# SINGULAR VECTORS UNDER RANDOM PERTURBATION 

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#### Abstract

Computing the first few singular vectors of a large matrix is a problem that frequently comes up in statistics and numerical analysis. Given the presence of noise, exact calculation is hard to achieve, and the following problem is of importance: How much a small perturbation to the matrix changes the singular vectors ?


Answering this question, classical theorems, such as those of Davis-Kahan and Wedin, give tight estimates for the worst-case scenario. In this paper, we show that if the perturbation (noise) is random and our matrix has low rank, then better estimates can be obtained. Our method relies on high dimensional geometry and is different from those used an earlier papers.

MSC indices: 65F15, 15A42, 62H30

## 1. Introduction

An important problem that appears in various areas of applied mathematics (in particular statistics, computer science and numerical analysis) is to compute the first few singular vectors of a large matrix. Among others, this problem lies at the heart of PCA (Principal Component Analysis), which has a very wide range of applications (for many examples, see [3, 5] and the references therein).

The basic setting of the problem is as follows:
Problem 1. Given a matrix $A$ of size $n \times n$ with singular values $\sigma_{1} \geq \cdots \geq \sigma_{n} \geq 0$. Let $v_{1}, \ldots, v_{n}$ be the corresponding (unit) singular vectors. Compute $v_{1}, \ldots, v_{k}$, for some $k \leq n$.

Typically $n$ is large and $k$ is relatively small. As a matter of fact, in many applications $k$ is a constant independent of $n$. For example, to obtain a visualization of a large set of data, one often sets $k=2$ or 3 . The assumption that $A$ is a square matrix is for convenience and our analysis can be carried out with nominal modification for rectangular matrices.

We use asymptotic notation such as $\Theta, \Omega, O$ under the assumption that $n \rightarrow \infty$. The vectors $v_{1}, \ldots, v_{k}$ are not unique. However, if $\sigma_{1}, \ldots, \sigma_{k}$ are different, then

[^0]they are determined up to the sign. We assume this is the case in all discussions. (In fact, as the reader will see, the gap $\delta_{i}:=\sigma_{i}-\sigma_{i+1}$ plays a crucial role.) For a vector $v,\|v\|$ denotes its $L_{2}$ norm. For a matrix $A,\|A\|=\sigma_{1}(A)$ denotes its spectral norm.
1.1. Classical perturbation bounds. The matrix $A$, which represents some sort of data, is often perturbed by noise. Thus, one typically works with $A+E$, where $E$ represents the noise. A natural and important problem is to estimate the influence of noise on the vectors $v_{1}, \ldots, v_{k}$. We denote by $v_{1}^{\prime}, \ldots, v_{k}^{\prime}$ the first $k$ singular vectors of $A+E$.

For sake of presentation, we restrict ourselves to the case $k=1$ (the first singular vector). Our analysis extends easily in the general case, discussed in Section 5 ,

The following question is of importance
Question 2. When is $v_{1}^{\prime}$ a good approximation of $v_{1}$ ?

A convenient way to measure the distance between two unit vectors $v$ and $v^{\prime}$ is to look at $\sin \angle\left(v, v^{\prime}\right)$, where $\angle\left(v, v^{\prime}\right)$ is the angle between the vectors, taken in $[0, \pi / 2]$. To make the problem more quantitative, let us fix a small parameter $\varepsilon>0$, which represents a desired accuracy. Our question now is to find a sufficient condition for the matrix $A$ which guarantees that $\sin \angle\left(v_{1}, v_{1}^{\prime}\right) \leq \epsilon$. It has turned out that the key parameter to look at is the gap (or separation)

$$
\delta:=\sigma_{1}-\sigma_{2}
$$

between the first and second singular values of $A$. Classical results in numerical linear algebra yield

Corollary 3. For any given $\varepsilon>0$, there is $C=C(\varepsilon)>0$ such that if $\delta \geq C\|E\|$, then

$$
\sin \angle\left(v_{1}, v_{1}^{\prime}\right) \leq \epsilon
$$

This follows from a well known result of Wedin
Theorem 4. (Wedin sin theorem) There is a positive constant $C$ such that

$$
\begin{equation*}
\sin \angle\left(v_{1}, v_{1}^{\prime}\right) \leq C \frac{\|E\|}{\delta} \tag{1}
\end{equation*}
$$

In the case when $A$ and $A+E$ are hermitian, this statement is a special case of the famous Davis-Kahan $\sin \theta$ theorem. Wedin [7] extended Davis-Kahan theorem to non-hermitian matrices, resulting in a general theorem that contains Theorem 4 as a special case (see [8, Chapter 8] for more discussion and history).

Let us consider the following simple, but illustrative example [2]. Let $A$ be the matrix

$$
\left(\begin{array}{cc}
1+\epsilon & 0 \\
0 & 1-\epsilon
\end{array}\right) .
$$

Apparently, the singular values of $A$ are $1+\epsilon$ and $1-\epsilon$, with corresponding singular vectors $(1,0)$ and $(0,1)$. Let $E$ be

$$
\left(\begin{array}{cc}
-\epsilon & \epsilon \\
\epsilon & \epsilon
\end{array}\right)
$$

where $\epsilon$ is a small positive number. The perturbed matrix $A+E$ has the form

$$
\left(\begin{array}{ll}
1 & \epsilon \\
\epsilon & 1
\end{array}\right)
$$

Obviously, the singular values $A+E$ are also $1+\epsilon$ and $1-\epsilon$. However, the corresponding singular vectors now are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, no matter how small $\epsilon$ is. This example shows that the consideration of the gap $\delta$ is necessary, and also that Theorem 4 is sharp, up to a constant factor.
1.2. Random perturbation. Noise (or perturbation) represents errors that come from various sources which are frequently of entirely different nature, such as errors occurring in measurements, errors occurring in recording and transmitting data, errors occurring by rounding etc. It is usually too complicated to model noise deterministically, so in practice, one often assumes that it is random. In particular, a popular model is that the entries of $E$ are independent random variables with mean 0 and variance 1 (the value 1 is, of course, just matter of normalization).

For simplicity, we restrict ourselves to a representative case when all entries of $E$ are iid Bernoulli random variables, taking values $\pm 1$ with probability half. For the treatment of more general models, see Section 5 .

Remark 5. We prefer the Bernoulli model over the gaussian one for two reasons. First, we believe that in many real-life applications, noise must have discrete nature (after all, data are finite). So it seems reasonable to use random variables with discrete support to model noise, and Bernoulli is the simplest such a variable. Second, as the reader will see, the analysis for the Bernoulli model easily extends to many other models of random matrices (including the gaussian one). On the other hand, the analysis for gaussian matrices often relies on special properties of the gaussian measure which are not available in other cases.

We say that an event $\mathcal{E}$ holds almost surely if $\mathbf{P}(\mathcal{E})=1-o(1)$; in other words, the probability that $\mathcal{E}$ holds tends to one as $n$ tends to infinity. It is well-known that the norm of a random Bernoulli matrix is of order $\sqrt{n}$, almost surely (see Lemma 12). Thus, Theorem 4 implies the following variant of Corollary 3.

Corollary 6. For any given $\varepsilon>0$, there is $C=C(\varepsilon)>0$ such that if $\delta \geq C \sqrt{n}$, then with probability $1-o(1)$

$$
\sin \angle\left(v_{1}, v_{1}^{\prime}\right) \leq \epsilon
$$

1.3. Low dimensional data and improved bounds. In a large variety of problems, the data is of small dimension, namely, $r:=\operatorname{rank} A \ll n$. The main point that we would like to make in this paper is that in this setting, the lower bound on $\delta$ can be significantly improved. Let us first present the following (improved) variant of Corollary 6

Corollary 7. For any positive constant $\epsilon$ there is a positive constant $C=C(\epsilon)$ such that the following holds. Assume that $A$ has rank $r \leq n^{.99}$ and $\frac{n}{\sqrt{r \log n}} \leq \sigma_{1}$ and $\delta \geq C \sqrt{r \log n}$. Then with probability $1-o(1)$

$$
\begin{equation*}
\sin \angle\left(v_{1}, v_{1}^{\prime}\right) \leq \epsilon . \tag{2}
\end{equation*}
$$

This result shows that (under the given circumstances) we can approximate $v_{1}$ closely (by $v_{1}^{\prime}$ ) provided $\delta \geq C \sqrt{r \log n}$, improving the previous assumption $\delta \geq$ $C \sqrt{n}$. Furthermore, the appearance of $\sigma_{1}$ in the statement is necessary. If $\sigma_{1} \ll \sqrt{n}$, then the noise dominates and we could not expect to recover any good information about $A$ from $A+E$.

Corollary 7 is an easy consequence of the following theorem.
Theorem 8. (Probabilistic sin-theorem) For any positive constants $\alpha_{1}, \alpha_{2}$ there is a positive constant $C$ such that the following holds. Assume that A has rank $r \leq n^{1-\alpha_{1}}$ and $\sigma_{1}:=\sigma_{1}(A) \leq n^{\alpha_{2}}$. Let $E$ be a random Bernoulli matrix. Then with probabilty $1-o(1)$

$$
\begin{equation*}
\sin ^{2} \angle\left(v_{1}, v_{1}^{\prime}\right) \leq C \max \left(\frac{\sqrt{r \log n}}{\delta}, \frac{n}{\delta \sigma_{1}}, \frac{\sqrt{n}}{\sigma_{1}}\right) . \tag{3}
\end{equation*}
$$

Furthermore, one can remove the term $\frac{\sqrt{n}}{\sigma_{1}}$ if $\delta \leq \frac{1}{2} \sigma_{1}$.

Let us know consider the general case when we try to approximate the first $k$ singular vectors. Set $\varepsilon_{k}:=\sin \angle\left(v_{k}, v_{k}^{\prime}\right)$ and $s_{k}:=\left(\varepsilon_{1}^{2}+\cdots+\varepsilon_{k}^{2}\right)^{1 / 2}$. We can bound $\varepsilon_{k}$ recursively as follows.

Theorem 9. For any positive constants $\alpha_{1}, \alpha_{2}, k$ there is a positive constant $C$ such that the following holds. Assume that A has rank $r \leq n^{1-\alpha_{1}}$ and $\sigma_{1}:=\sigma_{1}(A) \leq n^{\alpha_{2}}$. Let $E$ be a random Bernoulli matrix. Then with probabilty $1-o(1)$

$$
\begin{equation*}
\varepsilon_{k}^{2} \leq C \max \left(\frac{\sqrt{r \log n}}{\delta_{k}}, \frac{n}{\sigma_{k} \delta_{k}}, \frac{\sqrt{n}}{\sigma_{k}}, \frac{\sigma_{1}^{2} s_{k-1}^{2}}{\sigma_{k} \delta_{k}}, \frac{\left(\sigma_{1}+\sqrt{n}\right)\left(\sigma_{k}+\sqrt{n}\right) s_{k-1}}{\sigma_{k} \delta_{k}}\right) \tag{4}
\end{equation*}
$$

The first three terms in the RHS of (4) mirror those in (3). The last two terms represent the recursive effect.

To give the reader a feeling about this bound, let us consider the following example. Take $A$ such that $r=n^{o(1)}, \sigma_{1}=2 n^{\alpha}, \sigma_{2}=n^{\alpha}, \delta_{2}=n^{\beta}$, where $\alpha>1 / 2>\beta>1-\alpha$ are positive constants. Then $\delta_{1}=n^{\alpha}$ and $\epsilon_{1}^{2} \leq \max \left(n^{-\alpha+o(1)}, n^{1-2 \alpha+o(1)}\right)$, almost surely.

Assume that we want to bound $\sin \angle\left(v_{2}, v_{2}^{\prime}\right)$. The gap $\delta_{2}=n^{\beta}=o\left(n^{1 / 2}\right)$, so Wedin theorem (in the general form) does not apply. On the other hand, Theorem 9 implies that almost surely

$$
\varepsilon_{2}^{2} \leq \max \left(n^{-\beta+o(1)}, n^{1 / 2-\alpha+o(1)}, n^{-\alpha-\beta+1}\right)
$$

Thus, we have almost surely

$$
\sin \angle\left(v_{2}, v_{2}^{\prime}\right)=n^{-\Omega(1)}=o(1)
$$

The angle between two subspaces. Let us mention that if $\sin \angle\left(v_{j}, v_{j}^{\prime}\right) \leq \varepsilon$ for all $1 \leq j \leq k$, then $\sin \angle\left(V_{k}, V_{k}^{\prime}\right) \leq \varepsilon$, where $V_{k}\left(V_{k}^{\prime}\right)$ is the subspace spanned by $v_{1}, \ldots, v_{k}\left(v_{1}^{\prime}, \ldots, v_{k}^{\prime}\right.$, respectively). The formal (and a bit technical) definition of $\angle\left(V_{k}, V_{k}^{\prime}\right)$ can be seen in [8, 2]. It is important to know that for two subspaces $V, V^{\prime}$ of the same dimension

$$
\sin \angle\left(V, V^{\prime}\right)=\left\|P-P^{\prime}\right\|
$$

where $P\left(P^{\prime}\right)$ denotes the orthogonal projection onto $V\left(V^{\prime}\right)$. Moreover $\left\|P-P^{\prime}\right\|$ is frequently used as the distance between $V$ and $V^{\prime}$.

The rest of the paper is organized as follows. In the next section, we present tools from linear algebra and probability. The proofs of Theorems 8 and 9 follow in Sections 3 and 4, respectively. In Section 5, we extend these theorems for other models of random noise, including the gaussian one, and also to matrices $A$ which do not necessarily have low rank.

## 2. Preliminaries: Linear Algebra and Probability

2.1. Linear Algebra. Fix a system $v_{1}, \ldots, v_{n}$ of unit singular vectors of $A$. It is well-known that $v_{1}, \ldots, v_{n}$ form an orthonormal basis. (If $A$ has rank $r$, the choice of $v_{r+1}, \ldots, v_{n}$ will turn out to be irrelevant.)

For a vector $v$, if we decompose it as

$$
v:=\alpha_{1} v_{1}+\cdots+\alpha_{n} v_{n}
$$

then

$$
\begin{equation*}
\|A v\|^{2}=v \cdot A^{*} A v=\sum_{i=1}^{n} \alpha_{i}^{2} \sigma_{i}^{2} \tag{5}
\end{equation*}
$$

We will use the Courant-Fisher minimax principle for singular values, which asserts that

$$
\begin{equation*}
\sigma_{k}(M)=\max _{\operatorname{dim} H=k} \min _{v \in H,\|v\|=1}\|M v\|, \tag{6}
\end{equation*}
$$

where $\sigma_{k}(M)$ is the $k$ th largest singular value of $M$.
2.2. $\epsilon$-net lemma. Let $\epsilon$ be a positive number. A set $X$ is an $\epsilon$-net of a set $Y$ if for any $y \in Y$, there is $x \in X$ such that $\|x-y\| \leq \epsilon$.

Lemma 10. Let $H$ be a subspace and $S:=\{v\|v\|=1, v \in H\}$. Let $0<\varepsilon \leq 1$ be a number and $M$ a linear map. Let $\mathcal{N} \subset S$ be an $\epsilon$-net $\mathcal{N}$ of $S$. Then there is a vector $w \in \mathcal{N}$ such that

$$
\|M w\| \geq(1-\epsilon) \max _{\|v\| \in S}\|M v\| .
$$

Proof. Let $v$ be the vector where the maximum is attained and let $w$ be a vector in the net closest to $v$ (tights are broken arbitrarily). Then by the triangle inequality

$$
\|M w\| \geq\|M v\|-\|M(v-w)\| .
$$

As $\|v-w\| \leq \epsilon,\|M(v-w)\| \leq \epsilon \max _{\|v\| \in S}\|M v\|$, concluding the proof.

The following estimate for the minimum size of an $\epsilon$ of a sphere is well-known.
Lemma 11. A unit sphere in d dimension admits an $\epsilon$-net of size at most $\left(3 \epsilon^{-1}\right)^{d}$.

Proof. Let $S$ be the sphere in question, centered at $O$, and $\mathcal{N} \subset S$ be a finite subset of $S$ such that the distance between any two points is at least $\epsilon$. If $\mathcal{N}$ is maximal with respect to this property then $\mathcal{N}$ is an $\epsilon$-net. On the other hand, the balls of radius $\epsilon / 2$ centered at the points in $\mathcal{N}$ are disjoint subsets of the the ball of radius $(1+\varepsilon / 2)$, centered at $O$. Since

$$
\frac{1+\varepsilon / 2}{\varepsilon / 2} \leq 3 \varepsilon^{-1}
$$

the claim follows by a volume argument.
2.3. Probability. We need the following estimate on $\|E\|$ (see [1, 6]).

Lemma 12. There is a constant $C_{0}>0$ such that the following holds. Let $E$ be a random Bernoulli matrix of size n. Then

$$
\mathbf{P}(\|E\| \leq 3 \sqrt{n}) \leq \exp \left(-C_{0} n\right)
$$

Next, we present a lemma which roughly asserts that for any two vectors given $u$ and $v, u$ and $E v$ are, with high probability, almost orthogonal. We present the proof of this lemma in ??.

Lemma 13. Let E be a random Bernoulli matrix of size n. For any fixed unit vectors $u, v$ and positive number $t$

$$
\mathbf{P}\left(\left|u^{T} E v\right| \geq t\right) \leq 2 \exp \left(-t^{2} / 16\right)
$$

Now we are ready to state our key lemma.
Lemma 14. For any constants $0<\beta_{1}, 0<\beta_{2}<1$ there is a constant $C$ such that the following holds. Assume that $A$ is such that $\sigma_{1} \leq n^{\beta_{1}}$ and let $V:=$ $\operatorname{Span}\left\{v_{1}, \ldots, v_{d}\right\}$ for some $d \geq n^{1-\beta_{2}}$. Then the following holds almost surely. For any unit vector $v \in V$

$$
\|(A+E) v\|^{2} \leq \sum_{i=1}^{n}\left(v \cdot v_{i}\right)^{2} \sigma_{i}^{2}+C\left(n+\sigma_{1} \sqrt{d \log n}\right)
$$

Proof. It suffices to prove for $v$ belonging to an $\varepsilon$-net $\mathcal{N}$ of the unit sphere $S$ in $V$, with $\varepsilon:=\frac{1}{n+\sigma_{1}}$. With such small $\varepsilon$, the error coming from the term ( $1-\varepsilon$ ) (in Lemma 10) is swallowed into the error term $O\left(n+\sigma_{1} \sqrt{d \log n}\right)$.

By Lemma $10,|\mathcal{N}| \leq\left(\frac{3}{\varepsilon}\right)^{d} \leq \exp \left(C_{1} d \log n\right)$, for some constant $C_{1}$ (which depends on the exponent $\beta_{1}$ in the upper bound of $\sigma_{1}$ ). Thus, using the union bound, it suffices to show that if $C$ is large enough, then for any $v \in \mathcal{N}$

$$
\mathbf{P}\left(\|(A+E) v\|^{2} \geq \sum_{i=1}^{n}\left(v \cdot v_{i}\right)^{2}+C\left(n+\sigma_{1} \sqrt{d \log n}\right)\right) \leq \exp \left(-2 C_{1} d \log n\right)
$$

for any fixed $v \in \mathcal{N}$.
Fix $v \in \mathcal{N}$. By (5),
$\|(A+E) v\|^{2}=\|A v\|^{2}+\|E v\|^{2}+2(A v) \cdot(E v)=\sum_{i=1}^{n}\left(v \cdot v_{i}\right)^{2} \sigma_{i}^{2}+\|E v\|^{2}+2(A v) \cdot(E v)$.
Since $\|A v\| \leq \sigma_{1}$, we have, by Lemma 13 that with probability at least 1 $\exp \left(-C_{2} d \log n\right)$

$$
|(A v) \cdot(E v)| \leq C \sigma_{1} \sqrt{d \log n},
$$

where $C_{2}$ increases with $C$. Thus, by choosing $C$ sufficiently large, we can assume that $C_{2}>3 C_{1}$.

Furthermore, by Lemma $12,\|E v\| \leq 3 \sqrt{n}$ with probability at least $1-\exp (-\Omega(n))$. Combining this with the above bounds, we conclude that for a sufficiently large constant $C$

$$
\begin{aligned}
\mathbf{P}\left(\|(A+E) v\|^{2} \geq \sum_{i=1}^{n}\left(v \cdot v_{i}\right)^{2}+C\left(n+\sigma_{1}\right)\right) & \leq \exp \left(-3 C_{1} d \log n\right)+\exp (-\Omega(n)) \\
& \leq \exp \left(-2 C_{1} d \log n\right)
\end{aligned}
$$

completing the proof.

## 3. Proof of Theorem 8

Let $H$ be the subspace spanned by $\left\{v_{1}, v_{2}\right\}$ and $u_{i}(1 \leq i \leq n)$ be the singular vectors of the matrix $A^{*}$.

First, we give a lower bound for $\sigma_{1}^{\prime}:=\|A+E\|$. By the minimax principle, we have

$$
\sigma_{1}^{\prime}=\|A+E\| \geq\left|u_{1}^{T}(A+E) v_{1}\right|=\left|\sigma_{1}+u_{1}^{T} E v_{1}\right| .
$$

By Lemma 13, we have, with probability $1-o(1),\left|u_{1}^{T} E v_{1}\right| \leq \log \log n$. (The choice of $\log \log n$ is not important. One can replace it by any function that tends slowly to infinity with $n$.)

Thus, we have, with probability $1-o(1)$, that

$$
\begin{equation*}
\|A+E\| \geq \sigma_{1}-\log \log n \tag{7}
\end{equation*}
$$

Our main observation is that, with high probability, any $v$ that is far from $v_{1}$ would yield $\|(A+E) v\|<\sigma_{1}-\log \log n$. Therefore, the first singular vector $v_{1}^{\prime}$ of $A+E$ must be close to $v_{1}$.

Consider a unit vector $v$ and write it as

$$
v=c_{1} v_{1}+c_{2} v_{2}+\cdots+c_{r} v_{r}+c_{0} u
$$

where $u$ is a unit vector orthogonal to $H:=\operatorname{Span}\left\{v_{1}, \ldots, v_{r}\right\}$ and $c_{1}^{2}+\cdots+c_{r}^{2}+c_{0}^{2}=$ 1. Recall that $r$ is the rank of $A$, so $A u=0$. Setting $w:=c_{1} v_{1}+\cdots+c_{r} v_{r}$ and using Cauchy-Schwartz, we have

$$
\begin{aligned}
\|(A+E) v\|^{2} & =\left\|(A+E) w+c_{0} E u\right\|^{2} \leq\|(A+E) w\|^{2}+2 c_{0}\|(A+E) w\|\|E u\|+c_{0}^{2}\|E u\|^{2} \\
& \leq\left(1+\frac{c_{0}^{2}}{4}\right)\|(A+E) w\|^{2}+\left(4+c_{0}^{2}\right)\|E u\|^{2}
\end{aligned}
$$

By Lemma [12, we have, with probability $1-o(1)$, that $\|E u\| \leq 3 \sqrt{n}$ ) for every unit vector $u$. Furthermore, by Lemma 14, we have, with probability $1-o(1)$,

$$
\|(A+E) w\|^{2} \leq \sum_{i=1}^{r}\left(w \cdot v_{i}\right)^{2}+O\left(\sigma_{1} \sqrt{r \log n}+n\right)
$$

for every vector $w \in H$ of length at most 1 .
Since

$$
\sum_{i=1}^{r}\left(w \cdot v_{i}\right)^{2}=\sum_{i=1}^{r} c_{i}^{2} \sigma_{i}^{2} \leq\left(1-c_{0}^{2}\right) \sigma_{1}^{2}-\left(1-c_{0}^{2}-c_{1}^{2}\right)\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)
$$

we can conclude that with probability $1-o(1)$, the first singular vector of $A+E$, written in the form $v=c_{1} v_{1}+\cdots+c_{r} v_{r}+c_{0} u$, satisfies
(8) $\frac{1}{1+c_{0}^{2} / 4}\|(A+E) v\|^{2} \leq\left(1-c_{0}^{2}\right) \sigma_{1}^{2}-\left(1-c_{0}^{2}-c_{1}^{2}\right)\left(\sigma_{1}^{2}-\sigma_{2}^{2}\right)+O\left(\sigma_{1} \sqrt{r \log n}+n\right)$.

Notice that $c_{0} \leq 1$, so the term $c_{0} n$ is swallowed into $O(n)$. By (7) and the fact that $\frac{1}{1+c_{0}^{2}} \geq 1-\frac{c_{0}^{2}}{4}$, we have

$$
\frac{1}{1+c_{0}^{2} / 4}\|(A+E) v\|^{2} \geq\left(1-\frac{c_{0}^{2}}{4}\right)\left(\sigma_{1}-\log \log n\right)^{2}
$$

Comparing this with (8) and noticing that both $\sigma_{1} \log \log n$ and $(\log \log n)^{2}$ are $o\left(\sigma_{1} \sqrt{r \log n}\right)$, we obtain, for some properly chosen constant $C$, that

$$
\left(1-c_{1}^{2}\right) \sigma_{1} \delta-\frac{c_{0}^{2}}{4} \sigma_{1}^{2} \leq-c_{0}^{2} \sigma_{2}^{2}+C\left(\sigma_{1} \sqrt{r \log n}+n\right)
$$

Before concluding the proof, let us derive a bound on $c_{0}$. We can show that with probability $1-o(1)$

$$
\begin{equation*}
c_{0}^{2}=O\left(\frac{\sqrt{n}}{\sigma_{1}}\right) \tag{9}
\end{equation*}
$$

To verify this, we again used the bound $\|(A+E) v\| \geq \sigma_{1}-\log \log n$. Oh the other hand, by the triangle inequality and Lemma [12, we have with probability $1-o(1)$

$$
\|(A+E) v\| \leq\|A v\|+\|E v\| \leq \sqrt{1-c^{2}} \sigma_{1}+3 \sqrt{n}
$$

from which (9) follows by a simple computation.
Without loss of generality, we can assume that $C \geq 1$. If $\sigma_{2} \leq \frac{1}{2} \sigma_{1}$, then $\delta \geq \frac{1}{2} \sigma_{1}$ and

$$
\begin{equation*}
1-c_{1}^{2} \leq \frac{C\left(\sigma_{1} \sqrt{r \log n}+n\right)}{\sigma_{1}^{2} / 2}+\frac{c_{0}^{2}}{2}=O\left(\frac{\sqrt{r} \log n}{\sigma_{1}}\right)+O\left(\frac{n}{\sigma_{1}^{2}}\right)+O\left(\frac{\sqrt{n}}{\sigma_{1}}\right) \tag{10}
\end{equation*}
$$

In the case $\sigma_{2} \geq \frac{1}{2} \sigma_{1}, c_{0}^{2} \sigma_{2}^{2} \geq \frac{c_{0}^{2}}{4} \sigma_{1}^{2}$, so

$$
\left(1-c_{1}^{2}\right) \sigma_{1} \delta \leq C\left(\sigma_{1} \sqrt{r \log n}+n\right)
$$

which implies

$$
\begin{equation*}
\left(1-c_{1}^{2}\right) \leq C\left(\frac{\sqrt{r \log n}}{\delta}+\frac{n}{\sigma_{1} \delta}\right) \tag{11}
\end{equation*}
$$

Notice that $\sin \angle\left(v_{1}, v_{1}^{\prime}\right)^{2}=\sin \angle\left(v_{1}, v\right)^{2}=1-c_{1}^{2}$. The desired claim follows from (10) and (11).

Remark 15. One can improve the error term $\frac{\sqrt{n}}{\sigma_{1}}$ to $\left(\frac{\sqrt{n}}{\sigma_{1}}\right)^{3 / 2}$. However, this proof is more technical and harder to generalize.

## 4. Proof of Theorem 9

Similar to the previous proof, we start with a lower bound for $\sigma_{k}^{\prime}$, the $k$ th largest singular value of $A+E$. Using the minimax principle, we have

$$
\sigma_{k}^{\prime} \geq\left|u_{k}^{T}(A+E) v_{k}\right| \geq \sigma_{k}-\log \log n
$$

with probability $1-o(1)$.
We need to consider $\|(A+E) v\|$ for a unit vector $v$ orthogonal to $v_{1}^{\prime}, \ldots, v_{k-1}^{\prime}$. We write (as before)

$$
v:=c_{1} v_{1}+\cdots+c_{r} v_{r}+c_{0} u=w+c_{0} u
$$

If $v$ is the $k$ th singular vector of $A+E$, then $v \cdot v_{j}^{\prime}=0$ for $1 \leq j \leq k-1$, and we obtain

$$
\left|c_{j}\right|=\left|v \cdot v_{j}\right|=\left|v \cdot\left(v_{j}-v_{j}^{\prime}\right)\right| \leq\left|v_{j}-v_{j}^{\prime}\right| \leq 2 \sin \angle\left(v_{j} \cdot v_{j}^{\prime}\right)=2 \varepsilon_{j} .
$$

As in the previous proof, we consider the inequality

$$
\begin{aligned}
\|(A+E) v\|^{2} & =\left\|(A+E) w+c_{0} E u\right\|^{2} \leq\|(A+E) w\|^{2}+2 c_{0}\|(A+E) w\|\|E u\|+c_{0}^{2}\|E u\|^{2} \\
& \leq\left(1+\frac{c_{0}^{2}}{4}\right)\|(A+E) w\|^{2}+\left(4+c_{0}^{2}\right)\|E u\|^{2}
\end{aligned}
$$

We split $w=\bar{w}_{k}+w_{k}$, where $\bar{w}_{k}:=c_{1} v_{1}+\cdots+c_{k-1} v_{k-1}$ and $w_{k}:=c_{k} v_{k}+\ldots v_{r} c_{r}$. We have
$\|(A+E) w\|^{2}=\left\|(A+E)\left(\bar{w}_{k}+w_{k}\right)\right\|^{2} \leq\left\|(A+E) w_{k}\right\|^{2}+\left\|(A+E) \bar{w}_{k}\right\|^{2}+2\left\|(A+E) w_{k}\right\|\left\|(A+E) \bar{w}_{k}\right\|$.

Using Lemma 14 we have

$$
\begin{equation*}
\left\|(A+E) w_{k}\right\|^{2} \leq c_{k}^{2} v_{k}^{2}+\cdots+c_{r}^{2} v_{r}^{2}+O\left(\sigma_{k} \sqrt{r \log n}+n\right) \tag{12}
\end{equation*}
$$

The term $\left\|(A+E) \bar{w}_{k}\right\|^{2}$ can be bounded, rather generously, by

$$
\begin{equation*}
O\left(\left(\sigma_{1}+\sqrt{n}\right)^{2}\left(c_{1}^{2}+\cdots+c_{k-1}^{2}\right)=O\left(\sigma_{1}^{2}+n\right) s_{k-1}^{2}\right) \tag{13}
\end{equation*}
$$

Moreover,
$\left\|(A+E) w_{k}\right\|\left\|(A+E) \bar{w}_{k}\right\|=O\left(\left(\sigma_{k}+\sqrt{n}\right)\left(\sigma_{1}+\sqrt{n}\right)\left\|w_{k}\right\|\left\|b a r w_{k}\right\|=O\left(\left(\sigma_{1}+\sqrt{n}\right)\left(\sigma_{k}+\sqrt{n}\right) s_{k-1}\right.\right.$.

Repeating the calculations in the previous proof, we have, with probability $1-o(1)$
$\left(1-c_{k}^{2}\right)\left(\sigma_{k}^{2}-\sigma_{k+1}^{2}\right)-\frac{c_{0}^{2}}{4} \sigma_{k}^{2} \leq \sum_{j=1}^{k-1} c_{j}^{2}\left(\sigma_{j}^{2}-\sigma_{k+1}^{2}\right)-c_{0}^{2} \sigma_{k+1}^{2}+O\left(\sigma_{k} \sqrt{r \log n}+n\right)+O\left(\sigma_{1}^{2} s_{k-1}^{2}+\left(\sigma_{1}+\sqrt{n}\right)\left(\sigma_{k}+\sqrt{n}\right) s_{k-1}\right)$.

We can bound $c_{0}$ as follows

$$
\begin{equation*}
c_{0}^{2}+s_{k-1}^{2}=O\left(\frac{\sqrt{n}}{\sigma_{k}}+\frac{\sigma_{1} s_{k-1}}{\sigma_{k}}\right) \tag{15}
\end{equation*}
$$

By considering the two cases $\sigma_{k+1} \geq \frac{1}{2} \sigma_{k}$ and $\sigma_{k+1}<\frac{1}{2} \sigma_{k}$, the desired bound follows.

## 5. Extensions

In this section, we extend our results to other models of random matrices. It is easy to see that we did not rely ver heavily on properties of the Bernoulli random variable. All we need is a model of random matrices so that Lemmas 12 and 13 (or sufficiently strong variants) hold.

Both of these lemmas hold for the case where the noise is gaussian (instead of Bernoulli). In fact, Lemma 13 is trivial as $u^{T} E v$ has distribution $N(0,1)$.

Both lemmas hold in the case the entries of $E$ is bounded by a universal constant $K$. For the proof of Lemma [12, see [1, 6]. For the proof of Lemma 13, see Remark 17

Quite often, the boundedness condition can be replaced by the condition of having a rapidly decaying tail (such as sub-gaussian), using either more advanced concentration tools (see [9]) or a truncation argument (see [10]). We do not pursuit these matters here.

We can also extend our results for a matrix $A$ which does not have low rank, but can be well approximate by one. In this case, we consider $A=A^{\prime}+B$, where $A^{\prime}$ has small rank (say $r$ ) and $B$ is very small. In this case, we can apply, say, Theorem 8 to bound $\left\|v_{1}\left(A^{\prime}\right)-v_{1}\left(A^{\prime}+E\right)\right\|$ and Theorem 4 to bound $\left\|v_{1}\left(A^{\prime}\right)-v_{1}(A)\right\|$ and then use the triangle inequality. As a result, the RHS of (3) will have an extra term $\frac{\|B\|}{\delta}$. The reader is invited to work out the details.

Finally, our analysis also extends fairly easily to the case when $E$ is a hermitian random matrix (either Wigner or Wishart model) and $A$ is hermitian. The details and few applications will appear elsewhere.

## Appendix A. Proof of Lemma 13

As $u^{T} E v=\sum_{i, j} u_{j} v_{j} \xi_{i j}$ where $u=\left(u_{i}\right)_{i=1}^{n}, v=\left(v_{j}\right)_{j=1}^{n}$ and the $\xi_{i j}$ are the entries of $E$, Lemma 13 follows from

Lemma 16. Let $S:=c_{1} \xi_{1}+\cdots+c_{n} \xi_{n}$ where $\xi_{i}$ are iid Bernoulli random variables and $c_{i}$ are real numbers such as $\sum_{i=1}^{n} c_{i}^{2}=1$. Then for any number $t>0$

$$
\mathbf{P}(|S| \geq t) \leq 2 \exp \left(-t^{2} / 16\right)
$$

Proof. Without loss of generality, we can assume that $\left|c_{i}\right|$ decreases and $l$ is the last index such that $\left|c_{i}\right| \geq \frac{2}{T}$. As $\sum_{i=1}^{n} c_{i}^{2}=1, l \leq t^{2} / 4$. By Cauchy-Schwartz,

$$
\left|c_{1} \xi_{1}+\cdots+c_{l} \xi_{l}\right|^{2} \leq l^{2} \sum_{i=1}^{n} c_{i}^{2} \leq \frac{t^{2}}{4}
$$

which implies that with probability one $\left|c_{1} \xi_{1}+\ldots c_{l} \xi_{l}\right| \leq \frac{t}{2}$. Therefore,

$$
\mathbf{P}(|S| \geq t) \leq \mathbf{P}\left(\left|S^{\prime}\right| \leq \frac{t}{2}\right)
$$

where $S^{\prime}:=\sum_{i=l+1}^{n} c_{i} \xi_{i}$.
We can bound $\mathbf{P}\left(\left|S^{\prime}\right| \leq \frac{t}{2}\right)$ by the standard Laplace-transform argument. Set $z:=t / 4$. Thanks to independence, we have

$$
\mathbf{P}\left(S^{\prime} \geq \frac{t}{2}\right)=\mathbf{P}\left(\exp \left(z S^{\prime}\right) \geq e^{t z / 2}\right) \leq e^{-t z / 2} \mathbf{E}\left(\exp \left(z S^{\prime}\right)\right)=e^{-t z / 2} \prod_{i=l+1}^{n} \mathbf{E} \exp \left(z c_{i} \xi_{i}\right)
$$

On the other hand, as $\left|z c_{i}\right| \leq 1$, it is easy to show that

$$
\mathbf{E} \exp \left(z c_{i} \xi_{i}\right) \leq 1+\left(z c_{i}\right)^{2} \leq \exp \left(z^{2} c_{i}^{2}\right)
$$

Together, we obtain

$$
\mathbf{P}\left(S^{\prime} \geq t z / 2\right) \leq e^{-t z / 2} \exp \left(\sum_{i=l+1}^{n} z^{2} c_{i}^{2}\right) \leq \exp \left(z^{2}-\frac{t z}{2}\right)=\exp \left(-\frac{t^{2}}{16}\right)
$$

Similarly

$$
\mathbf{P}\left(S^{\prime} \leq-t z / 2\right)=\mathbf{P}\left(-S^{\prime} \geq t z / 2\right) \leq \exp \left(-\frac{t^{2}}{16}\right)
$$

concluding the proof.
Remark 17. The same proof works for $\xi$ being arbitrary independent random variable with mean 0 and variance 1 , uniformly bounded by a constant $K$. In this case, the constant 16 is replaced by a constant depending on $K$.

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