beta, \delta)$ with

$$
\begin{equation*}
\left|f_{n}(z)\right| \leq C_{2} n|1-z|^{-2}, \quad z \in \Delta(\beta, \delta) \tag{1.3}
\end{equation*}
$$

By standard singularity analysis (i.e., estimate of the Taylor coefficients of $f_{n}(z)$ using Cauchy's formula and a suitable contour in $\Delta(\beta, \delta)$ ), (1.3) implies $\mathbb{E} P_{k}\left(T_{n}\right)=O(n k)$, see Flajolet and Sedgewick [9], Theorem VI. 3 and (for the uniformity in $n$ ) Lemma IX. 2 (applied to the family $\left\{f_{n}(z) / n\right\}$ ). Hence, Theorem 1.1 follows from Theorem 1.3 ,

For each pair of vertices $v, w$ in a rooted tree, the path from $v$ to $w$ consists of two (possibly empty) parts, one going from $v$ towards the root, ending at the last common ancestor $v \wedge w$ of $v$ and $w$, and another part going from $v \wedge w$ to $w$ in the direction away from the root. We will also prove extensions of the results above for $T_{n}$, where we consider separately the lengths of these two parts. Define the corresponding bivariate generating function (now considering ordered pairs $v, w$ )

$$
\begin{equation*}
h_{n}(x, y):=\mathbb{E} \sum_{v, w \in T_{n}} x^{d(v, v \wedge w)} y^{d(w, v \wedge w)} . \tag{1.4}
\end{equation*}
$$

Theorem 1.4. For every $\xi$ there exist positive constants $C_{3}, \beta, \delta$ such that for all $n \geq 1$,

$$
\left|h_{n}(x, y)\right| \leq C_{3} n|1-x|^{-1}|1-y|^{-1}, \quad x, y \in \Delta(\beta, \delta) .
$$

Note that, by (1.4) and (1.3)

$$
h_{n}(z, z)=\mathbb{E} \sum_{v, w \in T_{n}} z^{d(v, w)}=n+2 f_{n}(z) .
$$

Hence Theorem 1.3 follows from Theorem [1.4. We prove Theorem 1.4 in Section 4.

If we define $\widetilde{P}_{\ell, m}(\tau):=\#\{(v, w) \in \tau: d(v, v \wedge w)=\ell, d(w, v \wedge w)=m\}$, then singularity analysis as above (but twice) shows that Theorem 1.4 implies the following. (Since $P_{k}=\frac{1}{2} \sum_{\ell=0}^{k} \widetilde{P}_{\ell, k-\ell}$, this too implies Theorem 1.1.)

Theorem 1.5. There exists a constant $C_{4}$ such that for all $\ell, m \geq 0$ and $n \geq 1, \mathbb{E} \widetilde{P}_{\ell, m}\left(T_{n}\right) \leq C_{4} n$.

One motivation for these results is that they (more precisely, Theorem 1.4) are used to prove the second type of result in this paper. For this, we assume that we are given a further random variable $\eta$. Given a rooted tree $\tau$, we take an independent copy $\eta_{e}$ of $\eta$ for every edge $e \in \tau$. We give each vertex
$v$ the label $L_{v}$ obtained by summing $\eta_{e}$ for all $e$ in the path from the root $o$ to $v$. (Thus, $L_{o}=0$.) We assume that

$$
\begin{equation*}
\mathbb{E} \eta=0 \quad \text { and } \quad 0<\sigma_{\eta}^{2}:=\operatorname{Var} \eta<\infty \tag{1.5}
\end{equation*}
$$

We further assume that $\eta$ is integer valued and with span 1 ; thus all labels are integers, and all integers are possible labels.

We let $X(j ; \tau)$ be the number of vertices in $\tau$ with label $j$; the sequence $(X(j ; \tau))_{j=-\infty}^{\infty}$ is the vertical profile of the labelled tree.

For the random tree $T_{n}$, we assume that the variables $\eta_{e}$ are independent of $T_{n}$. The vertical profile $X\left(j ; T_{n}\right)$ then is a random function defined for $j \in \mathbb{Z}$; we write $X_{n}(j):=X\left(j ; T_{n}\right)$ and extend the domain of $X_{n}$ to $\mathbb{R}$ by linear interpolation between the integer points. Our next theorem says that this function $X_{n}$, suitable normalized, converges in distribution in the space $C_{0}(\mathbb{R})$ of continuous functions on $\mathbb{R}$ that tend to 0 at $\pm \infty$; we equip $C_{0}(\mathbb{R})$ with the usual uniform topology defined by the supremum norm. Let, further, $f_{\text {ISE }}$ denote the density of the random measure ISE introduced by Aldous [3]; $f_{\text {ISE }}$ is a random continuous function with (random) compact support, see Bousquet-Mélou and Janson [5, Theorem 2.1].

Theorem 1.6. With the assumptions above, including (1.1) and (1.5), let $\gamma:=\sigma_{\eta}^{-1} \sigma^{1 / 2}$. Then, as $n \rightarrow \infty$,

$$
\begin{equation*}
\frac{1}{n} \gamma^{-1} n^{1 / 4} X_{n}\left(\gamma^{-1} n^{1 / 4} x\right) \xrightarrow{\mathrm{d}} f_{\mathrm{ISE}}(x), \tag{1.6}
\end{equation*}
$$

in the space $C_{0}(\mathbb{R})$ with the usual uniform topology. Equivalently,

$$
\begin{equation*}
n^{-3 / 4} X_{n}\left(n^{1 / 4} x\right) \xrightarrow{\mathrm{d}} \gamma f_{\mathrm{ISE}}(\gamma x) . \tag{1.7}
\end{equation*}
$$

Note that the random functions on the left and right hand sides of (1.6) and (1.7) are density functions, i.e., non-negative functions with integral 1.

Corollary 1.7. If $n \rightarrow \infty$ and $j_{n} / n^{1 / 4} \rightarrow x$, where $-\infty<x<\infty$, then $n^{-3 / 4} X\left(j_{n} ; T_{n}\right) \xrightarrow{\mathrm{d}} \gamma f_{\text {ISE }}(\gamma x)$.

The limit law is characterized in (4] by a formula for its Laplace transform.
Theorem 1.6 was conjectured in [5], and proved there in two special cases, viz. when $\xi$ has the Geometric distribution $\mathrm{Ge}(1 / 2)$ and thus $T_{n}$ is a random ordered tree, and $\eta$ is uniformly distributed on either $\{-1,1\}$ or $\{-1,0,1\}$. Moreover, it was shown there [5, Remark 3.7] that the proof given in [5] applies generally under the assumptions above, provided the following estimate holds.

Lemma 1.8. Under the assumptions above, there exists a constant $C_{5}$ such that for all $n \geq 1$ and $t \in[-\pi, \pi]$,

$$
\begin{equation*}
\mathbb{E}\left|\frac{1}{n} \sum_{j} X\left(j ; T_{n}\right) e^{\mathrm{i} j t}\right|^{2} \leq \frac{C_{5}}{1+n t^{4}} . \tag{1.8}
\end{equation*}
$$

We prove Lemma 1.8, and thus Theorem 1.6, in Section 3, assuming Theorem 1.4. Finally, we prove Theorem 1.4, using singularity analysis again, in Section 4, which completes the proof of all other results.

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## 2. First proof of Theorem 1.1

In a rooted tree $\tau$, let $Q_{k}(\tau), k \geq 1$, denote the number of (unordered) pairs of vertices at path distance $k$ from each other such that the path between them visits the root, and let $Q_{k}^{\prime}(\tau)$ be the number of such pairs where the root cannot be one of the two vertices in the pair; thus $Q_{k}(\tau)=$ $Q_{k}^{\prime}(\tau)+Z_{k}(\tau)$. Then, in the Galton-Watson tree $\mathcal{T}$, if $\xi$ is the number of children of the root, and the subtrees rooted at these children are denoted $\mathcal{T}_{1}, \ldots, \mathcal{T}_{\xi}$,

$$
\begin{equation*}
Q_{k}^{\prime}(\mathcal{T})=\sum_{(r, s): 1 \leq r<s \leq \xi} \sum_{j=0}^{k-2} Z_{j}\left(\mathcal{T}_{r}\right) Z_{k-2-j}\left(\mathcal{T}_{s}\right) \tag{2.1}
\end{equation*}
$$

and thus, since we assume $\mathcal{T}$ to be critical, i.e., $\mathbb{E} \xi=1$, so $\mathbb{E} Z_{k}(\mathcal{T})=1$ for every $k$,

$$
\begin{equation*}
\mathbb{E} Q_{k}(\mathcal{T})=\mathbb{E} Z_{k}(\mathcal{T})+\mathbb{E} Q_{k}^{\prime}(\mathcal{T})=1+\mathbb{E} \frac{\xi(\xi-1)}{2}(k-1)=1+(k-1) \frac{\sigma^{2}}{2} \tag{2.2}
\end{equation*}
$$

Let $\widehat{T}_{n}$ denote the random subtree of $T_{n}$ rooted at a uniformly selected random vertex. (Note the difference from $T_{n}^{*}$ in Corollary [1.2, in $T_{n}^{*}$ we keep all $n$ vertices, but in $\widehat{T}_{n}$ we keep only the vertices below the new root.) Then, clearly,

$$
\mathbb{E}\left\{P_{k}\left(T_{n}\right)\right\}=n \mathbb{E}\left\{Q_{k}\left(\widehat{T}_{n}\right)\right\}
$$

Consequently, Theorem 1.1 is equivalent to:
Theorem 2.1. There exists a constant $C_{6}$ such that for all $k \geq 1$ and $n \geq 1$, $\mathbb{E} Q_{k}\left(\widehat{T}_{n}\right) \leq C_{6} k$.

In order to prove this, we will need a related, but different, estimate for the conditioned Galton-Watson tree $T_{n}$.

Theorem 2.2. There exists a constant $C_{7}$ such that for all $k \geq 1$ and $n \geq 1$, $\mathbb{E} Q_{k}\left(T_{n}\right) \leq C_{7} k \sqrt{n}$.

It is easy to see $\mathbb{E} Q_{k}\left(T_{n}\right) \geq c_{1} n^{3 / 2}$ when $k \sim \sqrt{n}$, so the estimate in Theorem 2.2 then is of the right order; in particular, the estimate in Theorem 2.1 for $\widehat{T}_{n}$ does not hold for $T_{n}$.

To prove these theorems we use a few more or less standard estimates.

Lemma 2.3. Assume, as above, (1.1), and let $d$ be the span of $\xi$. Let $S_{n}:=\sum_{i=1}^{n} \xi_{i}$, where $\xi_{i}$ are independent copies of $\xi$. Then, for $n \equiv 1$ $(\bmod d)$,

$$
\begin{equation*}
\mathbb{P}(|\mathcal{T}|=n)=\frac{1}{n} \mathbb{P}\left(S_{n}=n-1\right) \sim \frac{d}{\sigma \sqrt{2 \pi} n^{3 / 2}} \quad \text { as } n \rightarrow \infty . \tag{2.3}
\end{equation*}
$$

More generally, let $W_{\ell}:=\sum_{i=1}^{\ell}\left|\mathcal{T}_{i}\right|$ be the size of the union of $\ell$ independent copies of $\mathcal{T}$, or equivalently, the total progeny of a Galton-Watson process started with $\ell$ individuals, with offspring distribution $\xi$. Then, for all $\ell \geq 1$ and $n \geq 1$,

$$
\begin{equation*}
\mathbb{P}\left(W_{\ell}=n\right)=\frac{\ell}{n} \mathbb{P}\left(S_{n}=n-\ell\right) \leq C_{8} \ell n^{-3 / 2} \exp \left(-c_{2} \ell^{2} / n\right) \tag{2.4}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\mathbb{P}\left(W_{\ell}=n\right) \leq C_{9} n^{-1} . \tag{2.5}
\end{equation*}
$$

Proof. The identity in (2.4) is well-known, see e.g., Dwass [8], Kolchin 12, Lemma 2.1.3, p. 105] and Pitman [14]. The identity in (2.3) is the special case $\ell=1$, and the well-known tail estimate in (2.3) then follows by the local central limit theorem, see, e.g., Kolchin [12, Lemma 2.1.4, p. 105].

Similarly, the inequality in (2.4) follows by the estimate $\mathbb{P}\left(S_{n}=n-\ell\right) \leq$ $C_{8} n^{-1 / 2} \exp \left(-c_{2} \ell^{2} / n\right)$ from [10, Lemma 2.1]. The inequality $e^{-x} \leq x^{-1 / 2}$ yields (2.5).
Lemma 2.4. For every $r>0$ there is a constant $C_{10}(r)$ such that for all $k \geq 0$ and $n \geq 1, \mathbb{E} Z_{k}\left(T_{n}\right)^{r} \leq C_{10}(r) n^{r / 2}$.
Proof. For any rooted tree $T$, let $T^{(k)}$ be the tree pruned at height $k$, i.e., the subtree consisting of all vertices of distance at most $k$ from the root. Let $\tau$ be a given rooted tree of height $k$, and let $m:=Z_{k}(\tau)$, the number of leaves at maximal depth. Note that if $\tau=T^{(k)}$ for some tree $T$, then $|T|=n$ if and only if $T$ has $n-|\tau|$ vertices at greater depth than $k$, and thus $N:=n-|\tau|+m$ vertices at depth $k$ or greater. Hence, with $W_{m}$ as in Lemma 2.3 and using (2.3) and (2.4), for any $r>0$ and assuming $N>0$ (otherwise the probability is 0 ),

$$
\begin{aligned}
\mathbb{P}\left(T_{n}^{(k)}=\tau\right) & =\frac{\mathbb{P}\left(\mathcal{T}^{(k)}=\tau,|\mathcal{T}|=n\right)}{\mathbb{P}(|\mathcal{T}|=n)}=\frac{\mathbb{P}\left(\mathcal{T}^{(k)}=\tau\right) \mathbb{P}\left(W_{m}=N\right)}{\mathbb{P}(|\mathcal{T}|=n)} \\
& \leq C_{11} n^{3 / 2} \mathbb{P}\left(\mathcal{T}^{(k)}=\tau\right) m N^{-3 / 2} e^{-c_{2} m^{2} / N} \\
& \leq C_{12}(r) n^{3 / 2} \mathbb{P}\left(\mathcal{T}^{(k)}=\tau\right) m N^{-3 / 2}\left(m^{2} / N\right)^{-r / 2} \\
& =C_{12}(r) n^{3 / 2} N^{r / 2-3 / 2} m^{1-r} \mathbb{P}\left(\mathcal{T}^{(k)}=\tau\right) .
\end{aligned}
$$

If $r \geq 3$, this yields, since $N \leq n$, the estimate

$$
\mathbb{P}\left(T_{n}^{(k)}=\tau\right) \leq C_{12}(r) n^{r / 2} m^{1-r} \mathbb{P}\left(\mathcal{T}^{(k)}=\tau\right)
$$

and summing over all $\tau$ of height $k$ with $Z_{k}(\tau)=m$ we obtain

$$
\mathbb{P}\left(Z_{k}\left(T_{n}\right)=m\right) \leq C_{12}(r) n^{r / 2} m^{1-r} \mathbb{P}\left(Z_{k}(\mathcal{T})=m\right) .
$$

Consequently,

$$
\begin{aligned}
\mathbb{E} Z_{k}\left(T_{n}\right)^{r} & =\sum_{m=1}^{\infty} m^{r} \mathbb{P}\left(Z_{k}\left(T_{n}\right)=m\right) \\
& \leq C_{12}(r) n^{r / 2} \sum_{m=1}^{\infty} m \mathbb{P}\left(Z_{k}(\mathcal{T})=m\right) \\
& =C_{12}(r) n^{r / 2} \mathbb{E} Z_{k}(\mathcal{T})=C_{12}(r) n^{r / 2}
\end{aligned}
$$

This proves the result for $r \geq 3$, and the result for $0<r<3$ follows by Lyapounov's (or Hölder's) inequality.

Lemma 2.5. For all $k \geq 1$ and $n \geq 1, \mathbb{E} Z_{k}\left(T_{n}\right) \leq C_{13}(k \wedge \sqrt{n})$. Equivalently, for all $k \geq 0$ and $n \geq 1, \mathbb{E} Z_{k}\left(T_{n}\right) \leq C_{14}((k+1) \wedge \sqrt{n})$.

Proof. The estimate $\mathbb{E} Z_{k}\left(T_{n}\right) \leq C_{15} k$ is (1.2), which is proved in 10, Theorem 1.13]. The estimate $\mathbb{E} Z_{k}\left(T_{n}\right) \leq C_{16} \sqrt{n}$ is proved in Lemma [2.4,

Remark. The estimate $\mathbb{E} Z_{k}\left(T_{n}\right) \leq C_{17} \sqrt{n}$ was proved by Drmota and Gittenberger [7], assuming that $\xi$ has an exponential moment; in fact, they then prove the stronger bound $\mathbb{E} Z_{k}\left(T_{n}\right) \leq C_{18} \sqrt{n} \exp \left(-c_{3} k / \sqrt{n}\right)$. The bound in Lemma 2.5 can be further improved to $\mathbb{E} Z_{k}\left(T_{n}\right) \leq C_{19} k \exp \left(-c_{4} k^{2} / n\right)$, but we do not know a reference for this estimate. (Details may appear elsewhere.)

Remark. Note that Lemma 2.4 yields an estimate $O\left(n^{r / 2}\right)$ of the $r$ th moment of $Z_{k}\left(T_{n}\right)$ for an arbitrary $r$ assuming only a second moment of $\xi$. This is in contrast to the estimate (1.2), where the corresponding estimate $\mathbb{E} Z_{k}\left(T_{n}\right)^{r}=O\left(k^{r}\right)$ is valid (for integer $r \geq 1$ at least) if $\xi$ has a finite $r+1$ :th moment, but not otherwise (not even for a fixed $k \geq 2$ ); one direction is by Theorem 1.13 in [10], and the converse follows from the discussion after Lemma 2.1 in [10].

Proof of Theorem [2.2. We have $Q_{k}\left(T_{n}\right)=Q_{k}^{\prime}\left(T_{n}\right)+Z_{k}\left(T_{n}\right)$, and $\mathbb{E} Z_{k}\left(T_{n}\right) \leq$ $C_{13} k$ by Lemma 2.5, so it suffices to show that $\mathbb{E} Q_{k}^{\prime}\left(T_{n}\right) \leq C_{20} k \sqrt{n}$.

We use (2.1), condition on $|\mathcal{T}|=n$ and take expectations. Using the symmetry and recalling that $\mathcal{T}_{1}, \ldots, \mathcal{T}_{\xi}$ are independent and $\left(\mathcal{T}_{i}| | \mathcal{T}_{i} \mid=\right.$ $\left.n_{i}\right) \stackrel{\mathrm{d}}{=}\left(\mathcal{T}\left||\mathcal{T}|=n_{i}\right) \stackrel{\mathrm{d}}{=} T_{n_{i}}\right.$ for any $n_{i}$, we obtain, with $p_{\ell}:=\mathbb{P}(\xi=\ell)$ and

$$
\begin{aligned}
& q_{m}:=\mathbb{P}(|\mathcal{T}|=m) \\
& \begin{aligned}
& \mathbb{E}\left\{Q_{k}^{\prime}\left(T_{n}\right)\right\}= \mathbb{E}\left\{\mathbf{1}_{[\xi \geq 2,|\mathcal{T}|=n]} \sum_{1 \leq r<s \leq \xi} \sum_{j=0}^{k-2} Z_{j}\left(\mathcal{T}_{r}\right) Z_{k-2-j}\left(\mathcal{T}_{s}\right)\right\} \\
& \mathbb{P}\{|\mathcal{T}|=n\} \\
&= \frac{\mathbb{E}\left\{\mathbf{1}_{[|\mathcal{T}|=n]}\binom{\xi}{2} \sum_{j=0}^{k-2} Z_{j}\left(\mathcal{T}_{1}\right) Z_{k-2-j}\left(\mathcal{T}_{2}\right)\right\}}{\mathbb{P}\{|\mathcal{T}|=n\}} \\
&= q_{n}^{-1} \sum_{\ell=2}^{\infty} p_{\ell}\binom{\ell}{2} \sum_{n_{1}, n_{2} \geq 1} q_{n_{1}} q_{n_{2}} \mathbb{P}\left(\sum_{i=3}^{\ell}\left|\mathcal{T}_{i}\right|=n-1-n_{1}-n_{2}\right) \\
& \times \sum_{j=0}^{k-2} \mathbb{E}\left\{Z_{j}\left(T_{n_{1}}\right)\right\} \mathbb{E}\left\{Z_{k-2-j}\left(T_{n_{2}}\right)\right\} .
\end{aligned}
\end{aligned}
$$

We begin with the inner sum over $j, \Sigma_{1}\left(n_{1}, n_{2}\right)$ say. By symmetry, we consider only $n_{1} \leq n_{2}$, and then we obtain from Lemma 2.5 the estimates $\mathbb{E} Z_{k-2-j}\left(T_{n_{2}}\right) \leq C_{14}\left((k-1-j) \wedge n_{2}^{1 / 2}\right) \leq C_{14}\left(k \wedge n_{2}^{1 / 2}\right)$ and

$$
\sum_{j=0}^{k-2} \mathbb{E}\left\{Z_{j}\left(T_{n_{1}}\right)\right\} \leq\left\{\begin{array}{l}
\sum_{j=0}^{k-2} C_{14}(j+1) \leq C_{14} k^{2} \\
\mathbb{E} \sum_{j=0}^{\infty} Z_{j}\left(T_{n_{1}}\right)=n_{1}
\end{array}\right.
$$

Hence

$$
\begin{equation*}
\Sigma_{1}\left(n_{1}, n_{2}\right) \leq C_{21}\left(k^{2} \wedge n_{1}\right)\left(k \wedge n_{2}^{1 / 2}\right) \tag{2.6}
\end{equation*}
$$

Let $m:=n_{1}+n_{2}$ and sum over $n_{1}, n_{2}$ with a given sum $m$. We have by (2.6) and (2.3),

$$
\begin{align*}
\Sigma_{2}(m) & :=\sum_{n_{1}+n_{2}=m} q_{n_{1}} q_{n_{2}} \Sigma_{1}\left(n_{1}, n_{2}\right) \\
& \leq 2 \sum_{n_{1}=1}^{m / 2} q_{n_{1}} q_{m-n_{1}} C_{21}\left(k^{2} \wedge n_{1}\right)\left(k \wedge\left(m-n_{1}\right)^{1 / 2}\right) \\
& \leq C_{22} \sum_{n_{1}=1}^{m / 2} n_{1}^{-3 / 2}\left(m-n_{1}\right)^{-3 / 2}\left(k^{2} \wedge n_{1}\right)\left(k \wedge\left(m-n_{1}\right)^{1 / 2}\right) \\
& \leq C_{23} \frac{k \wedge m^{1 / 2}}{m^{3 / 2}} \sum_{n_{1}=1}^{m / 2} \frac{k^{2} \wedge n_{1}}{n_{1}^{3 / 2}} \\
& \leq C_{24} \frac{k \wedge m^{1 / 2}}{m^{3 / 2}}\left(k \wedge m^{1 / 2}\right)=C_{24} \frac{k^{2} \wedge m}{m^{3 / 2}} \tag{2.7}
\end{align*}
$$

We define further

$$
\begin{equation*}
\Sigma_{3}(\ell):=\sum_{m=2}^{n-1} \Sigma_{2}(m) \mathbb{P}\left(\sum_{i=3}^{\ell}\left|\mathcal{T}_{i}\right|=n-1-m\right) \tag{2.8}
\end{equation*}
$$

thus

$$
\begin{equation*}
\mathbb{E} Q_{k}^{\prime}\left(T_{n}\right)=q_{n}^{-1} \sum_{\ell=2}^{\infty} p_{\ell}\binom{\ell}{2} \Sigma_{3}(\ell) \leq C_{25} n^{3 / 2} \sum_{\ell=2}^{\infty} p_{\ell} \ell^{2} \Sigma_{3}(\ell) \tag{2.9}
\end{equation*}
$$

We will show that $\Sigma_{3}(\ell) \leq C_{26} k / n$, uniformly in $\ell \geq 2$, and the result follows by (2.9), recalling that $\sum_{\ell} p_{\ell} \ell^{2}=\mathbb{E} \xi^{2}<\infty$. (The proof can be simplified in the case $\mathbb{E} \xi^{3}<\infty$, when it suffices to show that $\Sigma_{3}(\ell) \leq C_{27} k \ell / n$.)

First, if $\ell=2$, the only non-zero term in (2.8) is for $m=n-1$, which yields, by (2.7),

$$
\Sigma_{3}(2)=\Sigma_{2}(n-1) \leq C_{28} \frac{k^{2} \wedge n}{n^{3 / 2}} \leq C_{28} \frac{k \sqrt{n}}{n^{3 / 2}}
$$

For $\ell>2$, we split the sum in (2.8) into two parts, with $m \leq n / 2$ and $m>n / 2$. We have, by (2.7),

$$
\begin{aligned}
\sum_{m=n / 2}^{n-1} \Sigma_{2}(m) \mathbb{P}( & \left.\sum_{i=3}^{\ell}\left|\mathcal{T}_{i}\right|=n-1-m\right) \leq C_{29} \frac{k^{2} \wedge n}{n^{3 / 2}} \mathbb{P}\left(\sum_{i=3}^{\ell}\left|\mathcal{T}_{i}\right| \leq n / 2\right) \\
& \leq C_{29} \frac{k^{2} \wedge n}{n^{3 / 2}} \leq C_{29} \frac{k}{n}
\end{aligned}
$$

Similarly, using (2.7) and (2.5) (with $\ell$ replaced by $\ell-2$ ),

$$
\begin{aligned}
\sum_{m=1}^{n / 2} \Sigma_{2}(m) \mathbb{P}\left(\sum_{i=3}^{\ell}\left|\mathcal{T}_{i}\right|=n-1-m\right) & \leq C_{30} \sum_{m=1}^{n / 2} \frac{k^{2} \wedge m}{m^{3 / 2}} \cdot \frac{1}{n} \\
& \leq \frac{C_{30}}{n} \sum_{m=1}^{\infty} \frac{k^{2} \wedge m}{m^{3 / 2}} \leq C_{31} \frac{k}{n}
\end{aligned}
$$

Thus $\Sigma_{3}(\ell) \leq C_{32} k / n$, and the theorem follows by (2.9).
Proof of Theorems 2.1 and Theorem 1.1. Aldous [2] has studied the behavior of a random subtree $\widehat{T}_{n}$ in a conditional Galton-Watson tree $T_{n}$. In particular, he has the following identity [2, p. 242], for any fixed ordered tree $\tau$ of order at most $n$ (provided that the probabilities in the denominators are nonzero):

$$
\frac{\mathbb{P}\left\{\widehat{T}_{n}=\tau\right\}}{\mathbb{P}\{\mathcal{T}=\tau\}}=\frac{(n-|\tau|+1) \mathbb{P}\{|\mathcal{T}|=n-|\tau|+1\}}{n \mathbb{P}\{|\mathcal{T}|=n\}} \cdot \frac{\gamma}{p_{0}}
$$

where $\gamma$ is the expected proportion of leaves in $\widehat{T}_{n-|\tau|+1}$ and $p_{0}=\mathbb{P}\{\xi=0\}$. We will simply bound $\gamma \leq 1$, but it is well-known that as $n-|\tau|+1 \rightarrow \infty$, $\gamma \rightarrow p_{0}$, see e.g. Kolchin [12, Theorem 2.3.1, p. 113]. Thus, using the wellknown tail estimate (2.3), for all (permitted) $n$ and $\tau$

$$
\frac{\mathbb{P}\left\{\widehat{T}_{n}=\tau\right\}}{\mathbb{P}\{\mathcal{T}=\tau\}} \leq C_{33} \frac{(n-|\tau|+1) \mathbb{P}\{|\mathcal{T}|=n-|\tau|+1\}}{n \mathbb{P}\{|\mathcal{T}|=n\}} \leq C_{34} \sqrt{\frac{n}{n-|\tau|+1}}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left\{Q_{k}\left(\widehat{T}_{n}\right)\right\} & =\sum_{\tau} Q_{k}(\tau) \mathbb{P}\left\{\widehat{T}_{n}=\tau\right\} \\
& \leq C_{34} \sum_{\tau} Q_{k}(\tau) \sqrt{\frac{n}{n-|\tau|+1}} \mathbb{P}\{\mathcal{T}=\tau\} \\
& =C_{34} \mathbb{E}\left(Q_{k}(\mathcal{T}) \sqrt{\frac{n}{n-|\mathcal{T}|+1}}\right) \\
& \leq C_{35} \mathbb{E}\left\{Q_{k}(\mathcal{T})\right\}+C_{34} \sum_{n \geq \ell>n / 2} \sqrt{\frac{n}{n-\ell+1}} \mathbb{E}\left\{\mathbf{1}_{||\mathcal{T}|=\ell]} Q_{k}(\mathcal{T})\right\} .
\end{aligned}
$$

We have $\mathbb{E}\left\{Q_{k}(\mathcal{T})\right\} \leq C_{36} k$ by (2.2), and, using (2.3) and Theorem 2.2,

$$
\mathbb{E}\left\{\mathbf{1}_{[|\mathcal{T}|=\ell]} Q_{k}(\mathcal{T})\right\}=\mathbb{P}\{|\mathcal{T}|=\ell\} \mathbb{E}\left\{Q_{k}\left(T_{\ell}\right)\right\} \leq C_{37} \ell^{-3 / 2} k \ell^{1 / 2}=C_{37} k / \ell
$$

Consequently,

$$
\begin{aligned}
\mathbb{E}\left\{Q_{k}\left(\widehat{T}_{n}\right)\right\} & \leq C_{36} k+C_{38} \sum_{n \geq \ell>n / 2} \frac{k}{\ell} \sqrt{\frac{n}{n-\ell+1}} \\
& \leq C_{36} k+C_{39} \frac{k}{n} \sum_{j=1}^{n} \sqrt{\frac{n}{j}} \\
& \leq C_{40} k .
\end{aligned}
$$

This proves Theorem 2.1, which by the argument at the beginning of the section yields Theorem 1.1.

## 3. Proof of Lemma 1.8 and Theorem 1.6

Denote the left hand side of (1.8) by $\Psi(n, t)$. Since $\sum_{j} X\left(j ; T_{n}\right) e^{\mathrm{i} j t}=$ $\sum_{v \in T_{n}} e^{\mathrm{i} t L_{v}}$, we have

$$
\begin{equation*}
\Psi(n, t)=n^{-2} \mathbb{E}\left|\sum_{v \in T_{n}} e^{\mathrm{i} t L_{v}}\right|^{2}=n^{-2} \mathbb{E} \sum_{v, w \in T_{n}} e^{\mathrm{i} t\left(L_{v}-L_{w}\right)} \tag{3.1}
\end{equation*}
$$

Condition on $T_{n}$ and consider two vertices $v$ and $w$ in $T_{n}$. If $v \wedge w$ is the last common ancestor of $v$ and $w$, then $L_{v}-L_{v \wedge w}$ and $L_{w}-L_{v \wedge w}$ are (conditionally, given $\left.T_{n}\right)$ independent sums of $d(v, v \wedge w)$ and $d(w, v \wedge$ $w)$ copies of $\eta$, respectively. Consequently, letting $\varphi_{\eta}(t):=\mathbb{E} e^{\text {it } \eta}$ be the characteristic function of $\eta$,

$$
\begin{aligned}
\mathbb{E}\left(e^{\mathrm{i} t\left(L_{v}-L_{w}\right)} \mid T_{n}\right) & =\mathbb{E}\left(e^{\mathrm{it}\left(L_{v}-L_{v \wedge w}\right)} \mid T_{n}\right) \mathbb{E}\left(e^{-\mathrm{i} t\left(L_{w}-L_{v \wedge w}\right)} \mid T_{n}\right) \\
& =\varphi_{\eta}(t)^{d(v, v \wedge w)} \frac{\varphi_{\eta}(t)}{} d(w, v \wedge w)
\end{aligned}
$$

Hence, by (3.1) and (1.4),

$$
\Psi(n, t)=n^{-2} \mathbb{E} \sum_{v, w \in T_{n}} \varphi_{\eta}(t)^{d(v, v \wedge w)} \overline{\varphi_{\eta}(t)} d(w, v \wedge w)=n^{-2} h_{n}\left(\varphi_{\eta}(t), \overline{\varphi_{\eta}(t)}\right)
$$

and Theorem 1.4 yields

$$
\begin{equation*}
\Psi(n, t) \leq C_{3} n^{-1}\left|1-\varphi_{\eta}(t)\right|^{-2} . \tag{3.2}
\end{equation*}
$$

Since $\mathbb{E} \eta=0$ and $\mathbb{E} \eta^{2}=\sigma_{\eta}^{2}<\infty$, we have $\varphi_{\eta}(t)=\exp \left(-\frac{1}{2} \sigma_{\eta}^{2} t^{2}+o\left(t^{2}\right)\right)$ for small $|t|$; moreover, since $\eta$ has span $1, \varphi_{\eta}(t) \neq 1$ for $0<|t| \leq \pi$. It follows that $\psi(t):=\left(1-\varphi_{\eta}(t)\right) / t^{2}$ is a continuous non-zero function on $[-\pi, \pi]$ (with $\left.\psi(0):=\frac{1}{2} \sigma_{\eta}^{2}\right)$; hence, by compactness, $|\psi(t)| \geq c_{5}$ for some $c_{6}>0$, and thus

$$
\left|1-\varphi_{\eta}(t)\right| \geq c_{6} t^{2}, \quad|t| \leq \pi
$$

It now follows from (3.2), and the obvious fact that $\Psi(n, t) \leq 1$, that

$$
\left(1+n t^{4}\right) \Psi(n, t) \leq 1+n t^{4} \Psi(n, t) \leq 1+C_{3} \frac{t^{4}}{\left|1-\varphi_{\eta}(t)\right|^{2}} \leq C_{41}
$$

This proves Lemma 1.8, which as remarked in Section 1 implies Theorem 1.6 by [5, Remark 3.7].

## 4. Proof of Theorem 1.4

We use some further generating functions. Recall that $\mathcal{T}$ is the (unconditioned) Galton-Watson tree with offspring distribution $\xi$, and define

$$
\begin{aligned}
\Phi(z) & :=\mathbb{E} z^{\xi}, \\
F(z) & :=\mathbb{E} z^{|\mathcal{T}|}, \\
G(z, x) & :=\mathbb{E}\left(z^{|\mathcal{T}|} \sum_{v \in \mathcal{T}} x^{d(v, o)}\right), \\
H(z, x, y) & :=\mathbb{E}\left(z^{|\mathcal{T}|} \sum_{v, w \in \mathcal{T}} x^{d(v, v \wedge w)} y^{d(w, v \wedge w)}\right)=\sum_{n=1}^{\infty} \mathbb{P}(|\mathcal{T}|=n) h_{n}(x, y) z^{n} .
\end{aligned}
$$

These functions are defined and analytic at least for $|z|,|x|,|y|<1$.
Let us condition on the degree $d_{o}$ of the root of $\mathcal{T}$, recalling that $d_{o} \stackrel{\mathrm{~d}}{=} \xi$. If $d_{o}=\ell$, then $\mathcal{T}$ has $\ell$ subtrees $\mathcal{T}_{1}, \ldots, \mathcal{T}_{\ell}$ at the root $o$, and conditioned on $d_{o}=\ell$, these are independent and with the same distribution as $\mathcal{T}$; we denote their roots (the neighbours of $o$ ), by $o_{1}, \ldots, o_{\ell}$.

Assume $d_{o}=\ell$, and let $|z|,|x|,|y|<1$. First, $|\mathcal{T}|=1+\sum_{i=1}^{\ell}\left|\mathcal{T}_{i}\right|$ and thus $z^{|\mathcal{T}|}=z \prod_{i=1}^{\ell} z^{\left|\mathcal{T}_{i}\right|}$. Taking the expectation, we obtain, as is well-known, first

$$
\mathbb{E}\left(z^{|\mathcal{T}|} \mid d_{o}=\ell\right)=z \mathbb{E} \prod_{i=1}^{\ell} z^{\left|\mathcal{T}_{i}\right|}=z F(z)^{\ell}
$$

and then

$$
F(z)=\mathbb{E}\left(z^{|\mathcal{T}|}\right)=z \sum_{\ell=0}^{\infty} \mathbb{P}(\xi=\ell) F(z)^{\ell}=z \Phi(F(z))
$$

Similarly, separating the cases $v \in \mathcal{T}_{i}, i=1, \ldots, \ell$, and $v=o$,

$$
\sum_{v \in \mathcal{T}} x^{d(v, o)}=\sum_{i=1}^{\ell} \sum_{v \in \mathcal{T}_{i}} x^{d\left(v, o_{i}\right)+1}+1
$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left(z^{|\mathcal{T}|} \sum_{v \in \mathcal{T}} x^{d(v, o)} \mid d_{o}=\ell\right) & =\mathbb{E} \sum_{i=1}^{\ell} z z^{\left|\mathcal{T}_{i}\right|} \sum_{v \in \mathcal{T}_{i}} x^{d\left(v, o_{i}\right)+1} \prod_{j \neq i} z^{\left|\mathcal{T}_{j}\right|}+\mathbb{E}\left(z \prod_{i=1}^{\ell} z^{\left|\mathcal{T}_{i}\right|}\right) \\
& =\ell z x G(z, x) F(z)^{\ell-1}+z F(z)^{\ell}
\end{aligned}
$$

and
$G(z, x)=\sum_{\ell=0}^{\infty} \mathbb{P}(\xi=\ell) \ell z x G(z, x) F(z)^{\ell-1}+F(z)=z x \Phi^{\prime}(F(z)) G(z, x)+F(z)$
which gives

$$
\begin{equation*}
G(z, x)=\frac{F(z)}{1-z x \Phi^{\prime}(F(z))} \tag{4.1}
\end{equation*}
$$

Similarly,

$$
\begin{aligned}
& \mathbb{E}\left(z^{|\mathcal{T}|} \sum_{v, w \in \mathcal{T}} x^{d(v, v \wedge w)} y^{d(w, v \wedge w)} \mid d_{o}=\ell\right)= \\
& \mathbb{E} \sum_{i=1}^{\ell} z z^{\left|\mathcal{T}_{i}\right|} \sum_{v, w \in \mathcal{T}_{i}} x^{d(v, v \wedge w)} y^{d(w, v \wedge w)} \prod_{j \neq i} z^{\left|\mathcal{T}_{j}\right|} \\
& \quad+\mathbb{E} \sum_{i \neq j} z z^{\left|\mathcal{T}_{i}\right|} \sum_{v \in \mathcal{T}_{i}} x^{d\left(v, o_{i}\right)+1} z^{\left|\mathcal{T}_{j}\right|} \sum_{w \in \mathcal{T}_{j}} y^{d\left(w, o_{j}\right)+1} \prod_{k \neq i, j} z^{\left|\mathcal{T}_{k}\right|} \\
& \quad+\mathbb{E} \sum_{i=1}^{\ell} z z^{\left|\mathcal{T}_{i}\right|} \sum_{v \in \mathcal{T}_{i}} x^{d\left(v, o_{i}\right)+1} \prod_{k \neq i} z^{\left|\mathcal{T}_{k}\right|} \\
& \quad+\mathbb{E} \sum_{j=1}^{\ell} z z^{\left|\mathcal{T}_{j}\right|} \sum_{w \in \mathcal{T}_{j}} y^{d\left(w, o_{j}\right)+1} \prod_{k \neq j} z^{\left|\mathcal{T}_{k}\right|}+\mathbb{E}\left(z \prod_{i=1}^{\ell} z^{\left|\mathcal{T}_{i}\right|}\right)
\end{aligned}
$$

leading to

$$
\begin{aligned}
H(z, x, y)=z \Phi^{\prime}(F(z)) & H(z, x, y)+z x y \Phi^{\prime \prime}(F(z)) G(z, x) G(z, y) \\
& +z x \Phi^{\prime}(F(z)) G(z, x)+z y \Phi^{\prime}(F(z)) G(z, y)+F(z)
\end{aligned}
$$

which gives

$$
\begin{align*}
& H(z, x, y)= \\
& \frac{z x y \Phi^{\prime \prime}(F(z)) G(z, x) G(z, y)+z \Phi^{\prime}(F(z))(x G(z, x)+y G(z, y))+F(z)}{1-z \Phi^{\prime}(F(z))} \tag{4.2}
\end{align*}
$$

Assume now for simplicity that $\xi$ has span 1. (The case when the span is $d>1$ is treated similarly with the standard modification that we have to give special treatment to neighbourhoods of the $d$ :th unit roots.) Then, by [10, Lemma A.2], for some $\delta>0$ and $\beta \leq \pi / 4, F$ extends to an analytic function in $\Delta(\beta, \delta)$ with $|F(z)|<1$ for $z \in \Delta(\beta, \delta)$ and

$$
\begin{equation*}
F(z)=1-\sqrt{2} \sigma^{-1} \sqrt{1-z}+o\left(|z-1|^{1 / 2}\right), \quad \text { as } z \rightarrow 1 \text { with } z \in \Delta(\beta, \delta) . \tag{4.3}
\end{equation*}
$$

We will prove the following companion results.
Lemma 4.1. If $\xi$ has span 1 , then there exists $\beta, \delta>0$ such that $F$ extends to an analytic function in $\Delta(\beta, \delta)$ and, for some $c_{7}, c_{8}>0$, if $x, z \in \Delta(\beta, \delta)$, then

$$
\begin{align*}
\left|1-z \Phi^{\prime}(F(z))\right| & \geq c_{7}|1-z|^{1 / 2},  \tag{4.4}\\
\left|1-x z \Phi^{\prime}(F(z))\right| & \geq c_{8}|1-x| \tag{4.5}
\end{align*}
$$

Consequently, $G(z, x)$ and $H(z, x, y)$ extend to analytic functions of $x, y, z \in$ $\Delta(\beta, \delta)$, and, for all $x, y, z \in \Delta(\beta, \delta)$,

$$
\begin{align*}
|G(z, x)| & \leq C_{42}|1-x|^{-1},  \tag{4.6}\\
|H(z, x, y)| & \leq C_{43}|1-z|^{-1 / 2}|1-x|^{-1}|1-y|^{-1} . \tag{4.7}
\end{align*}
$$

Standard singularity analysis [9, Lemma IX.2] applied to (4.7) yields

$$
\left|\mathbb{P}(|\mathcal{T}|=n) h_{n}(x, y)\right| \leq C_{44} n^{-1 / 2}|1-x|^{-1}|1-y|^{-1}, \quad x, y \in \Delta(\beta, \delta)
$$

which proves Theorem 1.4 because, as is well known, a singularity analysis of (4.3) yields

$$
\mathbb{P}(|\mathcal{T}|=n) \sim c_{9} n^{-3 / 2}
$$

It thus remains only to prove Lemma 4.1.
Proof of Lemma 4.1. Since $\mathbb{E} \xi^{2}<\infty, \Phi^{\prime}$ and $\Phi^{\prime \prime}$ extend to continuous functions on the closed unit disc with $\Phi^{\prime}(1)=\mathbb{E} \xi=1$ and $\Phi^{\prime \prime}(1)=\mathbb{E} \xi(\xi-1)=$ $\sigma^{2}$. Hence, (4.3) yields, for $z \in \Delta(\beta, \delta)$,

$$
\begin{aligned}
\Phi^{\prime}(F(z)) & =\Phi^{\prime}(1)+\Phi^{\prime \prime}(1)(F(z)-1)+o(|F(z)-1|) \\
& =1-\sqrt{2} \sigma \sqrt{1-z}+o\left(|z-1|^{1 / 2}\right)
\end{aligned}
$$

and

$$
\begin{equation*}
z \Phi^{\prime}(F(z))=\Phi^{\prime}(F(z))+O(|z-1|)=1-\sqrt{2} \sigma \sqrt{1-z}+o\left(|z-1|^{1 / 2}\right) \tag{4.8}
\end{equation*}
$$

Let $B(1, \varepsilon):=\{z:|z-1|<\varepsilon\}$, and take $\beta<\pi / 4$. Since $z \in \overline{\Delta(\beta, \delta)} \backslash\{1\}$ entails $|\arg (1-z)| \leq \pi / 2+\beta$ and thus $|\arg \sqrt{1-z}| \leq \pi / 4+\beta / 2$, it follows from (4.8) that, for some small $\varepsilon>0$, if $z \in \overline{\Delta(\beta, \delta) \cap B(1, \varepsilon)}$ with $z \neq 1$, then (4.4) holds, $\left|z \Phi^{\prime}(F(z))-1\right|=O\left(\varepsilon^{1 / 2}\right)$,

$$
\begin{align*}
\left|\arg \left(z \Phi^{\prime}(F(z))-1\right)\right| & >|\arg (-\sqrt{1-z})|-\beta / 2 \\
& \geq \pi-(\pi / 4+\beta / 2)-\beta / 2=3 \pi / 4-\beta, \tag{4.9}
\end{align*}
$$

and consequently, since $3 \pi / 4-\beta>\pi / 2$, if $\varepsilon$ is small enough,

$$
\begin{equation*}
\left|z \Phi^{\prime}(F(z))\right|<1 . \tag{4.10}
\end{equation*}
$$

Similarly, if $x \in \Delta(\beta, \delta)$, then $|\arg (1-x)|<\pi / 2+\beta$ and

$$
x^{-1}=(1-(1-x))^{-1}=1+(1-x)+o(|1-x|), \quad x \rightarrow 1,
$$

so if $\varepsilon>0$ is small enough, then, for $x \in \Delta(\beta, \delta) \cap B(1, \varepsilon)$,

$$
\begin{equation*}
\left|\arg \left(x^{-1}-1\right)\right|<\pi / 2+2 \beta \tag{4.11}
\end{equation*}
$$

If we choose $\beta \leq \pi / 16$, it follows from (4.9) and (4.11) that the triangle with vertices in $1, x^{-1}$ and $z \Phi^{\prime}(F(z))$ has an angle at least $\pi / 4-3 \beta \geq \pi / 16$ at 1 , and thus by elementary trigonometry (the sine theorem),

$$
\left|x^{-1}-z \Phi^{\prime}(F(z))\right| \geq c_{10}\left|x^{-1}-1\right|
$$

and so (4.5) holds, when $z, x \in \Delta(\beta, \delta) \cap B(1, \varepsilon)$, provided $\beta, \delta, \varepsilon$ are small enough.

It remains to treat the case when $x$ or $z$ does not belong to $B(1, \varepsilon)$, i.e., $|x-1| \geq \varepsilon$ or $|z-1| \geq \varepsilon$. We do this by compactness arguments.

First, let

$$
\begin{aligned}
A & :=\left\{z \Phi^{\prime}(F(z)): z \in \Delta(\beta, \delta) \cap B(1, \varepsilon)\right\} \\
B_{\rho} & :=\left\{x^{-1}: x \in \overline{\Delta(\beta, \rho)} \backslash B(1, \varepsilon),|x| \geq 1 / 2\right\} .
\end{aligned}
$$

Then $B:=\bigcap_{\rho>0} B_{\rho} \subset\{\zeta:|\zeta| \geq 1\} \backslash\{1\}$, and it follows from (4.10) that $\bar{A} \cap B=\emptyset$. Since $\bar{A}$ and all $B_{\rho}$ are compact, it follows that $\bar{A} \cap B_{\rho}=\emptyset$ for some $\rho>0$, and thus, if $z \in \Delta(\beta, \delta) \cap B(1, \varepsilon)$ and $x \in \Delta(\beta, \rho) \backslash B(1, \varepsilon)$ with $|x| \geq 1 / 2$, then $\left|x^{-1}-z \Phi^{\prime}(F(z))\right| \geq c_{11}$ for some $c_{11}>0$, which implies (4.5) for such $z$ and $x$. Moreover, if $z \in \Delta(\beta, \delta) \cap B(1, \varepsilon)$ and $|x|<1 / 2$, (4.10) shows that $\left|1-x z \Phi^{\prime}(F(z))\right| \geq 1-|x| \geq 1 / 2$, so (4.5) then holds if $c_{8} \leq 1 / 3$.

Finally, if $z \in \Delta(\beta, \delta)$, then $|F(z)|<1$ [10, Lemma A.2] as stated above, and thus $\left|\Phi^{\prime}(F(z))\right|<1$. If $0<\beta_{1}<\beta$ and $0<\delta_{1}<\delta$, then $\overline{\Delta\left(\beta_{1}, \delta_{1}\right)} \subset$ $\Delta(\beta, \delta) \cup\{1\}$, and thus by compactness

$$
C_{\varepsilon}:=\sup \left\{\left|\Phi^{\prime}(F(z))\right|: z \in \overline{\Delta\left(\beta_{1}, \delta_{1}\right)} \backslash B(1, \varepsilon)\right\}<1
$$

Consequently, if $\delta_{2} \leq \delta_{1}$ is small enough and $x, z \in \Delta\left(\beta_{1}, \delta_{2}\right)$ with $|z-1| \geq \varepsilon$, then

$$
\left|x z \Phi^{\prime}(F(z))\right| \leq\left(1+\delta_{2}\right)^{2} C_{\varepsilon}<1
$$

Hence (4.5) holds in this case too for some $c_{8}>0$, and similarly (4.4) holds for $z \in \Delta\left(\beta_{1}, \delta_{2}\right) \backslash B(1, \varepsilon)$.

This completes the proof of (4.4) and (4.5), for some new $\beta, \delta>0$ (viz., $\beta_{1}$ and $\left.\min \left(\delta_{2}, \rho\right)\right) . G(z, x)$ now can be defined for all $x, z \in \Delta(\beta, \delta)$ by (4.1), and (4.6) holds by (4.5). Similarly, $H(z, x, y)$ can be defined for all $x, y, z \in$ $\Delta(\beta, \delta)$ by (4.2), and (4.7) holds by (4.4), (4.6), and the fact that $\Phi^{\prime}$ and $\Phi^{\prime \prime}$ are bounded on the unit disc. (Recall that $|F(z)|<1$ for $z \in \Delta(\beta, \delta)$.)

This completes the proof of Lemma 4.1, and thus of Theorem 1.4 and of all results in this paper.

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