Quasi-Random Hypergraphs Revisited

Fan Chung*
University of California, San Diego, California; e-mail: fan@ucsd.edu

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ABSTRACT: The quasi-random theory for graphs mainly focuses on a large equivalent class of graph properties each of which can be used as a certificate for randomness. For k-graphs (i.e., k-uniform hypergraphs), an analogous quasi-random class contains various equivalent graph properties including the *k-discrepancy property* (bounding the number of edges in the generalized induced subgraph determined by any given $(k-1)$-graph on the same vertex set) as well as the *k-deviation property* (bounding the occurrences of "octahedron", a generalization of 4-cycle). In a 1990 paper (Chung, Random Struct Algorithms 1 (1990) 363-382), a weaker notion of l-discrepancy properties for k-graphs was introduced for forming a nested chain of quasi-random classes, but the proof for showing the equivalence of l-discrepancy and l-deviation, for $2 \leq l < k$, contains an error. An additional parameter is needed in the definition of discrepancy, because of the rich and complex structure in hypergraphs. In this note, we introduce the notion of (l, s)-discrepancy for k-graphs and prove that the equivalence of the (k, s)-discrepancy and the s-deviation for $1 \leq s \leq k$. We remark that this refined notion of discrepancy seems to point to a lattice structure in relating various quasi-random classes for hypergraphs. Random Struct. Alg., 40, 39–48, 2012

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1. INTRODUCTION

The study of quasi-random graphs and hypergraphs explores the relationship among properties of graphs with special emphasis of finding equivalence classes and their classifications. For graphs, there is a large equivalence class that includes the *discrepancy property* and the *deviation property* [4]. The discrepancy property for a graph G is associated with bounding the difference between the number of edges in an induced subgraph S of G and the expected number of edges in S (which is basically $|S|^2/4$ for a graph G with edge density $1/2$). The discrepancy property for G is associated with bounding the difference between the number

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of four cycles containing an even number of edges in G and those with an odd number of edges in G. To extend the study of quasi-random graphs to k-uniform hypergraphs, (or k-graphs for short), there have been numerous attempts [1,3,5,7,8]. In the effort to extend the notion of deviation to k-graphs for $k \geq 3$, there is a nested sequences of l-deviation dev_l, $2 \leq l \leq k$ which concern the counts of so-called even "octahedra" and odd octahedra on $2l$ vertices. To generalize the notion of discrepancy for a k-graphs H with vertex set V, one of the ways is to consider the l-discrepancy $\mathrm{disc}_l H$, for a fixed $l, 2 \leq l \leq k$, which concerns the maximum difference of the edge counts in subgraph of H induced by any $(l-1)$-graph G from the expected value over all G on V. In [1,3] it was shown that for a k-graph H, the property disc $H = \mathrm{disc}_k H$ and dev $H = \mathrm{dev}_k H$ are equivalent in the sense that for any ϵ there exists δ such that disc $H \leq \delta$ implies dev $H \leq \epsilon$, (denoted by disc $\Rightarrow$ dev) and the reverse direction holds as well.

To further understand the structure for k-graphs, a natural approach is to establish a nested sequence of equivalence classes. In [1], it was shown that for $2 \leq l \leq k, \mathrm{dev}_l \Rightarrow \mathrm{disc}_l$. However, the proof for $\mathrm{disc}_l \Rightarrow \mathrm{dev}_l$ contains two cases, one of which, namely for $2 \leq l < k$, contains an erroneous statement. A counterexample was given in [6]. As it turns out, the hypergraphs have a richer and more intriguing structure than previously suspected (by the author). There are further extensions of the discrepancy property which we call (l, s)-discrepancy, denoted by $\mathrm{disc}_l^{(s)} H$, for a k-graph H with vertex set V, where $2 \leq l \leq k$ and $1 \leq s \leq \binom{k}{l}$. Roughly speaking, $\mathrm{disc}_l^{(s)} H$ concerns the subgraphs S_s of H which are induced by an l-graph G on V in the sense that an edge x in $E(H)$ is in S_s if the number of l-edges in G contained in x is at least s. The previous notion of disc_l is the special case of $\mathrm{disc}_l^{(s)}$ with $s = \binom{k}{l-1}$. The paper [6] examines the case of $\mathrm{disc}_2^{(k)}$ which was then shown to belong to a large equivalence class of hypergraph properties including counting the appearances of a fixed "linear" k-graph F in H where "linear" means the restriction that any two edges in F intersect at most one vertex.

With this refined notion of (l, s)-discrepancy for k-graphs, numerous questions arise. How are various known hypergraph properties related to $\mathrm{disc}_l^{(s)}$? For example, suppose we consider a generalization of linear k-graphs. We say a k-graph F is l-linear if any two edges in F intersect at no more than l vertices. We can then define the following subgraph containment property for a k-graph H on n vertices :

P_l: For every $(l-1)$-linear k-graph F on r vertices with r vertices and t edges with $r \geq k$, the number $N_F(H)$ of labelled embeddings of F in H satisfies

$$N_F(H) = (1/2)^t n^r + o(n^r).$$

It seems plausible to conjecture that P_l is equivalent to $\mathrm{disc}_l^{(s)}$ with $s = \binom{k}{l-1}$ by extending the techniques in [6] for the case of $l = 2$. Although the above formulation is mainly for k-graph H with edge density $1/2$, a general definition for P_l with graphs with edge density p can be obtained in a straightforward manner by replacing $1/2$ by p.

There are further questions just for the case of $l = 2$. Even for the special case of $k = 3$ and $s = 2$, the discrepancy property for a 3-graph H is reduced to the following: For any subset S of vertices, the number of edges in H containing at least 2 vertices in S is about as expected. Will this property be equivalent to some modified version of the deviation property (similar to some partial "doubling" as described in [6])?

In general, for various given hypergraph properties, can they be related to the $\mathrm{disc}_l^{(s)}$ in some way? Do they form quasi-random equivalence classes? What are the hierarchy of these quasi-random classes? And, how effective are these properties to be used as certificates for randomness? To partially answer some of these questions, we show that dev_s is equivalent to $\mathrm{disc}_k^{(s)}$ for k-graphs in the remaining part of this note. Further questions and remarks concerning the lattice structure of quasi-random classes for k-graphs will be discussed in the last section.

2. A REFINED NOTION OF THE DISCREPANCY PROPERTIES FOR HYPERGRAPHS

We follow the notation in [3]. A k-uniform hypergraph $H = (V, \mu_H)$ consists of a set of V of vertices of H together with a function $\mu_H : \binom{V}{k} \to \{1, -1\}$, called the multiplicative edge function of H. The set $E(H) = \mu_H^{-1}(-1)$ is called the edge set of H. When there is no confusion, we call H a k-graph. For a given function $\mu : \binom{V}{k} \to \{1, -1\}$, denote by $\bar{\mu}$ the extension $\bar{\mu} : V^k \to \{1, -1\}$ by $\bar{\mu}(v_1, \ldots, v_k) = \mu(\{v_1, \ldots, v_k\})$ where $v_1 \ldots, v_k$ are distinct elements of V and 1 otherwise.

Definition. *The l-deviation of a k-graph $H = H(E, V)$ with $|V| = n$, denoted by* $\mathrm{dev}_l H$*, is defined by*

$$\mathrm{dev}_l H = \frac{1}{n^{k+l}} \sum_{\substack{v_i(0), v_i(1) \in V \\ 1 \le i \le l}} \sum_{\substack{w_j \in V \\ l+1 \le j \le k}} \prod_{\substack{\epsilon_i \in \{0,1\} \\ 1 \le i \le l}} \bar{\mu}_H(v_1(\epsilon_1), \ldots, v_l(\epsilon_l), w_{l+1}, \ldots, w_k)$$

where $\bar{\mu}(x) = -1$ if x is an edge in H and $\bar{\mu}(x) = 1$ otherwise.

We remark that the above definition can be generalized to focus on graphs with edge density p by defining $\mu(x) = -p$ if x is an edge in H and $\mu(x) = 1 - p$ otherwise.

Definition. *For a k-graph H and a l-graph G on the same vertex set V, we define*

$$E(H, G) = \left\{ x \in E(H) : \binom{x}{l} \subseteq E(G) \right\},$$
$$e(H, G) = k!|E(H, G)|.$$

Namely $e(H, G)$ counts the number of ordered subsets in $E(H, G)$.

Definition. *For a k-graph H on vertex set V with $|V| = n$, we define* $\mathrm{disc}_l H$ *as follows:*

$$\mathrm{disc}_l H = \frac{1}{n^k} \max_G |e(H, G) - e(\bar{H}, G)|,$$

where the maximum is taken over all $(l-1)$-graphs G on V.

It was shown in [1, 3] that

$$\mathrm{dev}_l H \ge (\mathrm{disc}_l H)^{2^l}.$$

and for $l = k$,

$$\mathrm{dev}_k H \le 4^k (\mathrm{disc}_k H)^{2^{-k}}. \tag{1}$$

For a k-graph H, we use the notation that $\text{dev}H = \text{dev}_k H$ and $\text{disc}\, H = \text{disc}_k^{(k)} H$ which [3] mainly focused on.

It would have led to quasi-random classes for hypergraphs if a similar statement as follows holds for $2 \le l < k$.

$$\text{dev}_l H \le 4^l (disc_l H)^{2^{-l}}.$$

However, this inequality is not true for $l \ne k$ as evidenced by the example given int [6]. So, a natural question is to find the 'right' equivalent discrepancy property for dev_l.

Definition. *For a k-graph H and an l-graph G, we define*

$$E_s(H,G) = \left\{ x \in E(H) : \left| \binom{x}{l} \cap E(G) \right| \ge s \right\},$$
$$e_s(H,G) = k!|E_s(H,G)|.$$

Namely $e_s(H,G)$ counts the number of ordered subsets in $E_l(H,G)$. We note that for the case of $l = k-1$ and $s = k$, we have $e(H,G) = e_k(H,G)$.

Definition. *For a k-graph H on n vertices, we define* $\text{disc}_l^{(s)} H$ *as follows:*

$$\text{disc}_l^{(s)} H = \frac{1}{n^k} \max_G |e_s(H,G) - e_s(\bar{H},G)|,$$

where the max is taken over all $(l-1)$-graphs G on V.

Note that disc H is the special case $\text{disc}_k = \text{disc}_k^{(k)}$ and disc_l is the special case $\text{disc}_l = \text{disc}_l^{(s)}$ for $s = \binom{k}{l-1}$.

We remark that the above definition can be modified to focus on graphs with density p by defining $\text{disc}_l^{(s)} H = \frac{1}{\text{vol}H} \max_G |e_s(H,G) - p \cdot e_s(\binom{V}{k}, G)|$ where G ranges over all $(l-1)$-graphs. Here vol H denotes the number of edges in H. For simplicity, we will mainly deal with the case of $p = 1/2$ here.

Although we are far from fully understanding the relationship among properties $\text{disc}_l^{(s)}$, certain implications can be derived for the case of $l = k$. To simplify the notation, we write

$$\text{disc}^{(s)} H = \text{disc}_k^{(s)} H.$$

Note that for $\text{disc}^{(s)}$, the interesting range for s is for $s \le k$.

We will prove the following two theorems to establish the equivalence implications of dev_s and $\text{disc}^{(s)}$.

3. THE s-DEVIATION PROPERTY IMPLIES THE DISCREPANCY PROPERTY $\text{disc}^{(s)}$

Theorem 1. *For a k-graph H and $2 \le s \le k$, we have*

$$\text{dev}_s H \ge (\text{disc}^{(s)} H)^{2^s}.$$

Proof. It suffices to show that for any given $(k-1)$-graph G, we have

$$\text{dev}_s H \ge (E_s(H,G) - E_s(\bar{H},G))^{2^s}.$$

This can be proved by applying the Cauchy-Schwarz inequality on selected terms repeatedly as follows. We consider

$$\begin{aligned}
\mathrm{dev}_s H &= \frac{1}{n^{k+s}} \sum_{\substack{v_i(0),v_i(1)\in V\\ 1\le i\le s}} \sum_{\substack{w_j\in V\\ s+1\le j\le k}} \prod_{\substack{\epsilon_i\in\{0,1\}\\ 1\le i\le s}} \bar{\mu}_H(v_1(\epsilon_1),\ldots,v_s(\epsilon_s),w_{s+1},\ldots,w_k)\\
&= \frac{1}{n^{k+s}} \sum_{\substack{v_i(0),v_i(1)\in V\\ 1\le i\le s-1}} \sum_{\substack{w_j\in V\\ s+1\le j\le k}} \left(\sum_{v\in V} \prod_{\substack{\epsilon_i\in\{0,1\}\\ 1\le i\le s-1}} \bar{\mu}_H(v_1(\epsilon_1),\ldots,v_{s-1}(\epsilon_{s-1}),v,w_{s+1},\ldots,w_k)\right)^2\\
&\ge \frac{1}{n^{k+s}} \sum_{\substack{w_j\in V\\ s+1\le j\le k}} \sum^{G,s}_{\substack{v_i(0),v_i(1)\in V\\ 1\le i\le s-1}} \left(\sum_{v\in V} \prod_{\substack{\epsilon_i\in\{0,1\}\\ 1\le i\le s-1}} \bar{\mu}_H(v_1(\epsilon_1),\ldots,v_{s-1}(\epsilon_{s-1}),v,w_{s+1},\ldots,w_k)\right)^2
\end{aligned}$$

where $\sum^{G,s}$ denotes a partial sum with the restriction that the $v_i(\epsilon_i)$ satisfy the property that $(v_1(\epsilon_1),\ldots,v_{s-1}(\epsilon_{s-1}),w_{s+1},\ldots,w_k)$ are edges in G for all ϵ_i. Thus we have

$$\begin{aligned}
\mathrm{dev}_s H &\ge \frac{1}{n^{k+s-2}} \sum_{\substack{w_j\in V\\ s+1\le j\le k}} \sum^{G,s}_{\substack{v_i(0),v_i(1)\in V\\ 1\le i\le s-1,}} \left(\frac{1}{n}\sum_{v\in V} \prod_{\substack{\epsilon_i\in\{0,1\}\\ 1\le i\le s-1}} \bar{\mu}_H(v_1(\epsilon_1),\ldots,v_{s-1}(\epsilon_{s-1}),v,w_{s+1},\ldots,w_k)\right)^2\\
&\ge \left(\frac{1}{n^{k+s-1}} \sum_{\substack{w_j\in V\\ s\le j\le k}} \sum^{G,s}_{\substack{v_i(0),v_i(1)\in V\\ 1\le i\le s-1,}} \prod_{\substack{\epsilon_i\in\{0,1\}\\ 1\le i\le s-1}} \bar{\mu}_H(v_1(\epsilon_1),\ldots,v_{s-1}(\epsilon_{s-1}),w_s,w_{s+1},\ldots,w_k)\right)^2.
\end{aligned}$$

We will repeat the same methods using the notation that $\sum^{G,[j,s]}$ denotes a partial sum with the restriction that the $v_i(\epsilon_i)$ satisfy the property that $(v_1(\epsilon_1),\ldots,v_{j-1}(\epsilon_{j-1}),w_j,\ldots,w_{t-1},w_{t+1},\ldots,w_k)$ are edges in G for all ϵ_i and $t\in[j,s]$. Then we have

$$\begin{aligned}
\mathrm{dev}_s H &\ge \left(\frac{1}{n^{k+s-1}} \sum_{\substack{w_j\in V\\ s\le j\le k}} \sum^{G,[s,s]}_{\substack{v_i(0),v_i(1)\in V\\ 1\le i\le s-1,}} \prod_{\substack{\epsilon_i\in\{0,1\}\\ 1\le i\le s-1}} \bar{\mu}_H(v_1(\epsilon_1),\ldots,v_{s-1}(\epsilon_{s-1}),w_s,w_{s+1},\ldots,w_k)\right)^2\\
&\ge \left(\frac{1}{n^{k+s-2}} \sum_{\substack{w_j\in V\\ s-1\le j\le k}} \sum^{G,[s-1,s]}_{\substack{v_i(0),v_i(1)\in V\\ 1\le i\le s-2,}} \prod_{\substack{\epsilon_i\in\{0,1\}\\ 1\le i\le s-2}} \bar{\mu}_H(v_1(\epsilon_1),\ldots,v_{s-1}(\epsilon_{s-2}),w_s,w_{s+1},\ldots,w_k)\right)^4\\
&\ge \cdots
\end{aligned}$$

$$\geq \left(\frac{1}{n^{k+1}} \sum_{\substack{w_j \in V \\ 2 \leq j \leq k}} \sum_{v_1(0), v_1(1) \in V}^{G,[2,s]} \prod_{\epsilon_1 \in \{0,1\}} \bar{\mu}_H(v_1(\epsilon_1), w_2, w_{s+1}, \ldots, w_k) \right)^{2^{s-1}} .$$

Therefore we have

$$\begin{aligned} \mathrm{dev}_s H &\geq \left(\frac{1}{n^{k-1}} \sum_{\substack{w_j \in V \\ 2 \leq j \leq k}} \left(\frac{1}{n} \sum_{v \in V}^{G,[2,s]} \bar{\mu}_H(v, w_2, w_{s+1}, \ldots, w_k) \right)^2 \right)^{2^{s-1}} \\ &\geq \left(\frac{1}{n^k} \sum_{\substack{w_j \in V \\ 1 \leq j \leq k}}^{G,[1,s]} \bar{\mu}_H(w_1, w_2, w_{s+1}, \ldots, w_k) \right)^{2^s} \\ &= \left(\frac{1}{n^k} (e_s(H, G) - e_s(\bar{H}, G) \right)^{2^s} \end{aligned}$$

for any $(k-1)$-graph G. Thus we conclude that

$$\mathrm{dev}_s H \geq (\mathrm{disc}^{(s)} H)^{2^s}.$$

■

4. THE DISCREPANCY PROPERTY disc$^{(s)}$ IMPLIES THE s-DEVIATION PROPERTY

Theorem 2. *For a k-graph H and $2 \leq s \leq k$, suppose that for every $(k-1)$-graph G on V,*

$$|e_s(H, G) - e_s(\bar{H}, G)| \leq \epsilon n^k.$$

Then we have

$$\mathrm{dev}_s H \leq 2^{k+s} \epsilon^{1/((k-s)(k-s+1)2^s)}. \tag{2}$$

Proof. Assume that $k \geq 3$ (since the case of $k = 2$ is well understood [4]). We will first give a relative simple example for the case of $k = 3$ and $s = 2$ before proceeding to the general case.

Suppose that for every 2-graph G on V, we have

$$|e_2(H, G) - e_2(\bar{H}, G)| \leq \epsilon n^2.$$

We wish to show

$$\mathrm{dev}_2 H \leq 32\epsilon^{1/8}.$$

For a vertex w, we consider the 2-graph H_w with edge set $E(H_w) = \{y \in \binom{V}{2} : y \cup \{w\} \in E(H)\}$. From the definition of $\mathrm{dev}_2 H$, we have

$$\mathrm{dev}_2 H = \frac{1}{n} \sum_{w \in V} \mathrm{dev}_2 H_w.$$

We consider

$$S := \{w \in V : \mathrm{dev}_2 H_w \geq 30\epsilon^{1/8}\}.$$

If $|S| \leq 2\epsilon^{1/2} n$ then

$$\mathrm{dev}_2 H \leq \frac{1}{n}\big(|S| + 30\epsilon^{1/8} n\} \leq 32\epsilon^{1/8}$$

as desired. Thus, we may assume $|S| \geq 2\epsilon^{1/2} n$.

For each $w \in S$, the fact that $\mathrm{dev}_2 H_w \geq \epsilon' = 30\epsilon^{1/8}$ implies, by the induction hypothesis using (1) for 2-graphs, that there exists a subset G_w (which can be viewed as a 1-graph on V) satisfying

$$|e(H_w, G_w) - e(\bar{H}_w, G_w)| > \delta n^2$$

where δ satisfies $\delta \geq 16^{-4}\epsilon'^4 \geq 3\epsilon^{1/2}$. Thus, there is a subset S' of S with $|S'| = \epsilon^{1/2} n$ so that either

(a) $e(H_w, G_w) \geq \frac{1}{2} e(\binom{V}{l}), G_w) + 3\epsilon^{1/2} n^2/2$ for all $w \in S'$; or
(b) $e(H_w, G_w) \leq \frac{1}{2} e(\binom{V}{l}), G_w) - 3\epsilon^{1/2} n^2/2$ for all $w \in S'$.

We will treat case (a) and omit the similar treatment for case (b).

We proceed to define the following 2-graph G on V.

$$E(G) = \{w \cup y : y \in E(G_w)\} \setminus \binom{V \setminus S'}{2}.$$

For each $x \in E_2(H, G)$, there are three possibilities:

(i) x has at least two vertices in S'. There are at most ϵn^3 such edges in $E_2(H, G)$.
(ii) x has no vertex in S'. In this case, x can not contain a pair of vertices in G, contradicting $x \in E_2(H, G)$.
(iii) x has exactly one vertex w in S'. Say, $x = \{v, u, w\}$ and $u, v \in H_w$. Therefore, we have

$$\begin{aligned} E_2(H, G) - E_2(\bar{H}_G) &\geq \sum_{w \in S'} (E(H_w, G_w) - E(\bar{H}_w, G_w)) - \epsilon n^3 \\ &\geq |S'| 3\epsilon^{1/2} n^2/2 - \epsilon n^3 \\ &\geq 2\epsilon n^2 \end{aligned}$$

which is a contradiction. Thus we have proved (2) for the case of $k = 3$.

The proof for the general k is quite similar. For a k-graph H, suppose that for every $(k-1)$-graph G on V, we have

$$|e_s(H, G) - e_s(\bar{H}, G)| \leq \epsilon n^k.$$

We wish to show

$$\mathrm{dev}_s H \leq 2^{k+s}\epsilon^{1/((k-s)(k-s+1)2^s)}.$$

For a fixed string of $k-s$ vertices, say, $w = (w_1, w_2, \ldots, w_{k-s})$, we consider edges in $E(H)$ containing w_i for $1 \leq i \leq j$. We consider the $(k-i)$-graph $H_{(w_1,\ldots,w_i)}$ with edge set $E(H_{(w_1,\ldots,w_i)}) = \{y \in \binom{V}{s} : y \cup \{w_1, \ldots, w_i\} \in E(H)\}$. From the definition of $\mathrm{dev}_s H$, we have

$$\begin{aligned}
\mathrm{dev}_s H &= \frac{1}{n}\sum_{w_1} dev_s H_{(w_1)} \\
&= \frac{1}{n^2}\sum_{w_1,w_2} dev_s H_{(w_1,w_2)} \\
&= \ldots \\
&= \frac{1}{n^{k-s}}\sum_{w=(w_{s+1},\ldots,w_k)} \mathrm{dev}_s H_w.
\end{aligned}$$

For $w_1 \in V$, we consider

$$S_1 := \left\{ w_1 \in V : \sum_{w_1} \mathrm{dev}_s H_{w_1} \geq (2^{k+s}-2)\epsilon^{1/((k-s)(k-s+1)2^s)} \right\}.$$

If $|S| \leq 2\epsilon^{1/(k-s+1)}n$ then

$$\mathrm{dev}_s H \leq \frac{1}{n}(|S| + (2^{k+s}-2)\epsilon^{1/((k-s)(k-s+1)2^s)}n\} \leq 2^{k+s}\epsilon^{1/((k-s)(k-s+1)2^s)}$$

as desired. Thus, we may assume $|S_1| \geq 2\epsilon^{1/(k-s+1)}n$.

Similarly, it can be shown that for $i = 1, \ldots, k-s$, there are subsets S_j, with $j \leq i$, $|S_j| \geq 2\epsilon^{1/(k-s+1)}n$ such that for $\bar{w}_i = (w_1, \ldots, w_i)$ with $w_j \in S_j$ for all $j \leq i$, we have $\mathrm{dev}_s H_{\bar{w}_i} \geq (2^{k+s}-2^i)\epsilon^{i/((k-s)(k-s+1)2^s)}$. In particular, for $w = (w_1, \ldots, w_{k-s})$ with $w_i \in S_i$, $1 \leq i \leq k-s$, we have $\mathrm{dev}_s H_w \geq (2^{k+s}-2^{k-s})\epsilon^{1/2^{(k-s+1)2^s}}$.

For each $w \in S_1 \times S_2 \times \ldots \times S_{k-s}$, the induction hypothesis implies that there exists a $(s-1)$-graph G_w on V satisfying

$$|e(H_w, G_w) - e(\bar{H}_w, G_w)| > \delta n^s$$

where δ satisfies $\delta = 4^{-s}(2^{k+s-1})^{2^s}\epsilon^{1/(k-s+1)}$.

Thus, there are subsets S'_i of S_i, $1 \leq i \leq k-s$, with $|S'_i| = \epsilon^{1/(k-s+1)}n$ so that for $w \in S' = S'_1 \times S'_2 \times \ldots \times S'_{k-s}$ either

(a) $e(H_w, G_w) - e(\bar{H}_w, G_w) \geq \delta n^s$ for all $w \in S'$; or
(b) $e(H_w, G_w) - e(\bar{H}_w, G_w) \leq -\delta n^s$ for all $w \in S'$.

We will treat case (a) and omit the similar treatment for case (b).

We proceed to define the following $(k-1)$-graph G on V.

$$E(G) = \{w \cup y : y \in E(G_w)\} \setminus \binom{V \setminus (S'_1 \cup \ldots \cup S'_{k-s})}{s-1}.$$

For each $x \in E_s(H, G)$, there are three possibilities:

(i) x contains more than one vertex in some S_i'. There are at most ϵn^k such edges.
(ii) x has no vertex in S_i' for some i. In this case, x can not contain any edge in G, contradicting $x \in E_s(H, G)$.
(iii) x has exactly one vertex w_i in S_i' for $i = 1, \dots, k-s$. Say, $x = w \cup x'$, where for any vertex $u \in x'$ we have $x\{u\} \in E(G)$. Therefore, we have

$$\begin{aligned} E_s(H,G) - E_s(\bar{H}_G) &\geq \sum_{w \in S'} (E(H_w, G_w) - E(\bar{H}_w, G_w)) - \epsilon n^k \\ &\geq \prod_{i=1}^{k-s} |S_i'| \cdot \delta n^s - \epsilon n^k \\ &\geq \epsilon^{(k-s)/(k-s+1)} \cdot 3\epsilon^{1/(k-s+1)} n^k - \epsilon n^k \geq \epsilon n^k \end{aligned}$$

which is a contradiction. This completes the proof for (2).

Combining the above two theorem, we see that dev_s and $\text{disc}^{(s)}$ are equivalent.

5. CONCLUDING REMARKS

In a k-graph H, many questions can be asked concerning the (l, s)-discrepancy properties $\text{disc}_l^{(s)}$. For example, we have, for $s \geq 3$,

$$\text{disc}_k^{(s)} \Rightarrow \text{disc}_k^{(s-1)} \tag{3}$$

by using the fact that $\text{dev}_l \Rightarrow \text{dev}_{l-1}$ and the main theorem $\text{disc}_k^{(s)} \Leftrightarrow \text{dev}_s$. However, in a k-graph and $2 \leq l < k$, is it true that

$$\text{disc}_l^{(s)} \Rightarrow \text{disc}_l^{(s-1)}? \tag{4}$$

In the implication (3), the reversed direction does not hold (see [2]). For a general l with $l < k$, is it still true? Is it possible to have one equivalence class which includes $\text{disc}_l^{(s)}$ for some consecutive values of s for some l ? What is then the length of the chain of equivalence classes containing $\text{disc}_l^{(s)}$ as s ranges from 1 to $\binom{k}{l-1}$?

Recall that $\text{disc}_l = \text{disc}_l^{(s)}$ with $s = \binom{k}{l-1}$. From the definition, it is not hard to check that $\text{disc}_l \Rightarrow \text{disc}_{l-1}$ for $l \geq 3$. To further explore the relations among $\text{disc}_l^{(s)}$ and $\text{disc}_{l-1}^{(t)}$, we need more definitions.

We consider a k-graph H with vertex set $V = V(H)$ and $E = E(H)$. Let Q denote a fixed l-graph on k vertices and G denote a l-graph on vertex set V. In a k-graph H, an edges x in $E(H)$ is said to be Q-induced by G if there is an embedding π of Q into $\binom{x}{l}$, the set of l-subsets of x satisfying the property that for all $y \in E(Q)$, the images $\pi(y)$ are in $E(G)$. Let $e_Q(H, G)$ denote the total number of edges in H which are Q-induced by G.

Definition. *For a k-graph H on vertex set V with $|V| = n$ and a fixed $(l-1)$-graph Q, we define* $\text{disc}_l^Q H$ *by:*

$$\text{disc}_l^Q H = \frac{1}{n^k} \max_M |e_Q(H, G) - e_Q(\bar{H}, G)|,$$

where the maximum is taken over all $(l-1)$-graphs G on V.

It is of interest to examine possible necessary and sufficient conditions for a pair of graphs Q and Q' on k vertices such that $\text{disc}_l^Q \Leftrightarrow \text{disc}_l^{Q'}$. For a fixed k and a graph Q, how large is the family of graphs consisting of Q' satisfying the property that $\text{disc}_l^{Q'}$ is in the quasi-random class that includes disc_l^Q? How are properties disc_l^Q related to $\text{disc}_l^{(s)}$? Most of all, what is the lattice structure illustrating the relations among quasi-random classes of k-graphs? In this note, we only example some very special parts of this lattice. Numerous questions remain to be explored.

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