CLAIRVOYANT SCHEDULING OF RANDOM WALKS

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ABSTRACT. Two infinite walks on the same finite graph are called *compatible* if it is possible to introduce delays into them in such a way that they never collide. About 10 years ago, Peter Winkler asked the question: for which graphs are two independent random walks compatible with positive probability. Up to now, no such graphs were found. We show in this paper that large complete graphs have this property. The question is equivalent to a certain dependent percolation with a power-law behavior: the probability that the origin is blocked at distance n but not closer decreases only polynomially fast and not, as usual, exponentially.

1. INTRODUCTION

1.1. **The model.** Let us call any strictly increasing sequence $t = (t(0) = 0, t(1), \dots)$ of integers a *delay sequence*. For an infinite sequence $z = (z(0), z(1), \dots)$, the delay sequence t introduces a timing arrangement in which the value $z(n)$ occurs at time $t(n)$. Given infinite sequences z_d and delay sequences t_d, for $d = 0, 1$, we say that there is a *collision* at (d, n, k) if $t_d(n) \leqslant t_{1-d}(k) < t_d(n+1)$ and $z_{1-d}(k) = z_d(n)$. We call the two sequences z_0, z_1 *compatible* if there is a delay sequence for them that avoids collisions.

For a finite undirected graph, a Markov chain $Z(1), Z(2), \dots$ with values that are vertices in this graph is called a *random walk* over this graph if it moves, going from $Z(n)$ to $Z(n+1)$, from any vertex with equal probability to any one of its neighbors.

Take two infinite random sequences Z_d for $d = 0, 1$ independent from each other, both of which are random walks on the same finite undirected graph. Here, the delay sequence t_d can be viewed as causing the sequence Z_d to stay in state $z_d(n)$ between times $t_d(n)$ and $t_d(n+1)$. (See the example on the graph K_5 in Figure 1.) A collision occurs when the two delayed walks enter the same point of the graph. Our question is: are Z_0 and Z_1 compatible with positive probability? The question depends, of course, on the graph. Up to the present paper, no graph was known with an affirmative answer. Consider the case when the graph is the complete graph K_m of size m. It is known that if $m \leqslant 3$ then the two sequences are compatible only with zero probability. Simulations suggest that the walks do not collide if $m \geqslant 5$, and the simulations are inconclusive for $m = 4$. The present paper proves the following theorem.

Theorem 1 (Main). *If m is sufficiently large then on the graph K_m, the independent random walks Z_0, Z_1 are compatible with positive probability.*

The upper bound computable for m from the proof is very bad.

In what follows we will also use the simpler notation

$$X = Z_0, \quad Y = Z_1.$$

Date: February 22, 2019.

1991 *Mathematics Subject Classification.* 82B43; 60K35; 68Q85.

Key words and phrases. Percolation, random walk, renormalization.

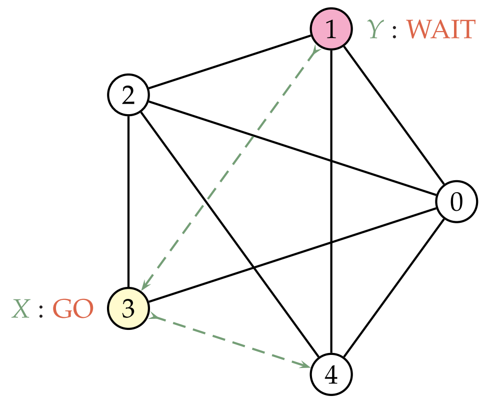

FIGURE 1. The clairvoyant demon problem. X, Y are "tokens" performing independent random walks on the same graph: here the complete graph K_5. A "demon" decides every time, whose turn it is. She is clairvoyant and wants to prevent collision.

The problem, called the clairvoyant demon problem, arose first in distributed computing. The original problem was to find a leader among a finite number of processes that form the nodes of a communication graph. There is a proposed algorithm: at start, let each process have a "token". The processes pass the tokens around in such a way that each token performs a random walk. However, when two tokens collide they merge. Eventually, only one token will remain and whichever process has it becomes the leader. The paper [2] examined the algorithm in the traditional setting of distributed computing, when the timing of this procedure is controlled by an adversary. Under the (reasonable) assumption that the adversary does not see the future sequence of moves to be made by the tokens, the work [2] gave a very good upper bound on the expected time by which a leader will be found. It considered then the question whether a clairvoyant adversary (a "demon" who sees far ahead into the future token moves) can, by controlling the timing alone, with positive probability, prevent two distinct tokens from ever colliding. The present paper solves Conjecture 3 of [2], which says that this is the case when the communication graph is a large complete graph.

1.2. **Related synchronization problems.** Let us define a notion of collision somewhat different from the previous section. For two infinite 0-1-sequences z_d ($d = 0, 1$) and corresponding delay sequences t_d we say that there is a *collision* at (d, n) if $z_d(n) = 1$, and there is no k such that $z_{1-d}(k) = 0$ and $t_d(n) = t_{1-d}(k)$. We say that the sequences z_d are *compatible* if there is a pair of delay sequences t_d without collisions. It is easy to see that this is equivalent to saying that 0's can be deleted from both sequences in such a way that the resulting sequences have no collisions in the sense that they never have a 1 in the same position.

Suppose that for $d = 0, 1$, $Z_d = (Z_d(0), Z_d(1), \dots)$ are two independent infinite sequences of independent random variables where $Z_d(j) = 1$ with probability p and 0 with probability $1 - p$. Our question is: are Z_0 and Z_1 compatible with positive probability? The question depends, of course, on the value of p: intuitively, it seems that they are compatible if p is small.

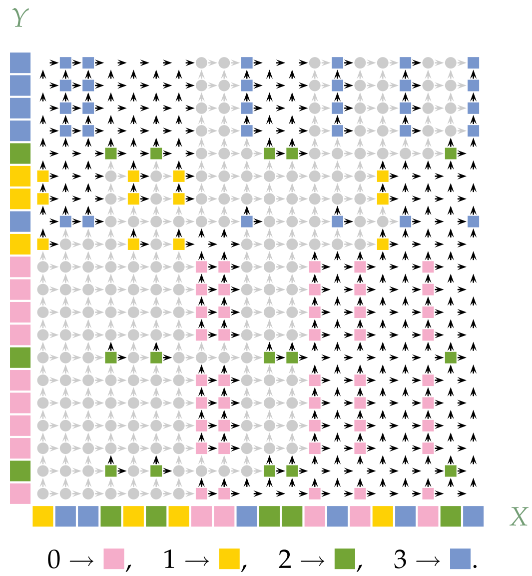

FIGURE 2. Percolation for the clairvoyant demon problem, for random walks on the complete graph K_4. Round light-grey dots mark the reachable points.

Peter Winkler and Harry Kesten [6], independently of each other, found an upper bound smaller than $1/2$ on the values p for which Z_0, Z_1 are compatible. Computer simulations by John Tromp suggest that when $p < 0.3$, with positive probability the sequences are compatible. The paper [4] proves that if p is sufficiently small then with positive probability, Z_0 and Z_1 are compatible.

The threshold for p obtained from the proof is only 10^{-400}, so there is lots of room for improvement between this number and the experimental 0.3.

1.3. **A percolation.** The clairvoyant demon problem has a natural translation into a percolation problem. Consider the lattice $\mathbb{Z}_+^2$, and a directed graph obtained from it in which each point is connected to its right and upper neighbor. For each i, j, let us "color" the ith vertical line by the state $X(i)$, and the jth horizontal line by the state $Y(j)$. The ingoing edges of a point (i, j) will be deleted from the graph if $X(i) = Y(j)$, if its horizontal and vertical colors coincide. We will also say that point (i, j) is *closed*; otherwise, it will be called *open*. (It is convenient to still keep the closed point (i, j) in the graph, even though it became unreachable from the origin.) The question is whether, with positive probability, an infinite path starting from $(0, 0)$ exists in the remaining random graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}).$$

In [4], we proposed to call this sort of percolation, where two infinite random sequences X, Y are given on the two coordinate axes and the openness of a point or edge at position (i, j) depends on the pair $(X(i), Y(j))$, a *Winkler percolation*. This problem permits an interesting variation: undirected percolation, where the the whole lattice $\mathbb{Z}^2$ is present, and the edges are undirected. This variation has been solved, independently, in [5] and [1]. On the other hand, the paper [3] shows that the directed problem has a different nature, since if there is percolation, it has power-law convergence (the undirected percolations have the usual exponential convergence).

2. Outline of the proof

2.1. **Renormalization.** The proof method used is *renormalization* (scale-up). An example of the ordinary renormalization method would be when, say, in an Ising model, the space is partitioned into blocks, spins in each block are summed into a sort of "superspin", and it is shown that the system of superspins exhibits a behavior that is in some sense similar to the original system. We will also map our model repeatedly into a series of higher-order models similar to each other. However the definition of the new models is more complex than just taking the sums of some quantity over blocks. The model which will scale up properly, may contain a number of new objects and restrictions more combinatorial than computational in character.

The method is messy, laborious, and rather crude (rarely leading to the computation of exact constants). However, it is robust and well-suited to "error-correction" situations. Here is a rough first outline.

1. Fix an appropriate sequence $\Delta_1 < \Delta_2 < \cdots$, of scale parameters with $\Delta_{k+1} > 4\Delta_k$. Let $\mathcal{F}_k$ be the event that point $(0, 0)$ is blocked in the square $[0, \Delta_k]^2$. (In other applications, it could be some other *ultimate bad event*.) Throughout the proof, we will denote the probability of an event E by $\mathbf{P}(E)$. We want to prove
$$\mathbf{P}\Big(\bigcup_k \mathcal{F}_k\Big) < 1.$$
This will be sufficient: if $(0, 0)$ is not blocked in any finite square then by compactness (or by what is sometimes called König's Lemma), there is an infinite path starting at $(0, 0)$.
2. Identify some events that you we call *bad events* and some others called *very bad events*, where the latter are much less probable.
3. Define a series $\mathcal{M}^1, \mathcal{M}^2, \dots$ of models similar to each other, where the very bad events of $\mathcal{M}^k$ become the bad events of $\mathcal{M}^{k+1}$. Let $\mathcal{F}'_k$ hold iff some bad event of $\mathcal{M}^k$ happens in the square $[0, \Delta_{k+1}]^2$.
4. Prove
$$\mathcal{F}_k \subseteq \bigcup_{i \leqslant k} \mathcal{F}'_i. \tag{2.1}$$
5. Prove $\sum_k \mathbf{P}(\mathcal{F}'_k) < 1$.

In later discussions, we will frequently delete the index k from $\mathcal{M}^k$ as well as from other quantities defined for $\mathcal{M}^k$. In this context, we will refer to $\mathcal{M}^{k+1}$ as $\mathcal{M}^*$.

2.2. **Application to our case.** The role of the "bad events" of Subsection 2.1 will be played by *traps* and *walls*. The simplest kind of trap is a point (i, j) in the plane such that $X(i) = Y(j)$; in other words, a closed point. More generally, traps will be certain rectangles in the plane. We want to view the occurrence of two traps close to each other as a very bad event; however, this is justified only if this is indeed very improbable. Consider the events
$$\mathcal{A}_5 = \{\, X(1) = X(2) = X(3) = Y(5) \,\}, \quad \mathcal{A}_{13} = \{\, X(1) = X(2) = X(3) = Y(13) \,\}.$$
(For simplicity, this example assumes that the random walk has the option of staying at the same point, that is loops have been added to the graph K_m.) The event $\mathcal{A}_5$ makes the rectangle $[1, 3] \times \{5\}$ a trap of size 3, and has probability is m^{-3}. Similarly for the event $\mathcal{A}_{13}$ and the rectangle $[1, 3] \times \{13\}$. However, these two events are not independent: the probability of $\mathcal{A}_5 \cap \mathcal{A}_{13}$ is only m^{-4}, not m^{-6}. The reason is that the event $\mathcal{E} = \{\, X(1) = X(2) = X(3) \,\}$ significantly increases the conditional probability that, say, the rectangle

$[1, 3] \times \{5\}$ becomes a trap. In such a case, we will want to say that event $\mathcal{E}$ creates a *vertical wall* on the segment $(0, 3]$.

Though our study only concerns the integer lattice, it is convenient to use the notations of the real line and Euclidean plane. In particular, walls will be right-closed intervals—though of course, $(a, b] \cap \mathbb{Z} = [a + 1, b] \cap \mathbb{Z}$. We will say that a certain rectangle *contains* a wall if the corresponding projection contains it, and that the same rectangle *intersects* a wall if the corresponding projection intersects it.

Traps will have low probability. If there are not too many traps, it is possible to get around them. On the other hand, to get through walls, one also needs extra luck: such lucky events will be called *holes*. Our proof systematizes the above ideas by introducing an abstract notion of traps, walls and holes. We will have walls of many different types. To each (say, vertical) wall of a given type, the probability that a (horizontal) hole goes through it at a given point will be much higher than the probability that a horizontal wall of this type occurred at that point. Thus, the "luck" needed to go through some wall type is still smaller than the "unluck" needed to create a wall of this type.

This model will be called a *mazery* $\mathcal{M}$ (a system for creating mazes). In any mazery, whenever it happens that walls and traps are well separated from each other and holes are not missing, then paths can pass through. (Formally, this claim will be called the Approximation Lemma—as the main combinatorial tool in a sequence of successive approximations.) Sometimes, however, unlucky events arise. These unlucky events can be classified in the types listed below. For any mazery $\mathcal{M}$, we will define a mazery $\mathcal{M}^*$ whose walls and traps correspond (essentially) to these typical unlucky events.

- A minimal rectangle enclosing two traps very close to each other, both of whose projections are disjoint, is an *uncorrelated compound trap*.
- For both directions $d = 0, 1$, a (essentially) minimal rectangle enclosing 3 traps very close to each other, whose d projections are disjoint, is a *correlated compound trap*.
- Whenever a certain horizontal wall W appears and at the same time there is a large interval without a vertical hole of $\mathcal{M}$ through W, this situation gives rise to an *emerging* trap of $\mathcal{M}^*$ of the *missing-hole* kind.
- A pair of very close walls of $\mathcal{M}$ gives rise to a wall of $\mathcal{M}^*$ called a *compound wall*.
- A segment of the X sequence such that conditioning on it, a correlated trap or a trap of the missing-hole kind occurs with too high conditional probability, is a new kind of wall called an *emerging* wall. (These are the walls that, indirectly, give rise to all other walls.)

(The exact definition of these objects involves some extra technical conditions: here, we are just trying to give the general idea.) There will be a constant

$$\chi = 0.015. \tag{2.2}$$

with the property that if a wall has probability p then a hole getting through it has probability lower bound p^{χ}. Thus, the "bad events" of the outline in Subsection 2.1 are the traps and walls of $\mathcal{M}$, the "very bad events" are (modulo some details that are not important now) the new traps and walls of $\mathcal{M}^*$. Let $\mathcal{F}, \mathcal{F}'$ be the events $\mathcal{F}_k, \mathcal{F}'_k$ formulated in Subsection 2.1. Thus, $\mathcal{F}'$ says that in $\mathcal{M}$ a wall or a trap is contained in the square $[0, \Delta^*]^2$.

We do not want to see all the details of $\mathcal{M}$ once we are on the level of $\mathcal{M}^*$: this was the reason for creating $\mathcal{M}^*$ in the first place. The walls and traps of $\mathcal{M}$ will indeed become transparent; however, some restrictions will be inherited from them: these are distilled in the concepts of a *clean point* and of a *slope constraint*. Actually, we distinguish the concept of *lower left clean* and *upper right clean*. Let

$$\mathcal{Q}$$

be the event that point $(0,0)$ is not upper right clean in $\mathcal{M}$.

We would like to say that in a mazery, if points (u_0,u_1), (v_0,v_1) are such that for $d=0,1$ we have $u_d < v_d$ and there are no walls and traps in the rectangle $[u_0,v_0]\times[u_1,v_1]$, then (v_0,v_1) is reachable from (u_0,u_1). However, this will only hold with some restrictions. What we will have is the following, with an appropriate parameter

$$0 \leqslant \sigma < 0.5.$$

Condition 2.1. Suppose that points $u=(u_0,u_1)$, $v=(v_0,v_1)$ are such that for $d=0,1$ we have $u_d < v_d$ and there are no traps contained in the rectangle between u and v, and no walls intersect it. If u is upper right clean, v is lower left clean and these points also satisfy the slope-constraint

$$\sigma \leqslant \frac{v_1-u_1}{v_0-u_0} \leqslant 1/\sigma$$

then v is reachable from u. ♢

We will also need sufficiently many clean points:

Condition 2.2. For every square $(a,b)+(0,\,3\Delta]^2$ that does not contain walls or traps, there is a lower left clean point in its middle third $(a,b)+(\Delta,\,2\Delta]^2$. ♢

Lemma 2.3. *We have $\mathcal{F}\subseteq\mathcal{F}'\cup\mathcal{Q}$.*

Proof. Suppose that $\mathcal{Q}$ does not hold, then $(0,0)$ is upper right clean.

Suppose also that $\mathcal{F}'$ does not hold: then by Condition 2.2, there is a point $u=(u_0,u_1)$ in the square $[\Delta,2\Delta]^2$ that is lower left clean in $\mathcal{M}$. This u also satisfies the slope condition $1/2\leqslant u_1/u_0\leqslant 2$ and is hence, by Condition 2.1, reachable from $(0,0)$. □

We will define a sequence of mazeries $\mathcal{M}^1,\mathcal{M}^2,\dots$ with $\mathcal{M}^{k+1}=(\mathcal{M}^k)^*$, with $\Delta_k\to\infty$. All these mazeries are on a common probability space, since $\mathcal{M}^{k+1}$ is a function of $\mathcal{M}^k$. All components of the mazeries will be indexed correspondingly: for example, the event $\mathcal{Q}_k$ that $(0,0)$ is not upper right clean in $\mathcal{M}^k$ plays the role of $\mathcal{Q}$ for the mazery$\mathcal{M}^k$. We will have the following property:

Condition 2.4. We have $\mathcal{Q}_k\subseteq\bigcup_{i<k}\mathcal{F}'_i$. ♢

This, along with Lemma 2.3 implies $\mathcal{F}_k\subseteq\bigcup_{i\leqslant k}\mathcal{F}'_i$, which is inequality (2.1). Hence the theorem is implied by the following lemma, which will be proved after all the details are given:

Lemma 2.5 (Main)**.** *If m is sufficiently large then the sequence $\mathcal{M}^k$ can be constructed, in such a way that it satisfies all the above conditions and also*

$$\sum_k \mathbf{P}(\mathcal{F}'_k) < 1. \tag{2.3}$$

2.3. **The rest of the paper.** The proof structure is quite similar to [4]. That paper is not simple, but it is still simpler than the present one, and we recommend very much looking at it in order to see some of the ideas going into the present paper in their simpler, original setting. Walls and holes, the general form of the definition of a mazery and the scale-up operation are similar. There are, of course, differences: traps are new.

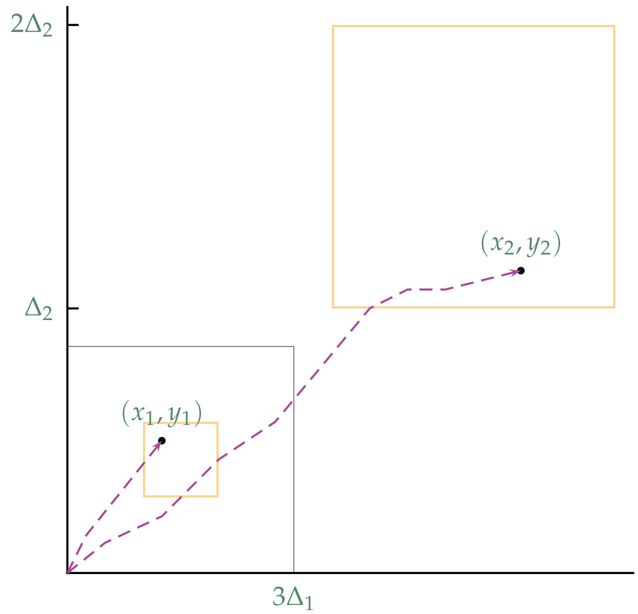

FIGURE 3. Proof of Theorem 1 from Lemma 2.5

3. MAZERIES

3.1. **Notation.** We will use

$$a \wedge b = \min(a, b), \quad a \vee b = \max(a, b).$$

As mentioned earlier, we will use intervals on the real line and rectangles over the Euclidean plane, even though we are really only interested in the lattice $\mathbb{Z}_+^2$. To capture all of $\mathbb{Z}_+$ this way, for our right-closed intervals $(a, b]$, we allow the left end a to range over all the values $-1, 0, 1, 2, \ldots$. For an interval $I = (a, b]$, we will denote

$$X(I) = (X(a+1), \ldots, X(b)).$$

The *size* of an interval I with endpoints a, b (whether it is open, closed or half-closed), is denoted by $|I| = b - a$. By the *distance* of two points $a = (a_0, a_1)$, $b = (b_0, b_1)$ of the plane, we mean

$$|b_0 - a_0| \vee |b_1 - a_1|.$$

The *size* of a rectangle

$$\mathrm{Rect}(a, b) = [a_0, b_0] \times [a_1, b_1]$$

in the plane is defined to be equal to the distance between a and b. For two different points $u = (u_0, u_1)$, $u = (u_0, u_1)$ in the plane, when $u_0 \leqslant v_0$, $u_1 \leqslant v_1$:

$$\mathrm{slope}(u, v) = \frac{v_1 - u_1}{v_0 - u_0},$$
$$\mathrm{minslope}(u, v) = \min\big(\mathrm{slope}(u, v), 1/\mathrm{slope}(u, v)\big).$$

We introduce the following partially open rectangles

$$\begin{aligned} \mathrm{Rect}^{\rightarrow}(a, b) &= (a_0, b_0] \times [a_1, b_1], \\ \mathrm{Rect}^{\uparrow}(a, b) &= [a_0, b_0] \times (a_1, b_1]. \end{aligned} \tag{3.1}$$

The relation
$$u \rightsquigarrow v$$
says that point v is reachable from point u (the underlying graph will always be clear from the context). For two sets A, B in the plane or on the line,
$$A + B = \{ a + b : a \in A,\ b \in B \}.$$

3.2. **The structure.**

3.2.1. *The tuple.* A *mazery*
$$\mathbb{M} = (\mathcal{M}, \Delta, \sigma, w, p(\cdot), q, R) \tag{3.2}$$
consists of a random process $\mathcal{M}$, the parameters $\Delta > 0$, $\sigma \geqslant 0$, the probability bounds $w > 0$, $p(\cdot), q$, and the rank lower bound R, all of which will be detailed below. Let us describe the random process
$$\mathcal{M} = (Z, \mathcal{T}, \mathcal{W}, \mathcal{C}).$$
Here,
$$Z = (X, Y) = (Z_0, Z_1)$$
is a pair of sequences of random variables $Z_d = (Z_d(0), Z_d(1), \dots)$ with $Z_d(t) \in \{1, \dots, m\}$: random walks on the set $\{1, \dots, m\}$ of nodes of the graph K_m for some fixed m. We have the random objects
$$\mathcal{T}, \quad \mathcal{W} = (\mathcal{W}_0, \mathcal{W}_1), \quad \mathcal{C} = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2).$$
all of which are functions of Z. The set $\mathcal{T}$ of random *traps* is a set of some closed rectangles of size $\leqslant \Delta$. For trap $\mathrm{Rect}(a, b)$, we will say that it *starts* at its lower left corner a.

To describe the process $\mathcal{W}$, we introduce the concept of a *wall value* $E = (B, r)$. Here B is the *body* which is a right-closed interval,[1] and *rank*
$$r \geqslant R > 0.$$
We write $\mathrm{Body}(E) = B$, $|E| = |B|$. We will sometimes denote the body also by E. Let Wvalues denote the set of all possible wall values. The random objects
$$\begin{aligned}
&\mathcal{W}_d \subseteq \text{Wvalues},\\
&\mathcal{C}_d \subseteq \mathbb{Z}_+^2 \times \{-1, 1\} \text{ for } d = 0, 1,\\
&\mathcal{C}_2 \subseteq \mathbb{Z}_+^4 \times \{-1, 1\} \times \{0, 1, 2\}
\end{aligned}$$
are also functions of Z. The elements of $\mathcal{W}_d$ are called *walls* of Z_d, where the set $\mathcal{W}_d$ is a function of Z_d. In particular, elements of $\mathcal{W}_0$ are called *vertical walls*, and elements of $\mathcal{W}_1$ are called *horizontal walls*. When we say that a certain interval contains a wall we mean that it contains its body.

Given an interval I, we say that a certain event $\mathcal{E} = \mathcal{E}(X, Y)$ holds (X, I)*-potentially* if there is an X' with $X(I) = X'(I)$ such that $\mathcal{E}(X', Y)$ holds. We define (Y, J)-potentiality similarly. For a rectangle $I \times J$ we say that $\mathcal{E}$ holds $I \times J$*-potentially* if there are X' with $X(I) = X'(I)$ and Y' with $Y(J) = Y'(J)$ such that $\mathcal{E}(X', Y')$ holds. We say that a right-closed interval I, is a *potential vertical wall* of rank r if it is (X, I)-potentially a wall. By definition, whether I is a potential vertical wall of rank r depends only on $X(I)$. Potential horizontal walls are defined similarly.

A right-closed interval is called *external* if it intersects no walls. A wall is called *dominant* if it contains every wall intersecting it.

Remarks 3.1.

[1] This is different from the definition in the paper [4], where walls were open intervals.

1. We will see below that, for any rectangle with projections $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$. It is, however, not true that for any interval I, the event that it is a (say, vertical) wall depends only on $X(I)$. If only $X(I)$ is known then we will need to speak about potential walls, etc.
2. Ranks are needed since in the definition of the mazery $\mathcal{M}^k$ from mazery $\mathcal{M}^{k+1}$, it would be too crude to treat all walls of $\mathcal{M}^k$ alike as we introduce walls for $\mathcal{M}^{k+1}$. As mentioned in the outline in Section 2, a pair of close walls of $\mathcal{M}^k$ will give rise to a compound wall of $\mathcal{M}^{k+1}$. In fact, such a compound wall will only be formed if at least one of its components has low rank. High-rank walls will remain also walls in $\mathcal{M}^{k+1}$; the inclusion of their combinations would increase, however, the range of possible wall probabilities too rapidly.
3. In what follows we will refer to $\mathcal{M}$ by itself also as a *mazery*, and will mention $\mathbb{M}$ only rarely. This should not cause confusion; though $\mathcal{M}$ is a component of $\mathbb{M}$, it relies implicitly on all the other components.

$\diamondsuit$

The following condition holds for the parts discussed above.

Condition 3.2. The constant Δ is a strict upper bound on the size of every wall and trap. $\diamondsuit$

3.2.2. *Cleanness.* Intuitively, a point x is clean in $\mathcal{M}^k$ when none of the mazeries $\mathcal{M}^i$ for $i < k$ has any bad events near x. This interpretation will become precise by the rescaling operation; at this point, we treat cleanness as a primitive, just like walls. Several kinds of cleanness are needed, depending on the direction in which the absence of lower-order bad events will be guaranteed.

The set $\mathcal{C}_d$ is a function of the process Z_d. For an interval $I = (a, b]$ or $I = [a, b]$, if $(a, b, -1) \in \mathcal{C}_d$ then we say that point b of $\mathbb{Z}_+$ is *clean* in I for the sequence Z_d. If $(a, b, 1) \in \mathcal{C}_d$ then we say that point a is clean in I. From now on, whenever we talk about cleanness of an element of $\mathbb{Z}_+$, it is always understood with respect to one of the sequences Z_d for $d = 0, 1$ (that is either for the sequence X or for Y).

Let us still fix a direction d and talk about cleanness, etc. with respect to the sequence Z_d. A point $x \in \mathbb{Z}_+$ is called left-clean (right-clean) if it is clean in all intervals of the form $(a, x]$ (all intervals of the form $(x, b]$). It is *clean* if it is both left- and right-clean. If both ends of an interval I are clean in I then we say I is *inner clean*. If its left end is left clean and its right end is right clean then we say that it is *outer-clean*.

For points $u = (u_0, u_1), v = (v_0, v_1), Q = \mathrm{Rect}^{\varepsilon}(u, v)$ where $\varepsilon = \rightarrow$ or $\uparrow$ or nothing, we say that point u is *clean in* Q (with respect to the pair of sequences (X, Y)) if $(u, v, 1, \varepsilon') \in \mathcal{C}_2$, where $\varepsilon' = 0, 1, 2$ depending on where $\varepsilon = \rightarrow$ or $\uparrow$ or nothing.

If u is clean in all such left-open rectangles then it is called *upper right rightward-clean*. We delete the “rightward” qualifier here if we have closed rectangles in the definition here instead of left-open ones. Cleanness with qualifier “upward” is defined similarly. Lower left cleanness of v is defined similarly, using $(u, v, -1, \varepsilon')$, except that the qualifier is unnecessary: all our rectangles are upper right closed.

A point is called *clean* if it is upper left clean and lower right clean. If both the lower left and upper right points of a rectangle Q are clean in Q then Q is called *inner clean*. If the lower left endpoint is lower left clean and the upper right endpoint is upper right rightward-clean then Q is called *outer rightward-clean*. Similarly for *outer upward-clean* and *outer-clean*.

3.2.3. *Hops.* A right-closed horizontal interval I is called a *hop* if it is inner clean and potentially contains no vertical wall. A closed interval $[a, b]$ is a hop if $(a, b]$ is a hop. Vertical hops are defined similarly.

We call a rectangle $I \times J$ a *hop* if it is inner-clean and potentially contains no trap or wall.

Remark 3.3. An interval or rectangle that is a hop can be empty: this is the case if the interval is $(a, a]$, or the rectangle is, say, $\mathrm{Rect}^{\rightarrow}(u, u)$. ♢

Two disjoint walls are called *neighbors* if the interval between them is a hop. A sequence $W_i \in \mathcal{W}$ of walls $i = 1, 2, \ldots$ along with the intervals $I_1, \ldots, I_{n-1}$ between them is called a *sequence of neighbor walls* if for all $i > 1$, W_i is a right neighbor of W_{i-1}. We say that an interval I is *spanned* by the sequence of neighbor walls $W_1, W_2, \ldots, W_n$ if $I = W_1 \cup I_1 \cup W_2 \cup \cdots \cup W_n$. We will also say that I is spanned by the sequence $(W_1, W_2, \ldots)$ if both I and the sequence are infinite and $I = W_1 \cup I_1 \cup W_2 \cup \ldots$. If there is a hop I_0 adjacent on the left to W_1 and a hop I_n adjacent on the right to W_n (or the sequence W_i is infinite) then this system is called an *extended sequence of neighbor walls*. We say that an interval I is *spanned* by this extended sequence if $I = I_0 \cup W_1 \cup I_1 \cup \cdots \cup I_n$ (and correspondingly for the ifinite case).

3.2.4. *Holes.* Let $a = (a_0, a_1)$, $b = (b_0, b_1)$, and let the interval $I = (a_0, b_0]$ be the body of a vertical wall B. For an interval $J = (a_1, b_1]$ with $|J| \leqslant |I|$ we say that J is a horizontal *hole passing through* B, or *fitting* B, if $a \rightsquigarrow b$ within the rectangle $[a_0, b_0] \times J$. This hole is called *lower left clean*, upper right clean, etc. if this rectangle is. Vertical holes are defined similarly.[2]

Remark 3.4. Note that the condition of passing through a wall depends on an interval slightly larger than the wall itself: it also depends on the left end of the left-open interval that is the body of the wall. ♢

3.3. **Conditions on the random process.** Most of our conditions on the distribution of process $\mathcal{M}$ are fairly natural; however, the need for some of them will be seen only later. For example, for Condition 3.5.3d, only its special case (in Remark 3.6.2) is well motivated now: it says that through every wall there is a hole with sufficiently large probability. The general case will be used in the inductive proof showing that the hole lower bound also holds on compound walls after renormalization (going from $\mathcal{M}^k$ to $\mathcal{M}^{k+1}$).

The function

$$p(r, l) \tag{3.3}$$

is defined as the supremum of probabilities (over all points t) that any potential wall with rank r and size l starts at t conditional over all possible conditions of the form $Z_d(t) = k$ for $k \in \{1, \ldots, m\}$. The function $p(r)$ will be an upper bound on $\sum_l p(r, l)$.

The constant χ has been introduced in (2.2). Its choice will be motivated in Section 6. We will use two additional constants, c_0 and c_1. Constant c_1 will be chosen at the end of the proof of Lemma 7.3, while c_0 will be chosen at the end of the proof of Lemma 7.10. For each rank r, let us define the function

$$h(r) = c_0 (r^{c_1} p(r))^{\chi}, \tag{3.4}$$

to be used as a lower bound for the probability of holes in walls of rank r. The factor $c_0 r^{c_1 \chi}$ will absorb some nuisance terms as they arise in the estimates.

Condition 3.5.

[2] The notion of hole in the present paper is different from that in [4]. Holes are not primitives; rather, they are defined with the help of reachability.

1. (Dependencies)
   a. For any rectangle with projections $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$.
   b. For every point x, integers $a < b$, the events $\{ (a, b, -1) \in \mathcal{C}_d \}$ and $\{ (a, b, 1) \in \mathcal{C}_d \}$ are functions of $Z_d((a, b])$.
   When Z is fixed, then for a fixed a, the cleanness of a in $(a, b]$ is decreasing as a function of $b - a$, and for a fixed b, the cleanness of b in $(a, b]$ is decreasing as a function of $b - a$. These functions reach their minimum at $b - a = \Delta$: thus, if x is left clean in $(x - \Delta, x]$ then it is left clean.
   c. For any rectangle $Q = I \times J$, the event that its lower left corner is clean in Q, is a function of the pair $X(I), Y(J)$.
   Among rectangles with a fixed lower left corner, the event that this corner is clean in Q is a decreasing function of Q (in the set of rectangles partially ordered by containment). In particular, the cleanness of u in $\mathrm{Rect}(u, v)$ implies its cleanness in $\mathrm{Rect}^{\rightarrow}(u, v)$ and in $\mathrm{Rect}^{\uparrow}(u, v)$. If u is upper right clean in the left-open or bottom-open or closed square of size Δ, then it is upper right clean in all rectangles Q of the same type. Similar statements hold if we replace lower left with upper right.
   d. Let Q be a rectangle. If point (x_0, x_1) of $\mathbb{Z}_+^2$ is upper right clean in Q with respect to the pair of sequences (Z_0, Z_1) then for both $d = 0, 1$, point x_d is right clean in the corresponding projection of Q with respect to the sequence Z_d. The same statement holds also if upper right is replaced with lower left and right is replaced with left.
2. (Combinatorial requirements)
   a. A maximal external interval is inner clean.
   b. Suppose that interval I is adjacent on the left to an external interval that either starts at -1 or has size $\geqslant \Delta$. Suppose also that it either adjacent on the right to a similar external interval or is infinite and contains no such external interval. Then it is spanned by a (finite or infinite) sequence of neighbor walls. In particular, the whole line is spanned by an extended sequence of neighbor walls.
   c. If a (not necessarily integer aligned) right-closed interval of size $\geqslant 3\Delta$ potentially contains no wall, then its middle third contains a clean point.
   d. Suppose that a rectangle $I \times J$ with (not necessarily integer aligned) right-closed I, J with $|I|, |J| \geqslant 3\Delta$ potentially contains no horizontal wall and no trap, and a is a right clean point in the middle third of I. There is an integer b in the middle third of J such that the point (a, b) is upper right clean. A similar statement holds if we replace lower left with upper right (and left with right). Also, if a is clean then we can find a point b in the middle third of J such that (a, b) is clean.
   There is also a similar set of statements if we vary a instead of b.
3. (Probability bounds)
   a. Given a string $x = (x(0), x(1), \dots)$, a point (a, b) and an interval $I \ni a$, let $\mathcal{F}$ be the event that a trap starts at (a, b), with projection lying in I. Let $k \in \{1, \dots, m\}$, then we have
   $$\mathbf{P}\big(cF \mid X(I) = x(I),\ Y(b-1) = k \big) \leqslant w.$$
   The same is required if we exchange horizontal and vertical.
   b. For all r we have $p(r) \geqslant \sum_l p(r, l)$.
   c. We require $q < 0.1$, and following inequalities for all $k \in \{1, \dots, m\}$, for all $a < b$ and all $u = (u_0, u_1)$, $v = (v_0, v_1)$, for all sequences y such that u_1 (resp. v_1) is clean in

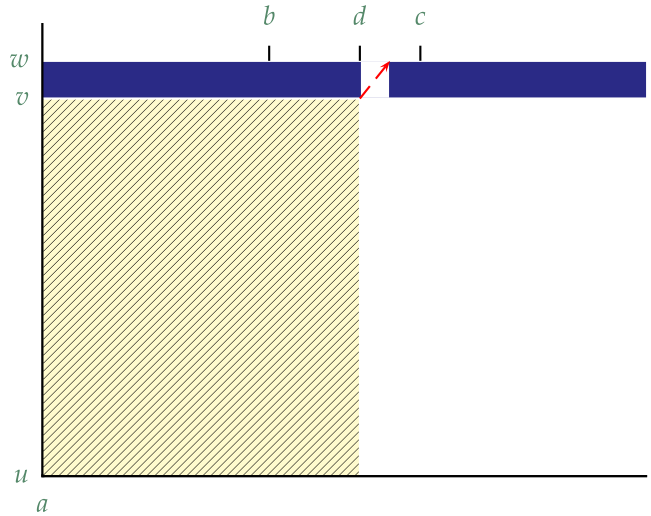

FIGURE 4. Condition 3.5.3d. The hatched rectangle is a hop.

$(u_1, v_1]$:

$$q \geqslant \mathbf{P}\{\, a \text{ (resp. } b) \text{ is not clean in } (a, b] \mid X(a) = k \,\},$$
$$q \geqslant \mathbf{P}\{\, u \text{ (resp. } v) \text{ is not clean in } \mathrm{Rect}^{\rightarrow}(u, v) \mid X(u_0) = k, Y = y \,\},$$
$$q \geqslant \mathbf{P}\{\, u \text{ (resp. } v) \text{ is not clean in } \mathrm{Rect}(u, v) \mid X(u_0 - 1) = k, Y = y \,\},$$

and similarly with X and Y reversed.

d. Let $u \leqslant v < w$, and a be given with $v - u \leqslant 12\Delta$, and define

$$b = a + \lceil (v - u)/2 \rceil, \qquad c = a + (v - u) + 1.$$

Assume that $Y = y$ is fixed in such a way that v is left-clean, the interval $(u, v]$ contains no walls, and B is a horizontal wall of rank r with body $(v, w]$. Let $E = E(u, v, w; a)$ be the event (a function of X) that there is a $d \in [b, c - 1]$ with the following properties for $Q = \mathrm{Rect}^{\rightarrow}((a, u), (d, v))$:

(i) a vertical hole fitting B starts at d;
(ii) Q contains no traps, potentially contains no walls, and (d, v) is clean in Q;
(iii) if also u is clean in $(u, v]$ then (a, u) is clean in Q;

Let $k \in \{1, \dots, m\}$. Then we have

$$\mathbf{P}\big(\, E \mid X(a) = k,\ Y = y \,\big) \geqslant (c - b)^{\chi} h(r).$$

The same is required if we exchange horizontal and vertical.

◇

Remarks 3.6.

1. Conditions 3.5.2c and 3.5.2d imply the following. Suppose that a right-upper closed square Q of size 3Δ contains no wall or trap. Then its middle third contains a clean point.
2. The most important special case of Condition 3.5.3d is $v = u$, implying $b = a$, $c = b + 1$: then it says that for any horizontal wall B of rank r, at any point a, the probability that there is a vertical hole passing through B at point a is at least $h(r)$.

◇

To each mazery $\mathcal{M}$ belongs a random graph

$$\mathcal{V} = \mathbb{Z}_+^2, \quad \mathcal{G} = (\mathcal{V}, \mathcal{E})$$

where $\mathcal{E}$ is determined by the above random processes as in Subsection 1.3. We say that point v is *reachable* from point u in $\mathcal{M}$ (and write $u \rightsquigarrow v$) if it is reachable in $\mathcal{G}$.

Remark 3.7. According to our definitions in Subsection 1.3, point u itself may be closed even if v is reachable from u. $\diamondsuit$

The graph $\mathcal{G}$ is required to satisfy the following conditions.

Condition 3.8 (Reachability). We require $0 \leqslant \sigma < 0.5$. Let u, v be points with $\operatorname{minslope}(u, v) \geqslant \sigma$. If they are the starting and endpoint of a rectangle that is a hop, then $u \rightsquigarrow v$. The rectangle in question is allowed to be bottom-open or left-open, but not both. $\diamondsuit$

Example 3.9. The clairvoyant demon problem can be seen as a special case of a mazery. There are no walls. Traps are points (i, j) with $X(i) = Y(j)$. We have $\Delta = 1$ and $\sigma = 0$. Condition 3.5.3a is satisfied if $m - 1 \geqslant 1/w$, where m is the size of the complete graph on which the random walks are performed. Every point is clean.

Note that the reachability condition is violated in the bottom-left open rectangle $(0, 1] \times (0, 1]$ if $X(0) = 1$, $X(1) = 2$, $Y(0) = 2$, $Y(1) = 1$. $\diamondsuit$

4. The scaled-up structure

In this section, we will define the scaling-up operation $\mathbb{M} \mapsto \mathbb{M}^*$: we still postpone the definition of several parameters and probability bounds for $\mathbb{M}^*$.

4.1. **The scale-up construction.**

Let Λ be a constant and f, g satisfying

$$\begin{aligned} \Lambda &= 500, \\ \Delta/g \leqslant g/f &< (0.5 - \sigma)/(2\Lambda). \end{aligned} \tag{4.1}$$

Here is the approximate meaning of f and g: We try not to permit walls closer than f to each other, and we try not to permit intervals larger than g without holes. Let

$$\sigma^* = \sigma + \Lambda g/f. \tag{4.2}$$

The value Δ^* will be defined later, but we will make sure that

$$3f \leqslant \Delta^* \tag{4.3}$$

holds. After defining the mazery $\mathcal{M}^*$, eventually we will have to prove the required properties. To be able to prove Condition 3.8 for $\mathcal{M}^*$, we will introduce some new walls and traps in $\mathcal{M}^*$ whenever some larger-scale obstacles prevent reachability. There will be two kinds of new walls, so-called *emerging* walls, and *compound* walls. A pair of traps too close to each other will define, under certain conditions, a compound trap, which becomes part of $\mathcal{M}^*$. A new kind of trap, called a trap of the missing-hole kind will arise when some long stretch of a low-rank wall is without a hole.

The following algorithm creates all these new objects. We will make use of parameter

$$\lambda = 2^{1/2} \tag{4.4}$$

whose meaning is that eventually, the probability bound on walls of rank r will be of the order of λ^{-r}. For the new value of R we require

$$R^* \leqslant 2R - \log_\lambda f. \tag{4.5}$$

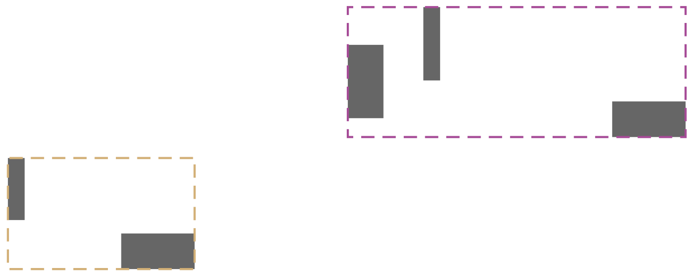

FIGURE 5. An uncorrelated and a horizontal correlated compound trap

Walls of rank lower than R^* are called *light*, the other ones are called *heavy*. Heavy walls of $\mathcal{M}$ will also be walls of $\mathcal{M}^*$. We will define walls only for either X or Y, but it is understood that they are also defined when the roles of X and Y are reversed.

Step 1 (Cleanness)**.** For an interval I, its right endpoint x will be called clean in I for $\mathcal{M}^*$ if
– it is clean in I for $\mathcal{M}$;
– Potentially, I contains no wall of $\mathcal{M}$ whose right end is closer to x than $f/3$.
Cleanness of the left endpoint is defined similarly. Let a point u be a starting point or endpoint of a rectangle Q. It will be called clean in Q for $\mathcal{M}^*$ if
– it is clean in Q for $\mathcal{M}$;
– its projections are clean in the projections of Q for $\mathcal{M}^*$;
– any trap contained in Q is at a distance $\geqslant g$ from u.

$\diamondsuit$

Step 2 (Uncorrelated traps)**.** A rectangle Q is called an *uncorrelated compound trap* if it contains two traps with disjoint projections, with a distance of their starting points at most f. $\diamondsuit$

Clearly, the size of an uncorrelated trap is bounded by $\Delta + f$.

Step 3 (Correlated trap)**.** Let

$$g' = 2.2g. \tag{4.6}$$

(Choice motivated by the proof of Lemmas 4.7 and 8.1.) Let $l_1 = 7\Delta$, $l_2 = g'$. For a $j \in \{1, 2\}$ let I be a closed interval with length $|I| = 3l_j$, and b a site, with $J = [b, b + 5\Delta]$. Let $x(I), y(J)$ be fixed. We say that event

$$\mathcal{L}_j(x, y, I, b)$$

holds if for all intervals $I' \subseteq I$ of size l_j, the rectangle $I' \times J$ contains a trap. We will say that $I \times J$ is a *horizontal correlated trap* of type j if $\mathcal{L}_j(x, y, I, b)$ holds and for all $k \in \{1, \dots, m\}$, we have the inequality

$$\mathbf{P}\big(\mathcal{L}_j(x, Y, I, b) \mid X(I) = x(I),\ Y(a-1) = k \big) \leqslant w^2. \tag{4.7}$$

$\diamondsuit$

Step 4 (Traps of the missing-hole kind)**.** Let I be a closed interval of size g, let b be a site with $J = [b, b + 3\Delta]$. Let $x(I), y(J)$ be fixed. We say that event

$$\mathcal{L}_3(x, y, I, b)$$

holds if, with $b_1 = b + \Delta$, there is a $b_2 > b_1$ such that $(b_1, b_2]$ is (Y, J)-potentially the body of a light outer-clean horizontal wall W, and no outer rightward-clean vertical hole $(a_1, a_2]$

with $(a_1 - \Delta, a_2 + \Delta] \subseteq I$ passes through W. (Recall that such a notion of cleanness for a hole $(a_1, a_2]$ was defined, in 3.2.4, to mean the corresponding notion for the rectangle $(a_1, a_2] \times [b_1, b_2]$.) We say that $I \times J$ is a *horizontal trap of the missing-hole kind* if $\mathcal{L}_3(x, y, I, b)$ holds and for all $k \in \{1, \dots, m\}$ we have

$$\mathbf{P}\big(\mathcal{L}_3(x, Y, I, b) \mid X(I) = x(I),\, Y(a-1) = k\big) \leqslant w^2. \tag{4.8}$$

$\diamondsuit$

Inequalities (4.1) and (4.3) bound the size of all new traps by Δ^*.

Step 5 (Emerging walls)**.** Let x be a particular value of the sequence X over an interval $I = (u, v]$. For any $u' \in (u, u + 2\Delta]$, $v' \in (v - 2\Delta, v]$, let us define the interval $I' = [u', v']$. We say that interval I is the body of a vertical *barrier* of the *emerging kind*, of type $j \in \{1, 2, 3\}$ if the following requirements hold:

(a) We have

$$\sup_{I', k} \mathbf{P}\big(\mathcal{L}_j(x, Y, I', 1) \mid X(I') = x(I'),\, Y(0) = k\big) \geqslant w^2. \tag{4.9}$$

(b) Either I is an external hop or it is the union of a dominant light wall and one or two external hops of $\mathcal{M}$, of size $\geqslant \Delta$, surrounding it.

(c) Each end of I is adjacent to either an external hop of size $\geqslant \Delta$ or a wall of $\mathcal{M}$.

Note that emerging barriers of type 1 are smallest, and those of type 2 are largest. More precisely, let

$$L_1 = 3l_1, \quad L_2 = 3l_2, \quad L_3 = g.$$

Then emerging barriers of type j have length in $L_j + [0, 4\Delta]$.

Now we will designate some of the emerging barriers as walls. For $j = 1, 2, 3$, list all barriers of type j in a sequence $(B_{j1}, B_{j2}, \dots)$. First process barriers $B_{11}, B_{12}, \dots$ one-by-one. Designate B_{1n} a wall if and only if it is disjoint of all emerging barriers designated as walls earlier. Next process the sequence $(B_{31}, B_{32}, \dots)$. Designate B_{3n} a wall if and only if it is disjoint of all emerging barriers designated as walls earlier. Finally process the sequence $(B_{21}, B_{22}, \dots)$. Designate B_{2n} a wall if and only if it is disjoint of all emerging barriers designated as walls earlier.

To emerging walls, we assign rank

$$\hat{R} > R^* \tag{4.10}$$

to be determined later.

$\diamondsuit$

Step 6 (Compound walls)**.** We make use of a certain sequence of integers:

$$d_i = \begin{cases} i & \text{if } i = 0, 1, \\ \lceil \lambda^i \rceil & \text{if } i \geqslant 2. \end{cases} \tag{4.11}$$

A *compound wall* occurs in $\mathcal{M}^*$ for X wherever neighbor walls W_1, W_2 occur (in this order) for X at a distance $d \in [d_i, d_{i+1})$, $d \leqslant f$, and W_1 is light. We denote the new compound wall by

$$W_1 + W_2.$$

Its body is the smallest right-closed interval containing the bodies of W_i. For r_j the rank of W_j, we will say that the compound wall in question has *type*

$$\langle r_1, r_2, i \rangle.$$

Its rank is defined as

$$r = r_1 + r_2 - i. \tag{4.12}$$

Thus, a shorter distance gives higher rank. This definition gives

$$r_1 + r_2 - \log_\lambda f \leqslant r \leqslant r_1 + r_2.$$

Inequality (4.5) will make sure that the rank of the compund walls is lowerbounded by R^*.

Now we repeat the whole compounding step, introducing compound walls in which now W_2 is required to be light. The wall W_1 can be any wall introduced until now, also a compound wall introduced in the first compounding step. ♢

The walls that will occur as a result of the compounding operation are of the type L-$*$, $*$-L, or L-$*$-L, where L is a light wall of $\mathcal{M}$ and $*$ is any wall of $\mathcal{M}$ or an emerging wall of $\mathcal{M}^*$. Thus, the maximum size of a compound wall is

$$\Delta + f + (3g' + 4\Delta) + f + \Delta < \Delta^*,$$

where we used (4.1) and (4.3).

Step 7 (Finish)**.** The graph $\mathcal{G}$ does not change in the scale-up: $\mathcal{G}^* = \mathcal{G}$. Remove all traps of $\mathcal{M}$.

Remove all light walls. If the removed light wall was dominant, remove also all other walls of $\mathcal{M}$ contained in it. ♢

4.2. **Combinatorial properties.** Let us prove some properties of $\mathcal{M}^*$ that can already be established. Note first that Conditions 2.1, 2.2 and 2.4 follow immediately from the conditions given in Section 3 and the definition of cleanness in $\mathcal{M}^*$ given in the present section.

Lemma 4.1. *The new mazery $\mathcal{M}^*$ satisfies Condition 3.5.1.*

Proof. We will see that all the properties in the condition follow essentially from the form of our definitions.

Condition 3.5.1a says that for any rectangle $I \times J$, the event that it is a trap is a function of the pair $X(I), Y(J)$. To check this, consider all possible traps of $\mathcal{M}^*$. We have the following kinds:

– Uncorrelated and correlated compound trap. The form of the definition shows that this event only depends on $X(I), Y(J)$.
– Trap of the missing-hole kind, for $J = [b, b + 3\Delta]$. This required that, potentially, some light outer-clean horizontal wall W starts at position $b + \Delta$ and that no outer rightward-clean vertical hole $(a_1, a_2]$ with $(a_1 - \Delta, a_2 + \Delta] \subseteq I$ passes through the barrier. Since all cleanness properties in $\mathcal{M}$ depend only on a Δ-neighborhood of a point, this event also only depends on $X(I), Y(J)$. The conditional probability inequality also depends only on $X(I)$.

Condition 3.5.1b says first that for every point interval $I = (a, b]$, $\{\, (x, r, -1) \in \mathcal{C}_d \,\}$ and $\{\, (x, r, 1) \in \mathcal{C}_d \,\}$ are functions of $Z_d(I)$. The property that a or b is clean in I in $\mathcal{M}^*$ is defined in terms of cleanness in $\mathcal{M}$ and the potential absence of certain walls contained in I. Therefore cleanness of a or b in I for $\mathcal{M}^*$ is a function of $Z_d(I)$. Since cleanness in I for $\mathcal{M}$ is a decreasing function of I, and the property stating the potential absence of walls is a decreasing function of I, cleanness for $\mathcal{M}^*$ is also a decreasing function of I. The inequality $f/3 + \Delta < \Delta^*$, implies that these functions reach their minimum for $|I| = \Delta^*$.

Condition 3.5.1c says first that for any rectangle Q with projections I, J, the event that its lower left corner is clean in Q, is a function of the pair $X(I), Y(J)$. If u is this point then, our definition of its $\mathcal{M}^*$-cleanness in rectangle Q required the following:

– It is clean in Q for $\mathcal{M}$;
– Its projections are clean in the projections of Q in $\mathcal{M}^*$;
– The starting point of any trap in Q is at a distance $\geqslant g$ from u.

All these requirements refer only to the projections of Q and depend therefore only on the pair $X(I), Y(J)$.

It can also be seen that, among rectangles with a fixed lower left corner, the event that this corner is $\mathcal{M}^*$-clean in Q is a decreasing function of Q (in the set of rectangles partially ordered by containment). And, since $g + \Delta$, $f/3 + \Delta < \Delta^*$, if (x, y) is upper right clean in a square of size Δ^*, then it is upper right clean.

Condition 3.5.1d follows immediately from the definition of cleanness in $\mathcal{M}^*$. □

Lemma 4.2. *The mazery $\mathcal{M}^*$ satisfies conditions 3.5.2a and 3.5.2b.*

Proof. We will prove the statement only for vertical walls; it is proved for horizontal walls the same way. In what follows, "wall", "hop", etc. mean vertical wall, horizontal hop, etc. Let $(U_1, U_2, \dots)$ be a (finite or infinite) sequence of disjoint walls of $\mathcal{M}$ and $\mathcal{M}^*$, and let $I_0, I_1, \dots$ be the (possibly empty) intervals separating them (interval I_0 is the interval preceding U_1). This sequence will be called *pure* if

(a) The intervals I_j are hops of $\mathcal{M}$.
(b) I_0 is an external interval of $\mathcal{M}$ starting at -1, while I_j for $j > 0$ is external if its size is $\geqslant 3\Delta$.

1. Let us build an initial pure sequence.

First we will use only elements of $\mathcal{M}$; however, later, walls of $\mathcal{M}^*$ will be added to it. Let $(E_1, E_2, \dots)$ be the (finite or infinite) sequence of maximal external intervals of size $\geqslant \Delta$, and let us add to it the maximal external interval starting at -1. Let $K_1, K_2, \dots$ be the intervals between them (or possibly after them, if there are only finitely many E_i). By Condition 3.5.2b of $\mathcal{M}$, each K_j can be covered by a sequence of neighbors W_{jk}. Each pair of these neighbors will be closer than 3Δ to each other. Indeed, each point of the hop between them belongs either to a wall intersecting one of the neighbors, or to a maximal external interval of size $\leqslant \Delta$, so the distance between the neighbors is $< 2\Delta + \Delta = 3\Delta$. The union of these sequences is a single infinite pure sequence of neighbor walls

$$\mathbf{U} = (U_1, U_2, \dots), \qquad \mathrm{Body}(U_j) = (a_j, b_j]. \tag{4.13}$$

Every wall of $\mathcal{M}$ intersects an element of $\mathbf{U}$.

A light wall in this sequence is called *isolated* if its distance from other elements of the sequence is greater than f. By our construction, all isolated light walls of the sequence $\mathbf{U}$ are dominant.

Let us change the sequence $\mathbf{U}$ using the sequence $(W_1, W_2, \dots)$ of all emerging walls (disjoint by definition) as follows. For $n = 1, 2, \dots$, add W_n to $\mathbf{U}$. If W_n intersects an element U_i then delete U_i.

2. (a) The result is a pure sequence $\mathbf{U}$ containing all the emerging walls.
(b) When adding W_n, if W_n intersects an element U_i then U_i is a dominant wall of $\mathcal{M}$ contained in W_n, and W_n intersects no other element U_j.

Proof. The proof is by induction. Suppose that we have already processed $W_1, \dots, W_{n-1}$, and we are about to process $W = W_n$. The sequence will be called $\mathbf{U}$ before processing W and $\mathbf{U}'$ after it.

Let us show (b) first. By the requirement (b) on emerging walls, either W is an external hop or it is the union of a dominant light wall and one or two external hops of $\mathcal{M}$, of size $\geqslant \Delta$, surrounding it. If W is an external hop then it intersects no elements of $\mathbf{U}$. Otherwise, the dominant light wall inside it can only be one of the U_i.

Let us show now (a), namely that if $\mathbf{U}$ is pure then so is $\mathbf{U}'$. Property (b) of the definition of purity follows immediately, since the intervals between elements of $\mathbf{U}'$ are subintervals of the ones between elements of $\mathbf{U}$. For the same reason, these intervals do not contain walls of $\mathcal{M}$. It remains to show that if $I'_{j-1} = (b'_{j-1}, a'_j]$ and $I'_j = (b'_j, a'_{j+1}]$ are the intervals around W in $\mathbf{U}'$ then a'_j is clean in I'_{j-1} and b'_j is clean in I'_j. Let us show that, for example, a'_j is clean in I'_{j-1}.

By the requirement (c) on emerging walls, a'_j is adjacent to either an external hop of size $\geqslant \Delta$ or a wall of $\mathcal{M}$. If the former case, it is left clean and therefore clean in I'_{j-1}. In the latter case, the external interval I'_{j-1} is empty.

3. Let us break up the pure sequence $\mathbf{U}$ containing all the emerging walls into subsequences separated by its intervals I_j of size $> f$. Consider one of these (possibly infinite) sequences, call it $W_1, \dots, W_n$, which is not just a single isolated light wall.

We will create a sequence of consecutive neighbor walls W'_i of $\mathcal{M}^*$ spanning the same interval as $W_1, \dots, W_n$. Assume that W_i for $i < j$ have been processed already, and a sequence of neighbors W'_i for $i < j'$ has been created in such a way that

$$\bigcup_{i<j} W_i \subseteq \bigcup_{i<j'} W'_i,$$

and W_j is not a light wall which is the last in the series. (This condition is satisfied when $j = 1$ since we assumed that our sequence is not an isolated light wall.) We show how to create $W'_{j'}$.

If W_j is the last element of the series then it is heavy, and we set $W'_{j'} = W_j$. Suppose now that W_j is not last.

Suppose that it is heavy. If W_{j+1} is also heavy, or light but not last then $W'_{j'} = W_j$. Else $W'_{j'} = W_j + W_{j+1}$, and W'_j replaces W_j, W_{j+1} in the sequence. In each later operation also, the introduced new wall will replace its components in the sequence.

Suppose now that W_j is light: then it is not last. If W_{j+1} is last or W_{j+2} is heavy then $W'_{j'} = W_j + W_{j+1}$

Suppose that W_{j+2} is light. If it is last then $W'_{j'} = (W_j + W_{j+1}) + W_{j+2}$; otherwise, $W'_{j'} = W_j + W_{j+1}$.

Remove all isolated light walls from $\mathbf{U}$ and combine all the subsequences created in part 3 above into a single infinite sequence $\mathbf{U}$ again. Consider an interval I between its elements. Then I is inner-clean for $\mathcal{M}$, and the only walls of $\mathcal{M}$ in I are covered by some isolated dominant light walls. Thus, I is inner-clean in $\mathcal{M}^*$. It does not contain any compound walls either, and by definition it does not contain emerging walls. Therefore it is a hop of $\mathcal{M}^*$.

4. Condition 3.5.2a holds for $\mathcal{M}^*$.

Proof. A maximal external interval J of $\mathcal{M}^*$ is an interval of size $> f$ separating two elements of $\mathbf{U}$ or the interval I_0. We have seen that it is a hop of $\mathcal{M}^*$.

5. Condition 3.5.2b holds for $\mathcal{M}^*$.

   Proof. By our construction, a maximal external interval of size $\geqslant \Delta^* > f$ is an interval separating two elements of $\mathbf{U}$. The segment between two such intervals (or one such and I_0) is spanned by elements of $\mathbf{U}$, separated by hops of $\mathcal{M}^*$.

□

Lemma 4.3. *Suppose that interval I is a hop of $\mathcal{M}^*$. Then it is either also a hop of $\mathcal{M}$ or it contains a sequence $W_1, \dots, W_n$ of dominant light neighbor walls $\mathcal{M}$ separated from each other by external hops of $\mathcal{M}$ of size $\geqslant f$, and from the ends by hops of $\mathcal{M}$ of size $\geqslant f/3$.*

Proof. If I contains no walls of $\mathcal{M}$ then it is a hop of $\mathcal{M}$.

Otherwise, let U be the union of all walls of $\mathcal{M}$ in I. The inner cleanness of I in $\mathcal{M}^*$ implies that U is farther than $f/3$ from its ends. The set $I \setminus U$ can be written as $I_0 \cup I_1 \cup \dots \cup I_n$. Here, I_j for $0 < j < n$ separates consecutive parts of U. It is a maximal external interval of $\mathcal{M}$, and as such it must be a hop of $\mathcal{M}$. The maximal external subinterval I'_0 of I_0 adjacent to U_1 is a hop of size $> \Delta$. This shows that I_0 is a hop. Proceed similarly with I_n.

Consider two neighboring such hops: the part of U between them must be a single dominant wall. Indeed, Condition 3.5.2b implies that it is spanned by a sequence of neighbor walls. They are closer than 3Δ to each other: if there is more than one, then I would contain a compound wall, which it cannot, since it is a hop of $\mathcal{M}^*$. □

The following lemma shows that there are "enough" emerging walls.

Lemma 4.4. *Let us be given intervals $I' \subset I$, and also $x(I)$, with the following properties for some $j \in \{1, 2, 3\}$.*

(a) *I is spanned by an extended sequence $W_1, \dots, W_n$ of dominant neighbor walls of $\mathcal{M}$ such that the W_i are at a distance $> f$ from each other and at a distance $> f/3$ from the ends of I.*
(b) *I' satisfies inequality (4.9) for emerging barriers.*
(c) *I' is at a distance $\geqslant L_j + 7\Delta$ from the ends of I.*

Then I contains an emerging wall.

Proof. Let $I = (a, b]$, $I' = (u', v']$. We will define an emerging wall $I'' = (u'', v'']$. The assumptions imply that the hops between the walls W_i are external. However, the hop $(a, c]$ between the left end of I and W_1 may not be. Let $(\hat{a}, c]$ be a maximal external subinterval of $(a, c]$ ending at c. Then $\hat{a} - a \leqslant \Delta$. Let us define $\hat{b}$ similarly on the right end of I, and let $\hat{I} = (\hat{a}, \hat{b}]$. We will find an emerging wall in $\hat{I}$, so let us simply redefine I to be $\hat{I}$. We now have the property that any wall $\mathcal{M}$ in I is at a distance $\geqslant f/3 - \Delta$ from the ends of I, and I' is at a distance $\geqslant L_j + 6\Delta$ from the ends of I.

Assume first that I is a hop of $\mathcal{M}$ (by the assumption, an external one). Let us define the interval I'' as follows. If $u' \geqslant a + 2\Delta$, then, since no wall is contained in $(u' - 2\Delta, u' + \Delta]$, by Condition 3.5.2c, there is a point $u'' \in (u' - \Delta, u']$ clean in $\mathcal{M}$. (Since $|I'| > \Delta$, there is no problem with walls on the right of v' when finding clean points on the left of u'.) Otherwise, set $u'' = a$. (This case cannot really occur, due to the assumption (c), but it is useful to pretend it can, for the argument of the next paragraph.) Similarly, if $b - v' \geqslant 2\Delta$, then there is a point $v'' \in (v', v' + \Delta]$ clean in $\mathcal{M}$. Otherwise, set $v'' = b$.

Assume now that I is not a hop of $\mathcal{M}$: then I is spanned by a nonempty extended sequence $W_1, \dots, W_n$ of neighbor walls of $\mathcal{M}$ such that the W_i are at a distance $> f$ from each other and at a distance $> f/3 - \Delta$ from the ends of I. We can assume that I' intersects one of these walls W_i, otherwise, if I' falls into one of the hops then we can apply the construction of the previous paragraph with I playing the role of one of these hops. Thus, assume that $(u', v']$ intersects $W_i = (c, d]$. Now, if $c \leqslant u' < d$ then take $u'' = c$. If $u' < c$ then there are no walls in the interval $(u' - 3\Delta, u']$, since it is in the hop on the left of W_i. Find a point u'' clean in $\mathcal{M}$ in the middle $(u' - 2\Delta, u' - \Delta]$ of this interval. The point v'' is defined similarly.

By this definition, interval I'' satisfies both requirements (b) and (c) of emerging barriers, and is at a distance $\geqslant 4\Delta + L_j$ from the ends of I.

If I contains no emerging walls then, in particular, it contains no walls of type i with $L_i \leqslant L_j$. Since I'' is at a distance $\geqslant 4\Delta + L_j$ (the bound on the size of emerging walls of type j) from the ends of I, it follows therefore that no wall of such type i intersects it. But then the process of designating walls in Step 5 of the scale-up construction would designate I'' a wall, contrary to the assumption that I contains no emerging walls. □

Lemma 4.5. *Let the rectangle Q, with X projection I, contain no walls of $\mathcal{M}^*$. Let $I' = [a, a+g]$, $J = [b, b+3\Delta]$ with $I' \times J \subseteq Q$ be such that I' is at a distance $\geqslant g + 7\Delta$ from the edges of I. Suppose that a light outer clean horizontal wall W starts at position $b + \Delta$. Then $[a+\Delta, a+g-\Delta]$ contains an outer rightward-clean vertical hole passing through W. The same holds if we interchange horizontal and vertical.*

Proof. Suppose that this is not the case. Then event $\mathcal{L}_3(x, y, I', b)$ holds, as defined in the introduction of missing-hole traps in Step 4 of the scale-up construction. Now, if inequality (4.8) holds then $I' \times J$ is a trap of the missing-hole kind; but this was excluded, since Q is a hop. On the other hand, if (4.8) does not hold then (due also to Lemma 4.3) Lemma 4.4 is applicable to the interval I' and the interval I that is the X projection of Q, and we can conclude that I contains an emerging wall. But this was also excluded. □

Lemma 4.6. *Let the rectangle Q, with X projection I, contain no walls of $\mathcal{M}^*$. For $j \in \{1, 2\}$, let l_j be as introduced in the definition of correlated traps in Step 3 of the scale-up construction. Let $I' = [a, a+3l_j]$, $J = [b, b+5\Delta]$ with $I' \times J \subseteq Q$ be such that I' is at a distance $\geqslant 3l_j + 7\Delta$ from the edges of I. Then there is an interval $I'' \subseteq I'$ of size l_j, such that the rectangle $I'' \times J$ contains no trap. The same holds if we interchange horizontal and vertical.*

Proof. The proof of this lemma is completely analogous to the proof of Lemma 4.5. □

Lemma 4.7. *The new mazery $\mathcal{M}^*$ defined by the above construction satisfies Conditions 3.5.2c and 3.5.2d.*

Proof.

1. Let us prove Condition 3.5.2c.

   Consider an interval I of size $3\Delta^*$ containing no walls of $\mathcal{M}^*$. Condition 3.5.2b says that the real line is spanned by an extended sequence $(W_1, W_2, \dots)$ of neighbor walls of $\mathcal{M}$ separated from each other by hops of $\mathcal{M}$. Since I contains no wall of $\mathcal{M}^*$, if two of these walls fall into I then they are separated by a hop of size $\geqslant f$.

   Let I' be the middle third of I. Then $|I'| \geqslant 2f + \Delta$ and removing the W_i from I' leaves a subinterval $(a, b] \subseteq I'$ of size at least f. (If at least two W_i intersect I' take the interval between consecutive ones, otherwise I' is divided into at most two pieces of total length at least $2f$.) Now $K = (a + \Delta + f/3,\ b - \Delta - f/3]$ is an interval of length at least $f/3 - 2\Delta > 3\Delta$ which has distance at least $f/3$ from any wall. There will be a clean point in the middle of K which will then be clean in $\mathcal{M}^*$.

2. Let us prove Condition 3.5.2d now for $\mathcal{M}^*$.

We will confine ourselves to the statement in which the point a is assumed clean and we find a b such that (a, b) is clean. The half clean cases are proved similarly. Let I, J be right-closed intervals of size $3\Delta^*$, suppose that the rectangle $I \times J$ contains no horizontal walls or traps of $\mathcal{M}^*$, and a is a point in the middle third of I that is clean in $\mathcal{M}^*$ for X. We need to prove that there is a b in the middle third of J such that the point (a, b) is clean in $\mathcal{M}^*$.

Just as in Part 1 above, we find K with $f/3 - 2\Delta \leqslant |K|$ in the middle of J which is at distance at least $f/3$ from any horizontal wall. Let $I' = (a - g - \Delta,\ a + g + \Delta]$, then $I' \subseteq I$. We will find an interval $K'' \subseteq K$ with $|K''| \geqslant g'$ such that $I' \times K''$ contains no trap. If there are no traps in $I' \times K$ let $K'' = K$. Assume now that $I' \times K$ contains a trap $T = \mathrm{Rect}(u, v)$ of $\mathcal{M}$, where $u = (u_0, u_1)$, $v = (v_0, v_1)$. Since there are no uncorrelated traps, any trap must meet either $[u_0,\ v_0] \times K$ or $I' \times [u_1,\ v_1]$ or be at a distance at least f from T (and hence outside $I' \times K$). Let K' be a subinterval of $K \setminus [u_1,\ v_1]$ of size $3g'$ (which exists since $|K| \geqslant 2 \cdot (3g') + \Delta$). By Lemma 4.6, there must exist a subinterval K'' of K' of length $g' \geqslant 2g + 3\Delta$ such that $[u_0 - 2\Delta,\ u_0 + 3\Delta] \times K''$ contains no trap. Then also $I' \times K''$ contains no trap.

Find a clean point (a, b) in the middle third of K''. Then (a, b) has distance at least $g + \Delta$ from the boundary of $I' \times K''$ and so has distance at least g from any trap. Since b is at distance at least $f/3$ from any wall it is clean in $\mathcal{M}^*$. Hence (a, b) is clean in $\mathcal{M}^*$.

□

5. Probability bounds

In this section, we derive all those bounds on probabilities in $\mathcal{M}^k$ that are possible to give without indicating the dependence on k.

5.1. **General bounds.** Recall the definitions needed for the hole lower bound condition 3.5.3d. Since $c - 1$ will be used often, we denote it by $\hat{c}$. Let $u \leqslant v < w$, and a be given with $v - u \leqslant 12\Delta$, and define $b = a + \lceil \frac{v-u}{2} \rceil$, $c = a + (v - u) + 1$. We need to extend the lower bound condition in several ways. Since we will hold Y fixed in this subsection, we take the liberty and omit the condition $Y = y$ from the probabilities: it is always assumed to be there. For the following lemma, remember Condition 3.5.3c.

Lemma 5.1. *In addition to the assumptions in Condition 3.5.3d, assume that w is clean in $(w,\ w + \Delta]$ for Y. Let F_t be the event that the point (t, w) is upper right rightward-clean. Let $\hat{E}$ be the event that E is realized with a hole $(d,\ t]$, and F_t holds. We have*

$$\mathbf{P}(\hat{E}) \geqslant (1 - q)\mathbf{P}(E). \tag{5.1}$$

Proof. For $b \leqslant t \leqslant c + \Delta$, let E_t be the event that E is realized by a hole ending at t but is not realized by any hole ending at any $t' < t$. Then $E = \bigcup_t E_t$, $\hat{E} \supseteq \bigcup_t (E_t \cap F_t)$. Due to the Markov chain property of X and the form of E_t, the fact that E_t depends only on $X(0), \dots, X(t)$ and Condition 3.5.3c, we have

$$\mathbf{P}(E_t \cap F_t) = \mathbf{P}(E_t)\mathbf{P}(F_t \mid E_t) \geqslant \mathbf{P}(E_t)(1 - q).$$

The events E_t are mutually disjoint. Hence

$$\mathbf{P}(\hat{E}) \geqslant \sum_t \mathbf{P}(E_t \cap F_t) \geqslant (1-q)\sum_t \mathbf{P}(E_t) = (1-q)\mathbf{P}(E).$$

□

Recall Remark 3.6.2, referring to the most important special case of the hole lower bound: for any horizontal wall B of rank r, at any point b, the probability that there is a vertical hole passing through B at point b is at least $h(r)$. We strengthen this observation in a way similar to Lemma 5.1.

Lemma 5.2. *Let $v < w$, and let us fix Y in such a way that there is an outer-clean horizontal wall B with body $(v, w]$. Let point b be given. Let E be the event that an outer rightward-clean hole $(b, b']$ passes through B. Let $k \in \{1, \dots, m\}$. Then we have*

$$\mathbf{P}(E \mid X(b-\Delta) = k) \geqslant (1-q)^2 h(r).$$

Proof. Condition 3.5.3c implies that the probability of everything but the upper right rightward-cleanness of (b', w) is at least $(1-q)h(r)$. This additional property comes at the price of an additional factor of $(1-q)$ as shown in the proof of Lemma 5.1. □

Now, we prove a version of the hole lower bound condition that will help proving the same bound for $\mathcal{M}^*$. This is probably the only part of the paper in which the probability estimates are somewhat tricky. Take the situation described above, possibly without the bound on $v - u$, but with the additional assumption that $(u, v]$ contains no walls of $\mathcal{M}^*$ and v is left-clean in $\mathcal{M}^*$. Let

$$E^* = E^*(u, v, w;\, a) \tag{5.2}$$

be the event (a function of X) that there is a $d \in [b, c-1]$ with the following properties for $Q = \mathrm{Rect}^{\rightarrow}((a, u), (d, v))$:

(i*) a vertical hole (of $\mathcal{M}$) fitting B starts at d;
(ii*) Q potentially contains no traps or vertical walls of $\mathcal{M}$ or $\mathcal{M}^*$, and (d, v) is clean in Q for $\mathcal{M}^*$;
(iii*) if also u is clean for $\mathcal{M}^*$ in $(u, v]$ then (a, u) is clean for $\mathcal{M}^*$ in Q.

The difference between $E^*(\cdot)$ and $E(\cdot)$ is only that E^* requires the cleanness for $\mathcal{M}^*$ and also absence of walls and traps for $\mathcal{M}^*$ whenever possible.

Every time we estimate $\mathbf{P}(E)$, the implicit assumption is that v is left-clean in $\mathcal{M}$ and $(u, v]$ contains no walls of $\mathcal{M}$; if we estimate $\mathbf{P}(E^*)$ the assumptions refer to $\mathcal{M}^*$ instead. Let

$$\overline{p} \tag{5.3}$$

be an upper bound of the probabilites over all possible points a of the line, and over all possible values of $X(a)$, that a wall of $\mathcal{M}$ starts at a and that a wall of $\mathcal{M}^*$ starts at a. Let

$$\overline{w} \tag{5.4}$$

be an upper bound of the conditional probabilites over X (with Y and $X(a-1)$ fixed in any possible way) over all possible points (a, b) of the plane, that a trap of $\mathcal{M}$ starts at (a, b) or that a trap of $\mathcal{M}^*$ starts there.

Lemma 5.3. *Suppose that the requirement $v - u \leqslant 12\Delta$ in the definition of the event E^* no longer holds, but the rest of the requirements does. We have*

$$\mathbf{P}(E^*) \geqslant 0.5 \wedge (1.1(c-b)^{\chi} h(r)) - U \tag{5.5}$$

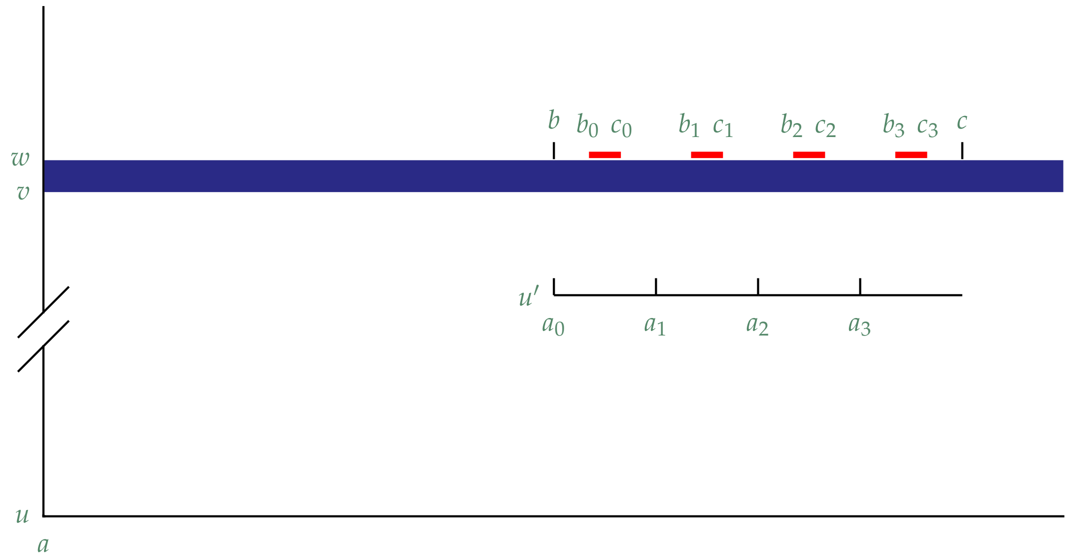

FIGURE 6. To the proof of Lemma 5.3.

with $U = 24\overline{p}\Delta^* + 312\overline{w}(\Delta^*)^2$. *Therefore the inequality*

$$\mathbf{P}(E^*) \geqslant 0.5 \wedge ((c-b)^\chi h(r)) - U \tag{5.6}$$

holds regardless of the size of $v - u$.

Proof. We will use the following inequality, which can be checked by direct calculation. Let $\alpha = 1 - 1/e = 0.632\ldots$, then for $x > 0$ we have

$$1 - e^{-x} \geqslant \alpha \wedge \alpha x. \tag{5.7}$$

The inequality $v - u > 12\Delta$ implies $\hat{c} - b \geqslant 6\Delta$.
Let $n = \lfloor (c-b)/(3\Delta) \rfloor$, then we have $n \geqslant 2$ and hence $(c-b)/(3\Delta) \leqslant n+1 \leqslant 1.5n$, implying

$$n\Delta \geqslant (c-b)/4.5. \tag{5.8}$$

Let

$$\begin{aligned} &u' = v - 2\Delta, \quad a_i = b + 3i\Delta, \\ &E'_i = E'(u', v, w;\, a_i) \quad \text{ for } i = 0, \ldots, n-1, \\ &E' = \bigcup_i E'_i. \end{aligned}$$

Let C be the event that (a, u) is upper right rightward-clean in $\mathcal{M}^*$. We will only prove the statement for the case when u is clean in $(u, v]$ for $\mathcal{M}^*$, and therefore event E^* requires C. In case when u is not right clean and E^* does not require this, the proof is the same, but the event C is not used.
Let D be the event that the rectangle $(a, c] \times [u, v]$ contains no trap or potential vertical wall of $\mathcal{M}$ or $\mathcal{M}^*$. By definition, we have

$$\begin{aligned} \mathbf{P}(\neg D) &\leqslant 2\overline{p}(c-a) + 2\overline{w}(c-a)(v-u+1) \\ &\leqslant 2 \cdot 12\overline{p}\Delta^* + 2 \cdot 12 \cdot 13\overline{w}(\Delta^*)^2 = 24\overline{p}\Delta^* + 312\overline{w}(\Delta^*)^2. \end{aligned}$$

1. Let us show $C \cap D \cap E' \subseteq E^*(u, v, w;\, a;\,)$.

Indeed, suppose that $C \cap D \cap E'_i$ holds with some hole starting at d. Then there is a rectangle $Q'_i = \mathrm{Rect}^{\to}((a_i, u'), (d, v))$ containing no traps or potential vertical walls of $\mathcal{M}$, such that (d, v) is clean in Q'_i. It follows from D that the rectangle

$$Q^*_i = \mathrm{Rect}^{\to}((a, u), (d, v)) \supseteq Q'_i$$

contains no traps or potential vertical walls of $\mathcal{M}$ or $\mathcal{M}^*$. The point (a, u) is clean for $\mathcal{M}$ in Q^*_i, The event E'_i implies that (d, v) is clean in Q^*_i, and a hole passing through the wall starts at d in X. The event D implies that there is no wall or trap of $\mathcal{M}$ or $\mathcal{M}^*$ in Q^*_i. Together with C, this implies E^*.

We have $\mathbf{P}(E^*) \geqslant \mathbf{P}(C)\mathbf{P}(E' \mid C) - \mathbf{P}(\neg D)$.

2. It remains to estimate $\mathbf{P}(E' \mid C)$.

Let us denote $s = \Delta^{\chi} h(r)$. Condition 3.5.3d is applicable to E'_i and we have $\mathbf{P}(E'_i \mid C) \geqslant s$ hence for each $k \in \{1, \dots, m\}$ we have

$$\mathbf{P}\big(\neg E'_i \mid C \cap \{X(a_i) = k\}\big) \leqslant 1 - s \leqslant e^{-s}.$$

Due to the Markov property, this implies $\mathbf{P}\big(\neg E'_i \mid C \cap \bigcap_{j<i} \neg E'_j\big)$, and hence

$$\mathbf{P}(E' \mid C) = 1 - \mathbf{P}\Big(\bigcap_i \neg E'_i \mid C\Big) \geqslant 1 - e^{-ns} \geqslant \alpha \wedge (\alpha n s), \tag{5.9}$$

where in the last step we used (5.7). By (5.8), we have

$$\alpha n \Delta^{\chi} = \alpha n^{1-\chi} (\Delta n)^{\chi} \geqslant \alpha 2^{1-\chi} (\Delta n)^{\chi} \geqslant \alpha (2^{1-\chi}/4.5^{\chi})(c-b)^{\chi} \geqslant 1.223(c-b)^{\chi},$$

where we used the value of χ. Substituting into (5.9):

$$\begin{aligned} \mathbf{P}(E' \mid C) &\geqslant \alpha \wedge (1.223(c-b)^{\chi} h(r)), \\ \mathbf{P}(C)\mathbf{P}(E' \mid C) &> 0.9 \cdot (\alpha \wedge (1.223(c-b)^{\chi} h(r))) > 0.5 \wedge (1.1(c-b)^{\chi} h(r)). \end{aligned}$$

□

5.2. **New traps.** Recall the definition of uncorrelated compound traps in Step 2 of the scale-up construction in Section 4.

Lemma 5.4 (Uncorrelated Traps)**.** *Given a string $x = (x(0), x(1), \dots)$, a point (a_1, b_1), and $v > a_1$, let $\mathcal{F}$ be the event that an uncorrelated compound trap of $\mathcal{M}^*$ starts at (a_1, b_1), with projection lying in the interval $I = [a_1, v)$. Let $k \in \{1, \dots, m\}$, then we have*

$$\mathbf{P}\big(\mathcal{F} \mid X(I) = x(I),\ Y(b_1 - 1) = k\big) \leqslant 2 f^2 w^2. \tag{5.10}$$

Proof. Let $\mathcal{G}(a, b)$ be the event that a trap of $\mathcal{M}$ starts at (a, b). Let $\mathcal{G}(a, b;\ a', b')$ be the event that a trap of $\mathcal{M}$ starts at (a, b), and is contained in $[a, a') \times [b, b')$. Since the new trap is the smallest rectangle containing two old traps, it must contain these in two of its opposite corners: let $\mathcal{E}$ be the event that one of these corners is (a_1, b_1).

Let $N = (a_1, b_1) + (0, f]^2$. Then

$$\mathcal{E} \subseteq \bigcup_{(a_2, b_2) \in N} \mathcal{G}(a_1, b_1; a_2, b_2) \cap \mathcal{G}(a_2, b_2).$$

Let $\mathcal{X}(I) = \mathcal{X}[a_1, v]$ be the event $X(I) = x(I)$. Thus, we can write

$$\mathcal{X}(I) \cap \mathcal{E} \subseteq \bigcup_{(a_2, b_2) \in N} (\mathcal{X}[a_1, a_2) \cap \mathcal{G}(a_1, b_1;\ a_2, b_2)) \cap (\mathcal{X}[a_2, v] \cap \mathcal{G}(a_2, b_2; v, \infty)).$$

The events $\mathcal{X}[a_1, a_2) \cap \mathcal{G}(a_1, b_1; a_2, b_2)$ and $\mathcal{X}[a_2, v] \cap \mathcal{G}(a_2, b_2; v, \infty)$ belong to rectangles whose projections are disjoint. Fixing $Y(b_1 - 1)$ arbitrarily, by Condition 3.5.3a and the Markov property we have:

$$\mathbf{P}\big(\mathcal{G}(a_1, b_1; a_2, b_2) \mid \mathcal{X}[a_1, v]\big) = \mathbf{P}\big(\mathcal{G}(a_1, b_1; a_2, b_2) \mid \mathcal{X}[a_1, a_2)\big) \leqslant w,$$
$$\mathbf{P}\big(\mathcal{G}(a_2, b_2; v, \infty) \mid \mathcal{G}(a_1, b_1; a_2, b_2) \cap \mathcal{X}[a_1, a_2) \cap \mathcal{X}[a_2, v]\big) \leqslant w.$$

Hence $\mathbf{P}(\mathcal{E} \mid \mathcal{X}(I)) \leqslant f^2 w^2$. If $\mathcal{F} \setminus \mathcal{E}$ holds then there is a random pair $(A, B) \in N$ such that $\mathcal{G}(a_1, B; a_2, \infty)$ and $\mathcal{G}(A, b_1; a_1, B)$ holds. A computation similar to the above one gives the upper bound $f^2 w^2$ for $\mathbf{P}\big(\mathcal{F} \setminus \mathcal{E} \mid \mathcal{X}(I)\big)$. □

Recall the definition of correlated traps in part 3 of the scale-up construction in Section 4.

Lemma 5.5 (Correlated Traps). *Let a site (a, b) be given. For $j = 1, 2$, let $\mathcal{F}_j$ be the event that a horizontal correlated trap of type j starts at (a, b).*

(a) *Let us fix a string $x = (x(0), x(1), \dots)$, and also $k \in \{1, \dots, m\}$ arbitrarily. We have*

$$\mathbf{P}\big(\mathcal{F}_j \mid X = x,\ Y(a-1) = k\big) \leqslant w^2. \tag{5.11}$$

(b) *Let us fix a string $y = (y(0), y(1), \dots)$, and also $k \in \{1, \dots, m\}$ arbitrarily. We have*

$$\mathbf{P}\big(\mathcal{F}_j \mid Y = y,\ X(a) = k\big) \leqslant (5\Delta l_j w)^3. \tag{5.12}$$

Proof. Part (a) is an immediate consequence of requirement (4.7) of the definition of correlated traps. It remains to prove part (b). Note that this result implies the same bounds also if we fix $X(a-1)$ arbitrarily. If there is a correlated trap with X-projection starting at some a then there must be traps with X-projections in $(a, a + l_j]$, $(a + l_j, a + 2l_j]$ and $(a + 2l_j, a + 3l_j]$. Due to the trap upper bound and the Markov property, the probability of a trap in any one of these is at most $5\Delta l_j w$, even conditioned on the values of X before. Hence the probability of such a compound trap happening is at most $5\Delta l_j w)^3$. □

Recall the definition of traps of the missing-hole kind in Step 4 of the scale-up algorithm in Section 4.

Lemma 5.6 (Missing-hole traps).
For $a, b \in \mathbb{Z}_+$, let $\mathcal{F}$ be the event that a horizontal trap of the missing-hole kind starts at (a, b).

(a) *Let us fix a string $x = (x(0), x(1), \dots)$, and also $k \in \{1, \dots, m\}$ arbitrarily. We have*

$$\mathbf{P}\big(\mathcal{F} \mid X = x,\ Y(a-1) = k\big) \leqslant w^2. \tag{5.13}$$

(b) *Let us fix a string $y = (y(0), y(1), \dots)$, and also $k \in \{1, \dots, m\}$ arbitrarily. Let $n = \lfloor \frac{g}{3\Delta} \rfloor$. We have*

$$\mathbf{P}\big(\mathcal{F} \mid Y = y,\ X(a) = k\big) \leqslant e^{-(1-q)^2 n h(R^*)}. \tag{5.14}$$

Proof. Part (a) is an immediate consequence of requirement (4.8) of the definition of missing-hole traps. It remains to prove part (b). Note that this result implies the same bounds also if we fix $X(a-1)$ arbitrarily. Let $J = [b, b + 3\Delta]$. According to the definition of missing-hole traps above, we can assume without loss of generality that, with $b_1 = b + \Delta$, there is a $b_2 > b_1$ such that $(b_1, b_2]$ is (Y, J)-potentially the body of a light outer clean horizontal wall W. For $i = 0, \dots, n-1$, let $\mathcal{A}(d, i)$ be the event that no outer rightward-clean vertical hole $(a_1, a_2]$ with $a_1 = a + 3i\Delta + \Delta$ passes through W. All these events must hold if a horizontal trap of the missing-hole kind starts at (a, b). Using the Markov property and Lemma 5.2, we have

$$\mathbf{P}\big(\,\mathcal{A}(d,i)\mid\bigcap_{j<i}\mathcal{A}(d,j)\,\big)\leqslant 1-(1-q)^2 h(R^*).$$

Therefore $\mathbf{P}\big(\bigcap_i \mathcal{A}(d,i)\big)\leqslant e^{-n(1-q)^2 h(R^*)}$. □

5.3. **Emerging walls.** Recall the definition of emerging walls in Step 5 of the scale-up algorithm in Section 4.

Lemma 5.7. *For any point u, let $\mathcal{F}$ be the event that a potential wall $(u, v]$ of X of the emerging kind starts at u. Let $k \in \{1,\dots,m\}$. We have, with $n = \lfloor \frac{g}{3\Delta}\rfloor$:*

$$\mathbf{P}(\mathcal{F}\mid X(u)=k)\leqslant 4\Delta^2 w\big(2\cdot(5\Delta g')^3+w^{-3}e^{-(1-q)^2 nh(R^*)}\big). \tag{5.15}$$

Proof. For interval $I' = [u', v']$ and $b \in \mathbb{Z}_+$, let $\mathcal{L}_j(x, Y, I', b)$ be defined as in Steps 3 and 4 of the scale-up algorithm in Section 4. Let us fix an arbitrary $k \in \{1,\dots,m\}$. By Lemma 5.5, for $j = 1, 2$ we have

$$\mathbf{P}\big(\,\mathcal{L}_j(X,Y,I',1)\mid Y(0)=k\,\big)\leqslant(5\Delta l_j w)^3=:U_j.$$

By Lemma 5.6:

$$\mathbf{P}\big(\,\mathcal{L}_3(X,Y,I',1)\mid Y(0)=k\,\big)\leqslant e^{-(1-q)^2 nh(R^*)}=:U_3.$$

Hence,

$$\sum_{x(I)}\mathbf{P}(X(I)=x(I))\mathbf{P}\big(\,\mathcal{L}_j(X,Y,I',1)\mid X(I)=x(I),\ Y(b-1)=k\,\big)\leqslant U_j.$$

The Markov inequality implies that for any k, the probability of those x for which

$$\mathbf{P}\big(\,\mathcal{L}_j(x,Y,I',1)\mid X(I)=x(I),\ Y(b-1)=k\,\big)\geqslant w^2,$$

will be upperbounded by $w^{-2}U_j$. Multiplying by the number $(2\Delta)^2$ of possible choices for I' upperbounds the probability of an emerging wall of type j. Adding up the three values gives

$$4\Delta^2 w^{-2}(U_1+U_2+U_3)<4\Delta^2 w\big(2\cdot(5\Delta g')^3+w^{-3}e^{-(1-q)^2 nh(R^*)}\big).$$

□

5.4. **Compound walls.** Let us use the definition of compound walls given in Step 6 of the scale-up algorithm of Section 4. Consider ranks r_1, r_2 at any stage of the scale-up construction. For the lemmas below, assume that Conditions 3.5.3b and 3.5.3d already hold for ranks r_j.

Lemma 5.8. *For a given point x_1, let us fix the $X(x_1) = k$ for some $k \in \{1,\dots,m\}$ arbitrarily. Then the sum, over all w, of the probabilities for the occurrence of a potential compound wall of type $\langle r_1, r_2, i\rangle$ and width w at x_1 is bounded above by*

$$\lambda^i p(r_1)p(r_2). \tag{5.16}$$

Proof. Noting $d_{i+1} - d_i \leqslant \lambda^i$ for all i, we will prove an upper bound $(d_{i+1} - d_i)p(r_1)p(r_2)$. For fixed r_1, r_2, x_1, d, let $B(d, l)$ be the event that a potential compound wall of any type $\langle r_1, r_2, i\rangle$ with distance d between the component (potential) walls, and size l appears at x_1.

For any l, let $A(x,r,l)$ be the event that a potential wall of rank r and size l starts at x. We can write

$$B(d,l) = \bigcup_{l_1+d+l_2=l} A(x_1,r_1,l_1) \cap A(x_1+l_1+d,\ r_2,\ l_2).$$

where events $A(x_1,r_1,l_1)$, $A(x_1+l_1+d,\ r_2,\ l_2)$ belong to disjoint intervals. Recall the definition of $p(r,l)$ in (3.3). By the Markov property,

$$\mathbf{P}(B(d,l)) \leqslant \sum_{l_1+d+l_2=l} p(r_1,l_1)p(r_2,l_2).$$

Hence Condition 3.5.3b implies $\sum_l \mathbf{P}(B(d,l)) \leqslant (\sum_{l_1} p(r_1,l_1)) \sum_{l_2} p(r_2,l_2) \leqslant p(r_1)p(r_2)$, which completes the proof. □

Lemma 5.9. *Let $u \leqslant v_1 < w_2$, and a be given with*

$$b = a + \lceil (v_1-u)/2 \rceil, \qquad c = b + (v_1-u) - (b-a) + 1.$$

Assume that $Y = y$ is fixed in such a way that v_1 is left-clean in $\mathcal{M}$, the interval $(u,\ v_1]$ contains no walls of $\mathcal{M}$, and that W is a compound horizontal wall with body $(v_1,\ w_2]$, and type $\langle r_1, r_2, i\rangle$, with rank r as given in (4.12). Let

$$E_2 = E_2(u,v_1,w_2;\ a) = E^*(u,v_1,w_2;\ a)$$

where E^* was defined in (5.2). Assume

$$(\Delta^*)^{\chi} h(r_j) \leqslant 0.07, \text{ for } j = 1,2. \tag{5.17}$$

Let $k \in \{1,\dots,m\}$. Then we have

$$\mathbf{P}(E_2 \mid X(a) = k) \geqslant (c-b)^{\chi}(\lambda^i/2)^{\chi} h(r_1)h(r_2)\cdot(1-V) \tag{5.18}$$

with $V = 2\cdot(24\overline{p}\Delta^* + 312\overline{w}(\Delta^*)^2)/h(r_1 \vee r_2)$.

Proof. Let D be the distance between the component walls. Let walls W_1, W_2 be the components of the wall W, where the body of W_i is $(v_i,\ w_i]$. Consider first passing through W_1. For each $x \in [b, c+\Delta-1]$, let A_x be the event that $E^*(u,v_1,w_1;\ a)$ holds with the vertical projection of the hole ending at x, and that x is the smallest possible number with this property.
Let $B_x = E^*(w_1,v_2,w_2;\ x)$.

1. We have $E_2 \supseteq \bigcup_x (A_x \cap B_x)$.

   Proof. If for some x we have A_x, then there is a rectangle $\mathrm{Rect}((a,u),(t_1,v_1))$ satisfying the requirements of $E^*(u,v_1,w_1;\ a)$ and also a hole $\mathrm{Rect}((t_1,v_1),(x,w_1))$ through the first wall. If also B_x holds, then there is a rectangle $\mathrm{Rect}((x,w_1),(t_2,v_2))$ satisfying the requirements of $E^*(w_1,v_2,w_2;\ x)$, and also a hole $\mathrm{Rect}((t_2,v_2),(x',w_2))$ through the second wall.

   Let us show $(d_1,v_1) \rightsquigarrow (x',w_2)$, meaning that the interval $(t_1,\ x']$ is a hole that passes through the compound wall W. We already know $(d_1,v_1) \rightsquigarrow (x,w_1)$ and $(d_2,v_2) \rightsquigarrow (x',w_2)$; we still need to prove $(x,w_1) \rightsquigarrow (d_2,v_2)$. The requirements imply that $\mathrm{Rect}((x,v_1),(t_2,v_2))$ is a hop of $\mathcal{M}$. According to B_x, this rectangle has the necessary slope constraints, hence by the reachability condition of $\mathcal{M}$, its endpoint is reachable from its starting point.

It remains to lowerbound $\mathbf{P}\big(\bigcup_x (A_x \cap B_x)\big)$. For each x, the events A_x, B_x belong to disjoint intervals, and the events A_x are disjoint of each other.

2. Let us lowerbound $\sum_x \mathbf{P}(A_x)$.

   We have, using the notation of Lemma 5.3: $\sum_x \mathbf{P}(A_x) = \mathbf{P}(E^*(u, v_1, w_1; a))$. Lemma 5.3 is applicable and we get $\mathbf{P}(E^*(u, v_1, w_1; a)) \geqslant F_1 - U$ with

$$\begin{aligned} F_1 &= 0.5 \wedge ((c-b)^\chi h(r_1)), \\ U &= 24\overline{p}\Delta^* + 312\overline{w}(\Delta^*)^2. \end{aligned} \tag{5.19}$$

   By the assumption (5.17): $(c-b)^\chi h(r_1) \leqslant (7\Delta^*)^\chi h(r_1) \leqslant 0.5$, hence the operation $0.5\wedge$ can be deleted from F_1:

$$F_1 = G_1 := (c-b)^\chi h(r_1). \tag{5.20}$$

3. Let us now lowerbound $\mathbf{P}(B_x)$, for an arbitrary condition $X(x) = k$ for $k \in \{1, \dots, m\}$.

   We have $B_x = E^*(w_1, v_2, w_2; x)$. The conditions of Lemma 5.3 are satisfied for $u = w_1$, $v = v_2$, $w = w_2$, $a = x$. It follows that $\mathbf{P}(B_x) \geqslant F_2 - U$ with

$$F_2 = 0.5 \wedge ((\lfloor D/2 \rfloor + 1)^\chi h(r_2)),$$

   which can again be simplified using assumption (5.17):

$$F_2 = G_2 := (\lfloor D/2 \rfloor + 1)^\chi h(r_2).$$

4. Let us combine these estimates, using $G = G_1 \wedge G_2 > h(r_1 \vee r_2)$.

   By the Markov property, we find that the lower bound on $\mathbf{P}(B_x)$ (for arbitrary $X(x) = k$) is also a lower bound on $\mathbf{P}(B_x \mid A_x)$:

$$\begin{aligned} \mathbf{P}(E_2) &\geqslant \sum_x \mathbf{P}(A_x)\mathbf{P}(B_x \mid A_x) \geqslant (G_1 - U)(G_2 - U) \\ &\geqslant G_1 G_2 (1 - U(1/G_1 + 1/G_2)) \geqslant G_1 G_2 (1 - 2U/G) \\ &= (c-b)^\chi (\lfloor D/2 \rfloor + 1)^\chi h(r_1) h(r_2)(1 - 2U/G). \\ &\geqslant (c-b)^\chi (\lfloor D/2 \rfloor + 1)^\chi h(r_1) h(r_2)(1 - 2U/h(r_1 \vee r_2)). \end{aligned}$$

5. We conclude by showing $\lfloor D/2 \rfloor + 1 \geqslant \lambda^i/2$.

   Recall $d_i \leqslant D < d_{i+1}$ where d_i was defined in (4.11). For $i = 0, 1$, we have $\lfloor D/2 \rfloor + 1 = 1 > \lambda^1/2$. For $i \geqslant 2$, we have $\lfloor D/2 \rfloor + 1 \geqslant D/2 \geqslant \lambda^i/2$.

□

6. THE SCALE-UP FUNCTIONS

Lemma 2.5 says that there is an m_0 such that if $m > m_0$ then the sequence $\mathcal{M}^k$ can be constructed in such a way that the claim (2.3) of the main lemma holds. If we computed m_0 explicitly then all parameters of the construction could be turned into constants: but this is unrewarding work and it would only make the relationships between the parameters less intelligible. We prefer to name all these parameters, to point out the necessary inequalities among them, and finally to show that if m is sufficiently large then all these inequalities can be satisfied simultaneously.

Recall that the slope lower bound σ must satisfy $\sigma < 1/2$. We set

$$\sigma_1 = 0. \tag{6.1}$$

Recall $\lambda = 2^{1/2}$, as defined in (4.4). To obtain the new rank lower bound, we multiply R by a constant:

$$R = R_k = R_0\tau^k, \quad R_{k+1} = R^* = R\tau, \quad 1 < \tau < 2, \quad 1 < R_0. \tag{6.2}$$

The rank of emerging walls, introduced in (4.10) is defined using a new parameter τ':

$$\hat{R} = \tau' R.$$

We require

$$\tau < \tau' < \tau^2. \tag{6.3}$$

We need some bounds on the possible rank values. Let

$$\overline{\tau} = 2\tau/(\tau - 1).$$

Lemma 6.1. *In a mazery, all ranks are upperbounded by $\overline{\tau}R$.*

Proof. Let $\overline{R}$ denote the maximum of all ranks. Since $\tau' < \tau^2 < \overline{\tau}$, emerging walls got a rank equal to $\tau' R < \overline{\tau}R$, and the largest rank produced by the compound operation is at most $\overline{R} + 2R^*$ (since the compound operation is applied twice):

$$\overline{R}^* \leqslant \overline{R} + 2R^*, \quad \overline{R}_k \leqslant 2\sum_{i=1}^{k} R_i = 2R_0\tau\frac{\tau^k - 1}{\tau - 1} \leqslant R_k\frac{2\tau}{\tau - 1}. \tag{6.4}$$

□

Corollary 6.2. *Every rank exists in $\mathcal{M}^k$ for at most $\lceil \log_\tau \frac{2\tau}{\tau-1} \rceil$ values of k.*

It can be seen from the definition of compound ranks in (4.12) and from Lemma 5.8 that the probability bound $p(r)$ of a wall should be approximately λ^{-r}. The actual definition makes the bound a little smaller:

$$p(r) = c_2 r^{-c_1}\lambda^{-r}. \tag{6.5}$$

The term $c_2 r^{-c_1}$, just like the factor in the function $h(r)$ defined for the hole lower bound condition serves for absorbing some lower-order factors that arise in estimates like (5.18). We have

$$h(r) = c_3\lambda^{-r\chi} \text{ with } c_3 = c_0 c_2^{\chi}. \tag{6.6}$$

We will choose c_0 implicitly, by choosing c_3 in the proof of Lemma 7.10.

It is convenient to express several other parameters of $\mathcal{M}$ and the scale-up in terms of T:

$$\begin{aligned}
&T = \lambda^R, \\
&\Delta = T^\delta, \quad f = T^\varphi, \quad g = T^\gamma, \quad w = T^{-\omega} \qquad\qquad \text{with } \omega = 4.
\end{aligned}$$

We require

$$0 < \delta < \gamma < \varphi < 1. \tag{6.7}$$

A bound on φ has been indicated in the requirement (4.5) which will be satisfied by

$$\tau \leqslant 2 - \varphi. \tag{6.8}$$

We choose $\tau = 2 - \varphi$. Let us estimate Δ^*. Emerging walls can have size as large as $3g' + 4\Delta$, and at the time of their creation, they are the largest existing ones. We get the largest new walls when the compound operation combines these with light walls on both sides, leaving the largest gap possible, so the largest new wall size is

$$3g' + 2f + 6\Delta < 3f,$$

where we used (4.1). Hence any value larger than $3f$ can be chosen as $\Delta^* = \Delta^\tau$. With R large enough, we always get this if

$$\varphi < \tau\delta. \tag{6.9}$$

As a reformulation of one of the inequalities of (4.1), we require

$$\gamma \leqslant \frac{\delta + \varphi}{2}. \tag{6.10}$$

We also need

$$3(\gamma + \delta) < \omega(3 - \tau), \tag{6.11}$$

$$1 < 3 - \tau, \tag{6.12}$$

$$3\gamma + 5\delta < \omega - \tau', \tag{6.13}$$

$$\tau(\delta + 1) < \tau'. \tag{6.14}$$

Using the exponent χ introduced in (2.2), we require

$$\tau\chi < \gamma - \delta, \tag{6.15}$$

$$\overline{\tau}\chi < 1 - \tau\delta, \tag{6.16}$$

$$\overline{\tau}\chi < \omega - 2\tau\delta. \tag{6.17}$$

Lemma 6.3. *The exponents $\delta, \varphi, \gamma, \tau, \chi$ can be chosen to satisfy the inequalities* (6.2),(6.3), (6.7)-(6.17).

Proof. It can be checked that the choices $\delta = 0.15$, $\gamma = 0.2$, $\varphi = 0.25$, $\tau' = 2.5$ satisfy all the inequalities in question. $\square$

Let us fix now all these exponents as chosen in the lemma. In order to satisfy all our requirements also for small k, we will fix c_2 sufficiently small, then c_1 sufficiently large, then c_0 sufficiently large, and finally R_0 sufficiently large.

7. Probability bounds after scale-up

7.1. **Bounds on traps.** The structures $\mathcal{M}^k$ are now defined but we have not proved that they are mazeries, since not all inequalities required in the definition of mazeries have been verified yet.

Lemma 7.1. *Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. Then $\mathcal{M}^*$ satisfies the trap upper bound 3.5.3a if R_0 is sufficiently large.*

Proof. For some string $x = (x(0), x(1), \dots)$, for a point (a, b) with interval $I \ni a$, let $\mathcal{E}$ be the event that a trap starts at (a, b) with X projection contained in I. We assume $Y(b-1)$ fixed arbitrarily. We need to bound $\mathbf{P}(\mathcal{E} \mid X(I) = x(I))$. There are three kinds of trap in $\mathcal{M}^*$: uncorrelated and correlated compound traps, and traps of the missing-hole kind. Let $\mathcal{E}_1$ be the event that an uncorrelated trap occurs. According to (5.10), we have, using $\tau = 2 - \varphi$:

$$\begin{aligned}\mathbf{P}(\mathcal{E}_1 \mid X(I) = x(I)) &\leqslant 2f^2w^2 = 2T^{2\varphi - 2\omega} \\ &= 2T^{-\tau\omega - (2-\tau)\omega + 2\varphi} = w^* \cdot 2/f^{\omega - 2}.\end{aligned}$$

This can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

Let $\mathcal{E}_2$ be the event that a vertical correlated trap appears. By Lemma 5.5, we have, using (4.6):

$$\begin{aligned}\mathbf{P}(\mathcal{E}_2 \mid X(I) = x(I)) &\leqslant \sum_{j=1}^{2}(5\Delta l_j w)^3 \leqslant 2\cdot(5\Delta g' w)^3 = 2\cdot 11^3 T^{3\gamma+3\delta-3\omega-\tau\omega+\tau\omega} \\ &= 2w^* \cdot 11^3 T^{3(\gamma+\delta)-\omega(3-\tau)}.\end{aligned}$$

Due to (6.11), this can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

Let $\mathcal{E}_3$ be the event that a vertical trap of the missing-hole kind appears at (a, b). Lemma 5.6 implies that for $n = \lfloor \frac{g}{3\Delta} \rfloor$, we have

$$\mathbf{P}(\mathcal{E}_3 \mid X(I) = x(I)) \leqslant e^{-(1-q)^2 n h(R^*)}.$$

Further, using inequality (4.1) and the largeness of R_0:

$$n > g/(3\Delta) - 2 > g/(4\Delta) = T^{\gamma-\delta}/4.$$

Now,

$$\begin{aligned}h(R^*) &= c_3 T^{-\tau\chi}, \\ (1-q)^2 n h(R^*) &> 0.8 n h(R^*) > 0.2 c_3 T^{\gamma-\delta-\tau\chi}, \\ \mathbf{P}(\mathcal{E}_3 \mid X(I) = x(I)) &\leqslant e^{-0.2 c_3 T^{\gamma-\delta-\tau\chi}}.\end{aligned}$$

Due to (6.15), this can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

For $j = 1, 2$, let $\mathcal{E}_{4,j}$ be the event that a horizontal trap of the correlated kind of type j starts at (a, b). Let $\mathcal{E}_{4,3}$ be the event that a trap of missing-hole kind starts at (a, b). Lemmas 5.5 and 5.6 imply

$$\mathbf{P}(\mathcal{E}_{4,j} \mid X(I) = x(I)) \leqslant w^2 = w^* T^{-\omega(2-\tau)}.$$

Due to (6.12), this can be made smaller than w^* by an arbitrarily large factor if R_0 is large.

Thus, if R_0 is sufficiently large then the sum of these six probabilities is still less than w^*. □

7.2. **Bounds on walls.**

Lemma 7.2. *For every possible value of c_0, c_1, if R_0 is sufficiently large then the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. Fixing any point a and fixing $X(a)$ in any way, the probability that a potential wall of the emerging kind starts at a is at most $p(\hat{R})/2 = p(\tau' R)/2$.*

Proof. We use the result and notation of Lemma 5.7, and also the estimate of $\mathbf{P}(\mathcal{E}_3)$ in the proof of Lemma 7.1:

$$\begin{aligned}\mathbf{P}(\mathcal{F}) &\leqslant 4\Delta^2 w\big(2\cdot(5\Delta g')^3 + w^{-3} e^{-(1-q)^2 n h(R^*)}\big), \\ 4\Delta^2 w^{-2} e^{-(1-q)^2 n h(R^*)} &\leqslant 4T^{2\omega+2\delta} e^{-0.2 c_3 T^{\gamma-\delta-\tau\chi}}.\end{aligned}$$

Due to (6.15), the last expression decreases exponentially in T, so for sufficiently large R_0 it is less than $p(\tau' R)$ by an arbitrarily large factor. On the other hand, using (4.6):

$$9\Delta^2 w^{-1}\cdot 2\cdot(5\Delta g')^3 = 18\cdot 11^3 T^{-\omega+3\gamma+5\delta} = 18T^{-\tau'}\cdot 11^3 T^{3\gamma+5\delta+\tau'-\omega}.$$

If R_0 is sufficiently large then, due to (6.13), this is less than $p(\tau' R)$ by an arbitrarily large factor. □

Lemma 7.3. *For a given value of c_2, if we choose the constants c_1, R_0 sufficiently large in this order then the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery. After one operation of forming compound walls, fixing any point a and fixing $X(a)$ in any way, for any rank r, the sum, over all widths w, of the probability that a potential compound wall of rank r and width w starts at a is at most $p(r)R^{-c_1/2}$.*

Proof. Let $r_1 \leqslant r_2$ be two ranks, and assume that r_1 is light: $r_1 < R^* = \tau R$. With these, we can form compound walls of type $\langle r_1, r_2, i\rangle$. The bound (5.16) and the definition of $p(r)$ in (6.5) shows that the contribution by this term to the sum of probabilites, over all sizes w, that a wall of rank $r = r_1 + r_2 - i$ and size w starts at x is at most

$$\lambda^i p(r_1)p(r_2) = c_2^2 \lambda^{-r}(r_1 r_2)^{-c_1} = c_2(r/r_1 r_2)^{-c_1} p(r).$$

Now we have $r_1 r_2 \geqslant R r_2 \geqslant (R/2)(r_1 + r_2) \geqslant rR/2$, hence the above bound reduces to $c_2(2/R)^{-c_1} p(r)$. The same rank r can be obtained by the compound operation at most the following number of times:

$$|\{\, (i, r_1) : i \leqslant R\varphi,\ r_1 < \tau R \,\}| \leqslant (\varphi R + 1)\tau R.$$

The total probability contributed to rank r is therefore at most

$$c_2(2/R)^{-c_1} p(r)(\varphi R + 1)\tau R < p(r)R^{-c_1/2}$$

if R_0 and c_1 are sufficiently large. □

Lemma 7.4. *Suppose that each structure $\mathcal{M}^i$ for $i \leqslant k$ is a mazery. Then Condition 3.5.3b holds for $\mathcal{M}^{k+1}$.*

Proof. By Corollary 6.2, each rank r occurs for at most a constant number $n = \lceil \log_\tau \frac{2\tau}{\tau - 1} \rceil$ values of $i \leqslant k$. For any rank, a potential wall can be formed only as a potential emerging wall or compound wall. The first can happen for one i only, and Lemma 7.2 bounds the probability contribution by $p(r)/2$. A compound wall can be formed for at most n values of i, and for each value in at most two steps. Lemma 7.3 bounds each contribution by $p(r)R^{-c_1/2}$. After these increases, the probability becomes at most $p(r)(1/2 + 2nR^{-c_1/2}) < p(r)$ if R_0 is sufficiently large. □

7.3. **Auxiliary bounds.** Let us give concrete value to the upper bounds $\overline{p}$, $\overline{w}$:

$$\overline{p} = T^{-1}, \quad \overline{w} = w. \tag{7.1}$$

The next two lemmas show that these choices satisfy the requirements imposed in (5.3),(5.4).

Lemma 7.5. *For small enough c_2, the probability of a potential wall of $\mathcal{M}$ starting at a given point b is bounded by $\overline{p}$.*

Proof. We have $\sum_{r \geqslant R} p(r) < c_2 \sum_{r \geqslant R} \lambda^{-r} = \lambda^{-R} c_2 (1 - 1/\lambda)^{-1} < \lambda^{-R}$ if $c_2 < 1 - 1/\lambda$. □

Lemma 7.6. *If R_0 is sufficiently large then we have $\sum_k (2\Delta_{k+1}\overline{p}_k + \Delta_{k+1}^2 w_k) < 0.5$.*

Proof. We have $\sum_k (2\Delta_{k+1}\overline{p}_k + \Delta_{k+1}^2 w_k) \leqslant 2\sum_k \lambda^{-R_0 \tau^k (1 - \delta\tau)} + \sum_k \lambda^{-R_0 \tau^k (4 - 2\delta\tau)}$ which because of (6.16), is less than 0.5 if R_0 is large. □

Note that for R_0 large enough, the relations

$$\Delta^* \overline{p} < 0.5(0.1 - q), \tag{7.2}$$

$$\Lambda g/f < 0.5(0.5 - \sigma) \tag{7.3}$$

hold for $\mathcal{M} = \mathcal{M}^1$ and $\sigma = \sigma_1$. This is clear for (7.2). For (7.3), since $\sigma_1 = 0$ according to (6.1), we only need $0.25 > \Lambda g/f = \Lambda T^{-(\varphi-\gamma)}$, which is satisfied if R_0 is large enough.

Lemma 7.7. *Suppose that the structure $\mathcal{M} = \mathcal{M}^k$ is a mazery and it satisfies (7.2) and (7.3). Then $\mathcal{M}^* = \mathcal{M}^{k+1}$ also satisfies these inequalities if R_0 is sufficiently large.*

Proof. The probability that a point a of the line is clean in $\mathcal{M}$ but not in $\mathcal{M}^*$ is upper-bounded by

$$(2f/3 + \Delta)\overline{p} + (g+\Delta)^2 w + g^2 w. \tag{7.4}$$

The first term upperbounds the probability that a potential horizontal wall of $\mathcal{M}$ starts in $(a - f/3 - \Delta,\ a + f/3]$. The two other terms upperbound the probability that a trap of $\mathcal{M}$ appears in $(a-g-\Delta,\ a] \times (b-g-\Delta,\ b]$ or in $(a,\ a+g] \times (b,\ b+g]$. The first term can be upperbounded by

$$f\overline{p} < T^{\varphi-1} < 0.5T^{\delta\tau-1} = 0.5\Delta^*\overline{p},$$

where the last inequality holds due to (6.9) if R_0 is sufficiently large. The rest of the sum in (7.4) can be upperbounded by

$$5g^2 w = 5T^{2\gamma-\omega} < 0.5T^{\tau\delta-1},$$

if R_0 is sufficiently large, showing $q^* - q \leqslant \Delta^*\overline{p}$.

For sufficiently large R_0, we have $\Delta^{**}\overline{p}^* < 0.5\Delta^*\overline{p}$. Indeed, this says $T^{(\tau\delta-1)(\tau-1)} < 0.5$. Hence using (7.2) we have

$$\Delta^{**}\overline{p}^* \leqslant 0.5\Delta^*\overline{p} \leqslant 0.5(0.1-q) - 0.5\Delta^*\overline{p} \leqslant 0.5(0.1-q) - 0.5(q^*-q) = 0.5(0.1-q^*).$$

This implies that if (7.2) holds for $\mathcal{M}$ then it also holds for $\mathcal{M}^*$.

For inequality (7.3), the scale-up definition (4.2) says $\sigma^* - \sigma = \Lambda g/f$. The inequality $g^*/f^* < 0.5g/f$ is guaranteed if R_0 is large. From here, we can conclude the proof as for q. □

7.4. **Lower bounds on holes.** We will make use of the following estimate.

Lemma 7.8. *Let $(a_0,\ b_0], (a_1,\ b_1]$ be intervals with length $\leqslant 7\Delta^*$. Suppose that X is fixed in such a way that $(a_0,\ b_0]$ is a hop of X, and $Y(a_1)$ is fixed arbitrarily. Then the (conditional) probability that the rectangle $Q = [a_0,\ b_0] \times (a_1,\ b_1]$ is a hop in $\mathcal{M}$ is at least 0.75.*

Proof. According to Condition 3.5.3c, the probability that one of the two cleanness conditions is not satisfied is at most 0.2. The probability that a potential horizontal wall of $\mathcal{M}$ is contained in Q is at most $7\Delta^*\overline{p} = 7T^{\tau\delta-1}$. The probability that a trap is contained in Q is at most $(7\Delta^*)(7\Delta^*+1)w < 56T^{2\tau\delta-\omega}$. If R_0 is sufficiently large then the sum of these two terms is at most 0.05. □

Lemma 7.9. *For emerging walls, the fitting holes satisfy Condition 3.5.3d if R_0 is sufficently large.*

Proof. Recall Condition 3.5.3d applied to the present case. Let $u \leqslant v < w, a$ be given with $v - u \leqslant 12\Delta^*$, and define $b = a + \lceil (v-u)/2 \rceil$, $c = a + (v-u) + 1$. Assume that $Y = y$ is fixed in such a way that v is left-clean, the interval $(u,\ v]$ contains no walls, and B is a horizontal wall of the emerging kind with body $(v,\ w]$. Let $E^* = E^*(u,v,w;\ a)$ be defined as after (5.2). Let $k \in \{1,\ldots,m\}$. We will prove

$$\mathbf{P}\big(\, E^* \mid X(a) = k,\ Y = y \,\big) \geqslant (c-b)^{\chi} h(\hat{R}).$$

Recall the definition of emerging walls in part 5 of the scale-up construction. The condition at the end says, in our case, that either $(v, w]$ is a hop of Y or it can be partitioned into a light (horizontal) wall $(v_1, v_2]$ of some rank r, and two hops surrounding it: so, $v \leqslant v_1 < v_2 \leqslant w$. Without loss of generality, assume this latter possibility. Let

$$a_1 = b + (v_1 - v), \quad c' = c + \Delta^*, \quad I = (a, c'].$$

Let $\mathcal{F}$ be the event that

(a) $Q = \mathrm{Rect}^{\rightarrow}((a,u),(b,v))$ potentially contains no vertical walls of $\mathcal{M}$ or $\mathcal{M}^*$, and (b,v) is clean for $\mathcal{M}^*$ in Q. If also u is clean in $(u, v]$ for $\mathcal{M}^*$ then (a,u) is clean for $\mathcal{M}^*$ in Q. $\mathrm{Rect}^{\rightarrow}((b,v),(a_1,v_1))$ is a hop.

(b) For an arbitrary $t \in (a_1, a_1 + \Delta]$, let

$$t' = t + (w - v_2).$$

We require that event $E(v_1, v_1, v_2; a_1)$ is realized with some hole $(a_1, t]$, and the rectangle $\mathrm{Rect}^{\rightarrow}((t,v_2),(t',w))$ is a hop.

1. Event $\mathcal{F}$ implies event E^*.

   Proof. Assume that $\mathcal{F}$ holds. Rectangle $\mathrm{Rect}((a,u),(b,v))$ has the necessary inner cleanness properties: it remains to show $(b,v) \rightsquigarrow (t',w)$. We have $(b,v) \rightsquigarrow (a_1,v_1)$ since $\mathrm{Rect}((b,v),(a_1,v_1))$ is a hop. For similar reasons, $(t,v_2) \rightsquigarrow (t',w)$. Also, since $(a_1, t]$ is a hole through $(v_1, v_2]$, we have $(a_1,v_1) \rightsquigarrow (t,v_2)$.

2. We have $\mathbf{P}(\mathcal{F}) \geqslant 0.75^3 c_3 T^{-\chi\tau}$.

   Proof. Without loss of generality, let us confine us to the case when u is clean in $(u, v]$ for $\mathcal{M}^*$. The condition (a) in the definition of $\mathcal{F}$ is coming from two rectangles with disjoint projections, therefore by the method used throughout the paper, we can multiply their probability lower bounds, which are given as 0.75 by Lemma 7.8.

   Condition (b) also refers to an event with a projection disjoint from the previous ones. The probability of the existence of a hole is lowerbounded via Condition 3.5.3d, by $h(r) \geqslant h(R^*) = c_3 T^{-\chi\tau}$. A reasoning similar to the proof of Lemma 5.2 shows that the whole condition (b) is satisfied at the expense of another factor 0.75.

The required lower bound of Condition 3.5.3d is

$$(c-b)^{\chi} h(\hat{R}) \leqslant (6\Delta^*)^{\chi} h(\hat{R}) = (6T^{\tau\delta})^{\chi} h(\tau' R) = c_3 6^{\chi} T^{\chi(\tau\delta - \tau')} < 0.75^3 c_3 T^{-\chi\tau}$$

if R_0 is sufficiently large, due to (6.14). □

Lemma 7.10. *After choosing c_1, c_0, R_0 sufficiently large in this order, the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery: then every compound wall satisfies the hole lower bound condition 3.5.3d.*

Proof.

1. Recall what is required.

   Let $u \leqslant v_1 < w_2, a$ be given with $v - u \leqslant 12\Delta^*$, and define

$$b = a + \lceil (v_1 - u)/2 \rceil, \qquad c = a + (v_1 - u) + 1.$$

   Assume that $Y = y$ is fixed in such a way that $(u, v_1]$ is a hop of Y in $\mathcal{M}^*$, and that there is a compound horizontal wall W with body $(v_1, w_2]$, and type $\langle r_1, r_2, i \rangle$, with rank

$$r = r_1 + r_2 - i$$

as in (4.12). Also, let $X(a)$ be fixed in an arbitrary way. Let $E_2 = E^*(u, v_1, w_2; a)$ as defined in (5.2). We need to prove $\mathbf{P}(E_2) \geqslant (c-b)^{\chi} h(r)$.

2. Let us apply Lemma 5.9.

   Assumption $(\Delta^*)^{\chi} h(r_i) \leqslant 0.07$ of the lemma holds since
$$h(r_i) = c_3 \lambda^{-\chi r_i} \leqslant c_3 T^{-\chi}, \quad (\Delta^*)^{\chi} h(r_i) \leqslant c_3 T^{-\chi(1-\delta\tau)}$$
   which, due to (6.9), is always smaller than 0.07 if R_0 is sufficiently large. We conclude
$$\mathbf{P}(E_2) \geqslant (c-b)^{\chi} (\lambda^i/2)^{\chi} h(r_1) h(r_2) \cdot (1 - V)$$
   with $V = 2 \cdot (24\overline{p}\Delta^* + 312\overline{w}(\Delta^*)^2)/h(r_1 \vee r_2)$.

3. Let us estimate the part of this expression before $1 - V$.

   We have, using the definition of $h(r)$ in (3.4):
$$(\lambda^i/2)^{\chi} h(r_1) h(r_2) = 2^{-\chi} c_3^2 \lambda^{-r\chi},$$
$$(c-b)^{\chi} (\lambda^i/2)^{\chi} h(r_1) h(r_2) \geqslant 2^{-\chi} c_3 (c-b)^{\chi} h(r) > 2(c-b)^{\chi} h(r)$$
   if c_3 is sufficiently large.

4. To complete the proof, we show that for large enough R_0 we have $1 - V \geqslant 0.5$.

   From Lemma 6.1 we have $r_1 \vee r_2 \leqslant \overline{\tau} R$, giving
$$h(r_1 \vee r_2) = c_3 \lambda^{-(r_1 \vee r_2)\chi} \geqslant c_3 \lambda^{-\chi \overline{\tau} R} = c_3 T^{-\chi \overline{\tau}}.$$
   Let us estimate both parts of V:
$$\overline{p}\Delta^*/h(r_1 \vee r_2) \leqslant c_3^{-1} T^{\chi\overline{\tau} + \tau\delta - 1}, \quad \overline{w}(\Delta^*)^2/h(r_1 \vee r_2) \leqslant c_3^{-1} T^{\chi\overline{\tau} + 2\tau\delta - \omega}.$$
   Conditions (6.16)-(6.17) imply that V can be made arbitrarily small if R_0 is sufficiently large.

□

In order to prove the hole lower bound condition for $\mathcal{M}^*$, there is one more case to consider.

Lemma 7.11. *After choosing c_1, c_0, R_0 sufficiently large in this order, the following holds. Assume that $\mathcal{M} = \mathcal{M}^k$ is a mazery: then every wall of $\mathcal{M}^{k+1}$ that is also a heavy wall of $\mathcal{M}^k$ satisfies the hole lower bound condition 3.5.3d.*

Proof. Recall Condition 3.5.3d applied to the present case. Let $u \leqslant v < w, a$ be given with $v - u \leqslant 12\Delta^*$, and define $b = a + \lceil (v-u)/2 \rceil$, $c = a + (v-u) + 1$. Assume that $Y = y$ is fixed in such a way that v is left-clean, the interval $(u, v]$ contains no walls, and B is a horizontal wall of $\mathcal{M}$ with body $(v, w]$, with rank $r \geqslant R$. Assume also that $X(a)$ is fixed arbitrarily. Let $E^* = E^*(u, v, w; a)$ be defined as after (5.2). We will prove
$$\mathbf{P}(E^* \mid Y = y) \geqslant (c-b)^{\chi} h(r).$$
Suppose first $v - u \leqslant 12\Delta$. Then the fact that $\mathcal{M}^k$ is a mazery implies
$$\mathbf{P}(E \mid Y = y) \geqslant (c-b)^{\chi} h(r).$$
In this case, however, the event E implies E^*. Indeed, the difference between E^* and E is only that E^* requires the cleanness for $\mathcal{M}^*$ and also absence of walls and traps for $\mathcal{M}^*$ in the rectangle Q in question. Compound walls and traps are excluded since their components are already excluded by the event E. And missing-hole traps, or emerging walls are simply too large to fit into Q.

It remains therefore to check the case $v - u > 12\Delta$. For this case, Lemma 5.3 says

$$\mathbf{P}(E^*) \geqslant 0.5 \wedge (1.1(c-b)^\chi h(r)) - U$$

with $U = 24\overline{p}\Delta^* + 312\overline{w}(\Delta^*)^2$. The operation $0.5\wedge$ can be omitted since $1.1(c-b)^\chi h(r) \leqslant 0.5$. Indeed, $c - b \leqslant 7\Delta^*$ implies

$$1.1(c-b)^\chi h(r) \leqslant 7.7 c_3 \lambda^{R\tau\delta\chi} \lambda^{-r\chi} = 7.7 c_3 \lambda^{\chi(R\tau\delta - r)}.$$

It follows from (6.16) that $\tau\delta < 1$. Since $r \geqslant R$, the right-hand side can be made < 0.5 for large enough R_0. Now, we have

$$1.1(c-b)^\chi h(r) - U \leqslant (c-b)^\chi h(r)(1.1 - U/h(r)).$$

The part subtracted from 1.1 is less than 0.1 if R_0 is sufficiently large, by the same argument as the estimate of V at the end of the proof of Lemma 7.10. □

8. The approximation lemma

The crucial combinatorial step in proving the main lemma is the following.

Lemma 8.1 (Approximation)**.** *The reachability condition 3.8 holds for $\mathcal{M}^*$ if R_0 is sufficently large.*

The name suggest to view our renormalization method as successive approximations: the lemma shows reachability in the absence of some less likely events (traps walls, and uncleanness in the corners of the rectangle).

The present section is taken up by the proof of this lemma. Recall that we have a bottom-open or left-open or closed rectangle Q with starting point u and endpoint v with

$$\operatorname{minslope}(u, v) \geqslant \sigma^* = \sigma + \Lambda g/f.$$

Denote $u = (u_0, u_1)$, $v = (v_0, v_1)$. We require Q to be a hop of $\mathcal{M}^*$. Thus, the points u, v are clean for $\mathcal{M}^*$ in Q, and Q contains no traps and potentially no walls of $\mathcal{M}^*$. Since reachability depends only on the inside of Q, we can choose X, Y outside of I_0, I_1 in such a way that Q actually contains no walls of $\mathcal{M}^*$.

We have to show $u \rightsquigarrow v$. Without loss of generality, assume

$$Q = I_0 \times I_1 = \operatorname{Rect}^{\varepsilon_0}(u, v)$$

with $|I_1| \leqslant |I_0|$, where $\varepsilon_0 = \rightarrow, \uparrow$ or nothing.

8.1. Walls and trap covers.

Let us determine the properties of the set of walls in Q.

Lemma 8.2. *Under conditions of Lemma 8.1, the following holds.*

(a) *For $d = 0, 1$, for some $n_d \geqslant 0$, there is a sequence $W_{d,1}, \dots, W_{d,n}$ of dominant light neighbor walls $\mathcal{M}$ separated from each other by external hops of $\mathcal{M}$ of size $\geqslant f$, and from the ends of I_d (if $n_d > 0$) by hops of $\mathcal{M}$ of size $\geqslant f/3$.*

(b) *For every wall W of $\mathcal{M}$ occurring in I_d, for every subinterval J of size g of the hops between and around walls of Z_{1-d} such that J is at a distance $\geqslant g + 7\Delta$ from the ends of I_d, there is an outer rightward- or upward-clean hole fitting W, its endpoints at a distance of at least Δ from the endpoints of J.*

Proof. This is a direct consequence of Lemmas 4.3 and 4.5. □

From now on, in this proof, whenever we mention a *wall* we mean one of the walls $W_{d,i}$, and whenever we mention a trap then, unless said otherwise, we mean only traps of $\mathcal{M}$ not intersecting any of these walls.

Let us limit the places where traps can appear in Q. A set of the form $I_0 \times J$ with $|J| \leqslant 4\Delta$ containing the starting point of a trap of $\mathcal{M}$ will be called a *horizontal trap cover*. Vertical trap covers are defined similarly.

In the following lemma, when we talk about the distance of two traps, we mean the distance of their starting points.

Lemma 8.3 (Trap Cover)**.** *Let T_1 be a trap of $\mathcal{M}$ contained in Q. Then there is a horizontal or vertical trap cover $U \supseteq T_1$ such that the starting point of every other trap in Q is either contained in U or is at least at a distance $f - \Delta$ from that of T_1. If the trap cover is vertical, it intersects none of the vertical walls $W_{0,i}$; if it is horizontal, it intersects none of the horizontal walls $W_{1,j}$.*

Proof. Let (a_1, b_1) be the starting point of T_1. If there is no trap $T_2 \subseteq Q$, with starting point (a_2, b_2), closer than $f - \Delta$ to T_1, such that $|a_2 - a_1| \geqslant 2\Delta$, then $U = [a_1 - 2\Delta,\ a_1 + 2\Delta] \times I_1$ will do. Otherwise, let T_2 be such a trap and let $U = I_0 \times [b_1 - 2\Delta,\ b_1 + 2\Delta]$. We have $|b_2 - b_1| < \Delta$, since otherwise T_1 and T_2 would form together an uncorrelated compound trap, which was excluded.

Consider now a trap $T_3 \subseteq Q$, with starting point (a_3, b_3), at a distance $< f - \Delta$ from (a_1, b_1). We will show $(a_3, b_3) \in U$. Suppose it is not so: then we have $|a_3 - a_1| < \Delta$, otherwise T_1 and T_3 would form an uncorrelated compound trap. Also, the distance of (a_2, b_2) and (a_3, b_3) must be at least f, since otherwise they would form an uncorrelated compound trap. Since $|a_2 - a_1| < f - \Delta$ and $|a_3 - a_1| < \Delta$, we have $|a_2 - a_3| < f$. Therefore we must have $|b_2 - b_3| \geqslant f$. Since $|b_2 - b_1| < \Delta$, it follows $|b_3 - b_1| > f - \Delta$, so T_3 is at a distance at least $f - \Delta$ from T_1, contrary to our assumption.

If the trap cover thus constructed is vertical and intersects some vertical wall, just decrease it so that it does not intersect any such walls. Similarly with horizontal trap covers. □

Define, for a point $a = (a_0, a_1)$:

$$d(a) = (a_1 - u_1) - \mathrm{slope}(u, v)(a_0 - u_0)$$

to be the distance of a above the diagonal of Q, then we have, for $w = (x, y)$, $w' = (x', y')$:

$$d(w') - d(w) = y' - y - \mathrm{slope}(u, v)(x' - x). \tag{8.1}$$

We define the strip

$$C^{\varepsilon}(u, v, h_1, h_2) = \{\, w \in \mathrm{Rect}^{\varepsilon}(u, v) : h_1 < d(w) \leqslant h_2 \,\},$$

a channel of vertical width $h_2 - h_1$ parallel to the diagonal of $\mathrm{Rect}^{\varepsilon}(u, v)$.

Lemma 8.4. *Assume that points u, v are clean for $\mathcal{M}$ in $Q = \mathrm{Rect}^{\varepsilon}(u, v)$, with*

$$\mathrm{slope}(u, v) \geqslant \sigma + 6g/f.$$

If $C = C^{\varepsilon}(u, v, -g, g)$ contains no traps or walls of $\mathcal{M}$ then $u \rightsquigarrow v$.

Proof. If $|I_0| < g$ then there is no trap in Q, therefore we are done. Suppose that $|I_0| \geqslant g$. Let

$$n = \left\lceil \frac{|I_0|}{0.9g} \right\rceil, \quad h = |I_0|/n.$$

Then $g/2 \leqslant h \leqslant 0.9g$. Indeed, the second inequality is immediate. For the first one note that h is a monotonically decreasing function of n, and if $n \leqslant 2$, we have $g \leqslant |I_0| = nh \leqslant 2h$. For $i = 1, 2, \dots, n-1$, let

$$a_i = u_0 + ih, \quad b_i = u_1 + ih \cdot \mathrm{slope}(u, v), \quad w_i = (a_i, b_i), \quad S_i = w_i + [-\Delta, 2\Delta]^2.$$

Let us show $S_i \subseteq C$. For all elements w of S_i, we have $|d(w)| \leqslant 2\Delta$, and we know $2\Delta < g$ from (4.1). To see that $S_i \subseteq \mathrm{Rect}^{\varepsilon}(u, v)$, we need (from the worst case $i = n-1$) $\mathrm{slope}(u, v)h > 2\Delta$. Using (4.1) and the assumptions of the lemma:

$$\frac{2\Delta}{h} \leqslant \frac{2\Delta}{g/2} \leqslant 4\Delta/g \leqslant 4g/f \leqslant \mathrm{slope}(u, v).$$

By Remark 3.6.1, there is a clean point $w'_i = (a'_i, b'_i)$ in the middle third $w_i + [0, \Delta]^2$ of S_i. Let $w'_0 = u$, $w'_n = v$. By their definition, each rectangle $\mathrm{Rect}^{\varepsilon}(w'_i, w'_{i+1})$ has size $< 0.9g + 2\Delta < g$. They fall into the channel C and hence none of them contains a trap.

Let us show $\mathrm{minslope}(w'_i, w'_{i+1}) \geqslant \sigma$: this will imply that $w'_i \rightsquigarrow w'_{i+1}$. It is sufficient to show $\mathrm{slope}(w'_i, w'_{i+1}) \geqslant \sigma$. Let $x \geqslant h - \Delta > g/3$ be the horizontal projection of $w'_{i+1} - w'_i$. We have

$$\begin{aligned}\mathrm{slope}(w'_i, w'_{i+1}) &\geqslant \frac{x \cdot \mathrm{slope}(u, v) - 2\Delta}{x} = \mathrm{slope}(u, v) - 2\Delta/x \\ &\geqslant \mathrm{slope}(u, v) - 6\Delta/g \geqslant \mathrm{slope}(u, v) - 6g/f \geqslant \sigma.\end{aligned}$$

□

Let

$$H = 12, \tag{8.2}$$

$$C = C^{\varepsilon}(u, v, -3Hg, 3Hg). \tag{8.3}$$

Then (4.1) implies

$$\Lambda \geqslant 33H + 7. \tag{8.4}$$

Let us define a sequence of trap covers $U_1, U_2, \dots$ as follows. If some trap T_1 is in C, then let U_1 be a (horizontal or vertical) trap cover covering it according to Lemma 8.3. If U_i has been defined already and there is a trap T_{i+1} in C not covered by $\bigcup_{j \leqslant i} U_j$ then let U_{i+1} be a trap cover covering this new trap. To each trap cover U_i we assign a real number a_i as follows. Let (a_i, a'_i) be the intersection of the diagonal of Q and the left or bottom edge of U_i (if U_i is vertical or horizontal respectively). Let (b_i, b'_i) be the intersection of the diagonal and the left edge of the vertical wall $W_{0,i}$ introduced in Lemma 8.2, and let (c'_i, c_i) be the intersection of the diagonal and the bottom edge of the horizontal wall $W_{1,i}$. Let us define the finite set

$$\{s_1, s_2, \dots\} = \{a_1, a_2, \dots\} \cup \{b_1, b_2, \dots\} \cup \{c'_1, c'_2, \dots\}$$

where $s_i \leqslant s_{i+1}$.

Let us call the objects (trap covers or walls) belonging to the points s_i our our *obstacles*.

Lemma 8.5. *If s_i, s_j belong to the same obstacle category among the three (horizontal wall, vertical wall, trap cover), then $|s_i - s_j| \geqslant 3f/4$. Hence for every i at least one of the three numbers $(s_{i+1} - s_i)$, $(s_{i+2} - s_{i+1})$, $(s_{i+3} - s_{i+2})$ is larger than $f/4$.*

Proof. If both s_i and s_j belong to walls of the same orientation then they are farther than f from each other, since the walls from which they come are at least f apart. (For the numbers c'_i, this uses $\mathrm{slope}(u, v) \leqslant 1$.)

Suppose that both belong to the set $\{a_1, a_2, \dots\}$, say they are $a_1 \leqslant a_2$, coming from U_1 and U_2. Let (x_j, y_j) be the starting point of some trap T_j in $U_j \cap C$ (with C defined in (8.3)). If U_j is vertical then $|x_j - a_j| \leqslant 4\Delta$, and $|y_j - a'_j| \leqslant 3Hg + 4\Delta$. If U_j is horizontal then $|x_j - a_j| \leqslant (3Hg + 4\Delta)/\mathrm{slope}(u, v)$, and $|y_j - a'_j| \leqslant 4\Delta$.

Suppose that $a_2 - a_1 \leqslant 0.75f$, then also $a'_2 - a'_1 \leqslant 0.75f$. From the above estimates it follows that

$$\begin{aligned}|x_2 - x_1| \vee |y_2 - y_1| &\leqslant 0.75f + (2 \cdot 3Hg + 8\Delta)/\mathrm{slope}(u, v) \leqslant 0.75f + 2.1 \cdot 3Hf/\Lambda \\ &= f - 0.05f - (0.2 - 2.1 \cdot 3H/\Lambda)f \leqslant f - 0.05f < f - \Delta,\end{aligned}$$

where we used $\mathrm{slope}(u, v) \geqslant \sigma^* \geqslant \Lambda g/f$, (8.4) and $\Delta < 0.05f$ which follows from (4.1). But this would mean that the starting points of the traps T_j are closer than $f - \Delta$, in contradiction to Lemma 8.3. □

8.2. **Passing through the obstacles.** Let us prove a weaker form of the Approximation Lemma first.

Lemma 8.6. *Assume* $\mathrm{slope}(u, v) \leqslant 1$, $\sigma + \Lambda g/f < 1/2$, *and let* u, v *be points with*

$$\sigma + (\Lambda - 1)g/f \leqslant \mathrm{slope}(u, v). \tag{8.5}$$

Assume that the set $\{s_1, s_2, \dots\}$ *defined above consists of at most three elements, with the consecutive elements less than* $f/4$ *apart. Assume also*

$$v_0 - s_i,\ s_i - u_0 \geqslant 0.1f. \tag{8.6}$$

Then if $\mathrm{Rect}^{\rightarrow}(u, v)$ *or* $\mathrm{Rect}^{\uparrow}(u, v)$ *is a hop of* $\mathcal{M}^*$ *then* $u \rightsquigarrow v$ *holds.*

Proof. We can assume without loss of generality that there are indeed three points s_1, s_2, s_3. By Lemma 8.5, they must then come from three obstacles of different categories: $\{s_1, s_2, s_3\} = \{a, b, c'\}$ where b comes from a vertical wall, c' from a horizontal wall, and a from a trap cover.

If the index $i \in \{1, 2, 3\}$ of a trap cover is adjacent to the index of a wall of the same orientation, then this pair will be called a *parallel pair*. A parallel pair is either horizontal or vertical. It will be called a *trap-wall pair* if the trap cover comes first, and the *wall-trap pair* if the wall comes first. If we have $s_i - s_{i-1} < 1.1g$ for a vertical pair or $(s_i - s_{i-1})\mathrm{slope}(u, v) < 1.1g$ for a horizontal pair then we say that the pair is *bound*. Thus, a pair is bound if the distance between the starting edges of its obstacles is less than $1.1g$. We will call an obstacle i *free*, if it is not part of a bound pair. Consider the three disjoint channels

$$C(u, v, K, K + 2Hg),\ \text{for}\ K = -3Hg,\ -Hg,\ Hg.$$

The three lines (bottom or left edges) of the trap covers or walls corresponding to s_1, s_2, s_3 can intersect in at most two places, so at least one of the above channels does not contain such an intersection. Let K belong to such a channel. For $i \in \{1, 2, 3\}$, we shall choose points

$$w_i = (x_i, y_i), \quad w'_i = (x'_i, y'_i), \quad w''_i = (x''_i, y''_i)$$

in the channel $C(u, v, K + 2g, K + (H - 2)g)$ in such a way that w_i is on the (horizontal or vertical) line corresponding to s_i. Not all these points will be defined. The points w_i, w'_i, w''_i will always be defined if i is free. Their role in this case is the following: w'_i and w''_i are points on the two sides of the trap cover or wall with $w'_i \rightsquigarrow w''_i$. Point w_i will be on the starting edge of the obstacle, and it will direct us in locating w'_i, w''_i. However, w_i by itself will not determine w'_i, w''_i: other factors are involved. If two obstacles form a bound pair then their crossing will be determined in a single operation. For each free obstacle, we will distinguish a *forward* way of crossing (when $d(w_i)$ will be made equal to $d(w''_j)$ for some $j < i$) and a *backward* way of crossing (when $d(w_i)$ will be made equal to $d(w'_j)$ for some $j > i$).

1. Consider crossing a free obstacle s_i, assuming that w_i has been defined already.

We have cases corresponding to whether the obstacle is a trap cover or a wall, and whether it is vertical or horizontal. Backward crossings are quite similar to forward ones.

1.1. Consider crossing a trap cover s_i.

1.1.1. Assume that the trap cover is vertical.

Consider crossing a vertical trap cover forward. Let us apply Lemma 4.6 to vertical correlated traps $J \times I'$, with $J = [x_i,\, x_i + 5\Delta]$, $I' = [y_i,\, y_i + 3l_1]$. (Recall $l_1 = 7\Delta$.) The lemma is applicable since $w_i \in C(u, v, K + 2g, K + (H-2)g)$ implies $u_1 < y_i - 3l_1 - 7\Delta < y_i + 6l_1 + 7\Delta < v_1$. It implies that there is a region $[x_i,\, x_i + 5\Delta] \times [y,\, y + l_1]$ containing no traps, with $[y,\, y + l_1) \subseteq [y_i,\, y_i + 3l_1)$. Thus, there is a y in $[y_i,\, y_i + 14\Delta)$ such that $[x_i,\, x_i + 5\Delta] \times [y,\, y + 7\Delta]$ contains no traps. (In the present proof, all other arguments finding a region with no traps in trap covers are analogous, so we will not mention Lemma 4.6 explicitly again.) However, all traps must start in a trap cover, so the region $[x_i - 2\Delta,\, x_i + 6\Delta] \times [y,\, y + 7\Delta]$ contains no trap. Thus there are clean points $w'_i \in (x_i - \Delta, y + \Delta) + [0,\, \Delta]^2$ and $w''_i \in (x_i + 4\Delta, y + 5\Delta) + [0,\, \Delta]^2$. Note that $\operatorname{minslope}(w'_i, w''_i) \geqslant 1/2$, so that $w'_i \rightsquigarrow w''_i$ holds. We have, using (8.1) and $\operatorname{slope}(u, v) \leqslant 1$:

$$
\begin{aligned}
-\Delta &\leqslant x'_i - x_i \leqslant 0, & 4\Delta &\leqslant x''_i - x_i \leqslant 5\Delta,\\
\Delta &\leqslant y'_i - y_i \leqslant 16\Delta, & 5\Delta &\leqslant y''_i - y_i \leqslant 20\Delta,\\
\Delta &\leqslant d(w'_i) - d(w_i) \leqslant 17\Delta, & 0 &\leqslant d(w''_i) - d(w_i) \leqslant 20\Delta.
\end{aligned}
\tag{8.7}
$$

Consider crossing a vertical trap cover backward. There is a y in $[y_i - 21\Delta,\, y_i - 14\Delta)$ such that the region $[x_i - 2\Delta,\, x_i + 6\Delta] \times [y,\, y + 7\Delta]$ contains no trap. There are clean points $w'_i \in (x_i - \Delta, y + \Delta) + [0,\, \Delta]^2$ and $w''_i \in (x_i + 4\Delta, y + 5\Delta) + [0,\, \Delta]^2$ with $\operatorname{minslope}(w'_i, w''_i) \geqslant 1/2$, so that $w'_i \rightsquigarrow w''_i$ holds. We have

$$
\begin{aligned}
-\Delta &\leqslant x'_i - x_i \leqslant 0, & 4\Delta &\leqslant x''_i - x_i \leqslant 5\Delta,\\
-20\Delta &\leqslant y'_i - y_i \leqslant -5\Delta, & -16\Delta &\leqslant y''_i - y_i \leqslant -\Delta,\\
-20\Delta &\leqslant d(w'_i) - d(w_i) \leqslant -4\Delta, & -21\Delta &\leqslant d(w''_i) - d(w_i) \leqslant -\Delta.
\end{aligned}
$$

1.1.2. Assume that the trap cover is horizontal.

Consider crossing a horizontal trap cover forward. There is an x in $[x_i - 21\Delta,\, x_i - 7\Delta)$ such that $[x,\, x + 7\Delta] \times [y_i - 2\Delta,\, y_i + 6\Delta]$ contains no trap. Thus there are clean points $w'_i \in (x + \Delta, y_i - \Delta) + [0,\, \Delta]^2$ and $w''_i \in (x + 5\Delta, y_i + 4\Delta) + [0,\, \Delta]^2$ with $w'_i \rightsquigarrow$

w_i''. We have similarly to the above, the inequalitites

$$\begin{aligned} -20\Delta &\leqslant x_i' - x_i \leqslant -5\Delta, & -16\Delta &\leqslant x_i'' - x_i \leqslant -\Delta, \\ -\Delta &\leqslant y_i' - y_i \leqslant 0, & 4\Delta &\leqslant y_i'' - y_i \leqslant 5\Delta, \\ -\Delta &\leqslant d(w_i') - d(w_i) \leqslant 20\Delta, & 4\Delta &\leqslant d(w_i'') - d(w_i) \leqslant 21\Delta. \end{aligned}$$

Consider crossing a horizontal trap cover backward. There is an x in $[x_i,\ x_i + 14\Delta)$ such that $[x,\ x+7\Delta] \times [y_i - 2\Delta,\ y_i + 6\Delta]$ contains no trap. Thus there are clean points $w_i' \in (x+\Delta, y_i - \Delta) + [0,\ \Delta]^2$ and $w_i'' \in (x+5\Delta, y_i + 4\Delta) + [0,\ \Delta]^2$ with $w_i' \rightsquigarrow w_i''$. We again have

$$\begin{aligned} \Delta &\leqslant x_i' - x_i \leqslant 16\Delta, & 5\Delta &\leqslant x_i'' - x_i \leqslant 20\Delta, \\ -\Delta &\leqslant y_i' - y_i \leqslant 0, & 4\Delta &\leqslant y_i'' - y_i \leqslant 5\Delta, \\ -17\Delta &\leqslant d(w_i') - d(w_i) \leqslant 0, & -16\Delta &\leqslant d(w_i'') - d(w_i) \leqslant 5\Delta. \end{aligned}$$

1.2. Consider crossing a wall.

1.2.1. Assume that the wall is vertical.

Consider crossing a vertical wall forward. Let us apply Lemma 4.5, with $I' = [y_i,\ y_i + g]$. The lemma is applicable since $w_i \in C(u, v, K + 2g, K + (H-2)g)$ implies $u_1 \leqslant y_i - g - 7\Delta < y_i + 2g + 7\Delta < v_1$. It implies that our wall contains an outer upward-clean hole $(y_i',\ y_i''] \subseteq y_i + (\Delta,\ g - \Delta]$ passing through it. (In the present proof, all other arguments finding a hole through walls are analogous, so we will not mention Lemma 4.5 explicitly again.) Let $w_i' = (x_i, y_i')$, and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. We have

$$\begin{aligned} x_i' &= x_i, & 0 &\leqslant x_i'' - x_i \leqslant \Delta, \\ \Delta &\leqslant y_i' - y_i \leqslant y_i'' - y_i \leqslant g - \Delta, \\ \Delta &\leqslant d(w_i') - d(w_i) \leqslant g - \Delta, & 0 &\leqslant d(w_i'') - d(w_i) \leqslant g - \Delta. \end{aligned} \tag{8.8}$$

Consider crossing a vertical wall backward. This wall contains an outer upward-clean hole $(y_i',\ y_i''] \subseteq y_i + (-g + \Delta,\ -\Delta]$ passing through it. Let $w_i' = (x_i, y_i')$, and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. We have

$$\begin{aligned} x_i' &= x_i, & 0 &\leqslant x_i'' - x_i \leqslant \Delta, \\ -g + \Delta &\leqslant y_i' - y_i \leqslant y_i'' - y_i \leqslant -\Delta, \\ -g + \Delta &\leqslant d(w_i') - d(w_i) \leqslant -\Delta, & -g &\leqslant d(w_i'') - d(w_i) \leqslant -\Delta. \end{aligned}$$

1.2.2. Assume that the wall is horizontal.

Consider crossing a horizontal wall forward. Similarly to above, this wall contains an outer righwards-clean hole $(x_i',\ x_i''] \subseteq x_i + (-g + \Delta,\ -\Delta]$ passing through it. Let $w_i' = (x_i', y_i)$ and let $w_i'' = (x_i'', y_i'')$ be the point on the other side of the wall reachable from w_i'. We have

$$\begin{aligned} -g + \Delta &\leqslant x_i' - x_i \leqslant x_i'' - x_i \leqslant -\Delta, \\ y_i' &= y_i, & 0 &\leqslant y_i'' - y_i \leqslant \Delta, \\ 0 &\leqslant d(w_i') - d(w_i) \leqslant g - \Delta, & 0 &\leqslant d(w_i'') - d(w_i) \leqslant g. \end{aligned}$$

Consider crossing a horizontal wall backward. This wall contains an outer rightward-clean hole $(x'_i, x''_i] \subseteq x_i + (\Delta, g - \Delta]$ passing through it. Let $w'_i = (x'_i, y_i)$ and let $w''_i = (x''_i, y''_i)$ be the point on the other side of the wall reachable from w'_i. We have

$$\begin{aligned} &\Delta \leqslant x'_i - x_i \leqslant x''_i - x_i \leqslant g - \Delta, \\ &y'_i = y_i, && 0 \leqslant y''_i - y_i \leqslant \Delta, \\ -g + \Delta \leqslant{}& d(w'_i) - d(w_i) \leqslant 0, && -g + \Delta \leqslant d(w''_i) - d(w_i) \leqslant \Delta. \end{aligned}$$

1.3. Let us summarize some of the inequalities proved above, with

$$D = d(w) - d(w_i),$$

where w is equal to any one of the defined w'_i, w''_i.

$$\begin{aligned} \text{trap covers going forward:}\quad & -\Delta \leqslant D \leqslant 21\Delta, \\ \text{trap covers going backward:}\quad & -21\Delta \leqslant D \leqslant 5\Delta, \\ \text{walls going forward:}\quad & 0 \leqslant D \leqslant g, \\ \text{walls going backward:}\quad & -g \leqslant D \leqslant \Delta. \end{aligned} \tag{8.9}$$

Further

$$\begin{aligned} \text{vertical obstacles:}\quad & -2\Delta \leqslant x'_i - x_i \leqslant x''_i - x_i \leqslant 5\Delta, \\ \text{horizontal obstacles:}\quad & -2\Delta \leqslant y'_i - y_i \leqslant y''_i - y_i \leqslant 5\Delta, \\ \text{horizontal trap covers:}\quad & -20\Delta \leqslant x'_i - x_i \leqslant x''_i - x_i \leqslant 20\Delta, \\ \text{horizontal walls:}\quad & -g + \Delta \leqslant x'_i - x_i \leqslant x''_i - x_i \leqslant g - \Delta. \end{aligned} \tag{8.10}$$

Let $\pi_x w, \pi_y w \in \mathbb{R}$ be the X and Y projections of a point, and let $\pi_i w \in \mathbb{R}^2$ be the projection of point w onto the (horizontal or vertical) line corresponding to s_i. Then the above inequalities and (8.1) imply, with $\hat{w} = \pi_i(w) - w$ where $w = w'_i, w''_i$:

$$-5\Delta \leqslant d(\hat{w}), \pi_x \hat{w}, \pi_y \hat{w} \leqslant 5\Delta. \tag{8.11}$$

For crossing a wall we have

$$-\Delta \leqslant d(w''_i) - d(w'_i) \leqslant \Delta. \tag{8.12}$$

2. Assume that there is no bound pair: then we have $u \rightsquigarrow v$.

Proof.

2.1. Assume that there is no horizontal trap-wall pair.

We choose w_1 with $d(w_1) = d(u) + K + 3g$. For each $i > 1$ we choose w_i with $d(w_i) = d(w''_{i-1})$, and we cross each obstacle in the forward direction.

2.1.1. For all i, the points we created are inside a certain channel:

$$d(w), d(\pi_i w) \in K + [2g,\ (2H-2)g], \tag{8.13}$$

where w is any one of w_i, w'_i, w''_i.

Proof. It follows from (8.9) that, for $w \in \{w_i', w_i''\}$ we have

$$-\Delta \leqslant d(w) - d(w_i) \leqslant 21\Delta \qquad \text{for trap covers,} \tag{8.14}$$

$$0 \leqslant d(w) - d(w_i) \leqslant g \qquad \text{for walls.} \tag{8.15}$$

Since we have two walls and a trap cover, adding up $d(w_1) = K + 3g + \Delta$, inequality (8.14) once and (8.15) twice gives

$$K + 3g - \Delta \leqslant d(w_i), d(w_i'), d(w_i'') \leqslant K + 5g + 21\Delta.$$

Then (8.11) implies $K + 3g - 6\Delta \leqslant d(w), d(\pi_i w) \leqslant K + 5g + 26\Delta$, where w is any one of w_i, w_i', w_i''.

2.1.2. For $i = 2, 3$ the inequality $x_i' - x_{i-1}'' \geqslant g$ holds.

Proof. If s_{i-1}, s_i come from trap covers or walls in different orientations, then the intersection of their lines lies outside $C(u, v, K, K + 2Hg)$. Part 2.1.1 above says

$$\pi_i w_i', \pi_{i-1} w_{i-1}'' \in C(u, v, K + 2g, K + (2H - 2)g).$$

Now if two segments A, B of different orientation (horizontal and vertical) are contained in $C(u, v, K + 2g, K + (2H - 2)g)$ and are such that A is to the left of B and their lines intersect outside $C(u, v, K, K + 2Hg)$, then for any points $a \in A, b \in B$ we have $\pi_x(b - a) \geqslant 2g$. In particular, $\pi_x \pi_i w_i' - \pi_x \pi_{i-1} w_{i-1}'' \geqslant 2g$. Using (8.11) we get:

$$x_i' - x_{i-1}'' = \pi_x \pi_i w_i' - \pi_x \pi_{i-1} w_{i-1}'' + (x_i' - \pi_x \pi_i w_i') - (x_{i-1}'' - \pi_x \pi_{i-1} w_{i-1}'') \geqslant 2g - 10\Delta.$$

If s_{i-1}, s_i come from a vertical trap-wall or wall-trap pair, then freeness implies that elements of this pair are farther than $1.1g$ from each other. Then we get $x_i' - x_{i-1}'' \geqslant 1.1g - 7\Delta > g$.

If s_{i-1}, s_i come from a horizontal wall-trap pair then, using $\mathrm{slope}(u, v) \leqslant 1$ and (8.10) we have

$$x_i - x_{i-1}'' = (y_i - y_{i-1}'')/\mathrm{slope}(u, v) \geqslant y_i - y_{i-1}'' = y_i - y_{i-1} - (y_{i-1}'' - y_{i-1}) \geqslant 1.1g - 5\Delta.$$

By (8.10) we have $x_i' - x_i \geqslant -20\Delta$. Combination with the above estimate and (4.1) gives $x_i' - x_{i-1}'' \geqslant 1.1g - 25\Delta > g$.

2.1.3. Let us show $u \rightsquigarrow v$.

Proof. We defined all w_i', w_i'' as clean points with $w_i' \rightsquigarrow w_i''$ and the sets $C^{\varepsilon_i}(w_i'', w_{i+1}', -g, g)$ are trap-free, where $\varepsilon_i = \rightarrow$ for horizontal walls, $\uparrow$ for vertical walls and nothing for trap covers. If we are able to show that the minslopes between the endpoints of these sets are lowerbounded by $\sigma + 6g/f$ then Lemma 8.4 will imply $u \rightsquigarrow v$. For this, it will be sufficient to show that the slopes are lowerbounded by $\sigma + 6g/f$ and upperbounded by 2, since (4.1) implies $1/(\sigma + 6g/f) > 2$. We will make use of the following relation for arbitrary $a = (a_0, a_1)$, $b = (b_0, b_1)$:

$$\mathrm{slope}(a, b) = \mathrm{slope}(u, v) + \frac{d(b) - d(a)}{b_0 - a_0}. \tag{8.16}$$

Let us bound the end slopes first. We have

$$\mathrm{slope}(u, w_1') = \mathrm{slope}(u, v) + \frac{d(w_1')}{x_1' - u_0}, \quad \mathrm{slope}(w_3'', v) = \mathrm{slope}(u, v) - \frac{d(w_3'')}{v_0 - x_3''}. \tag{8.17}$$

By (8.13), we have $|d(w'_1)|, |d(w''_3)| \leqslant K + 2Hg \leqslant 3Hg$. By (8.6) and (4.1), we have, using also (8.10):

$$(x'_1 - u_0),\ (v_0 - x''_3) \geqslant 0.1f - g + \Delta \geqslant f/11.$$

This shows

$$|\mathrm{slope}(u, w'_1) - \mathrm{slope}(u, v)|,\ |\mathrm{slope}(w''_3, v) - \mathrm{slope}(u, v)| \leqslant 33Hg/f. \tag{8.18}$$

Using $1 \geqslant \mathrm{slope}(u, v) \geqslant \sigma + (\Lambda - 1)g/f$:

$$\begin{aligned} 2 > 1 + 33Hg/f &\geqslant \mathrm{slope}(u, w'_1),\ \mathrm{slope}(w''_3, v) \\ &\geqslant \sigma + (\Lambda - 1 - 33H)g/f \geqslant \sigma + 6g/f \end{aligned}$$

by (4.1) and (8.4).

Let us proceed to lowerbounding $\mathrm{minslope}(w''_{i-1}, w'_i)$ for $i = 2, 3$. We have, using (8.16):

$$\mathrm{slope}(w''_{i-1}, w'_i) = \mathrm{slope}(u, v) + \frac{d(w'_i) - d(w''_{i-1})}{x'_i - x''_{i-1}}. \tag{8.19}$$

Using (8.9) and Part 2.1.2 above, we get

$$-21\Delta/g \leqslant \mathrm{slope}(w''_{i-1}, w'_i) - \mathrm{slope}(u, v) \leqslant 1.$$

By the conditions of the lemma and (4.1) we have

$$\begin{aligned} \mathrm{slope}(w''_{i-1}, w'_i) &\geqslant \mathrm{slope}(u, v) - 21\Delta/g \geqslant \mathrm{slope}(u, v) - 21g/f \\ &\geqslant \sigma + (\Lambda - 22)g/f \geqslant \sigma + 6g/f, \\ \mathrm{slope}(w''_{i-1}, w'_i) &\leqslant \mathrm{slope}(u, v) + 1 < 2. \end{aligned}$$

2.2. Assume now that there is a horizonal trap-wall pair.

What has been done in part 2.1 will be repeated, going backward through $i = 3, 2, 1$ rather than forward. Thus, we choose w_3 with $d(w_3) = d(v) + (2H - 3)g$. Assuming that w'_{i+1} has been chosen already, we choose w_i with $d(w_i) = d(w'_{i+1})$, and we cross each obstacle in the backward direction.

It follows from (8.9), that for all i we have (8.13) again.

2.2.1. The inequality $x'_i - x''_{i-1} \geqslant g$ holds.

Proof. If s_{i-1} and s_i come from trap covers or walls in different orientations, then we can reason as in Part 2.1.2. If s_{i-1}, s_i come from a horizontal trap-wall pair then $d(w'_i) = d(w_{i-1})$ and $y'_i = y_i$ imply

$$x'_i - x_{i-1} = (y_i - y_{i-1})/\mathrm{slope}(u, v) \geqslant 1.1g.$$

By (8.10), we have $x_{i-1} - x''_{i-1} \geqslant -20\Delta$, hence $x'_i - x''_{i-1} \geqslant 1.1g - 20\Delta \geqslant g$.

2.2.2. We have $u \rightsquigarrow v$.

Proof. Let us estimate the minslopes as in part 2.1.3, using (8.17) again. The estimates for $\mathrm{slope}(u, w'_1)$ and $\mathrm{slope}(w''_3, v)$ are as before. We conclude for $\mathrm{minslope}(w''_{i-1}, w'_i)$ using (8.19) and Part 2.2.1 just as we did in Part 2.1.3 above.

3. Consider crossing a bound pair.

A bound trap-wall or wall-trap pair will be crossed with an approximate slope 1 rather than $\mathrm{slope}(u, v)$. We first find a big enough (size g') hole in the trap cover, and then locate a hole in the wall that allows to pass, with slope 1, through the big hole of the trap cover. There are cases according to whether we have a trap-wall pair or a wall-trap pair, and whether it is vertical or horizontal.

We will prove

$$-1.2g \leqslant d(w) - d(w_i) \leqslant 8g, \tag{8.20}$$

for w equal to any one of the defined w'_j, w''_j where $j \in \{i, i-1\}$ (wall-trap) and $j \in \{i, i+1\}$ (trap-wall). The inequalities (8.10) and (8.11) will hold also if the obstacle is within a bound pair.

3.1. Consider crossing a trap-wall pair $(i,\ i+1)$, assuming that w_i has been defined already.

3.1.1. Assume that the trap-wall pair is vertical.

Let us apply Lemma 4.6 with $j = 2$, so taking $l_2 = g' = 2.2g$, similarly to the forward crossing in Part 1.1.1. As there, we find a $y^{(1)}$ in $[y_i,\ y_i + 2g')$ such that the region $[x_i - 2\Delta,\ x_i + 5\Delta] \times [y^{(1)},\ y^{(1)} + g']$ contains no trap.

Let w_{i+1} be defined by $y_{i+1} = y^{(1)} + (s_{i+1} - s_i) + 2\Delta$. Thus, it is the point on the left edge of the wall if we intersect it with a slope 1 line from $(s_i, y^{(1)})$ and then move up 2Δ. Similarly to the forward crossing in Part 1.2.1, the wall starting at s_{i+1} contains an outer upward-clean hole $(y'_{i+1},\ y''_{i+1}] \subseteq y_{i+1} + (\Delta,\ g - \Delta]$ passing through it. Let $w'_{i+1} = (x_{i+1}, y'_{i+1})$, and let $w''_{i+1} = (x''_{i+1}, y''_{i+1})$ be the point on the other side of the wall reachable from w'_{i+1}.

Let $w = (x_i, y^{(2)})$ be defined by $y^{(2)} = y'_{i+1} - (s_{i+1} - s_i)$. Thus, it is the point on the left edge of the trap cover if we intersect it with a slope 1 line from w'_{i+1}. Then $3\Delta \leqslant y^{(2)} - y^{(1)}$, therefore $w + [-3\Delta,\ 0]^2$ contains no trap, and there is a clean point $w'_i \in w + [-2\Delta,\ -\Delta]^2$. (Point w''_i is not needed.)

Let us estimate $d(w'_i) - d(w_i)$ and $d(w''_{i+1}) - d(w_i)$. We have

$$\begin{aligned}
y^{(2)} &\in y^{(1)} + 2\Delta + (\Delta,\ g - \Delta], \\
3\Delta \leqslant y^{(2)} - y_i &= d(w) - d(w_i) \leqslant 2g' + g + \Delta, \\
-2\Delta &\leqslant d(w'_i) - d(w) \leqslant 0, \\
0 &\leqslant d(w'_{i+1}) - d(w) \leqslant 1.1g.
\end{aligned} \tag{8.21}$$

Combining the last inequalities with (8.21) gives

$$\begin{aligned}
\Delta &\leqslant d(w'_i) - d(w_i) \leqslant 2g' + g + \Delta, \\
3\Delta &\leqslant d(w'_{i+1}) - d(w_i) \leqslant 2g' + 2.1g + \Delta.
\end{aligned}$$

These and (8.12) prove (8.20) for our case. Let us show $w'_i \rightsquigarrow w'_{i+1}$. We apply Condition 3.8. It is easy to see that the rectangle $\mathrm{Rect}^{\varepsilon_i}(w'_i, w'_{i+1})$ is trap-free. Consider the slope condition. We have $1/2 \leqslant \mathrm{slope}(w'_i, w) \leqslant 2$, and $\mathrm{slope}(w, w'_{i+1}) = 1$. Hence, $1/2 \leqslant \mathrm{slope}(w'_i, w'_{i+1}) \leqslant 2$, which implies $\mathrm{minslope}(w'_i, w'_{i+1}) \geqslant \sigma + 6g/f$ as needed.

3.1.2. Assume that the trap-wall pair is horizontal.

There is an $x^{(1)}$ in $[x_i - 3g', x_i - g')$ such that the region $[x^{(1)}, x^{(1)} + g'] \times [y_i - 2\Delta, y_i + 5\Delta]$ contains no trap. Let w_{i+1} be defined by $x_{i+1} = x^{(1)} + (s_{i+1} - s_i) + 2\Delta$. The wall starting at s_{i+1} contains an outer rightward-clean hole $(x'_{i+1}, x''_{i+1}] \subseteq x_{i+1} + (\Delta, g - \Delta]$ passing through it. Let $w'_{i+1} = (x'_{i+1}, y_{i+1})$, and let $w''_{i+1} = (x''_{i+1}, y''_{i+1})$ be the point on the other side of the wall reachable from w'_{i+1}. Let $w = (x^{(2)}, y_i)$ be defined by $x^{(2)} = x'_{i+1} - (s_{i+1} - s_i)$. Then there is a clean point $w'_i \in w + [-2\Delta, -\Delta]^2$ as before. We have

$$x^{(2)} \in x^{(1)} + 2\Delta + (\Delta, g - \Delta], \quad -3g' + 3\Delta \leqslant x^{(2)} - x_i \leqslant -g' + g + \Delta.$$

Using this, $d(w) - d(w_i) = -(x^{(2)} - x_i)\mathrm{slope}(u, v)$ and $\mathrm{slope}(u, v) \leqslant 1$ we get

$$0 \leqslant d(w) - d(w_i) \leqslant 3g' - 3\Delta.$$

As in Part 3.1.1, this gives

$$\begin{aligned} -2\Delta \leqslant d(w'_i) - d(w_i) &\leqslant 3g' - 3\Delta, \\ 0 \leqslant d(w'_{i+1}) - d(w_i) &\leqslant 3g' + 1.1g - 3\Delta. \end{aligned}$$

These and (8.12) prove (8.20) for our case. We show $w'_i \rightsquigarrow w'_{i+1}$ similarly to Part 3.1.1.

3.2. Consider crossing a wall-trap pair $(i - 1, i)$, assuming that w_i has been defined already.

3.2.1. Assume that the wall-trap pair is vertical.

This part is somewhat similar to Part 3.1.1. There is a $y^{(1)}$ in $[y_i + g', y_i + 3g')$ such that the region $[x_i, x_i + 6\Delta] \times [y^{(1)} - g', y^{(1)}]$ contains no trap. Let w_{i-1} be defined by $y_{i-1} = y^{(1)} - (s_i - s_{i-1}) - 5\Delta$. The wall starting at s_{i-1} contains an outer upward-clean hole $(y'_{i-1}, y''_{i-1}] \subseteq y_{i-1} + (-g + \Delta, -\Delta]$ passing through it. We define w'_{i-1}, and w''_{i-1} accordingly. Let $w = (x_i, y^{(2)})$ where $y^{(2)} = y'_{i-1} + (s_i - s_{i-1})$. There is a clean point $w''_i \in w + (4\Delta, 4\Delta) + [0, \Delta]^2$. We have

$$y^{(2)} \in y^{(1)} - 5\Delta + [-g + \Delta, -\Delta],$$

$$g' - g - 4\Delta \leqslant y^{(2)} - y_i = d(w) - d(w_i) \leqslant 3g' - 6\Delta. \tag{8.22}$$

$$-\Delta \leqslant d(w''_i) - d(w) \leqslant 5\Delta, \tag{8.23}$$

$$-1.1g \leqslant d(w'_{i-1}) - d(w) \leqslant 0. \tag{8.24}$$

Combining the last inequalities with (8.22) gives

$$\begin{aligned} g' - 2.1g - 4\Delta \leqslant d(w'_{i-1}) - d(w_i) &\leqslant 3g' - 6\Delta, \\ g' - g - 5\Delta \leqslant d(w''_i) - d(w_i) &\leqslant 3g' - \Delta. \end{aligned}$$

These and (8.12) prove (8.20) for our case. The reachability $w''_{i-1} \rightsquigarrow w''_i$ is shown similarly to Part 3.1.1. For this note

$$\begin{aligned} y_{i-1} &\geqslant y^{(1)} - 1.1g - 5\Delta, \\ y''_{i-1} &\geqslant y^{(1)} - 2.1g - 4\Delta \geqslant y^{(1)} - 2.2g = y^{(1)} - g'. \end{aligned}$$

This shows that the rectangle $\mathrm{Rect}^{\varepsilon}(w''_{i-1}, w''_i)$ is trap-free. The bound $1/2 \leqslant \mathrm{minslope}(w''_{i-1}, w''_i)$ is easy to check.

3.2.2. Assume that the wall-trap pair is horizontal.

This part is somewhat similar to Parts 3.1.2 and 3.2.1. There is an $x^{(1)}$ in $[x_i - 2g', x_i)$ such that the region $[x^{(1)} - g', x^{(1)}] \times [y_i, y_i + 6\Delta]$ contains no trap. Let w_{i-1} be defined by $x_{i-1} = x^{(1)} - (s_i - s_{i-1}) - 5\Delta$. The wall starting at s_{i-1} contains an outer rightward-clean hole $(x'_{i-1}, x''_{i-1}] \subseteq x_{i-1} + (-g + \Delta, -\Delta]$ passing through it. We define w'_{i-1}, w''_{i-1} accordingly. Let $w = (x^{(2)}, y_i)$ where $x^{(2)} = x'_{i-1} + (s_i - s_{i-1})$. There is a clean point $w''_i \in w + (4\Delta, 4\Delta) + [0, \Delta]^2$. We have

$$x^{(2)} \in x^{(1)} - 5\Delta + (-g + \Delta, -\Delta], \quad -2g' - g - 4\Delta \leqslant x^{(2)} - x_i \leqslant -6\Delta.$$

This gives $0 \leqslant d(w) - d(w_i) \leqslant 2g' + g + 4\Delta$. Combining with (8.23) and (8.24) which holds just as in Part 3.2.1, we get

$$\begin{aligned} -1.1g \leqslant d(w'_{i-1}) - d(w_i) &\leqslant 2g' + g + 4\Delta, \\ -\Delta \leqslant d(w''_i) - d(w_i) &\leqslant 2g' + g + 9\Delta. \end{aligned}$$

These and (8.12) prove (8.20) for our case. The reachability $w''_{i-1} \rightsquigarrow w''_i$ is shown similarly to Part 3.2.1.

4. Assume that there is a bound pair: then $u \rightsquigarrow v$.

Proof. We define w_i with $d(w_i) = d(u) + K + 5g$ if we have a trap-wall pair $(i, i+1)$ or a wall-trap pair $(i-1, i)$. Note that the third obstacle, outside the bound pair, is a wall.

4.1. Assume that the bound pair is $(1, 2)$.

4.1.1. Assume that we have a trap-wall pair.

We defined

$$d(w_1) = d(u) + K + 5g, \tag{8.25}$$

further define w'_1, w''_2 as in Part 3.1, and w_3, w'_3, w''_3 as in Part 2.1. Let us show that these points do not leave $C(u, v, K + 2g, K + (2H - 2)g)$: for all i, we have (8.13). Inequalities (8.20) imply

$$-1.2g \leqslant d(w'_1) - d(w_1),\ d(w''_2) - d(w_1) = d(w_3) - d(w_1) \leqslant 8g,$$

while inequalities (8.9) imply $0 \leqslant d(w'_3) - d(w_3),\ d(w''_3) - d(w_3) \leqslant g$. Combining with (8.25) gives for $w \in \{w'_1, w''_2, w'_3, w''_3\}$:

$$K + 3.8g \leqslant d(w) - d(u) \leqslant K + 13g < K + (2H - 2)g$$

according to (8.2).

We have shown $w'_1 \rightsquigarrow w''_2$ and $w'_3 \rightsquigarrow w''_3$, further such that the sets $C^{\varepsilon_1}(u, w'_1, -g, g)$, $C^{\varepsilon_2}(w''_2, w'_3, -g, g)$ and $C^{\varepsilon_3}(w''_3, v, -g, g)$ for the chosen ε_i are trap-free. It remains to show that the minslopes between the endpoints of these sets are lowerbounded by $\sigma + 6g/f$: then a reference to Lemma 8.4 will imply $u \rightsquigarrow v$. This is done for all three pairs (u, w'_1), (w''_3, v) and (w''_2, w'_3) just as in Part 2.1.3.

4.1.2. Assume that we have a wall-trap pair.

We defined $d(w_2) = d(u) + K + 5g$; we further define w'_1, w''_2 as in Part 3.2, and w_3, w'_3, w''_3 as in Part 2.1. The proof is finished similarly to Part 4.1.1.

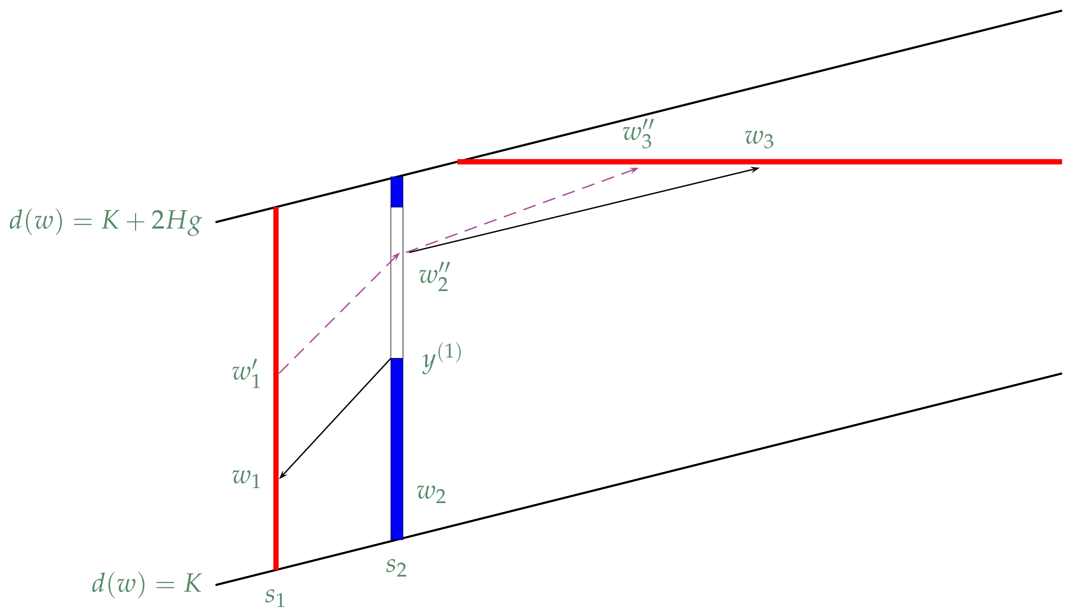

FIGURE 7. Approximation Lemma: the case of a bound wall-trap pair $(1, 2)$. The arrows show the order of selection. First w_2 is defined. Then the trap-free segment of size g' above w_2 is found. Its starting point is projected back by a slope 1 line onto the vertical wall to find w_1 after moving up by 2Δ. The hole starting with w_1' is found within g above w_1. Then w_2'' is found near the back-projection of w_1''. Then w_2'' is projected forward, by a slope(u, v) line onto the horizontal wall, to find w_3. Finally, the hole ending in w_3'' is found within g backward from w_3.

4.2. Assume now that the bound pair is $(2, 3)$.

4.2.1. Assume that we have a trap-wall pair.

We defined $d(w_2) = d(u) + K + 5g$; we further define w_2', w_3'' as in Part 3.1, and w_1, w_1', w_1'' as in Part 2.2. Let us show that these points do not leave $C(u, v, K + 2g, K + (2H - 2)g)$. Inequalities (8.20) imply

$$-1.2g \leqslant d(w_2') - d(w_2) = d(w_1) - d(w_2),\ d(w_3'') - d(w_2) \leqslant 8g,$$

while inequalities (8.9) imply $-g \leqslant d(w_1') - d(w_1),\ d(w_1'') - d(w_1) \leqslant \Delta$. Combining with (8.25) gives for $w \in \{w_1', w_1'', w_2', w_3''\}$:

$$K + 2.8g \leqslant d(w) - d(u) \leqslant K + 13g + \Delta < K + (2H - 2)g.$$

Reachability is proved as in Part 4.1.1.

4.2.2. Assume that we have a wall-trap pair.

We defined $d(w_3) = d(u) + K + 5g$; we further define w_2', w_3'' as in Part 3.2, and w_1, w_1', w_1'' as in Part 2.2. The proof is finished as in Part 4.2.1.

$\square$

Proof of Lemma 8.1(Approximation). For each pair of numbers s_i, s_{i+1} with $s_{i+1} - s_i \geqslant f/4$, define its midpoint $(s_i + s_{i+1})/2$. Let $t_1 < t_2 < \cdots < t_n$ be the sequence of all these midpoints. Let $t_0 = u_0$, $t_{n+1} = v_0$. Let us define the square

$$S_i = (t_i,\, u_1 + \mathrm{slope}(u,v)(t_i - u_0)) + [0,\, \Delta] \times [-\Delta,\, 0].$$

By Remark 3.6.1, each of these squares contains a clean point p_i.

1. For $1 \leqslant i < n$, the rectangle $\mathrm{Rect}(p_i, p_{i+1})$ satisfies the conditions of Lemma 8.6, and therefore $p_i \rightsquigarrow p_{i+1}$. The same holds also for $i = 0$ if the first obstacle is a wall, and for $i = n$ if the last obstacle is a wall.

   Proof. By Lemma 8.5, there are at most three points of $\{s_1, s_2, \dots\}$ between t_i and t_{i+1}. Let these be $s_{j_i}, s_{j_i+1}, s_{j_i+2}$. Let t'_i be the x coordinate of p_i, then $0 \leqslant t'_i - t_i \leqslant \Delta$. The distance of each t'_i from the closest point s_j is at most $f/8 - \Delta \geqslant 0.1f$. It is also easy to check that p_i, p_{i+1} satisfy (8.5), so Lemma 8.6 is indeed applicable.

2. We have $u \rightsquigarrow p_1$ and $p_n \rightsquigarrow v$.

   Proof. If $s_1 \geqslant 0.1f$, then the statement is proved by an application Lemma 8.6, so suppose $s_1 < 0.1f$. Then s_1 belongs to a trap cover. By the same reasoning used in Lemma 8.5, we then find that $s_2 - s_1 > f/4$, and therefore there is only s_1 between u and t_1, and also $t_1 - s_1 > 0.1f$.

   If the trap cover belonging to s_1 is at a distance $\geqslant g - 6\Delta$ from u then we can pass through it, going from u to p_1 just like in Part 2 of the proof of Lemma 8.6. If it is closer than $g - 6\Delta$ then the fact that u is clean in $\mathcal{M}^*$ implies that it contains a large trap-free region where it is easy to get through.

   The relation $p_n \rightsquigarrow v$ is shown similarly.

□

9. Proof of Lemma 2.5 (Main)

The construction of $\mathcal{M}^k$ is complete by the algorithm of Section 4, and the fixing of all parameters in Section 6. We will prove, by induction, that every structure $\mathcal{M}^k$ is a mazery. We already know that the statement is true for $k = 1$. Assuming that it is true for all $i \leqslant k$, we prove it for $k + 1$. Condition 3.5.1 is satisfied according to Lemma 4.1. Condition 3.5.2 has been proved in Lemmas 4.2 and 4.7.

Condition 3.5.3a has been proved in Lemma 7.1. Condition 3.5.3b has been proved in Lemma 7.4. Condition 3.5.3c has been proved in Lemma 7.7. Condition 3.5.3d has been proved in Lemmas 7.9, 7.10 and 7.11.

Condition 3.8 is satisfied via Lemma 8.1 (the Approximation Lemma). There are some conditions on the parameters f, g, Δ used in this lemma. Of these, condition (4.1) holds if R_0 is sufficiently large; the rest follows from our choice of parameters and Lemma 7.7.

Let us show that the conditions preceding the Main Lemma 2.5 hold. Condition 2.1 is implied by Condition 3.8. Condition 2.2 is implied by Remark 3.6.1. Condition 2.4 follows immediately from the definition of cleanness.

Finally, inequality (2.3) of the Main Lemma follows from Lemma 7.6.

10. Conclusions

It was pointed out in [4] that the clairvoyant demon does not really have to look into the infinite future, it is sufficient for it to look as far ahead as maybe n^3 when scheduling $X(n), Y(n)$. This is also true for the present paper.

Another natural question is: how about three independent random walks? The methods of the present paper make it very likely that three independent random walks on a very large complete graph can also be synchronized, but it would be nice to have a very simple, elegant reduction.

It is seems possible to give a common generalization of the model of the paper [4] and the present paper. Let us also mention that we have not used about the independent Markov processes X, Y the fact that they are homogenous: the transition matrix could depend on i. We only used the fact that for some small constant w, the inequality $\mathbf{P}(X(i+1) = j \mid X(i) = k) \leqslant w$ holds for all i, j, k (and similarly for Y).

What will strike most readers as the most pressing open question is how to decrease the number of elements of the smallest graph for which scheduling is provably possible from super-astronomical to, say, 5. Doing some obvious optimizations on the present renormalization method is unlikely to yield impressive improvement: new ideas are needed.

Maybe computational work can find the better probability thresholds needed for renormalization even on the graph K_5, introducing supersteps consisting of several single steps.

Acknowledgement. I am grateful to Peter Winkler and Márton Balázs for valuable comments, to the anonymous referee of paper [4] and emphatically to the referee of the present paper for repeated very careful reading and thorough criticism. I also thank János Komlós and Endre Szemerédi, who first challenged me to solve this problem.

References

[1] P.N. Balister, B. Bollobás, and A.N. Stacey. Dependent percolation in two dimensions. *Probab. Theory Relat. Fields*, **117** #4 (2000), 495–513 1.3

[2] D. Coppersmith, P. Tetali, and P. Winkler. Collisions among random walks on a graph. *SIAM J. Discrete Math*, **6** (1993), no. 3, 363–374. 1.1

[3] P. Gács. The clairvoyant demon has a hard task. *Combinatorics, Probability and Computing*, 9:421–424, 2000. 1.3

[4] P. Gács. Compatible sequences and a slow Winkler percolation. math.PR/0011008 1.2, 1.3, 2.3, 1, 2, 10, 10

[5] P. Winkler. Dependent percolation and colliding random walks. *Random Structures & Algorithms*, **16** #1 (2000), 58–84. 1.3

[6] P. Winkler. Personal communication. 1.2

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