GRAPH DIAMETER IN LONG-RANGE PERCOLATION

MAREK BISKUP

Department of Mathematics, University of California at Los Angeles, USA
School of Economics, University of South Bohemia, Czech Republic

Abstract: We study the asymptotic growth of the diameter of a graph obtained by adding sparse "long" edges to a square box in $\mathbb{Z}^d$. We focus on the cases when an edge between x and y is added with probability decaying with the Euclidean distance as $|x-y|^{-s+o(1)}$ when $|x-y| \to \infty$. For $s \in (d, 2d)$ we show that the graph diameter for the graph reduced to a box of side L scales like $(\log L)^{\Delta+o(1)}$ where $\Delta^{-1} := \log_2(2d/s)$. In particular, the diameter grows about as fast as the typical graph distance between two vertices at distance L. We also show that a ball of radius r in the intrinsic metric on the (infinite) graph will roughly coincide with a ball of radius $\exp\{r^{1/\Delta+o(1)}\}$ in the Euclidean metric.

1. Main result

Consider the d-dimensional hypercubic lattice $\mathbb{Z}^d$ and add a random collection of edges to $\mathbb{Z}^d$ according to the following rule: An edge between distinct sites x and y occurs with probability p_{xy}, independently of all other edges, where p_{xy} depends only on the difference $x-y$ and decays like $|x-y|^{-s+o(1)}$ as the Euclidean norm $|x-y|$ tends to infinity. Let $D(x,y)$ denote the graph distance between x and y which is defined as the length of the shortest path that connects x to y using only edges that are available in the present (random) sample.

In [8] we studied the asymptotic of $D(x,y)$ as $|x-y| \to \infty$. In particular, it was shown that for $s \in (d, 2d)$ this distance behaves like

$$D(x,y) = \bigl(\log|x-y|\bigr)^{\Delta+o(1)}, \qquad |x-y| \to \infty, \tag{1.1}$$

where

$$\Delta := \frac{\log 2}{\log\bigl(\frac{2d}{s}\bigr)}. \tag{1.2}$$

Technically, (1.1) is established with "$o(1)$ tending to zero in probability" and thus represents the *typical* behavior for fixed x and y. The result allows for the possibility that even the nearest-neighbor edges are randomized — x and y are then restricted to the unique infinite connected component.

The main purpose of this note is to determine the corresponding asymptotic for the *maximal* graph distance between any two sites in a large, finite set. Explicitly, let us consider the box $\Lambda_L := [-L,L]^d \cap \mathbb{Z}^d$ and let $\mathbb{G}_L$ denote the restriction of the above random graph to vertices, and edges with both endpoints, in Λ_L. Let $D_L(x,y)$ denote graph-theoretical (a.k.a. intrinsic or chemical) distance between $x, y \in \Lambda_L$ as measured on $\mathbb{G}_L$. The diameter of $\mathbb{G}_L$ is then given by

$$D_L := \max_{x,y\in\Lambda_L} D_L(x,y). \tag{1.3}$$

The following settles a question that was left open in [8]:

Theorem 1.1 *Suppose that p_{xy} can be written as $p_{xy} := 1 - \mathrm{e}^{-q(x-y)}$, where $q\colon \mathbb{Z}^d \to [0,\infty]$ is an even function for which the limit*

$$s := -\lim_{|x|\to\infty} \frac{\log q(x)}{\log |x|} \tag{1.4}$$

exists and satisfies $s \in (d, 2d)$. Then for all $\epsilon > 0$,

$$\lim_{L\to\infty} \mathbb{P}\bigl((\log L)^{\Delta-\epsilon} \le D_L \le (\log L)^{\Delta+\epsilon}\bigr) = 1, \tag{1.5}$$

where Δ is as in (1.2).

It is clear that the asymptotic in (1.1) serves as a lower bound on D_L. However, a matching upper bound — the main contribution of this note — turns out to be much less immediate. The point is that the bounds from [8] for $\mathbb{P}(D(x,y) \ge (\log|x-y|)^{\Delta+\epsilon})$ are much too weak to imply the same upper bound on D_L. This is because the strategy employed in [8] is based on the fact that one can find, with overwhelming probability, long edges within up to $(\log L)^{o(1)}$-distance from any given site. Unfortunately, this does not hold for every site of Λ_L; in fact, Λ_L *will* contain a translate of Λ_ℓ with $\ell \approx (\log L)^{1/d}$ whose vertices have *no other* edges than those inherited from $\mathbb{Z}^d$.

The restriction of the above result to hypercubic lattice is mostly a matter of convenience; no part of the proof depends essentially on the details of the underlying graph. (What we need is that the graph is embedded in $\mathbb{R}^d$ so that *any* square block of side L contains order L^d sites.) Similarly, we could also work with more general sequences of sets than cubic boxes. In fact, we could even accommodate non-translation invariant distributions and/or diluted lattices, e.g., work (as was done in [8]) with long-range percolation under the sole assumption that there is an infinite connected component. Notwithstanding, such generalizations tend to obscure the main ideas of the proof and so we settle to a translation-invariant and *a priori* connected setting.

The control of the diameter provided by Theorem 1.1 allows for some control of the volume growth of the percolation graph. Consider a realization of the long-range percolation on $\mathbb{Z}^d$ — still including the edges of $\mathbb{Z}^d$ — and let

$$B(0,r) := \bigl\{x \in \mathbb{Z}^d \colon D(0,x) \le r\bigr\} \tag{1.6}$$

denote the ball of radius r in the intrinsic metric. Trapman [15, Theorem 1.1(c)] has recently shown that the volume of this ball grows subexponentially with the radius, i.e.,

$$s > d \quad \Rightarrow \quad \lim_{r\to\infty} \big|B(0,r)\big|^{1/r} = 1, \quad \mathbb{P}\text{-a.s.} \tag{1.7}$$

Here we derive the leading order of the growth of $|B(0,r)|$ with r:

Theorem 1.2 *Under the conditions of Theorem 1.1, for each $\epsilon > 0$,*

$$\lim_{r\to\infty} \mathbb{P}\big(\Lambda_{\exp\{r^{-\epsilon+1/\Delta}\}} \subset B(0,r) \subset \Lambda_{\exp\{r^{\epsilon+1/\Delta}\}}\big) = 1. \tag{1.8}$$

In particular,

$$\frac{\log\log|B(0,r)|}{\log r} \underset{r\to\infty}{\longrightarrow} \frac{1}{\Delta} \tag{1.9}$$

in probability.

As $\Delta \in (1,\infty)$ for $s \in (d,2d)$, the leading-order volume growth takes a stretched-exponential form, i.e.,

$$\big|B(0,r)\big| = \exp\big\{r^{\frac{1}{\Delta}+o(1)}\big\}, \qquad r \to \infty. \tag{1.10}$$

While the left inclusion in (1.8) is implied directly by Theorem 1.1, for that on the right we will have to invoke — and, in fact, prove again in order to accommodate for a more general setting — a result due to Trapman (see Theorem 3.1).

The rest of this note is organized as follows: In Sect. 2 we discuss various motivations for, and further results related to this work. In Sect. 3 we prove Theorem 1.2 concerning volume growth estimates on the infinite graph. Sect. 4 gives the proof of Theorem 1.1 on graph diameter subject to some technical claims; these are then established in Sect. 5.

2. Related work

Long-range percolation, of which our model is an example, originated in the mathematical-physics literature as a model that exhibits a phase transition even in spatial dimension one (e.g., Newman and Schulman [13], Schulman [14], Aizenman and Newman [1], Imbrie and Newman [11]). It soon became clear that $s = d$ and $s = 2d$ are two distinguished values; for $s < d$ the model is essentially mean-field (or complete-graph) alike, for $s > 2d$ the behavior is more or less as for the nearest-neighbor percolation. The regime $d < s < 2d$ turned out to be quite interesting; indeed, it is the only general class of percolation models with Euclidean (or amenable) geometry where one can prove absence of percolation at the percolation threshold (Berger [6]). In all dimensions, the model with $s = 2d$ has a natural continuum scaling limit.

Recently, long-range percolation has been invoked as a fruitful source of graphs with non-trivial growth properties. Our interest was stirred by the work of Benjamini and Berger [3] who proposed (and studied) long-range percolation as a model of social networks. It is this context where the graph distance scaling, and volume growth, are particularly of much interest. Thanks to numerous contributions that followed [3], this scaling is now known for most values of s and d. Explicitly, for $s < d$, a corollary to the main result of Benjamini, Kesten, Peres and Schramm [5] asserts that

$$D_L \underset{L\to\infty}{\longrightarrow} \left\lceil \frac{s}{d-s} \right\rceil, \tag{2.1}$$

almost surely. As $s \uparrow d$, the right-hand side tends to infinity and so, at $s = d$, we expect $D_L \to \infty$. And, indeed, the precise growth rate in this case has been established by Coppersmith, Gamarnik and Sviridenko [9],

$$D_L \asymp \frac{\log L}{\log\log L}, \qquad L \to \infty, \tag{2.2}$$

where "$\asymp$" means that the ratio of left and right-hand side is a random variable that is bounded away from zero and infinity with probability tending to one.

For $s \in (d, 2d)$, the present paper states $D_L = (\log L)^{\Delta+o(1)}$. Here we note that $\Delta \downarrow 1$ as $s \downarrow d$ which, formally, is in agreement with (2.2). For $s \uparrow 2d$ we in turn have $\Delta \to \infty$ and so, at $s = 2d$, a polylogarithmic growth is no longer sustainable. Instead, for the case of the decay $p_{xy} \sim \beta|x-y|^{-2d}$ one expects that

$$D_L = L^{\theta(\beta)+o(1)}, \qquad L \to \infty, \tag{2.3}$$

where $\theta(\beta)$ varies through $(0,1)$ as β sweeps through $(0,\infty)$. This claim is supported by upper and lower bounds in somewhat restricted one-dimensional cases (Benjamini and Berger [3], Coppersmith, Gamarnik and Sviridenko [9]). However, even the existence of a sharp exponent $\theta(\beta)$ has been elusive so far.

For $s > 2d$ one expects [3] the same behavior as for the original graph. And indeed, the linear asymptotic,

$$D_L \asymp L, \tag{2.4}$$

has been established by Berger [7]. For the nearest-neighbor percolation case, this statement goes back to the work of Antal and Pisztora [2].

Further motivation comes from the recent interest in diffusive properties of graphs arising via long-range percolation. An early work in this respect was that of Berger [6] who characterized regimes of recurrence and transience for the simple random walk on such graphs. Benjamini, Berger and Yadin [4] later showed that the mixing time τ_L of the random walk on $\mathbb{G}_L$ in $d = 1$ scales like

$$\tau_L \sim \begin{cases} L^{s-1}, & \text{if } 1 < s < 2, \\ L^2, & \text{if } s = 2, \end{cases} \tag{2.5}$$

with an apparent jump in the exponent when s passes through 2. Misumi [12] found estimates on the effective resistance in $\Lambda_{2L} \setminus \Lambda_L$ that exhibit a similar transition.

Very recently, precise bounds for the heat kernel and spectral gap of such random walks have been derived by Crawford and Sly [10]. These are claimed to lead to the proof that the law of such random walks scales to α-stable processes for $d < s < d+2$ in $d \geq 2$ and $1 < s < 2$ in $d = 1$. For s on the increasing side of these regimes, the random walk is expected to scale to Brownian motion.

3. VOLUME GROWTH

The goal of this section is to prove Theorem 1.2. As already mentioned, while the left inclusion in (1.8) is a direct consequence of Theorem 1.1, the proof of the right inclusion will be based on ideas underlying the proof of Theorem 1.2 in Trapman [15]. Unfortunately, Trapman's setting is too stringent for our purposes and so we restate (and prove) the relevant result in a more suitable form:

Theorem 3.1 *Under the conditions of Theorem 1.1, for each $s' \in (d,s)$ there are constants $c_1, c_2 \in (0,\infty)$ such that, for $\Delta' := 1/\log_2(2d/s')$,*

$$\mathbb{P}\big(D(0,x) \le n\big) \le c_1 \left(\frac{\mathrm{e}^{c_2\, n^{1/\Delta'}}}{|x|}\right)^{s'}, \qquad n \ge 1. \tag{3.1}$$

Before we provide a proof of this result, let us see how it fits into our proof of the volume growth estimate:

Proof of Theorem 1.2. Notice first that, by the structure of the expressions, it suffices to prove both limits in the statement along a single sequence of r's that tends to infinity at most exponentially fast. In fact, we will do this for r being of the form $(\log L)^\theta$ with $\theta \approx \Delta$ and L is running through positive integers.

We begin with the right inclusion in (1.8). Let $\epsilon > 0$ and pick $s' \in (d,s)$ so that $\Delta' := 1/\log_2(2d/s')$ satisfies $\Delta' > \Delta - \epsilon$. Setting $\beta := (\Delta-\epsilon)/\Delta'$, a union bound and Theorem 3.1 then give

$$\begin{aligned}\mathbb{P}\big(\exists x \in \Lambda_{2^{k+1}L} \setminus \Lambda_{2^k L} \colon D(0,x) \le (\log L)^{\Delta-\epsilon}\big)& \\ \le c_3 \left(\frac{\exp\{c_2(\log L)^\beta\}}{2^k L}\right)^{s'} \big(2^{k+1}L\big)^d &= L^{d-s'+o(1)}(2^k)^{d-s'},\end{aligned} \tag{3.2}$$

where we used that $\beta < 1$ by our assumptions and where $o(1) \to 0$ in the limit as $L \to \infty$. Since $s' > d$, the right-hand side is summable on k and so we conclude

$$\mathbb{P}\big(\exists x \notin \Lambda_L \colon D(0,x) \le (\log L)^{\Delta-\epsilon}\big) \le L^{d-s'+o(1)}, \tag{3.3}$$

which tends to zero as $L \to \infty$. A moment's thought shows that

$$\big\{\exists x \notin \Lambda_L \colon D(0,x) \le (\log L)^{\Delta-\epsilon}\big\} \supset \big\{B(0,r) \not\subset \Lambda_{\exp\{r^{\epsilon'+1/\Delta}\}}\big\} \tag{3.4}$$

for $r := (\log L)^{\Delta-\epsilon}$ and $\epsilon' := (\Delta-\epsilon)^{-1} - \Delta^{-1}$. The right inclusion in (1.8) thus holds with probability tending to one for all $\epsilon > 0$.

As to the left inclusion in (1.8) we notice that for $r := (\log L)^{\Delta+\epsilon}$,

$$\big\{D_L \le (\log L)^{\Delta+\epsilon}\big\} \subset \big\{\Lambda_{\exp\{r^{-\epsilon'+1/\Delta}\}} \subset B(0,r)\big\} \tag{3.5}$$

where $\epsilon' := \Delta^{-1} - (\Delta+\epsilon)^{-1}$. By Theorem 1.1, the event on the left occurs with probability tending to one as $L \to \infty$. Therefore, so does the left inclusion in (1.8). □

In order to prove Theorem 3.1, we will follow Trapman's remarkable simplification of the proof from Biskup [8] for the lower bound on the graph distance in infinite-volume setting. Fix $s' \in (d,s)$ and let $R = R(s') \geq 1$ be the number such that

$$p_{xy} \leq |x-y|^{-s'}, \qquad |x-y| \geq R. \tag{3.6}$$

This number exists by our assumption (1.4). The key steps of Trapman's argument can be encapsulated into two lemmas:

Lemma 3.2 *Abbreviate $B_k := B(0,k)$. If $|x|/k \geq R$, then*

$$\mathbb{P}\bigl(D(0,x) \leq k\bigr) \leq \Bigl(\frac{|x|}{k}\Bigr)^{-s'} \sum_{j=0}^{k} \mathbb{E}|B_j|\,\mathbb{E}|B_{n-j}| \tag{3.7}$$

Proof. If $D(0,x) \leq k$, then there exists a (vertex) self-avoiding path from 0 to x such that at least one edge has length at least $|x|/k$. If this edge occurs at the j-th step and it goes from vertex y to vertex z, then we must have $D(0,y) \leq j$ and $D(z,x) \leq k-j$. Conditioning on j and (y,z) thus yields

$$\mathbb{P}\bigl(D(0,x) \leq k\bigr) \leq \sum_{j=1}^{k} \sum_{\substack{y,z \in \mathbb{Z}^d \\ |y-z| \geq |x|/k}} \mathbb{P}\bigl(D(0,y) \leq j\bigr)\, p_{yz}\, \mathbb{P}\bigl(D(z,x) \leq k-j\bigr). \tag{3.8}$$

Under the assumption that $|x|/k \geq R$ we can bound $p_{yz} \leq (|x|/k)^{-s'}$. Dropping the condition on $|y-z|$ we can now sum over y and z to get the right-hand side of (3.7). $\square$

Lemma 3.3 *There is a constant $a = a(d,s')$ such that, given $j \geq 1$, if there is a $K \geq Rj$ such that $\mathbb{P}(D(0,x) \leq j) \leq [K/|x|]^{s'}$ for all $x \in \mathbb{Z}^d$ with $|x|/j \geq R$, then $\mathbb{E}|B_j| \leq aK^d$.*

Proof. Note that $|x| > K$ implies $|x|/j \geq R$. Thus

$$\mathbb{E}|B_j| = \sum_{x \in \mathbb{Z}^d} \mathbb{P}\bigl(D(0,x) \leq j\bigr) \leq \sum_{x\colon |x| \leq K} 1 + \sum_{x\colon |x| > K} \Bigl(\frac{K}{|x|}\Bigr)^{s'}. \tag{3.9}$$

It is easy to check that the first term is bounded by a constant $a_1 = a_1(d)$ times K^d, while the sum over $|x|^{-s'}$ over $|x| > K$ is at most a constant $a_2 = a_2(d,s')$ times $K^{d-s'}$. Putting all terms together, the desired claim follows. $\square$

In addition to the above lemmas, the proof will require one unpleasant calculation that we formalize as follows:

Lemma 3.4 *For each $p > \frac{s'+1}{2d-s'}$ and each $c_0 > 0$ there is $C = C(p,c_0) \in (0,\infty)$ such that for each $c \geq c_0$ the quantity*

$$K(n) := \frac{1}{C}(n+1)^{-p} \mathrm{e}^{c\, n^{1/\Delta'}}, \tag{3.10}$$

where Δ' is as above, obeys

$$\sum_{j=0}^{n} K(j)^d K(n-j)^d \leq n^{-s'} K(n)^{s'}, \qquad n \geq 1. \tag{3.11}$$

Proof. Consider the function $\varphi(x) := x^{1/\Delta'} + (1-x)^{1/\Delta'}$ and note that the exponentials in $K(j)^d K(n-j)^d$ combine into $\exp\{cn^{1/\Delta'}\varphi(j/n)d\}$. Note also that φ is maximized at $x := 1/2$ at where it equals $2^{1-1/\Delta'} = s'/d$. Let

$$\delta := s' - d \max_{0\le x\le 1/4} \varphi(x) \tag{3.12}$$

and observe that $\delta > 0$. Splitting the sum over j into the part when $|j - n/2| \le n/4$ or not, and using the symmetry $j \leftrightarrow n-j$ we thus get

$$\begin{aligned}\sum_{j=0}^{n} K(j)^d K(n-j)^d &\le 2\sum_{j\le n/4} K(j)^d K(n-j)^d + \sum_{j\colon |n/2-j|\le n/4} K(j)^d K(n-j)^d \\ &\le 2\sum_{j\le n/4} C^{-2d} \frac{\mathrm{e}^{c\,n^{1/\Delta'}(s'-\delta)}}{(j+1)^{pd}(n-j+1)^{pd}} \\ &\qquad + \sum_{j\colon |n/2-j|\le n/4} C^{-2d} \frac{\mathrm{e}^{c\,n^{1/\Delta'}s'}}{(j+1)^{pd}(n-j+1)^{pd}}.\end{aligned} \tag{3.13}$$

Using that $j+1 \ge (n+1)/8$ and $n-j+1 \ge (n+1)/8$ for all integers j such that $|j - n/2| \le n/4$, we now get

$$\begin{aligned}\text{LHS of (3.11)} &\le 2(n+1)C^{-2d}\mathrm{e}^{c\,n^{1/\Delta'}(s'-\delta)} + C^{-2d}8^{2pd}(n+1)^{1-2pd}\mathrm{e}^{c\,n^{1/\Delta'}s'} \\ &\le h(n)\, n^{-s'} \Big[\frac{1}{C}(n+1)^{-p}\mathrm{e}^{c\,n^{1/\Delta'}}\Big]^{s'}\end{aligned} \tag{3.14}$$

where

$$h(n) := C^{s'-2d}\big(8^{2pd}(n+1)^{1-2pd} + 2(n+1)\mathrm{e}^{-c\delta\, n^{1/\Delta'}}\big)(n+1)^{s'+ps'}. \tag{3.15}$$

It is easy to check that $1 - 2pd + s' + ps' < 0$ under the assumed condition on p and so the term multiplying $C^{s'-2d}$ is bounded uniformly in n for all $c \ge c_0$. Choosing C sufficiently small, we get $h(n) \le 1$ for all $c \ge c_0$. This proves the claim. □

Proof of Theorem 3.1. Let $p > \frac{s'+1}{2d-s'}$, set $q := \frac{2}{2d-s'}$ and let $a = a(d,s')$ be as in Lemma 3.3. Pick $c_0 > 0$ and let $C(p,c_0)$ be as in Lemma 3.4. Finally, pick $c \ge c_0$ so large that

$$K(n) := \frac{1}{C(p,c_0)}(n+1)^{-p}\mathrm{e}^{c\,n^{1/\Delta'}} \ge a^q R n, \qquad n \ge 1. \tag{3.16}$$

We will show by induction that, for each $n \ge 1$,

$$\mathbb{P}\big(D(0,x) \le n\big) \le \left(\frac{a^{-q}K(n)}{|x|}\right)^{s'}. \tag{3.17}$$

Notice that this is trivially true for $|x| < a^{-q}K(n)$ and so we may thus always suppose that $|x| \ge a^{-q}K(n)$ which by (3.16) implies $|x| \ge Rn$. Also, we may assume that $x \neq 0$; otherwise the right-hand side is infinity.

To start the induction we note that (3.17) holds for $n = 1$ as, for x away from the origin, $\mathbb{P}(D(0,x) \le 1) = p_{0x}$ which is less than the right-hand side by (3.6) and the bound $a^{-q}K(1) \ge R \ge 1$. So let us now suppose (3.17) holds for all $n \le m \in \{1,2,\dots\}$ and let us prove it for $n := m+1$. Notice that as we may assume $|x| \ge R(m+1) \ge Rj$ for $j = 0,\dots,m+1$, Lemma 3.3 can be used for $\mathbb{E}|B_j|$ with $K := a^{-q}K(j)$ for all $j = 1,\dots,m+1$. By Lemma 3.2 and Lemma 3.3 we thus get

$$\mathbb{P}\bigl(D(0,x) \le m+1\bigr) \le \Bigl(\frac{|x|}{m+1}\Bigr)^{-s'} a^{2-2dq} \sum_{j=0}^{m+1} K(j)^d K(m+1-j)^d. \tag{3.18}$$

Invoking Lemma 3.4, the sum can be further bounded with the result

$$\mathbb{P}\bigl(D(0,x) \le m+1\bigr) \le a^{2-2dq}\Bigl(\frac{K(m+1)}{|x|}\Bigr)^{s'}. \tag{3.19}$$

Since $2-2dq = -s'q$, we get (3.17) for $n = m+1$. Thus (3.17) holds for all $n \ge 1$; choosing $c_1 := a^{-qs'}C(p,c_0)^{-s'}$ and $c_2 := c$ we then get also (3.1). □

Remark 3.5 Notice that summing (3.3) over L along powers of 2 yields

$$\liminf_{|x|\to\infty} \frac{\log D(0,x)}{\log\log|x|} \ge \Delta, \qquad \mathbb{P}\text{-a.s.} \tag{3.20}$$

i.e., a lower bound on the growth of the graph distance proved along a far more elegant argument than the original proof in [8].

4. Diameter control

We now pass to the proof of Theorem 1.1. As remarked earlier, the lower bound in (1.5) is an easy consequence of the asymptotic (1.1).

Proof of Theorem 1.1, lower bound. Recall that $D_L(x,y)$ is the graph distance between x and y measured on $\mathbb{G}_L$ and let $D(x,y)$ be the distance measured on the full long-range percolation graph on $\mathbb{Z}^d$. Then we have

$$D_L \ge D_L(x,y) \ge D(x,y), \qquad x,y \in \mathbb{G}_L. \tag{4.1}$$

Now by (3.20) (or [8, Theorem 1.1]), for every $\epsilon > 0$, we have $D(0,x) \ge (\log L)^{\Delta-\epsilon}$ once L is sufficiently large and $|x| \approx L$. The lower bound in (1.5) follows. □

The key is thus to prove the corresponding upper bound. A natural idea is to follow the strategy of [8] which is based on the following observation: Let x and y be two vertices and let $L := |x-y|$. Abbreviate

$$B_\ell(x) := x + [-\ell,\ell]^d \cap \mathbb{Z}^d. \tag{4.2}$$

The probability that $B_\ell(x)$ and $B_\ell(y)$ are directly connected by an edge is then

$$1 - \exp\{-\ell^{2d}L^{-s+o(1)}\}. \tag{4.3}$$

Thus, choosing $\ell := L^\gamma$, the aforementioned edge with be present with very high probability as long as $\gamma \gtrsim s/2d$.

Denoting $z_{00} := x$ and $z_{11} := y$, and letting z_{01} and z_{10} be the endpoints of the primary edge (z_{01}, z_{10}), we can now find two secondary edges of length order L^γ spanning the "gaps" (z_{00}, z_{01}) and (z_{10}, z_{11}) to within L^{γ^2} from the respective endpoints. Next we identify 4 tertiary edges of length L^{γ^2} that leave behind 8 "gaps" of length L^{γ^3}, etc. In [8] it was shown that this edge-identification procedure can be iterated k-times with $k \approx (\log\log L)/\log(1/\gamma)$ until we are down to 2^k "gaps" of size $(\log L)^{o(1)}$. Using the underlying $\mathbb{Z}^d$-lattice structure, we readily extract a path from x to y of length $2^k(\log L)^{o(1)}$. Taking $\gamma \uparrow s/2d$ along with $L \to \infty$, this behaves like $(\log L)^{\Delta+o(1)}$.

Unfortunately, as remarked earlier, this is *not* going to work for controlling the length of paths between *all* pairs of vertices $x, y \in \Lambda_L$. The reason is that, to construct a path between two fixed points we only need to ensure the presence of $(\log L)^{\Delta+o(1)}$ edges but, to do this uniformly for all pairs of points in Λ_L we would need to control order $L^{d+o(1)}$ of them. This entropy cannot be beaten since there are blocks of side $(\log L)^{1/d+o(1)}$ with no incident long edges at all. We will thus have to deal with the cases where the requisite connections fail to occur by methods of nearest-neighbor percolation.

For better understanding of what is to follow, it is actually worth noting that the above strategy is in the least capable of proving a polylogarithmic bound on D_L. Indeed, the identification of successive levels can possibly fail at stage k only if *somewhere* in Λ_L there are two vertices at distance L^{γ^k} whose neighborhoods of size $L^{\gamma^{k+1}}$ are not connected by an edge in $\mathbb{G}_L$. By (4.3), this has probability bounded by

$$L^{2d} \exp\bigl\{-L^{\gamma^k(2d\gamma - s + o(1))}\bigr\}. \tag{4.4}$$

Thus, as long as $L^{\gamma^k} \ge (\log L)^\theta$, where $1/\theta < 2d\gamma - s$, this will not happen with probability tending to one. Halting the procedure at this step shows that

$$D_L \le (\log L)^{\Delta+\theta+o(1)}. \tag{4.5}$$

Further improvement can be achieved if from this point on we make the successive scales related not by exponent γ, but by an exponent ζ which is taken close to one. The procedure can then be made to work up to the point when the gaps are at most of size $(\log L)^{1/(2d-s)+o(1)}$. This is still way too large to infer the desired bound on D_L, but now the gaps are themselves much smaller than $(\log L)^\Delta$ — see Lemma 4.1 below. It thus remains to show that such bad regions will not come close to one another. This is the content of Proposition 4.5 below and this is where methods of nearest-neighbor percolation need to be brought into play.

Having outlined the general strategy, we now turn to the details. Fix an $\epsilon > 0$. We will need numbers, s', γ, ζ and η subject to the restrictions:

$$s < s' < 2d \quad \text{and} \quad \frac{s'}{2d} < \gamma < 1, \tag{4.6}$$

$$\frac{\log 2}{\log(1/\gamma)} < \Delta + \epsilon, \tag{4.7}$$

$$\gamma < \zeta < 1 \quad \text{and} \quad \Delta > \frac{1}{2d\zeta - s'} \tag{4.8}$$

and

$$\Delta > \eta > \frac{1}{2d\zeta - s'}. \tag{4.9}$$

To see that such choices can be made, we note the following relation:

Lemma 4.1 *Let $s \in (d, 2d)$ and let Δ be as in in* (1.2)*. Then $\Delta > \frac{1}{2d-s}$.*

Proof. It suffices to note that $s \mapsto (2d - s)\Delta$ is strictly increasing on $(d, 2d)$ and equal to d (which is at least one) at $s = d$. For this, write $(2d - s)\Delta = (2d \log 2)/f({}^{s}/_{2d})$ with $f(x) := \frac{1}{1-x}\log({}^{1}/_{x})$. A computation shows that $f'(x) < 0$ for $0 < x < 1$. $\square$

Using $\vee$ to denote the maximum (and $\wedge$ the minimum), with the above s', γ, η and ζ fixed, we also choose a quantity θ such that

$$\theta > \frac{1}{2d\gamma - s'} \vee \eta \tag{4.10}$$

and define

$$k_0 := \max\bigl\{k \ge 1 \colon \lfloor L^{\gamma^k} \rfloor > (\log L)^{\theta}\bigr\}. \tag{4.11}$$

For any (large) positive integer L we now define a family of scales (L_k) as follows:

$$L_k := \begin{cases} \lfloor L^{\gamma^k} \rfloor, & \text{if } k \le k_0, \\ \lfloor L^{\gamma^{k_0}\zeta^{k-k_0}} \rfloor, & \text{otherwise.} \end{cases} \tag{4.12}$$

Thus, for $k \le k_0$ the subsequent scales are related by exponent γ, while beyond k_0 the corresponding exponent is "only" ζ. In particular, the subsequent scales for $k > k_0$ are far closer than for $k \le k_0$. For later purposes we will need to introduce other two distinguished values:

$$k_1 := \max\bigl\{k \ge 1 \colon L_k > (\log L)^{\eta}\bigr\} \tag{4.13}$$

and

$$k_2 := \min\bigl\{k \ge 1 \colon L_k < (\log L)^{\epsilon}\bigr\}. \tag{4.14}$$

We will only need to consider the scales L_k up to $k = k_2$. A forthcoming definition (of good blocks) will also depend on a $\delta > 0$ that is picked so small that

$$(1 - \delta)^{k_2 - k_1} > {}^{1}/_{2}. \tag{4.15}$$

This is possible since $k_2 - k_1$ is bounded by a constant times $\log({}^{\eta}/_{\epsilon})/\log(1/\zeta)$.

Now consider the cubic box Λ_L. We wish to partition Λ_L into blocks of scale L_1 which in turn should be partitioned into blocks of scale L_2, etc. Unfortunately, the subsequent scales may not be divisible by one another and so we will have to work with rectangular boxes of uneven dimensions. For an integer $\ell \ge 1$, we call an *ℓ-block* any translate of

$$\bigl\{(n_1, \dots, n_d) \colon 1 \le n_i \le \ell_i,\ i = 1, \dots, d\bigr\}, \tag{4.16}$$

where $\ell_1, \dots, \ell_d$ are numbers such that $\ell/2 \le \ell_i \le \ell$ for all $i = 1, \dots, d$. We note:

Lemma 4.2 *Let $0 < \ell' < \ell$ be integers. Then any ℓ-block can be partitioned into ℓ'-blocks.*

Proof. Since the partitioning can be done independently in each lattice direction, we may assume $d = 1$. Without loss of generality, let ℓ be the actual size of the larger block. Let n be the unique integer such that $n\ell' < \ell \leq (n+1)\ell'$. If $(n+\frac{1}{2})\ell' < \ell$, then $\ell - n\ell' \in [\ell'/2, \ell']$ and we may decompose the ℓ-block into n blocks of side ℓ' and one block of side $\ell - n\ell'$. If instead $(n+\frac{1}{2})\ell' \geq \ell$, then we use only $n-1$ blocks of side ℓ' and two blocks of about the same size between $\ell'/2$ and ℓ' whose combined side-length is $\ell - (n-1)\ell'$ — a number between ℓ' and $\frac{3}{2}\ell'$. $\square$

Given L, we will now choose a partitioning of Λ_L into L_1-blocks, a partitioning of these L_1-blocks into L_2-blocks, etc, for all scales L_k with $k \leq k_2$. The hierarchical decomposition will be fixed for the remainder of the argument.

Next we designate good and bad blocks as follows:

Definition 4.3 (Good/bad blocks) *For the above hierarchical decomposition of Λ_L, define good blocks as follows:*

(1) *Any L_{k_2}-block is good.*

If $k < k_2$, an L_k-block is said to be good if

(2a) *at least $1-\delta$ fraction of the L_{k+1}-blocks contained therein are good, and*

(2b) *any two distinct good L_{k+1}-subblocks are linked by an edge from $\mathbb{G}_L$ whose endpoints are contained only in good $L_{k'}$-blocks, for all $k' = k+1, \dots, k_2$.*

An L_k-block is bad *if it is not good.*

Let $\mathscr{B}_k$ be the union of all bad L_k-blocks and let

$$\mathscr{B} := \bigcup_{k=k_1}^{k_2} \mathscr{B}_k. \tag{4.17}$$

We will refer to vertices in $\Lambda_L \setminus \mathscr{B}$ as *good* and those in $\mathscr{B}$ as *bad*. The nearest-neighbor structure on $\mathbb{Z}^d$ induces a decomposition of $\mathscr{B}$ into connected components; let $C(x)$ denote the connected component of $\mathscr{B}$ that contains x and define

$$T_L(x) := \operatorname{diam} C(x) \quad \text{and} \quad T_L := \max_{x \in \Lambda_L} T_L(x). \tag{4.18}$$

Next, consider the restriction $\mathbb{G}'_L$ of $\mathbb{G}_L$ to vertex set $\Lambda_L \setminus \mathscr{B}$ and let $D'_L(x,y)$ be the graph-theoretical distance between x and y as measured on $\mathbb{G}'_L$. Define

$$D'_L := \max_{x,y \in \Lambda_L \setminus \mathscr{B}} D'_L(x,y). \tag{4.19}$$

Notice that T_L and D'_L depend on the choices of s', γ, η, δ and ϵ. Our proof of the upper bound in (1.5) is now reduced to the following propositions:

Proposition 4.4 *For ϵ as above,*

$$\lim_{L\to\infty} \mathbb{P}\big(D'_L \leq (\log L)^{\Delta+2\epsilon} \big) = 1. \tag{4.20}$$

Proposition 4.5 *For ϵ as above,*

$$\lim_{L\to\infty} \mathbb{P}\big(T_L \le (\log L)^{\Delta+\epsilon}\big) = 1. \tag{4.21}$$

These are proved in the next section. Subject to these propositions, we are now ready to establish the main result of this work:

Proof of Theorem 1.1, upper bound. Pick $x, y \in \Lambda_L$. If x is contained in a bad block then, within $\mathbb{Z}^d$-distance $T_L(x)$, there is a vertex $x' \in \Lambda_L \setminus \mathscr{B}$, and similarly we find a vertex $y' \in \Lambda_L \setminus \mathscr{B}$ within distance $T_L(y)$ of y. By concatenating the shortest path between x' and y' on $\mathbb{G}_L'$ with shortest paths connecting x to x' and y to y' on $\mathbb{Z}^d$, we have

$$D_L(x,y) \le T_L(x) + T_L(y) + D_L'(x',y'). \tag{4.22}$$

Therefore,

$$D_L \le 2T_L + D_L'. \tag{4.23}$$

By Propositions 4.4 and 4.5, the right hand side is bounded by $3(\log L)^{\Delta+2\epsilon}$ with probability tending to one. As ϵ was arbitrary positive, the claim follows. $\square$

5. Taming the bad blocks

To finish the proof of Theorem 1.1 we have to provide proofs of Propositions 4.4 and 4.5. We begin by a lemma. Recall that a vertex $x \in \Lambda_L$ is good if it is contained only in good L_k-blocks, for all $k = k_1, \dots, k_2$. Then we have:

Lemma 5.1 *For each $k = k_1, \dots, k_2$, each good L_k-block contains at least half of good vertices. In addition, if $L_k \ge 32dR$, at least quarter of the vertices in the L_k-block with distance at least R from the boundary are good.*

Proof. We claim that, in fact, at least $(1-\delta)^{k_2-k}$ fraction of all vertices in a good L_k-block are good. This is obviously true for $k = k_2$, as all L_{k_2}-blocks are good by Definition 4.3(1). For $k = k_1, \dots, k_2 - 1$ this is proved by induction using Definition 4.3(2a). The first part of the claim now follows by invoking the bound (4.15).

To get the second part, we note that for each lattice direction, at least $4R/L_k$-fraction of all vertices are closer than R to the sides of the block in this direction. Thus less than $8dR/L_k \le 1/4$ of all vertices in the L_k-block are at least R-away from any side. If half of all vertices in the L_k-block are good, at least quarter of all vertices at least R-away from the boundary must be good. $\square$

For the probability estimates that are to follow, it will be useful to note that by (1.4), for each $s' \in (s, 2d)$ there is a number $R = R(s') < \infty$ such that

$$p_{xy} \ge 1 - \mathrm{e}^{-|x-y|^{-s'}}, \qquad |x-y| \ge R. \tag{5.1}$$

Note that this is different from (3.6), where we cared for an upper bound on p_{xy}.

Proposition 5.2 *Given an L_k-block, let $\mathcal{E}_k$ be the event that this block is good. There are constants $c_1, c_2 \in (0,\infty)$ such that, whenever L is so large that $L_{k_2} \ge 8dR$, we have*

$$\mathbb{P}(\mathcal{E}_k^{\mathrm{c}}) \le c_1 \mathrm{e}^{-c_2 L_k^{2d\zeta - s'}}, \qquad k = k_1, \dots, k_2. \tag{5.2}$$

Remark 5.3 The above estimate shows why we need to make the subsequent scales L_k related by exponent ζ — which can be taken arbitrarily close to one — and not γ (as is done for scales for $k < k_1$). Indeed, the bound (5.2) permits the existence of bad L_k blocks already when $L_k^{2d\zeta - s'} = (\log L)^{1+o(1)}$. When ζ obeys (4.8), this rules out existence of bad L_k blocks with $L_k \approx (\log L)^{\Delta}$, but if we worked with $\zeta = \gamma$ this (and consequently, Proposition 4.5) would fail once γ is not sufficiently close to one. Another instance where the difference between ζ and γ shows up is the derivation (5.13–5.15).

Proposition 5.2 will be established by proving a recursive estimate on the probability that an L_k-block is bad:

Lemma 5.4 *Let a_k be the maximum value of $\mathbb{P}(\mathcal{E}_k^{\mathrm{c}})$ over all L_k-blocks. There are constants $c_3, c_4, c_5 \in (0,\infty)$ such that when $L_{k_2} \ge 32dR$, the sequence (a_k) obeys the recursive bound*

$$a_k \le (2a_{k+1})^{c_3 L_k^{2d(1-\zeta)}} + c_4 L_k^{2d} \mathrm{e}^{-c_5 L_k^{2d\zeta - s'}}, \qquad k = k_1, \dots, k_2 - 1, \tag{5.3}$$

with terminal condition

$$a_{k_2} := 0. \tag{5.4}$$

Proof. Pick an L_k-block and let $\mathcal{A}_k$ be the event that at least $1-\delta$ fraction of all L_{k+1}-blocks therein is good. Then we can bound $\mathbb{P}(\mathcal{E}_k^{\mathrm{c}})$ by

$$\mathbb{P}(\mathcal{E}_k^{\mathrm{c}}) \le \mathbb{P}(\mathcal{A}_k^{\mathrm{c}}) + \mathbb{P}(\mathcal{E}_k^{\mathrm{c}} | \mathcal{A}_k). \tag{5.5}$$

We will now prove that the probabilities on the right-hand side are bounded, respectively, by the two terms in (5.3).

Let n_k denote the number of L_{k+1}-blocks in the given L_k-block. By induction assumption $\mathbb{P}(\mathcal{E}_{k+1}^{\mathrm{c}})$ is bounded by a_{k+1} for each L_{k+1}-block and the events $\mathcal{E}_{k+1}$ for distinct blocks are independent. It follows that the number of bad L_{k+1}-blocks in the given L_k-block is stochastically dominated by a binomial random variable with parameters n_k and a_{k+1}. In particular, $\mathbb{P}(\mathcal{A}_k^{\mathrm{c}})$ is less than the probability that this random variable is at most δn_k. The exponential Chebyshev bound now gives

$$\mathbb{P}(\mathcal{A}_k^{\mathrm{c}}) \le \mathrm{e}^{-\lambda \delta n_k} (1 - a_{k+1} + \mathrm{e}^{\lambda} a_{k+1})^{n_k}, \qquad \lambda \ge 0. \tag{5.6}$$

Choosing $\mathrm{e}^{-\lambda} := a_{k+1}$ and noting that n_k is at least a constant times $(L_k/L_{k+1})^d$ which is bounded by a constant times $L_k^{2d(1-\zeta)}$ then yields

$$\mathbb{P}(\mathcal{A}_k^{\mathrm{c}}) \le (2a_{k+1}^{\delta})^{n_k} \le (2a_{k+1})^{c_3 L_k^{2d(1-\zeta)}}. \tag{5.7}$$

This proves the first term on the right-hand side of (5.3).

To get the second term, we note that $\mathcal{A}_k$ is determined only by the edges with both endpoints in the same L_{k+1}-block. Thus, conditioning $\mathcal{E}_k^{\mathrm{c}}$ on $\mathcal{A}_k$ means that the set of good vertices in one of the good L_{k+1}-blocks contained therein is not joined by an edge to the set of good vertices in another such good L_{k+1}-block. As, by Lemma 5.1, at least a quarter of all vertices in each such good L_{k+1}-block that are R-away from its boundary are good, the use of (5.1) permissible and so we have

$$\mathbb{P}(\mathcal{E}_k^{\rm c}|\mathcal{A}_k) \le \binom{n_k}{2} \exp\Bigl\{-\frac{\bigl(\frac14(L_{k+1}/2)^d\bigr)^2}{(dL_k)^{-s'}}\Bigr\}. \tag{5.8}$$

Here the binomial coefficient counts the number of pairs of L_{k+1}-blocks, $(L_{k+1}/2)^d$ is a lower bound on the size of any L_{k+1}-block, the factor $1/4$ accounts for the number of good vertices in such L_{k+1}-block that are at least distance R from the boundary and dL_k is the maximum of $|x-y|$ for any pair of such good vertices in the L_k-block. Using that $L_{k+1} \ge c^{-1}L_k^{\zeta}$ and $n_k \le cL_k^d$ for some constant $c \in (0,\infty)$, the second term on the right-hand side of (5.3) is proved too. □

Proof of Proposition 5.2. We have to show how to get (5.2) from (5.3). Here we we invoke the inequality

$$L_{k+1}^{2d\zeta-s'} L_k^{2d(1-\zeta)} \ge cL_k^{2d\zeta-s'}, \tag{5.9}$$

valid for some constant $c \in (0,\infty)$ for all k, to check that an upper bound of the form

$$a_k \le c_6^{k_2-k} \mathrm{e}^{-c_2 L_k^{2d\zeta-s'}} \tag{5.10}$$

propagates under this recursion once c_2 and c_6 are taken sufficiently small but positive. As this bound holds for $k = k_2$ by (5.4), it holds for all $k = k_1, \dots, k_2$. Noting that $k_2 - k \le k_2 - k_1$ is bounded, we have (5.2) with $c_1 := c_6^{k_2-k_1} \vee 1$. □

An immediate consequence of the bound in Proposition 5.2 is:

Corollary 5.5 *Let $\mathcal{F}_L$ be the event that all L_{k_1}-blocks are good. Then* $\lim_{L\to\infty} \mathbb{P}(\mathcal{F}_L) = 1$.

Proof. As the number of L_{k_1}-blocks is at most cL^d for some $c < \infty$, a union bound yields

$$\mathbb{P}(\mathcal{F}_L^{\rm c}) \le cL^d\, c_1 \mathrm{e}^{-c_2 L_{k_1}^{2d\zeta-s'}}. \tag{5.11}$$

By (4.9) and the definition of k_1, the exponent is much larger than $\log L$. □

Lemma 5.6 *Let $\mathcal{G}_k$ be the event that in every L_k-block, any two distinct L_{k+1}-blocks are connected by an edge in $\mathbb{G}_L$ with both endpoints at good vertices. Then*

$$\lim_{L\to\infty} \mathbb{P}\Bigl(\bigcap_{k=0}^{k_1-1} \mathcal{G}_k\Bigr) = 1. \tag{5.12}$$

Proof. Consider the event $\mathcal{G}_k^{\rm c} \cap \mathcal{F}_L$ for $0 \le k \le k_1$. On this event, Lemma 5.1 ensures that at least half of all vertices in L_{k+1}-blocks are good. Moreover, for L sufficiently large, at least a quarter of the vertices in L_{k+1}-block are further than R from its boundary. The probability that two such blocks at distance at most L_k are not connected by an edge with both endpoints in $\mathscr{B}^{\mathrm{c}}$ is bounded by $\exp\{-cL_{k+1}^{2d}/L_k^{-s'}\}$. In particular,

$$\mathbb{P}(\mathcal{G}_k^{\mathrm{c}} \cap \mathcal{F}_L) \le \tilde{c}L^{2d}\exp\{-cL_{k+1}^{2d}/L_k^{-s'}\}. \tag{5.13}$$

for some $\tilde{c} < \infty$. Let

$$\alpha := \min\big\{(2d\zeta - s')\eta, (2d\gamma - s')\eta\big\}. \tag{5.14}$$

Examining separately the cases $k \ge k_0$ and $k < k_0$, we find $L_{k+1}^{2d}/L_k^{-s'} \ge a(\log L)^{\alpha}$ for some constant $a > 0$. Plugging this into (5.13) we infer

$$\mathbb{P}\bigg(\mathcal{F}_L \cap \bigcup_{k=0}^{k_1-1} \mathcal{G}_k^{\mathrm{c}}\bigg) \le k_1\tilde{c}L^{2d}\mathrm{e}^{-ac(\log L)^{\alpha}}. \tag{5.15}$$

As $k_1 = O(\log\log L)$ and, by (4.10) and (4.9), $\alpha > 1$, the right-hand side tends to zero as $L \to \infty$. The claim now follows by invoking Corollary 5.5. $\square$

We are now ready to provide the necessary control of the distance function on $\mathbb{G}_L'$. The key observation we will need is as follows:

Lemma 5.7 *Assume $\mathcal{G} := \bigcap_{k=0}^{k_1-1}\mathcal{G}_k$ occurs and let $k(z,z')$ denote the maximal k such that z and z' belong to the same L_k-block. If $z, z' \notin \mathscr{B}$, then for each $k = k(z,z')+1,\dots,k_2$, the L_k-blocks Λ and Λ' containing z and z', respectively, are connected by an edge in $\mathbb{G}_L'$.*

Proof. For $k \ge k_1$ this follows by Definition 4.3 (and the fact that z and z' are good), for $k < k_1$ this is implied by the fact that $\mathcal{G}$ occurs. $\square$

Proof of Proposition 4.4. Pick $x, y \in \Lambda_L \setminus \mathscr{B}$ and assume that the event $\mathcal{G}$ occurs. We will show that then $\mathbb{G}_L'$ contains a path from x to y of length at most $(\log L)^{\Delta+\epsilon}$.

Let Λ_0 and Λ_1 be the L_1-blocks containing x and y, respectively. If $\Lambda_0 \ne \Lambda_1$, by Lemma 5.7, there is an edge in $\mathbb{G}_L'$ with endpoints $z_{01} \in \Lambda_0$ and $z_{10} \in \Lambda_1$. If $\Lambda_0 = \Lambda_1$, we set $z_{01} = z_0$ and $z_{10} = z_1$. This defines the first level of a hieararchy of vertices and edges. For the next level, denote $z_{00} := z_0$ and $z_{11} := z_1$ and for $\sigma \in \{00, 01, 10, 11\}$ let Λ_σ be the L_2-blocks containing z_σ, respectively. As all of the vertices z_σ are good, Lemma 5.7 ensures the existence of edges (z_{001}, z_{010}) and (z_{101}, z_{110}) from $\mathbb{G}_L'$ between "good" vertices z_{001}, z_{010}, z_{101} and z_{110} in the L_3-blocks containing $z_{000} := x$, $z_{011} := z_{01}$, $z_{100} := z_{10}$ and $z_{111} := z_{11}$, respectively.

Proceeding by induction along scales until we get to level k_2, we will thus identify a collection of vertices (z_σ), indexed by $\sigma \in \{0,1\}^{k_2+1}$, such that the following properties hold for each $k \le k_2$:

1. $z_\sigma := x$ if $\sigma = (0,\dots,0)$ while $z_\sigma := y$ if $\sigma = (1,\dots,1)$.
2. For each $\sigma \in \{0,1\}^{k-1}$, the pair $(z_{\sigma 01}, z_{\sigma 10})$ is connected by an edge from $\mathbb{G}_L'$.
3. For each $\sigma \in \{0,1\}^{k-1}$, the vertices $z_{\sigma 00}, z_{\sigma 01}$ lie in one of the (good) L_{k+1}-blocks, and similarly for the pair $(z_{\sigma 10}, z_{\sigma 11})$.

Here σ is a hierarchical index and "$\sigma01$" denotes a concatenation of the string σ with "01." The subsequent "generations" of the hierarchy are nested via the "cancellation rules:" $z_{\sigma00} = z_{\sigma0}$ and $z_{\sigma11} = z_{\sigma1}$. The vertices z_σ are not required to be distinct.

The pair of pairs of vertices $(z_{\sigma00}, z_{\sigma01})$ and/or $(z_{\sigma10}, z_{\sigma11})$, $\sigma \in \{0,1\}^{k_2-1}$, are contained in the same L_{k_2}-block; joining them by shortest paths on $\mathbb{Z}^d$ we thus construct a path on $\mathbb{G}'_L$ from x to y. This path has at most $2^{k_2}-1$ long edges and at most 2^{k_2} nearest-neighbor paths on $\mathbb{Z}^d$ each of which is of length at most dL_{k_2}. Hence we get

$$D'_L(x,y) \le 2^{k_2} - 1 + 2^{k_2} d L_{k_2}. \tag{5.16}$$

Invoking the explicit definitions of k_2 and L_{k_2}, we find

$$2^{k_2} \ll (\log L)^{\Delta+\epsilon} \quad \text{and} \quad L_{k_2} \le (\log L)^{\epsilon}. \tag{5.17}$$

The right-hand side of (5.16) is thus at most $(\log L)^{\Delta+2\epsilon}$, uniformly for all good vertices x and y. As $\mathcal{G}$ occurs with probability tending to one, the desired claim follows. $\square$

The proof is finished by providing a control of the maximal size of connected components of bad vertices.

Proof of Proposition 4.5. We will derive a uniform bound on the probability $\mathbb{P}(T_L(x) \ge (\log L)^{\Delta+\epsilon})$. Suppose $x \in \mathscr{B}$ and consider the connected component $C(x)$. Then $C(x)$ is the disjoint union of L_k-blocks, $k = k_1, \dots, k_2$, all of which are bad. If we fix one such possible collection, $\bigcup_k \bigcup_{i=1}^{m_k} \Lambda_k^{(i)}$ containing disjoint L_k-blocks $\Lambda_k^{(i)}$, $i = 1, \dots, m_k$, Proposition 5.2 and the fact that disjoint blocks are independent yields

$$\mathbb{P}\biggl(C(x) = \bigcup_{k=k_1}^{k_2-1} \bigcup_{i=1}^{m_k} \Lambda_k^{(i)} \biggr) \le \prod_{k=k_1}^{k_2-1} \Bigl\{ c_1 \mathrm{e}^{-c_2 L_k^{2d\zeta - s'}} \Bigr\}^{m_k} \tag{5.18}$$

Now, if $\operatorname{diam} C(x) \ge t$, then $\sum_k m_k L_k \ge t$ and so

$$\sum_{k=k_1}^{k_2-1} m_k L_k^{2d\zeta - s'} \ge t^{1 \wedge (2d\zeta - s')} \tag{5.19}$$

As the number of distinct collections of $m := \sum_k m_k$ blocks that may give rise to $C(x)$ is bounded by $[2d(k_2-k_1)]^{2m}$, we may thus borrow half of the exponent in (5.18) and use the rest to control the entropy. This yields

$$\mathbb{P}\bigl(T_L(x) \ge t\bigr) \le \mathrm{e}^{-\frac12 c_2 t^{1\wedge(2d\zeta - s')}} \sum_{m \ge 1} \Bigl\{ 4d^2 (k_2-k_1)^2 \, c_1 \mathrm{e}^{-\frac12 c_2 L_{k_2}^{2d\zeta - s'}} \Bigr\}^m. \tag{5.20}$$

As $k_2 - k_1 = O(\log\log L)$ while $L_{k_2} \ge (\log L)^{\zeta\epsilon}$, the term in the large braces is small as soon as L is sufficiently large. Setting $t := (\log L)^{\Delta+\epsilon}$ and noting that

$$(\Delta + \epsilon)\bigl(1 \wedge (2d\zeta - s')\bigr) > 1 \tag{5.21}$$

by (4.8) and/or $\Delta > 1$, the probability that $T_L(x) \ge (\log L)^{\Delta+\epsilon}$ is $o(L^{-d})$ uniformly in x. The claim is finished by a standard union bound. $\square$

ACKNOWLEDGMENTS

This research was, at various stages, partially supported by the NSF under the grants DMS-0306167 and DMS-0806198/DMS-0949250. I wish to thank Jianyang Zeng who spotted an error in the initial write-up of this proof and kindly brought it to my attention.

REFERENCES

[1] M. Aizenman and C. Newman, Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models, *Commun. Math. Phys.* **107** (1986) 611–647.
[2] P. Antal and A. Pisztora, On the chemical distance for supercritical Bernoulli percolation, *Ann. Probab.* **24** (1996) 1036–1048.
[3] I. Benjamini and N. Berger, The diameter of long-range percolation clusters on finite cycles, *Random Structures Algorithms* **19** (2001), no. 2, 102–111.
[4] I. Benjamini, N. Berger and A. Yadin, Long range percolation mixing time, *Comb. Probab. Comp.* **17** (2008), 487-494.
[5] I. Benjamini, H. Kesten, Y. Peres and O. Schramm, The geometry of the uniform spanning forests: transitions in dimensions $4, 8, 12, \dots$, *Ann. Math.* (2) **160** (2004), no. 2, 465–491.
[6] N. Berger, Transience, recurrence and critical behavior for long-range percolation, *Commun. Math. Phys.* **226** (2002) 531–558.
[7] N. Berger, A lower bound for the chemical distance in sparse long-range percolation models, preprint (arxiv:math.PR/0409021).
[8] M. Biskup, On the scaling of the chemical distance in long range percolation models, *Ann. Probab.* **32** (2004), no. 4, 2938–2977.
[9] D. Coppersmith, D. Gamarnik and M. Sviridenko, The diameter of a long-range percolation graph, *Random Structures Algorithms* **21** (2002), no. 1, 1–13.
[10] N. Crawford and A. Sly, Heat-kernel upper bounds on long-range percolation cluster, arXiv:0907.2434.
[11] J. Imbrie and C. Newman, An intermediate phase with slow decay of correlations in one-dimensional $1/|x-y|^2$ percolation, Ising and Potts models, *Commun. Math. Phys.* **118** (1988), no. 2, 303–336.
[12] J. Misumi, Estimates of effective resistances in a long-range percolation on $\mathbb{Z}^d$, *J. Math. Kyoto Univ.*, **48** (2008), no. 2, 389–40.
[13] C.M. Newman and L.S. Schulman, One-dimensional $1/|j-i|^s$ percolation models: the existence of a transition for $s \leq 2$, *Commun. Math. Phys.* **104** (1986) 547–571.
[14] L.S. Schulman, Long-range percolation in one dimension, *J. Phys. A* **16** (1983), no. 17, L639-L641.
[15] P. Trapman, The growth of the infinite long-range percolation cluster. arXiv:0901.0661.