TURÁNNICAL HYPERGRAPHS

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Abstract. This paper is motivated by the question of how global and dense restriction sets in results from extremal combinatorics can be replaced by less global and sparser ones. The result we consider here as an example is Turán's theorem, which deals with graphs $G = ([n], E)$ such that no member of the *restriction set* $\mathcal{R} = \binom{[n]}{r}$ induces a copy of K_r.

Firstly, we examine what happens when this restriction set is replaced by $\mathcal{R} = \{X \in \binom{[n]}{r} : X \cap [m] \neq \emptyset\}$. That is, we determine the maximal number of edges in an n-vertex such that no K_r hits a given vertex set.

Secondly, we consider sparse random restriction sets. An r-uniform hypergraph $\mathcal{R}$ on vertex set $[n]$ is called *Turánnical* (respectively ε-*Turánnical*), if for any graph G on $[n]$ with more edges than the Turán number $t_r(n)$ (respectively $(1+\varepsilon)t_r(n)$), no hyperedge of $\mathcal{R}$ induces a copy of K_r in G. We determine the thresholds for random r-uniform hypergraphs to be Turánnical and to be ε-Turánnical.

Thirdly, we transfer this result to sparse random graphs, using techniques recently developed by Schacht [Extremal results for random discrete structures] to prove the Kohayakawa-Łuczak-Rödl Conjecture on Turán's theorem in random graphs.

1. Introduction

Turán's theorem [22], whose proof in 1941 marks the birth of extremal graph theory, determines the maximal number of edges in an n-vertex graph without cliques of size r. Let $\mathrm{T}_r(n)$ denote the complete *balanced* $(r-1)$-partite graph on n vertices (i.e., the part sizes of $\mathrm{T}_r(n)$ are as equal as possible) and $t_r(n)$ the number of its edges.

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Theorem 1 (Turán [22])**.** *Given n and r, let G be an n-vertex graph that contains no copy of K_r. Then G has at most $t_r(n)$ edges.*

Since 1941, many extensions of Turán's theorem have been established. Highlights certainly include the Erdős-Stone theorem [4] which generalises the result from cliques to arbitrary r-chromatic graphs, and the recent proofs by Schacht [17] and Conlon and Gowers [3] of the Kohayakawa-Łuczak-Rödl conjecture on Turán's theorem in random graphs.

These extensions, however, do not deviate from the original result as far as the following aspect is concerned. The restrictions they impose on the class of objects under study are *global* and *dense*. More concretely, they require for *every* k-tuple of vertices that these vertices do not host a copy of a given graph K on k vertices. In this paper we are interested in the question of how weakening these restrictions to less global or sparser ones (that is, forbidding K-copies only for certain k-tuples but not all) can influence the conclusion of the original Turán theorem.

To make a first move, let us investigate the following natural question which replaces the global restriction of Turán's theorem by a non-global one. *How many edges can an n-vertex graph have such that no K_r intersects a given set of m vertices in this graph?* Our first result states that the answer is

$$t_r(n,m) := \begin{cases} t_r(n) & \text{if } n \le (r-1)m\,, \\ \binom{n}{2} - nm + (r-1)\binom{m+1}{2} & \text{otherwise}\,. \end{cases} \tag{1}$$

Theorem 2. *Given $r \ge 3$ and $m \le n$, let G be any n-vertex graph and $M \subseteq V(G)$ contain m vertices. If no copy of K_r in G intersects M, then $e(G) \le t_r(n,m)$. Moreover, if $n \le (r-1)m$ and $e(G) = t_r(n,m)$ then G is isomorphic to $\mathrm{T}_r(n)$.*

This means that for fixed n, as m decreases from n (the original scenario of Turán's theorem) to 0 (no restrictions at all) the extremal number $t_r(n,m)$ stays equal to $t_r(n)$ until $m = n/(r-1)$ and then slowly increases (as a quadratic function in m) to $\binom{n}{2}$.

A natural way of formalising this deviation from Turán's theorem is to introduce a hypergraph which contains a hyperedge for every restriction and then ask for the maximal number k of edges in a graph respecting these restrictions. The following definition makes this precise. We shall distinguish between the case when k is still the Turán number and when it is bigger by a certain percentage.

Definition 3 (Turánnical)**.** *Let $r \ge 3$ be an integer. Let $\mathcal{F} = (V, \mathcal{E})$ be an n-vertex, r-uniform hypergraph with vertex set V, which we also occasionally call* restriction hypergraph*. The hypergraph $\mathcal{F}$* detects *a graph $G = (V, E)$ if some $F \in \mathcal{E}$ induces a copy of K_r in G. We say that $\mathcal{F}$ is* exactly Turánnical *or simply* Turánnical*, if for all graphs $G = (V, E)$ with $e(G) > t_r(n)$ the hypergraph $\mathcal{F}$ detects G. In addition, $\mathcal{F}$ is* ε-approximately Turánnical *or simply* ε-Turánnical *if for all graphs* $G = (V, E)$ *with* $e(G) > (1+\varepsilon)t_r(n)$ *the hypergraph* $\mathcal{F}$ *detects* G.

In other words, a restriction hypergraph is Turánnical if it detects all graphs whose density is large enough that *one* copy of K_r is forced to exist, and it is approximately Turánnical if it detects all graphs whose density forces a *positive density* of copies of K_r to exist (cf. the so-called supersaturation theorem, Theorem 14, by Erdős and Simonovits [5]).

In this language Turán's theorem states that the complete r-uniform hypergraph is Turánnical and Theorem 2 concerns restriction hypergraphs with all hyperedges meeting a specified set of vertices M (see also the reformulation in Theorem 4).

Another natural question is whether the dense complete r-uniform restriction hypergraph from Turán's theorem may be replaced by a much sparser one. Here, hypergraphs formed by *random restrictions* might appear promising candidates: A random r-uniform hypergraph $\mathcal{R}^{(r)}(n,p)$ with hyperedge probability p is a hypergraph on vertex set $[n]$ where hyperedges from $\binom{[n]}{r}$ exist independently from each other with probability p. And in fact, we will show that $\mathcal{R}^{(r)}(n,p)$ for appropriate values of $p = p_n$ produces the Turánnical hypergraphs and ε-Turánnical hypergraphs with the fewest number of hyperedges, up to constant factors (compare Proposition 5 with Theorems 6 and 7). In addition, building on the aforementioned work of Schacht [17] we obtain a corresponding result for the random graphs version of Turán's theorem (see Theorem 11).

Before we state and explain these results in detail in the following section, let us remark that the observed behaviour concerning the *evolution* of $\mathcal{R}^{(r)}(n,p)$ as we decrease the density of the random restrictions is somewhat different from the one described for Theorem 2 above: When p decreases from 1 to 0, then $\mathcal{R}^{(r)}(n,p)$ stays (asymptotically almost surely) Turánnical for a long time, until $p_n \sim n^{3-r}$. Then, between $p_n \sim n^{3-r}$ and $p_n \sim n^{2-r}$ the hypergraph $\mathcal{R}^{(r)}(n,p)$ is ε-Turánnical for *arbitrarily* small (but fixed) $\varepsilon > 0$, and for even smaller p_n the hypergraph $\mathcal{R}^{(r)}(n,p)$ fails to be ε-Turánnical for *any* non-trivial ε. As we shall see later, this sudden change of behaviour is caused by the supersaturation property of graphs (cf. Theorem 14). Put differently, there is a qualitative difference between random restriction sets detecting graphs with enough edges to force a single K_r to exist and restriction sets detecting graphs with enough edges to force a positive K_r-density, but the value of this density is not of big influence.

Organisation. The remainder of this paper is organised as follows. In Section 2 we state our results. In Section 3 we then prove Theorem 2 and some general deterministic lower bounds on the number of hyperedges in Turánnical and approximately Turánnical hypergraphs. The proofs for our results concerning random restrictions for general graphs are contained in Sections 4 and 5 and those concerning random restrictions for random graphs in Section 6. In Section 7 we argue that the hypergraph property of being

Turánnical has a sharp threshold; that is, the threshold determined in one of our main theorems, Theorem 6, is sharp. In Section 8, finally, we explain how the concept of random restrictions generalises to other problems besides Turán's theorem. We provide an outlook on which phenomena may be observed with regard to questions of this type and the corresponding evolution of random restrictions, and how they may differ from the Turán case treated in this paper.

2. Results

In this section we give our results. We start with non-global but dense restrictions and then turn to sparse restrictions. Finally we consider sparse restrictions for sparse random graphs.

2.1. Restrictions that are not global.

For completeness, let us start with a formulation of the problem on non-global restrictions addressed in Theorem 2 in the hypergraph terms introduced in Definition 3. We define $\mathcal{I}^{(r)}(n,m) = ([n],\mathcal{E})$ as the r-uniform hypergraph with hyperedges $\mathcal{E} := \left\{K \in \binom{n}{r} \colon K \cap [m] \neq \emptyset\right\}$.

Theorem 4. *Let $r \geq 3$ and n and $m \leq n$ be positive integers.*

(*a*) *The hypergraph $\mathcal{I}^{(r)}(n,m)$ is Turánnical if and only if $n \leq (r-1)m$.*
(*b*) *For every $\delta > 0$ there exists $\varepsilon > 0$ such that if $n \geq (1+\delta)(r-1)m$, then $\mathcal{I}^{(r)}(n,m)$ is not ε-Turánnical.*

It is easy to deduce Theorem 4 from Theorem 2, which determines the maximum number of edges of a graph G which is not detected by $\mathcal{I}^{(r)}(n,m)$ exactly, also for the case $n > (r-1)m$. We prove Theorem 2 in Section 3.

2.2. Sparse restrictions.

Next we consider sparser hypergraphs. An easy counting argument (which we defer to Section 3) gives the following lower bounds for the density of Turánnical and approximately Turánnical hypergraphs.

Proposition 5. *Let $r \geq 3$ and $n \geq 5$ be integers, let ε be a real with $0 < \varepsilon \leq 1/(2r)$, and let $\mathcal{F} = ([n],\mathcal{E})$ be an r-uniform hypergraph.*

(*a*) *If $|\mathcal{E}| < \frac{n(n-1)(n-2)}{r(r-1)^2(r-2)}$ then $\mathcal{F}$ is not Turánnical.*
(*b*) *If $|\mathcal{E}| \leq (1-r\varepsilon)\frac{1}{4r}n^2$, then $\mathcal{F}$ is not ε-Turánnical.*

These density bounds are sharp up to constant factors. In fact, in random r-uniform hypergraphs their magnitudes provide thresholds for being Turánnical and approximately Turánnical, respectively, as the following two results show. We first state the result concerning the threshold for being approximately Turánnical.

Theorem 6. *For every integer $r \geq 3$ and every $0 < \varepsilon \leq 1/(2r)$ there are $c = c(r,\varepsilon) > 0$ and $C = C(r,\varepsilon) > 0$ such that for any sequence $p = p_n$ of* *probabilities*

$$\lim_{n\to\infty} \mathbb{P}\big(\mathcal{R}^{(r)}(n,p) \text{ is } \varepsilon\text{-Turánnical}\big) = \begin{cases} 0, & \text{if } p_n \le cn^{2-r} \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } p_n \ge Cn^{2-r} \text{ for all } n \in \mathbb{N}. \end{cases}$$

Clearly, a random r-uniform hypergraph with hyperedge probability $p = cn^{2-r}$ asymptotically almost surely (a.a.s.) has less than $\frac{3c}{r!}\binom{n}{2}$ hyperedges. Thus part (*b*) of Proposition 5 does indeed imply the 0-statement in Theorem 6. A proof of the 1-statement is provided in Section 4.

Using part (*a*) of Proposition 5, a similar calculation shows that a random r-uniform hypergraph with hyperedge probability $p = cn^{3-r}$ with $c > 0$ sufficiently small is asymptotically almost surely not Turánnical. The corresponding 1-statement is given in the following theorem. For the case $r = 3$ the threshold probability is a constant, which we determine precisely.

Theorem 7. *For $r = 3$ and p constant we have*

$$\lim_{n\to\infty} \mathbb{P}\big(\mathcal{R}^{(3)}(n,p) \text{ is Turánnical}\big) = \begin{cases} 0, & \text{if } p \le 1/2, \\ 1, & \text{if } p > 1/2. \end{cases}$$

For every integer $r > 3$ there are $c = c(r) > 0$ and $C = C(r) > 0$ such that for any sequence $p = p_n$ of probabilities

$$\lim_{n\to\infty} \mathbb{P}\big(\mathcal{R}^{(r)}(n,p) \text{ is Turánnical}\big) = \begin{cases} 0, & \text{if } p_n \le cn^{3-r} \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } p_n \ge Cn^{3-r} \text{ for all } n \in \mathbb{N}. \end{cases}$$

This theorem is proven in Section 5. As a side remark we mention that, for its proof we shall need a structural lemma (Lemma 18) which classifies graphs with at least $t_r(n)$ edges and has the following direct consequence which might be of independent interest.

Lemma 8. *For every integer $r \ge 3$ and real $\tilde{\varepsilon} > 0$ there exists $\delta > 0$ such that for all n-vertex graphs G with $e(G) > t_r(n)$ one of the the following is true.*

(*i*) *Some vertex in G is contained in at least δn^{r-1} copies of K_r.*
(*ii*) *Some edge in G is contained in at least $(1-\tilde{\varepsilon})(n/(r-1))^{r-2}$ copies of K_r.*

An edge contained in b triangles is sometimes called a *book of size* b. Lemma 8 in the case $r = 3$ thus states that if $e(G) > t_3(n)$ and no vertex of G is contained in many K_3-copies, then G contains a book of size almost $\frac{n}{2}$. We remark that Mubayi [14] recently showed that for every $\alpha \in (\frac{1}{2}, 1)$, if G has $e(G) > t_3(n)$ and less than $\alpha(1-\alpha)n^2/4 - o(n^2)$ triangles, then G contains a book of size at least $\alpha n/2$. This result is harder, but does not imply Lemma 8.

Finally, it follows from Friedgut's celebrated result [7] that the property of being Turánnical considered in Theorem 7 has a sharp threshold. This is detailed in Section 7.

2.3. **Sparse restrictions for sparse random graphs.** In the previous subsection we examined the effect of random restrictions on Turán's theorem. A version of Turán's theorem for the Erdős-Rényi random graph $G(n,q)$ was recently proved by Schacht [17], and independently by Conlon and Gowers [3]. To understand this theorem, one should view Turán's theorem as the statement that the fraction of the edges one must delete from the complete graph K_n to remove all copies of K_r is approximately $\frac{1}{r-1}$. One can replace K_n with any graph G, and ask which graphs G have the property that deletion of a fraction of approximately $\frac{1}{r-1}$ of the edges is necessary to remove all copies of K_r.

Theorem 9 (Schacht [17], Conlon & Gowers [3])**.** *Given $\varepsilon > 0$ and r there exists a constant C such that the following is true. For $q \geq Cn^{-2/(r+1)}$, a.a.s. $G = G(n,q)$ has the property that every subgraph of G with at least $(1+\varepsilon)\frac{r-2}{r-1}e(G)$ edges contains a copy of K_r.*

Prior to the recent breakthroughs [17] and [3], Theorem 9 was known for $r = 3, 4, 5$ (see [6, 12, 10], respectively). We remark that this is also closely related to the more general line of research concerning the local and global resilience of graphs, which recently received increased attention, after the work of Sudakov and Vu [21].

Theorem 9 is best possible in the sense that it ceases to be true for values of q growing more slowly than $n^{-2/(r+1)}$. Moreover, ε cannot be replaced by 0.

Again, the restriction set in Theorem 9 is the complete r-partite hypergraph (sequence). So, extending Theorem 6, we would like to analyse what happens when this is replaced by a sparser set of random restrictions and investigate the influence of the two independent probability parameters (coming from the random restrictions and the random graph) on each other. Thus, we will be dealing with two random objects: namely a random r-uniform hypergraph $\mathcal{R}^{(r)}(n,p)$ and a random graph $G(n,q)$, picked at the same time. Furthermore, since we wish to prove asymptotically almost sure results, we need to refer not to single n-vertex hypergraphs but to sequences of hypergraphs and graphs.

Before we can formulate our result, we first need to generalise the concept of being Turánnical or approximately Turánnical from (copies of K_r in) the complete graph K_n to arbitrary graphs G. Observe that, in Theorem 9 we are interested in graphs G for which any subgraph with at least $(1+\varepsilon)\frac{r-2}{r-1} \cdot e(G)$ edges contains a copy of K_r. Hence it is natural to say that the r-uniform hypergraph $\mathcal{F}$ is ε-Turánnical for G when $\mathcal{F}$ detects every such subgraph.

For finding a similarly suitable definition of Turánnical hypergraphs for G we need some additional observations. Recall that ε cannot be 0 in Theorem 9. In other words an *exact* version of Turán's theorem for random graphs cannot be expressed in terms of the number of its edges. Instead it has to utilise the structure provided by Turán's theorem: the maximal

K_r-free subgraph of $G = G(n,q)$ should have exactly as many edges as the biggest $(r-1)$-partite subgraph of G. Accordingly, we will call a hypergraph Turánnical for G if it detects all subgraphs with more edges. The following definition summarises this.

Definition 10 (Turánnical for G). *Let $r \geq 3$ be an integer, G an n-vertex graph, and $\mathcal{F}$ an r-uniform hypergraph on the same vertex set. Then we call $\mathcal{F}$* exactly Turánnical for G *when the following holds. Every subgraph of G with more edges than are contained in a maximum $(r-1)$-partition of G has a copy of K_r induced by an edge of $\mathcal{F}$. We say that $\mathcal{F}$ is* ε-approximately Turánnical for G, *or simply* ε-Turánnical for G, *if every subgraph of G with more than $(1+\varepsilon)\frac{r-2}{r-1}e(G)$ edges has a copy of K_r induced by an edge of $\mathcal{F}$.*

In this language, Theorem 9 becomes the statement that, given r and $\varepsilon > 0$, there exists C such that the complete r-uniform hypergraph is a.a.s. ε-Turánnical for $G(n,q)$, whenever $q \geq Cn^{-2/(r+1)}$. Moreover, according to a result of Brightwell, Panagiotou and Steger [2], for every r there exists $\mu > 0$ such that the complete r-uniform hypergraph is a.a.s. exactly Turánnical for $G(n,q)$ whenever $q > n^{-\mu}$.[1]

In our last theorem we determine the relationship between r, $\varepsilon > 0$, p and q such that the random r-uniform hypergraph $\mathcal{R}^{(r)}(n,p)$ is a.a.s. ε-Turánnical for $G(n,q)$. Not surprisingly, a suitable combination of the two threshold probabilities from Theorem 6 and Theorem 9 determines the threshold in this case.

Theorem 11. *Given $r \in \mathbb{N}$, $r \geq 3$ and $\varepsilon \in (0, 1/(r-2))$, there exist $c = c(r,\varepsilon) > 0$ and $C = C(r,\varepsilon) > 0$ such that for any pair of sequences $p = p_n$ and $q = q_n$ of probabilities and for $\vartheta_q(n) := (nq^{(r+1)/2})^{2-r}$ we have*

$$\lim_{n\to\infty} \mathbb{P}\big(\mathcal{R}^{(r)}(n,p) \text{ is } \varepsilon\text{-Turánnical for } G(n,q)\big) = \begin{cases} 0, & \text{if } p_n \leq c\vartheta_q(n) \text{ for all } n \in \mathbb{N}, \\ 1, & \text{if } p_n \geq C\vartheta_q(n) \text{ for all } n \in \mathbb{N}. \end{cases}$$

This theorem states that for a fixed q_n the threshold probability for $\mathcal{R}^{(r)}(n,p)$ to be ε-Turánnical for $G(n,q)$ is $\vartheta_q(n)$. Equivalently, if instead we fix the hyperedge probability p_n then $\vartheta_p(n) := (np^{1/(r-2)})^{-2/(r+1)}$ is the threshold probability for $G(n,q)$ such that $\mathcal{R}^{(r)}(n,p)$ is ε-Turánnical for $G(n,q)$. In particular, $\vartheta_q(n)$ is constant when q_n is the threshold probability from Theorem 6 and $\vartheta_p(n)$ is constant when p_n is the threshold probability from Theorem 9.

We note that the requirement $\varepsilon < 1/(r-2)$ in Theorem 11 is necessary for the 0-statement. Indeed, if $\varepsilon > 1/(r-2)$ then $(1+\varepsilon)\frac{r-2}{r-1}e(G) > e(G)$.

[1] However, Brightwell, Panagiotou and Steger do not believe that their result is best possible: for example, for $r = 3$ their proof works for $\mu = 1/250$, but they suggest the result might hold for any $\mu < 1/2$.

Therefore the premise in Definition 10 is never met, and consequently every hypergraph is ε-Turánnical.

In order to establish Theorem 11 we employ in Section 6 Schacht's machinery from [17]. However we need to modify this machinery to allow working with two sources of randomness: graphs $G(n,q)$ and hypergraphs $\mathcal{R}^{(r)}(n,p)$. We believe that this might prove useful in the future.

We believe that a similar result as Theorem 11 should be true if ε-Turánnical is replaced by exactly Turánnical in this theorem. More precisely, we think that for $r \geq 3$ the hypergraph $\mathcal{R}^{(r)}(n,p)$ is a.a.s. exactly Turánnical for $G(n,q)$, if p and q are both sufficiently large. For obtaining a result of this type, possibly a modification of the methods used in [2] may be of assistance.

3. Deterministic constructions

In this section we provide the proofs for Theorem 2 and Proposition 5. We start with the latter.

Let $\mathcal{F} = (V, \mathcal{E})$ be an r-uniform hypergraph and X be a subset of its vertices of size $|X| = s < r$. The *link hypergraph* $\mathrm{Link}_{\mathcal{F}}(X) = (V, \mathcal{E}')$ of X is the $(r-s)$-uniform hypergraph with hyperedges $\mathcal{E}' = \{Y \in \binom{V}{r-s} : Y \cup X \in \mathcal{E}\}$. If $X = \{x_1, \dots, x_s\}$ we also write $\mathrm{Link}_{\mathcal{F}}(x_1, \dots, x_s)$ for $\mathrm{Link}_{\mathcal{F}}(X)$. When the underlying hypergraph $\mathcal{F}$ is clear from the context we write $\mathrm{Link}(X)$ instead of $\mathrm{Link}_{\mathcal{F}}(X)$.

Proof of Proposition 5. Let the r-uniform hypergraph $\mathcal{F} = ([n], \mathcal{E})$ be given. We start with the proof of (a) and first consider the case $r > 3$. We have

$$\sum_{\{u,v\} \in \binom{[n]}{2}} e\big(\mathrm{Link}(u,v)\big) = \binom{r}{2} |\mathcal{E}| < \binom{r}{2} \frac{n(n-1)(n-2)}{r(r-1)^2(r-2)} \leq \frac{\binom{n}{2} n}{(r-2)(r-1)},$$

Accordingly there are two vertices $u, v \in [n]$ such that $(r-2) e\big(\mathrm{Link}(u,v)\big) \leq n/(r-1)$. Let

$$L := \Big\{ w \in [n] \colon w \in Y \text{ for some } Y \in E\big(\mathrm{Link}(u,v)\big) \Big\}$$

be the set of vertices covered by the hyperedges of $\mathrm{Link}(u,v)$. Because $\mathrm{Link}(u,v)$ is an $(r-2)$-uniform hypergraph, it follows from the choice of u and v that $|L| \leq n/(r-1)$. Now suppose the graph $G = ([n], E)$ is a copy of the $(r-1)$-partite Turán graph $\mathrm{T}_r(n)$ such that u and v are in the same partition class of $\mathrm{T}_r(n)$ and L is entirely contained in another partition class. The graph G exists because some partition class of $\mathrm{T}_r(n)$ has at least $n/(r-1)$ vertices, and at least two partition classes of $\mathrm{T}_n(r)$ have at least two vertices (unless $n \leq r$, in which case $L = \emptyset$). As $r > 3$, we can add the edge uv to G without creating a copy of K_r on any hyperedge of $\mathcal{F}$. Therefore $G + uv$ witnesses that $\mathcal{F}$ is not Turánnical.

For the case $r = 3$ of (a) we proceed similarly and infer from $|\mathcal{E}| < \frac{1}{2}\binom{n}{3}$ that there are distinct vertices $u, v \in [n]$ with $e\big(\mathrm{Link}(u,v)\big) < \frac{n}{2} - 1$

(observe that the hyperedges in $\mathrm{Link}(u,v)$ are singletons). Accordingly we can place the vertices u,v together with $E\big(\mathrm{Link}(u,v)\big)$ into one partition class of the bipartite graph $\mathrm{T}_3(n)$ and subsequently add the edge uv. $\mathcal{F}$ does not detect G, even thought $e(G)=t_3(n)+1$.

For (b) an even simpler construction for $G=([n],E)$ suffices. We start with the complete graph $K_n=: G$. Then, for each hyperedge Y of $\mathcal{F}$ we pick two arbitrary vertices $u,v\in Y$ and delete the edge uv from G (if it is still present). Using $|\mathcal{E}|\le(1-r\varepsilon)\frac{1}{4r}n^2$ and $r\ge 3$, $n\ge 5$, it is easy to check that the resulting graph G has more than $(1+\varepsilon)t_r(n)$ edges, and by construction G contains no copies of K_r on hyperedges of $\mathcal{F}$. Hence $\mathcal{F}$ is not ε-Turánnical. □

Now we turn to the proof of Theorem 2, which provides an upper bound on the number of edges in a graph on n vertices with the property that no r-clique intersects a fixed set M of m vertices. Theorem 2 states that the following graphs $\mathrm{T}_r(n,m)$ are extremal for this problem. For $n\le(r-1)m$ let $\mathrm{T}_r(n,m)=\mathrm{T}_r(n)$ be a Turán graph on n vertices. For $n>(r-1)m$ we construct $T=\mathrm{T}_r(n,m)$ as follows. Initially, we take $T=\mathrm{T}_r((r-1)m)$. We then fix an arbitrary set $M\subseteq V(T)$ of size m and add $n-(r-1)m$ new vertices to T. Finally, for each of the new vertices we add edges to all other vertices except those in M. By construction, it is clear that $\mathrm{T}_r(n,m)$ has n vertices and no copy of K_r intersects M. Moreover, observe that the number of edges of $\mathrm{T}_r(n,m)$ is given by the function $t_r(n,m)$ defined in (1) since

$$m^2\tbinom{r-1}{2}+m(r-2)\big(n-(r-1)m\big)+\tbinom{n-(r-1)m}{2}=\binom{n}{2}-nm+(r-1)\binom{m+1}{2}.$$

We shall use the following notation. Let G be a graph, X and Y be disjoint subsets of its vertices, and u be a vertex. Then we write $G[X]$ for the subgraph of G induced by X and $G[X,Y]$ for the bipartite subgraph of G on vertex set $X\cup Y$ which contains exactly those edges of G which run between X and Y. Moreover, we write $\Gamma(u,X)$ for the set of neighbours of u in X, and set $\deg(u,X):=|\Gamma(u,X)|$.

Proof of Theorem 2. Let r, n, m be fixed and let G and M satisfy the conditions of the theorem. Assume moreover, that G has a maximum number of edges, subject to these conditions. The definition of $t_r(n,m)$ suggests the following case distinction. We shall first proof the theorem for $n\le(r-1)m$ and then for $n>(r-1)m$. In fact, for the second case we use the correctness of the first case.

First assume $n\le(r-1)m$. In this case we start by iteratively finding vertex disjoint cliques $Q_1,\dots,Q_k$ with at least r vertices in G as follows. Assume, that $Q_1,\dots,Q_{i-1}$ have already been defined for some i. Then let Q_i be an arbitrary maximum clique on at least r vertices in $G-\bigcup_{j<i}Q_j$. If no such clique exists, then set $k=i-1$ and terminate.

Now, let us establish some simple bounds on the number of edges between these cliques and the rest of G. For this purpose, set $q_i := v(Q_i) \geq r$ to be the number of vertices of the clique Q_i for all $i \in [k]$ and $q := \sum_{i=1}^k q_i$. Clearly, the graph $G - \bigcup_{i=1}^k V(Q_i)$ is K_r-free, and therefore

$$e\Big(G - \bigcup_{i=1}^k V(Q_i)\Big) \leq t_r(n-q)\,.$$

Moreover, $M \subseteq V(G) \setminus \bigcup_{i=1}^k V(Q_i)$ and we have $\deg(v, Q_i) \leq r-2$ for each $v \in M$, as v is not contained in a copy of K_r by assumption. In addition, the maximality of $Q_1, \dots, Q_k$ implies that $\deg(v, Q_i) \leq q_i - 1$ for any $v \in V(G) \setminus (M \cup \bigcup_{j=1}^i V(Q_i))$. Putting these three estimates together we obtain

$$\begin{aligned} e(G) \leq \sum_{i=1}^k \binom{q_i}{2} + \sum_{1\leq i<j\leq k} (q_i-1)q_j + t_r(n-q) + mk(r-2) \\ + (q-k)(n-m-q) =: g(q_1, \dots, q_k)\,. \end{aligned} \tag{2}$$

Observe that (2) defines a function $g(q_1, \dots, q_\ell)$ for each number of arguments ℓ. In particular, we also allow $\ell = 0$, in which case (2) asserts that $g() = t_r(n)$. In the remainder of this case of the proof we shall investigate the family of functions $g(q_1, \dots, q_\ell)$. We shall show, that for all $\ell > 0$ we have $g() > g(q_1, \dots, q_\ell)$, which is a consequence of the following claim.

Claim 12. Assuming that $q = \sum_{i=1}^k q_i \leq n-m$ and $q_i \geq r$ for all $i \in [k]$ we have

$$g(q_1, \dots, q_{k-1}, q_k) < g(q_1, \dots, q_{k-1}, q_k - 1) \qquad \text{if } q_k > r\,, \quad \text{and} \tag{3}$$

$$g(q_1, \dots, q_{k-1}, q_k) < g(q_1, \dots, q_{k-1}) \qquad \text{if } q_k = r\,. \tag{4}$$

Proof of Claim 12. Adding one or r vertices to a Turán graph $\mathrm{T}_r(n')$ to create a bigger Turán graph and counting the additionally created edges gives

$$t_r(n'+1) - t_r(n') = n' - \left\lfloor \frac{n'}{r-1} \right\rfloor, \quad \text{and} \tag{5}$$

$$t_r(n'+r) - t_r(n') = (r-1)n' + \binom{r}{2} - \left\lfloor \frac{n'+r-1}{r-1} \right\rfloor. \tag{6}$$

Observe that $m > 1$, or otherwise $r \leq q \leq n-1 \leq (r-1)m - 1$ would lead to a contradiction. If $q_k > r$ then plugging (5) (with $n' = n - q$) into the definition of g in (2) we obtain

$$g(q_1, \dots, q_{k-1}, q_k - 1) - g(q_1, \dots, q_{k-1}, q_k) = m - \left\lfloor \frac{n-q}{r-1} \right\rfloor - 1 > 0\,,$$

proving (3). Similarly, if $q_k = r$ then (6) implies

$$g(q_1, \dots, q_{k-1}) - g(q_1, \dots, q_{k-1}, q_k) = m - \left\lfloor \frac{n-q}{r-1} \right\rfloor - 1 > 0\,,$$

proving (4). □

Clearly, applying Claim 12 for sequentially decreasing or discarding the last argument of $g(q_1,\dots,q_\ell)$ gives that

$$g\big(v(Q_1),v(Q_2),\dots,v(Q_k)\big)=g(q_1,\dots,q_k)\le g()=t_r(n)\,.$$

Moreover, equality holds only when $k=0$, that is, when G does not contain any K_r. This proves the theorem in the case $n\le(r-1)m$.

Now assume $n>(r-1)m$. Let $X\subseteq V(G)-M$ be the vertices of $V(G)-M$ which possess at least one neighbour in M. Let $Y:=V(G)-M-X$. We start by transforming G into a graph with the same number of edges, which satisfies the assumptions of the theorem, and which has the clear structure described in the following claim.

Claim 13. We may assume without loss of generality that

(a) For each $x\in X$ we have $\deg(x)\ge n-m$.

(b) $G[M]$ is a complete s-partite graph with parts $M_1,\dots,M_s$, for some $s\le r-1$. Moreover, $\Gamma(u,X)=\Gamma(u',X)$ for all $u,u'\in M_i$ and $1\le i\le s$.

(c) $G[X]$ is a complete t-partite graph with parts $X_1,\dots,X_t$, for some t.

(d) For each M_i and X_j with $i\in[s]$ and $j\in[t]$, either $G[M_i,X_j]$ is complete or empty, which we denote by $M_i\sim X_j$ and $M_i\nsim X_j$, respectively. For each $i\in[s]$ we have $M_i\sim X_j$ for at most $r-2$ values of j.

Proof of Claim 13. To see (a), observe that, if some $x\in X$ were adjacent to fewer than $n-m$ vertices of G, then deleting all edges adjacent to x and inserting edges from x to all vertices in $X\cup Y$ (except x) would yield a modified graph with no K_r intersecting M, and with at least as many edges as G. Note that x gets removed from the set X of neighbours of M to Y during this modification.

Now we turn to (b). Suppose that u and v are two non-adjacent vertices of M. If $\deg(u)\ge\deg(v)$, then the graph G' obtained from G by deleting all edges emanating from v and inserting all edges from v to $\Gamma(u)$ certainly does not have fewer edges than G, and further G' does not have any copy of K_r intersecting M. Clearly, repeating this process for every pair of non-adjacent vertices of M gives a graph with the desired property.

Applying an analogous process to non-adjacent vertices in X we infer (c). Note that these deletion and insertion processes in M and X moreover guarantee the first part of (d). The second part follows since otherwise we would obtain a K_r intersecting M. □

In the following we assume that G has the partite structure described in Claim 13 and use it to infer some further properties of G which in turn will allow us to obtain the desired bound on the edges in G. By (a) of Claim 13 we have $|X_j|+\sum_{i:M_i\nsim X_j}|M_i|\le m$, and hence

$$|X|=\sum_j|X_j|\le\sum_j\Big(m-\sum_{i:M_i\nsim X_j}|M_i|\Big)=\sum_j\sum_{i:M_i\sim X_j}|M_i|\le(r-2)m\,,\quad(7)$$

where the last inequality follows from (d) of Claim 13.

Clearly, this implies $|Y| = n-|X|-|M| \geq n-(r-1)m > 0$ which allows us to conclude that the inequality in Claim 13(a) is in fact an equality: Suppose for contradiction that $\deg(x) \geq n-m+1$ for some $x \in X$. Then we may select any $y \in Y$ and obtain a graph G' by deleting all edges incident to y and inserting all edges from y to the neighbours of x. This graph continues to satisfy the conditions of the theorem and has at least one more edge. It follows that for each $x \in X$ we have $\deg(x) = n-|M|$.

For each $i \in [s]$ we also have that $M_i \sim X_j$ for exactly $r-2$ values of j (otherwise we could set all vertices of M_i adjacent to y for some $y \in Y$ and gain edges, since $|Y| > 0$). It follows that in fact equality must hold in (7) and hence $|X| = (r-2)m$. This implies that $|X \cup M| = (r-1)m$. Hence we may apply the first case of the proof on the graph $G[X \cup M]$ and conclude that $e(G[X \cup M]) \leq t_r((r-1)m) = m^2\binom{r-1}{2}$. Therefore,

$$\begin{aligned} e(G) &= e(G[X \cup M]) + |X||Y| + \tbinom{|Y|}{2} \\ &\leq m^2\tbinom{r-1}{2} + m(r-2)(n-(r-1)m) + \tbinom{n-(r-1)m}{2} = t_r(n,m) \ , \end{aligned}$$

as desired. □

4. Approximately Turánnical random hypergraphs

In this section we prove Theorem 6. As noted in Section 1, the simple deterministic part (b) of Proposition 5, that no too sparse hypergraph $\mathcal{F}$ can be ε-approximately Turánnical, gives the 0-statement. We therefore focus on the proof of the 1-statement. To this end we use the following theorem of Erdős and Simonovits [5].

Theorem 14 (Erdős & Simonovits [5]). *Given any $r \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that the following is true. If G is any n-vertex graph with $e(G) \geq (1+\varepsilon)t_r(n)$, then there are at least δn^r copies of K_r in G.*

Proof of Theorem 6. Given $\varepsilon > 0$, by Theorem 14, there exists $\delta > 0$ such that if G is any graph with $e(G) \geq (1+\varepsilon)t_r(n)$, then G contains at least δn^r copies of K_r.

Let $p \geq \binom{n}{2}n^{-r}/\delta$. Given one graph G with at least δn^r copies of K_r, the probability that G is not detected by $\mathcal{R}^{(r)}(n,p)$ is at most

$$(1-p)^{\delta n^r} \ .$$

Summing over the at most $2^{\binom{n}{2}}$ such graphs G, we see that the probability that there exists an n-vertex graph G, with at least δn^r copies of K_r, which is undetected by $\mathcal{R}^{(r)}(n,p)$, is at most

$$2^{\binom{n}{2}}(1-p)^{\delta n^r} < 2^{\binom{n}{2}}e^{-p\delta n^r} \leq 2^{\binom{n}{2}}e^{-\binom{n}{2}} \ ,$$

which tends to zero as n tends to infinity. In particular, with probability tending to 1, any graph G with $e(G) \geq (1+\varepsilon)t_r(n)$ is detected by $\mathcal{R}^{(r)}(n,p)$. □

5. Exactly Turánnical random hypergraphs

In this section we prove Theorem 7. The 0-statement of Theorem 7 follows from Proposition 5 (a) for $r > 3$, and from Lemma 15 below for $r = 3$.

Lemma 15. *For $p \leq \frac{1}{2}$, we have $\mathbb{P}(\mathcal{R}^{(3)}(n,p)$ is Turánnical $) = o(1)$.*

Proof. By monotonicity, we may assume that $p = \frac{1}{2}$. As in the proof of Proposition 5 it suffices to show that there is a.a.s. a pair of vertices $u, v \in V(\mathcal{R}^{(3)}(n,p))$ with $e(\mathrm{Link}(u,v)) \leq \frac{n}{2} - 2$ (we remark that the hypergraph $\mathrm{Link}(u,v)$ is 1-uniform in this case). So choose two arbitrary vertices u and v. Observe that from the properties binomial distribution $\mathbb{P}\left(e(\mathrm{Link}(u,v)) > \frac{n}{2} - 2\right) \leq 0.6$, for large enough n. Let $\{u_1, v_1\}, \dots, \{u_{\lfloor \frac{n}{2} \rfloor}, v_{\lfloor \frac{n}{2} \rfloor}\}$ be disjoint pairs of vertices. Using the independence of the variables $e(\mathrm{Link}(u_i, v_i))$, we obtain that $\mathbb{P}\left(\forall i\colon e(\mathrm{Link}(u_i, v_i) > \frac{n}{2} - 2\right) \leq 0.6^{\lfloor \frac{n}{2} \rfloor} = o(1)$. □

For the 1-statement of Theorem 7 we shall, in Lemma 18, investigate the structural properties of graphs with more edges than a Turán graph has, and classify them into three possible categories. We then treat these three types of graphs separately, and show for each of them that with high probability a random restriction hypergraph $\mathcal{R}^{(r)}(n,p)$ detects *each* of the graphs of this type. Let us first take a small detour.

The Erdős-Simonovits theorem, Theorem 14, states that graphs G with many more edges than a Turán graph $\mathrm{T}_r(n)$ contain a positive fraction of the possible r-cliques. This is not true anymore when G has just one edge more than $\mathrm{T}_r(n)$. However, as the well-known stability theorem of Simonovits [19] shows, we can still draw the same conclusion when we know in addition that G looks very different from $\mathrm{T}_r(n)$. To state the result of Simonovits we need the following definition. Let ε be a positive constant and G and H be graphs on n vertices. If G cannot be obtained from H by adding and deleting together at most εn^2 edges, then we say that G is *ε-far* from H.

Theorem 16 (Simonovits [19])**.** *For every $r \geq 3$ and $\varepsilon > 0$ there exists $\delta > 0$ such that any n-vertex graph G with $e(G) \geq t_r(n)$ which is ε-far from $\mathrm{T}_r(n)$ contains at least δn^r copies of K_r.*

If a graph G is not far from a Turán graph, on the other hand, we have a lot of structural information about G: we know that its vertex set can be partitioned into $r-1$ sets which are almost of the same size and almost independent, such that most of the edges between these sets are present. If in addition almost all vertices of G have many neighbours in all partition classes other than their own, then we say that G has an ε-close $(r-1)$-partition. The following definition makes this precise.

Definition 17 (ε-close $(r-1)$-partition)**.** *Let $G = (V, E)$ be a graph. An* ε-close $(r-1)$-partition *of G is a partition $V = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_{r-1}$ of its vertex set such that* (*i*) $|V_0| \le \varepsilon^2 n$ *and* $|V_i| \ge (1-\varepsilon)\frac{n}{r-1}$ *for all* $i \in [r-1]$,

(*ii*) *for all* $v \in V_0$ *we have* $\deg(v) \le (1-\varepsilon^2)\frac{r-2}{r-1}n$, *and for all* $i, j \in [r-1]$ *with* $i \neq j$ *and for all* $v \in V_i$ *we have* $\deg(v, V_j) \ge (1-\varepsilon)|V_j|$.

The edges (non-edges) in such a partition that run between two different parts V_i *and* V_j *with* $1 \le i, j \le r-1$, *are called* crossing, *and those that lie within a partition class* V_i *with* $1 \le i \le r-1$, *are* non-crossing.

The following lemma states that a graph which has at least as many edges as $\mathrm{T}_r(n)$ either contains a vertex whose neighbourhood has a positive K_{r-1}-density, or has an ε-close $(r-1)$-partition.

Lemma 18. *For every integer* $r \ge 3$ *and real* $0 < \varepsilon \le 1/(16r^2)$ *there exists a positive constant* δ *such that for every* n*-vertex graph* G *with* $e(G) \ge t_r(n)$ *one of the the following is true.*

(*i*) *Some vertex in* G *is contained in at least* δn^{r-1} *copies of* K_r.

(*ii*) G *has an* ε*-close* $(r-1)$*-partition.*

We postpone the proof of Lemma 18 and first sketch that it implies Lemma 8.

Proof of Lemma 8. Suppose we are given r and $\tilde{\varepsilon}$. By monotonicity we may assume that $\tilde{\varepsilon} < 1/16$. Let δ be given by Lemma 18 with input parameters r and $\varepsilon := \tilde{\varepsilon}/r^2$. By Lemma 18 it suffices to show that in each n-vertex graph G with

$$e(G) > t_r(n) \tag{8}$$

which possesses an ε-close $(r-1)$-partition $V(G) = V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_{r-1}$ there is an edge contained in at least $(1-\tilde{\varepsilon})(n/(r-1))^{r-2}$ copies of K_r. First observe that by (8) and (*ii*) of Definition 17 we have $e(G - V_0) > t_r(n - |V_0|)$. Thus, by Turán's Theorem, there is an edge $uv \subseteq V_i$ for some $i \in [r-1]$. The edge uv has at least $(1-2\varepsilon)|V_j|$ common neighbours in each V_j, $j \neq i$, creating at least

$$\left(\big(1-(r-1)\varepsilon\big)(1-\varepsilon)\frac{n}{r-1}\right)^{r-2} \ge (1-r\varepsilon)^{r-2}\left(\frac{n}{r-1}\right)^{r-2} \ge (1-\tilde{\varepsilon})\left(\frac{n}{r-1}\right)^{r-2}$$

copies of K_r. $\square$

Proof of Lemma 18. Given r and ε, let G be an n-vertex graph with $e(G) \ge t_r(n)$. By Theorem 16, there exists $\gamma = \gamma(\varepsilon, r) > 0$ such that if G is $\varepsilon^3/(16r^3)$-far from $\mathrm{T}_r(n)$, then G contains γn^r copies of K_r. We set

$$\delta := \min\left\{\gamma, \frac{1}{r!2^r r^r}, \frac{\varepsilon}{4^r r^r}, \left(\frac{\varepsilon}{2r}\right)^{r-1}\right\}.$$

Since $e(G) \ge t_r(n)$, either $G = \mathrm{T}_r(n)$, which clearly has an ε-close $(r-1)$-partition, or G contains a copy of K_r. Observe that the last term in this minimum ensures that if $n < \frac{2r}{\varepsilon}$, then $\delta n^{r-1} < 1$, and thus that one copy of K_r in G is enough to satisfy the Lemma. It follows that we may henceforth assume $n \ge \frac{2r}{\varepsilon}$.

As G contains γn^r copies of K_r then there is a vertex lying in $\gamma n^{r-1} \geq \delta n^{r-1}$ copies of K_r. Thus we may assume that G is not $\varepsilon^3/(16r^3)$-far from $\mathrm{T}_r(n)$. So there exists a balanced partition $V(G) = U_1 \dot\cup \dots \dot\cup U_{r-1}$ such that the total number of non-edges between the parts is at most $\varepsilon^3 n^2/(16r^3)$.

Now for each $1 \leq i \leq r-1$, we define

$$V_i = \left\{ v \in V(G)\colon\ \deg(v, V(G) \setminus U_i) \geq \left(\frac{r-2}{r-1} - \frac{\varepsilon}{4r} \right) n \right\} . \tag{9}$$

We let $V_0 := V(G) \setminus (V_1 \cup \dots \cup V_{r-1})$. We aim to show that either there is some vertex of G which lies in at least δn^{r-1} copies of K_r, or that $V_0 \dot\cup V_1 \dot\cup \dots \dot\cup V_{r-1}$ is an ε-close $(r-1)$-partition.

For each $1 \leq i \leq r-1$, every vertex in $U_i \setminus V_i$ lies in at least $\varepsilon n/(4r)$ non-edges crossing the partition $(U_1, \dots, U_{r-1})$. It follows that

$$|U_i \setminus V_i| \leq \frac{\varepsilon^2 n}{4r^2} , \tag{10}$$

since there are at most $\varepsilon^3 n^2/(16r^3)$ such non-edges. Summing over $i = 1, \dots, r-1$ we get

$$|V_0| \leq \frac{(r-1)\varepsilon^2 n}{4r^2} < \frac{\varepsilon^2 n}{4r} < \varepsilon^2 n . \tag{11}$$

Since $n \geq 2r/\varepsilon$ we also have, for each $1 \leq i, j \leq r-1$ with $i \neq j$, and each $v \in V_i$, that

$$\begin{aligned} |V_i| \geq |U_i| - \frac{\varepsilon^2 n}{4r^2} > (1-\varepsilon)\frac{n}{r-1} , \qquad \text{and} \\ \deg(v, V_j) \overset{(9),(10)}{\geq} |U_j| - 1 - \frac{\varepsilon n}{4r} - \frac{\varepsilon^2 n}{4r^2} \\ \geq |V_j| - 1 - (r-2)\frac{\varepsilon^2 n}{4r^2} - \frac{\varepsilon n}{4r} - \frac{\varepsilon^2 n}{4r^2} \geq (1-\varepsilon)|V_j| , \end{aligned} \tag{12}$$

where we use $\varepsilon \leq \frac{1}{10}$ to obtain the last inequality.

We claim that a vertex u lying in more than one of the sets $V_1, \dots, V_{r-1}$ must lie in at least δn^{r-1} copies of K_r. To see this, observe that u must have at least $(1-\varepsilon)|V_i|$ neighbours in V_i for each $1 \leq i \leq r-1$. Now consider the following method of constructing a copy of K_r in G using u. We choose a neighbour v_1 of u in V_1, a common neighbour v_2 of u and v_1 in V_2, and so on. Since $\varepsilon \leq 1/(16r)$, the common neighbourhood of $u, v_1, \dots, v_{i-1}$ in V_i contains at least $(1 - i\varepsilon)|V_i| > \frac{n}{2(r-1)}$ vertices for each i, there are at least $\frac{n}{2(r-1)}$ choices at each of the $r-1$ steps (and in particular this construction is possible). This procedure may construct the same copy of K_r more than once (since at this point we do not yet know that the sets $V_1, \dots, V_{r-1}$ are disjoint), but not more than $(r-1)!$ times. It follows that u lies in at least

$$\frac{1}{(r-1)!} \left(\frac{n}{2(r-1)} \right)^{r-1} \geq \delta n^{r-1}$$

copies of K_r.

Hence, we can assume from now on that the sets $V_1, \dots, V_{r-1}$ are disjoint. Next we claim that a vertex u in V_0 whose degree exceeds $(1-\varepsilon^2)\frac{r-2}{r-1}n$ must lie in at least δn^{r-1} copies of K_r. Without loss of generality, we may assume that we have $\deg(u, V_1) \le \deg(u, V_2) \le \dots \le \deg(u, V_{r-1})$. Since $u \notin V_1$, we have

$$
\begin{aligned}
\deg(u, V_1) &= \deg(u) - \deg(u, V(G) \setminus V_1) \\
&\ge \deg(u) - \deg(u, U_2 \dot\cup \dots \dot\cup U_{r-1}) - |U_1 \setminus V_1| \\
&\overset{(9),(10)}{>} (1-\varepsilon^2)\frac{r-2}{r-1}n - \left(\frac{r-2}{r-1} - \frac{\varepsilon}{4r}\right)n - \frac{\varepsilon^2 n}{4r^2} \qquad (13)\\
&\ge -\varepsilon^2 n + \frac{\varepsilon n}{4r} - \frac{\varepsilon^2 n}{4r^2} \ge \frac{\varepsilon n}{16r},
\end{aligned}
$$

where the last inequality follows from $\varepsilon \le 1/(16r)$. Since $\deg(u, V_2) \ge \deg(u, V_1)$ and u has at most $\frac{n}{r-1} + \varepsilon^2 n$ non-neighbours by assumption, we infer that $\deg(u, V_2) \ge \frac{n}{3(r-1)}$, using again $\varepsilon \le 1/(16r)$. Hence

$$
\deg(u, V_i) \ge \frac{n}{3(r-1)} \qquad \text{for each } 2 \le i \le r-1. \tag{14}
$$

Now consider the same inductive construction of copies of K_r containing u as before. This time we know that there are at least $\frac{\varepsilon n}{16r}$ choices for v_1, and at least

$$
\frac{n}{3(r-1)} - (i-1)\varepsilon|V_i| > \frac{n}{4(r-1)}
$$

choices for v_i, for each $2 \le i \le r-1$. Since the sets $V_1, \dots, V_{r-1}$ are disjoint, each copy of K_r can be constructed in only one way. Thus u does indeed lie in at least

$$
\frac{\varepsilon n}{16r}\left(\frac{n}{4(r-1)}\right)^{r-2} \ge \delta n^{r-1}
$$

copies of K_r.

Accordingly, we can assume that $\deg(u) \le (1-\varepsilon^2)\frac{r-2}{r-1}n$, for all u in V_0. Together with (11) and (12) this implies that the partition $V_0 \dot\cup \dots \dot\cup V_{r-1}$ satisfies (i) and (ii) of Definition 17 and hence is an ε-close $(r-1)$-partition of G. $\square$

We need a more precise structural result to handle the case $r = 3$ of Theorem 7. As we shall see, this is a simple consequence of the above proof.

Corollary 19. *For every $0 < \varepsilon \le 1/144$ there exists a positive constant δ such that for all n-vertex graphs G with $e(G) \ge t_3(n)$ one of the the following is true.*

(i) G contains at least δn^3 triangles.
(ii) There is a vertex u of G such that $\Gamma(u) \supset X \dot\cup Y$, where $|X||Y| \ge \varepsilon n^2/288$ and $e(X,Y) \ge (1-4\varepsilon)|X||Y|$.
(iii) G has an ε-close 2-partition.

Proof. We follow the previous proof with $r = 3$, using the same value for δ. If G contains less than δn^3 triangles we obtain the three sets V_0, V_1, V_2 (as defined in (9)). If these sets do not form a partition of $V(G)$, then there is a vertex v in both V_1 and V_2. Then we let $X := \Gamma(v) \cap V_1$ and $Y := \Gamma(v) \cap V_2$. By (12) we have $|X||Y| \geq (1-\varepsilon)^2|V_1||V_2| \geq (1-\varepsilon)^4 n^2/4 > \varepsilon n^2/32$ because $\varepsilon \leq 1/2$. Since each vertex of X is adjacent to all but at most $\varepsilon|V_2|$ vertices of Y by (12), we also have $e(X,Y) \geq (1-4\varepsilon)|X||Y|$ as required.

Hence we may assume that V_0, V_1, V_2 form a partition of $V(G)$. The only remaining barrier to V_0, V_1, V_2 being an ε-close 2-partition of G is the existence of a vertex v in V_0 with degree more than $(1-\varepsilon^2)\frac{n}{2}$. As in the previous proof, if this vertex exists we may without loss of generality presume by (13) that it has at least $\varepsilon n/48$ neighbours in V_1, and by (14) that it has at least $n/6$ neighbours in V_2. Again we let $X := \Gamma(v) \cap V_1$, and $Y := \Gamma(v) \cap V_2$, and get $|X||Y| \geq \varepsilon n^2/288$ as required. Now since $|Y| > |V_2|/4$, and since every vertex in X is adjacent to all but at most $\varepsilon|V_2|$ vertices of Y, we have $e(X,Y) \geq (1-4\varepsilon)|X||Y|$ as required. $\square$

Our next lemma counts the number of graphs with an ε-close $(r-1)$-partition and a given number of non-crossing edges. In addition it estimates the number of r-cliques in such a graph.

Lemma 20. *Let $\ell \geq 0$ and $r \geq 3$ be integers, $0 < \varepsilon < 1/(2r)$ be a real and $n \geq 2r^3/\varepsilon^2$ be an integer. Let $\mathcal{G}$ be the family of all graphs on a fixed vertex set of size n with $e(G) > t_r(n)$ which have an ε-close $(r-1)$-partition with exactly ℓ non-crossing edges. Then*

(*a*) *if $\ell = 0$ then $|\mathcal{G}| = 0$,*
(*b*) *$|\mathcal{G}| \leq r^{5\ell n}$, and*
(*c*) *every $G \in \mathcal{G}$ contains at least $\ell\big(\frac{n}{2r-2}\big)^{r-2}$ copies of K_r.*

Proof. In the following, let $G \in \mathcal{G}$. We fix an ε-close $(r-1)$-partition $V_0, \ldots, V_{r-1}$ of G with ℓ non-crossing edges. Let the number of crossing non-edges be k.

First we show (*c*). Let e be a non-crossing edge of G. Without loss of generality, we may presume e lies in V_1. We can construct an r-clique using e as follows: we choose any common neighbour v_2 of e in V_2, then a common neighbour v_3 of e and v_2 in V_3, and so on. By definition of an ε-close $(r-1)$-partition, for each $2 \leq i \leq r-1$, the common neighbourhood of $e, v_2, \ldots, v_{i-1}$ in V_i has size at least $(1-i\varepsilon)|V_i| > \frac{1}{2}n/(r-1)$ because $\varepsilon < 1/(2r)$. It follows that e lies in at least $(n/(2r-2))^{r-2}$ copies of K_r in G. Further, if e' is a second non-crossing edge of G, then no r-clique of G using e' can be one of the r-cliques through e given by the above construction. It follows that G contains $\ell(n/(2r-2))^{r-2}$ copies of K_r.

Now we prove (*a*) and (*b*). We first show that

$$\ell \geq |V_0| + k + 1\,. \tag{15}$$

If $V_0 = \emptyset$, then we have $t_r(n) + 1 \le e(G) \le t_r(n) + \ell - k$, and therefore $\ell \ge |V_0| + k + 1$. If $V_0 \ne \emptyset$ on the other hand, then, since every vertex in V_0 has degree at most $(1-\varepsilon^2)\frac{r-2}{r-1}n$, we have

$$t_r(n) + 1 \le e(G) \le (1-\varepsilon^2)\frac{r-2}{r-1}n|V_0| + \left(\frac{n - |V_0|}{r-1}\right)^2 \binom{r-1}{2} + \ell - k\,.$$

Using the facts $|V_0| \le \varepsilon^2 n$ and $\left(\frac{n}{r-1}\right)^2\binom{r-1}{2} \le t_r(n) + r^2$, we infer

$$\begin{aligned} t_r&(n) + 1 \\ &\le (1-\varepsilon^2)\tfrac{r-2}{r-1}n|V_0| + \left(\tfrac{n}{r-1}\right)^2\tbinom{r-1}{2} - \tfrac{r-2}{r-1}n|V_0| + \tfrac{(r-2)}{2(r-1)}|V_0|^2 + \ell - k \\ &\le t_r(n) + r^2 - \varepsilon^2\tfrac{r-2}{r-1}n|V_0| + \varepsilon^2\tfrac{r-2}{2(r-1)}n|V_0| + \ell - k \\ &= t_r(n) + r^2 - \varepsilon^2\tfrac{r-2}{2(r-1)}n|V_0| + \ell - k\,. \end{aligned}$$

It follows from $n \ge 2r^3/\varepsilon^2$ that $\varepsilon^2\frac{r-2}{2(r-1)}n|V_0| \ge r^2 + |V_0|$, and so we again obtain $\ell \ge |V_0| + k + 1$.

Now, if $G \in \mathcal{G}$ exists, then (15) clearly implies $\ell > 0$, proving (a). It remains to show (b). We can construct any graph G in $\mathcal{G}$ as follows. We choose $k \in \{0, \dots, \ell - 1\}$. We partition $[n]$ into r sets $V_0, \dots, V_{r-1}$ such that V_0 satisfies (15). For each pair of vertices intersecting V_0, we choose whether or not to make it an edge of G; there are at most $2^{|V_0|n} \le 2^{\ell n}$ such choices. Then we choose k pairs of vertices crossing the partition to be non-edges of G, and make all other crossing pairs edges of G. Finally, we choose ℓ pairs of vertices within partition classes to be the ℓ non-crossing edges of G. The total number of choices in this process is at most

$$\sum_{0 \le k \le \ell - 1} r^n 2^{\ell n} \binom{\binom{n}{2}}{k}\binom{\binom{n}{2}}{\ell} \overset{(15)}{\le} \ell r^n 2^{\ell n} n^{2\ell + 2\ell} \le r^{5\ell n}\,,$$

as required. □

With these tools at hand we can proceed to the proof of Theorem 7. For a binomially distributed random variable X we will use the following Chernoff bound which can be found, e.g., in [11, Theorem 2.1]. For each $\gamma \in (0, \frac{1}{3})$ we have

$$\mathbb{P}\big(X \le (1-\gamma)\mathbb{E}X\big) \le \exp(-\gamma^2\mathbb{E}X/2)\,. \tag{16}$$

Proof of the 1*-statements of Theorem 7.* We shall first prove the case $r = 3$ and then turn to the case $r > 3$. In both cases we will consider the class $\mathcal{G}_r$ of all n-vertex graphs G with $e(G) > t_r(n)$. In the case $r = 3$, $\mathcal{G}_3$ can be written as the union of three sub-classes $\mathcal{G}_{\mathrm{A}}$, $\mathcal{G}_{\mathrm{B}}$, and $\mathcal{G}_{\mathrm{C}}$ defined by the properties in (i), (ii), and (iii) of Corollary 19, respectively. Similarly, for $r > 3$ Lemma 18 allows us to write $\mathcal{G}_r = \mathcal{G}_{\mathrm{D}} \cup \mathcal{G}_{\mathrm{E}}$, where the graphs $\mathcal{G}_{\mathrm{D}}$ and $\mathcal{G}_{\mathrm{E}}$ enjoy properties given by Lemma 18(i) and Lemma 18(ii), respectively. We will prove that for each of these sub-classes a.a.s. the random hypergraph $\mathcal{R}^{(r)}(n,p)$ with p as required detects all graphs in this sub-class. The result then follows from the union bound.

<u>*Case $r = 3$:*</u> Let $p > 1/2$ be fixed and set

$$\varepsilon := \min\left\{\frac{1}{144}, \frac{p}{8}, \frac{2p-1}{4p+3}\right\}.$$

Let $\delta > 0$ be guaranteed by Corollary 19 for this ε. Observe that this choice of ε and n allows the application of Corollary 19. Further, let $\mathcal{G}_3 = \mathcal{G}_{\mathrm{A}} \cup \mathcal{G}_{\mathrm{B}} \cup \mathcal{G}_{\mathrm{C}}$ be as defined above. We will now show for each of the graph classes $\mathcal{G}_{\mathrm{A}}$, $\mathcal{G}_{\mathrm{B}}$, and $\mathcal{G}_{\mathrm{C}}$ that a.a.s. $\mathcal{R}^{(3)}(n,p)$ detects all their members.

Suppose a graph $G \in \mathcal{G}_{\mathrm{A}}$ is given. Then Corollary 19(i) the graph G contains at least δn^3 triangles. The probability that $\mathcal{R}^{(3)}(n,p)$ does not detect G is at most

$$(1-p)^{\delta n^3} \leq e^{-p\delta n^3} \leq e^{-\delta n^3/2} ,$$

and since $|\mathcal{G}_{\mathrm{A}}| < 2^{\binom{n}{2}}$, applying the union bound, the probability that there is a graph in $\mathcal{G}_{\mathrm{A}}$ which $\mathcal{R}^{(3)}(n,p)$ does not detect is at most

$$2^{\binom{n}{2}} e^{-\delta n^3/2} ,$$

which tends to zero as n tends to infinity.

Recall that $\mathcal{G}_{\mathrm{B}}$ is the sub-class of $\mathcal{G}_3$ with graphs in which there is a vertex u and disjoint set $X, Y \subseteq \Gamma(u)$ with both $|X||Y| \geq \varepsilon n^2/288$ and $e(X,Y) \geq (1-4\varepsilon)|X||Y|$. Suppose that a 3-uniform n-vertex hypergraph $\mathcal{H}$ has the property that for every vertex v and disjoint sets W and Z with $|W||Z| \geq \varepsilon n^2/288$, there are more than $4\varepsilon|W||Z|$ hyperedges of $\mathcal{H}$, each consisting of v, a vertex of W, and a vertex of Z. Then, clearly for any $G \in \mathcal{G}_{\mathrm{B}}$ the hypergraph $\mathcal{H}$ detects G. Hence it remains to show that a.a.s. $\mathcal{R}^{(3)}(n,p)$ has this property.

Given one vertex v and pair of disjoint vertex sets X and Y of $\mathcal{R}^{(3)}(n,p)$ with $|X||Y| \geq \varepsilon n^2/288$ the expected size of $E\big(\mathrm{Link}_{\mathcal{R}^{(3)}(n,p)}(v)\big) \cap (X \times Y)$ in $\mathcal{R}^{(3)}(n,p)$ is $p|X||Y|$. Using the Chernoff bound (16), the probability that we have

$$e\big(\mathrm{Link}_{\mathcal{R}^{(3)}(n,p)}(v) \cap (X \times Y)\big) < 4\varepsilon|X||Y| \leq p|X||Y|/2$$

is at most $e^{-p|X||Y|/8} \leq e^{-\varepsilon n^2/5000}$. By the union bound, the probability that there exists any such vertex and pair of disjoint subsets in $\mathcal{R}^{(3)}(n,p)$ is at most

$$n 2^n 2^n e^{-\varepsilon n^2/5000}$$

which tends to zero as n tends to infinity.

Finally, $\mathcal{G}_{\mathrm{C}}$ is the class of n-vertex graphs $G \in \mathcal{G}_3$ which possess an ε-close 2-partition $V_0 \dot\cup V_1 \dot\cup V_2$. Since $e(G) \geq t_r(n) + 1$ there is at least one non-crossing edge e in this partition by Lemma 20(a). Without loss of generality, we may presume e lies in V_1. Then the common neighbourhood of e contains more than $(1-2\varepsilon)|V_2| \geq (1-3\varepsilon)\frac{n}{2}$ vertices. In particular, if $\mathcal{R}^{(3)}(n,p)$ has the property that every pair of vertices is in at least $(1+3\varepsilon)\frac{n}{2}$

hyperedges, then $\mathcal{R}^{(3)}(n,p)$ detects every graph in $\mathcal{G}_{\mathrm{C}}$. We will show that a.a.s. $\mathcal{R}^{(3)}(n,p)$ has this property.

Given one pair of vertices u,v, we have

$$\mathbb{E}\big(e(\mathrm{Link}_{\mathcal{R}^{(3)}(n,p)}(u,v))\big) = p(n-2)\ .$$

Using the fact that $\varepsilon \leq \frac{2p-1}{4p+3}$ we note that

$$(1+3\varepsilon)\frac{n}{2} \leq \Big(1+3\frac{2p-1}{4p+3}\Big)\frac{n}{2} = \Big(1-2\frac{2p-1}{4p+3}\Big)pn < (1-\varepsilon)p(n-2)\ ,$$

for large enough n. The Chernoff bound (16) then gives

$$\mathbb{P}\left(e(\mathrm{Link}_{\mathcal{R}^{(3)}(n,p)}(u,v)) \leq (1+3\varepsilon)\frac{n}{2}\right) \leq$$
$$\mathbb{P}\left(e(\mathrm{Link}_{\mathcal{R}^{(3)}(n,p)}(u,v)) \leq (1-\varepsilon)p(n-2)\right) \leq e^{-\varepsilon^2 p(n-2)/2}\ .$$

By the union bound, the probability that there exists any such pair of vertices in $\mathcal{R}^{(3)}(n,p)$ is at most $\binom{n}{2}e^{-\varepsilon^2 p(n-2)/2}$, which tends to zero as n tends to infinity.

Case $r>3$: Let $\varepsilon := 1/(16r^2)$, and let $\delta > 0$ be the positive constant guaranteed by Lemma 18 for this ε. Let $\mathcal{G}_r = \mathcal{G}_{\mathrm{D}} \cup \mathcal{G}_{\mathrm{E}}$ be classes of n-vertex graphs satisfying (i) and (ii) of Lemma 18, respectively. Set

$$C := \max\left\{\frac{1}{\delta}, 6r(2r-2)^{r-2}\right\}\ , \qquad \text{and let} \quad p \geq Cn^{3-r}\ .$$

Again, we will prove that a.a.s. $\mathcal{R}^{(r)}(n,p)$ detects all graphs in $\mathcal{G}_{\mathrm{D}}$ and $\mathcal{G}_{\mathrm{E}}$.

The class $\mathcal{G}_{\mathrm{D}}$ contains the graphs from $\mathcal{G}_r$ in which there is a vertex contained in at least δn^{r-1} copies of K_r. Given one such graph G, the probability that G is not detected by $\mathcal{R}^{(r)}(n,p)$ is at most

$$(1-p)^{\delta n^{r-1}} < e^{-Cn^{3-r}\delta n^{r-1}} = e^{-C\delta n^2} \leq e^{-n^2}\ ,$$

and since there are at most $2^{\binom{n}{2}}$ graphs in $\mathcal{G}_{\mathrm{D}}$, the probability that there is a graph in $\mathcal{G}_{\mathrm{D}}$ undetected by $\mathcal{R}^{(r)}(n,p)$ is at most

$$2^{\binom{n}{2}}e^{-n^2}\ ,$$

which tends to zero as n tends to infinity.

It remains to consider the class $\mathcal{G}_{\mathrm{E}}$ of graphs $G \in \mathcal{G}_r$ with ε-close $(r-1)$-partition. For $1 \leq \ell \leq \binom{n}{2}$ let $\mathcal{G}_{\mathrm{E}}(\ell) \subseteq \mathcal{G}_{\mathrm{E}}$ be the class of graphs that have an ε-close $(r-1)$-partition with exactly ℓ non-crossing edges. By Lemma 20(a) we have

$$\bigcup_{1\leq\ell\leq\binom{n}{2}} \mathcal{G}_{\mathrm{E}}(\ell) = \mathcal{G}_{\mathrm{E}}\ . \tag{17}$$

Now fix $\ell \in \{1,\dots,\binom{n}{2}\}$. Lemma 20$(b)$ asserts that $|\mathcal{G}_{\mathrm{E}}(\ell)| \leq r^{5\ell n}$. Moreover, each graph in $\mathcal{G}_{\mathrm{E}}(\ell)$ contains at least $\ell(n/(2r-2))^{r-2}$ copies of K_r by

Lemma 20(*c*). Hence, by the union bound, the probability that $\mathcal{R}^{(r)}(n,p)$ fails to detect at least one graph in $\mathcal{G}_{\mathrm{E}}(\ell)$ is at most

$$r^{5\ell n}(1-p)^{\left(\frac{n}{2r-2}\right)^{r-2}\ell} < r^{5\ell n}\exp\left(-Cn^{3-r}\ell\left(\frac{n}{2r-2}\right)^{r-2}\right) \le r^{5\ell n}e^{-6r\ell n} < e^{-\ell n}\,.$$

Finally, applying the union bound in conjunction with (17), we conclude that $\mathcal{R}^{(r)}(n,p)$ detects all graphs in $\mathcal{G}_{\mathrm{E}}$ with probability at least $1-\binom{n}{2}e^{-n}$, which tends to one as n tends to infinity. □

6. Turánnical hypergraphs for random graphs

In this section we prove Theorem 11. For this purpose we shall use the machinery developed by Schacht [17] for proving Theorem 9. Conlon and Gowers [3] obtained independently (using different methods) a result very similar to Schacht's. While either result is equally suited for proving 11 we follow notation introduced in [17]. Schacht formulates a powerful abstract result, a so-called *transference theorem* (Theorem 3.3 in [17]; see also Theorem 4.5 in [3]), which is phrased in the language of hypergraphs and gives very general conditions under which a result from extremal combinatorics may be transferred to an analogue for sparse random structures. Actually, Theorem 9 mentioned above is only one of several results where the transference theorem applies. Schacht, and Conlon and Gowers, give further applications to transfer the multidimensional Szemerédi theorem, a result on Schur's equation, and others. Here we are interested in a transference of Theorem 6.

Below we will state a special version of Schacht's transference theorem, tailored to our situation. For formulating this theorem we need some definitions. We remark that in these definitions we slightly deviate from Schacht's setting. More precisely, the transference theorem uses a certain sequence of hypergraphs which encode the classical extremal problem under consideration. In the case of Turán's problem for K_r, the n-th hypergraph in this sequence has vertex set $E(K_n)$ and a hyperedge for every $\binom{r}{2}$-tuple of elements from $E(K_n)$ which form a copy of K_r in K_n in Schacht's setting. Instead, we shall work with r-uniform hypergraphs $\mathcal{H}_n$ on vertex set $V(K_n)$, making use of the fact that a copy of K_r is uniquely identified by its vertices. The corresponding modifications of the definitions and of the transference theorem are straightforward.

The transference theorem requires the sequence of hypergraphs to satisfy two conditions. The first one is a requirement upon the extremal problem to be transferred, namely, that it has a certain 'super-saturation' property (similar to the one given in Theorem 14). The following definition makes this precise.

Definition 21 ((α,ε,ζ)-dense)**.** *Let* $\mathbf{H}=(\mathcal{H}_n)_{n\in\mathbb{N}}$ *be a sequence of* n*-vertex* r*-uniform hypergraphs,* $\alpha\ge 0$ *and* $\varepsilon,\zeta>0$ *be constants. We say* $\mathbf{H}$ *is*

$(\alpha, \varepsilon, \zeta)$-dense *if the following is true. There exists* n_0 *such that for every* $n \geq n_0$ *and every graph* G *on the vertex set* $V(\mathcal{H}_n)$ *with at least* $(\alpha+\varepsilon)\binom{n}{2}$ *edges, the number of copies of* K_r *in* G *induced by hyperedges of* $\mathcal{H}_n$ *is at least* $\zeta e(\mathcal{H}_n)$.

The second condition determines the sparseness of a random graph to which one may transfer the extremal result. Given an r-uniform hypergraph $\mathcal{H}$, a graph G on the same vertex set, and a pair of distinct vertices u and v of $V(G)$, we let $\deg_i(u,v,G)$ be the number of hyperedges of $\mathcal{H}$ containing u, v and at least i edges of G, not counting the possible edge uv. If $u = v$ we let $\deg_i(u,v,G) := 0$. The hypergraph $\mathcal{H}$ itself is suppressed from the notation as it will be clear from the context. We set

$$\mu_i(\mathcal{H}, q) := \mathbb{E}\Big[\sum\nolimits_{u,v} \deg_i^2\big(u,v,G(n,q)\big)\Big]\,,$$

where the expectation is taken over the space of random graphs $G(n,q)$.

Definition 22 ($(K,\mathbf{q})$-bounded)**.** *Let* $\mathbf{H} = (\mathcal{H}_n)_{n\in\mathbb{N}}$ *be a sequence of* n-*vertex* r-*uniform hypergraphs,* $\mathbf{q} = (q_n)_{n\in\mathbb{N}}$ *be a sequence of probabilities, and* $K \geq 1$ *be a constant. We say that* $\mathbf{H}$ *is* $(K,\mathbf{q})$-bounded *if the following holds. For each* $i \in [\binom{r}{2}-1]$ *there exists* n_0 *such that for each* $n \geq n_0$ *and* $q \geq q_n$ *we have*

$$\mu_i(\mathcal{H}_n, q) \leq K q^{2i} \cdot \frac{e(\mathcal{H}_n)^2}{n^2}\,.$$

We can now state (a special case of) Schacht's transference theorem.

Theorem 23 (transference theorem, Schacht [17])**.** *For all* $r \geq 3$, $K \geq 1$, $\delta > 0$, $\zeta > 0$ *and* $(\omega_n)_{n\in\mathbb{N}}$ *with* $\omega_n \to \infty$ *as* $n \to \infty$, *there exists* $C > 1$ *such that the following holds. Let* $\varepsilon := 8^{-r(r-1)/2}\delta$, *and let* $\mathbf{H} = (\mathcal{H}_n)_{n\in\mathbb{N}}$ *be a sequence of* n-*vertex* r-*uniform hypergraphs which is* $(\frac{r-2}{r-1}, \varepsilon, \zeta)$-*dense. Let* $\mathbf{q} = (q_n)_{n\in\mathbb{N}}$ *be a sequence of probabilities with* $q_n^{r(r-1)/2} \cdot e(\mathcal{H}_n) \to \infty$ *and* $Cq_n < 1/\omega_n$ *such that* $\mathbf{H}$ *is* $(K,\mathbf{q})$-*bounded.*

Then the following holds a.a.s. for $G = G(n, Cq_n)$. *Every subgraph of* G *with at least* $(\frac{r-2}{r-1}+\delta)\cdot e(G)$ *edges contains an* r-*clique induced by a hyperedge of* $\mathcal{H}_n$.

We remark that the quantification in this theorem and the $(\alpha,\varepsilon,\zeta)$-denseness condition given here is not the same as in [17] (in fact, in [17] the two parameters ε and ζ are not made explicit in the concept of α-denseness used in [17]). The statement in [17] is certainly cleaner, but for our purposes it is necessary that we check the denseness condition only for a special ε (as opposed to all $\varepsilon > 0$, which is necessary for the original definition of α-denseness), and that the constant C does not depend on the sequences $\mathbf{H}$ or $\mathbf{q}$. That Theorem 23 is valid, however, follows easily from the proof of [17, Theorem 3.3]. This can be checked as follows. It is clearly stated in the proof of [17, Theorem 3.3] that the requirement of $(\alpha,\varepsilon,\zeta)$-denseness is necessary only once, namely for the base case of the induction performed there, with the value $\varepsilon = 8^{-r(r-1)/2}\delta$ given above. The values of the various constants are also explicitly stated in the proof. In particular, the value of C does indeed depend only upon r, K, δ and ζ as claimed.

To prove the 1-statement of Theorem 11, we need to further modify the setting from [17]: we do not have a sequence of fixed hypergraphs, but instead a sequence of random objects $\mathcal{R}^{(r)}(n, p_n)$. We describe how to modify the above definitions appropriately, and explain why the transfer result we require, Corollary 26, follows from Theorem 23.

Definition 24 ($(\alpha, \varepsilon, \zeta)$-dense for random hypergraphs)**.** *Let $\mathbf{p} = (p_n)_{n\in\mathbb{N}}$ be a sequence of probabilities, and let $\alpha, \varepsilon, \zeta \geq 0$ be constants. We say the random hypergraph $\mathcal{R}^{(r)}(n, p_n)$ is* a.a.s. $(\alpha, \varepsilon, \zeta)$-dense *if a.a.s. for $\mathcal{R}_n = \mathcal{R}^{(r)}(n, p_n)$, the following is true. For every n-vertex graph G on $[n]$ with at least $(\alpha+\varepsilon)\binom{n}{2}$ edges, the number of copies of K_r in G induced by hyperedges of $\mathcal{R}_n$ is at least $\zeta e(\mathcal{R}_n)$.*

Next, we modify the definition of boundedness.

Definition 25 ($(K, \mathbf{q})$-bounded for random hypergraphs)**.** *Let $\mathbf{p} = (p_n)_{n\in\mathbb{N}}$ and $\mathbf{q} = (q_n)_{n\in\mathbb{N}}$ be sequences of probabilities and $K \geq 1$ be a constant. We say that the random hypergraph $\mathcal{R}^{(r)}(n, p_n)$ is* a.a.s. $(K, \mathbf{q})$-bounded *if the following holds a.a.s. for $\mathcal{R}_n = \mathcal{R}^{(r)}(n, p_n)$. For each $i \in [\binom{r}{2} - 1]$ and $\tilde{q} \geq q_n$, we have*

$$\mu_i(\mathcal{R}_n, \tilde{q}) \leq K\tilde{q}^{2i} \cdot \frac{e(\mathcal{R}_n)^2}{n^2} .$$

Using these definitions we obtain the following transference result using random hypergraphs as a corollary to Theorem 23.

Corollary 26. *Given $r \geq 3$, $K \geq 1$, $\delta > 0$, $\zeta > 0$ and $(\omega_n)_{n\in\mathbb{N}}$ with $\omega_n \to \infty$ as $n \to \infty$, let $\varepsilon := \delta/8^{\binom{r}{2}}$. There exists $C > 1$ such that the following is true. Let $\mathbf{p} = (p_n)_{n\in\mathbb{N}}$ be a sequence of probabilities such that $\mathcal{R}^{(r)}(n, p_n)$ is a.a.s. $\big(\frac{r-2}{r-1}, \varepsilon, \zeta\big)$-dense. Let $\mathbf{q} = (q_n)_{n\in\mathbb{N}}$ be a sequence of probabilities such that $Cq_n < 1/\omega_n$, such that for every integer L, a.a.s. $q_n^{r(r-1)/2} \cdot e\big(\mathcal{R}^{(r)}(n, p_n)\big) > L$, and such that $\mathcal{R}^{(r)}(n, p_n)$ is a.a.s. $(K, \mathbf{q})$-bounded. Then for $G = G(n, Cq_n)$ and $\mathcal{R}_n = \mathcal{R}^{(r)}(n, p_n)$ a.a.s. $\mathcal{R}_n$ is δ-Turánnical for G.*

Proof. Given $r \geq 3$, $K \geq 1$, $\delta > 0$, $\zeta > 0$ and $(\omega_n)_{n\in\mathbb{N}}$ with $\omega_n \to \infty$ as $n \to \infty$, let C be the constant returned by Theorem 23. Let $\mathbf{p}$ and $\mathbf{q}$ be sequences of probabilities satisfying the conditions of the corollary.

We define a property $\mathcal{A}_n$ of r-uniform hypergraphs as follows. An n-vertex hypergraph $\mathcal{H}_n$ has property $\mathcal{A}_n$ if for all n-vertex graphs H with $V(H) = V(\mathcal{H}_n)$ and $e(H) \geq \big(\frac{r-2}{r-1} + \varepsilon\big)\binom{n}{2}$ the number of copies of K_r in H induced by hyperedges of $\mathcal{H}_n$ is at least $\zeta e(\mathcal{H}_n)$.

We claim that there is a monotone function $\nu(n)$ tending to zero as n tends to infinity with the following properties.

(a) Let $P_1(n)$ be the probability that $\mathcal{R}_n = \mathcal{R}^{(r)}(n, p_n)$ has the property $\mathcal{A}_n$. Then $P_1(n) \ge 1 - \nu(n)$.

(b) There is a function $L(n)$ tending to infinity such that the probability $P_2(n)$ that for $\mathcal{R}_n = \mathcal{R}^{(r)}(n, p_n)$

$$q_n^{r(r-1)/2} \cdot e\big(\mathcal{R}_n\big) > L(n) \tag{18}$$

is at least $1 - \nu(n)$.

(c) The probability $P_3(n)$ that, for $\mathcal{R}_n = \mathcal{R}^{(r)}(n, p_n)$, we have for each $i \in [\binom{r}{2} - 1]$ and $\tilde{q} \ge q_n$

$$\mu_i(\mathcal{R}_n, \tilde{q}) \le K \tilde{q}^{2i} \cdot \frac{e(\mathcal{R}_n)^2}{n^2}\,, \tag{19}$$

is at least $1 - \nu(n)$.

Items (a) and (c) are immediate from the definitions of $(\frac{r-2}{r-1}, \varepsilon, \zeta)$-denseness and $(K, \mathbf{q})$-boundedness, respectively. Item (b) is immediate from the fact that for each L, a.a.s. $q_n^{r(r-1)/2} \cdot e\big(\mathcal{R}_n\big) > L$ holds.

Let n_0 be such that $\nu(n_0) < \frac{1}{3}$. We fix a sequence $\mathbf{R} = (\mathcal{R}_n)_{n \in \mathbb{N}}$ of hypergraphs in the following way. For each $n \ge n_0$, consider the set of all n-vertex hypergraphs satisfying Property $\mathcal{A}_n$, (18), and (19). This set is non-empty by choice of n_0. Now let $\mathcal{R}_n$ be the element of this set which maximises the probability $P_4(n)$ that the random graph $G = G(n, Cq_n)$ possesses a subgraph with at least $(\frac{r-2}{r-1} + \delta) \cdot e(G)$ edges which is undetected by $\mathcal{R}_n$. For $n < n_0$ let $\mathcal{R}_n$ be an arbitrary n-vertex hypergraph.

We deduce from Property $\mathcal{A}_n$ that $\mathbf{R}$ is $\big(\frac{r-2}{r-1}, \varepsilon, \zeta\big)$-dense (in the sense of Definition 21), from (19) that $\mathbf{R}$ is $(K, \mathbf{q})$-bounded (in the sense of Definition 22), and from (18) that $\mathbf{R}$ satisfies $q_n^{r(r-1)/2} \cdot e(\mathcal{R}_n) \to \infty$. It follows that we can apply Theorem 23 to $\mathbf{R}$, which implies that the probability $P_4(n)$ tends to zero as n tends to infinity. Consequently, with probability at least $1 - \big((1 - P_1(n)) + (1 - P_2(n)) + (1 - P_3(n))\big) - P_4(n) \ge 1 - 3\nu(n) - P_4(n) = 1 - o(1)$, the random hypergraph $\mathcal{R}^{(r)}(n, p_n)$ detects every subgraph of $G = G(n, Cq_n)$ with at least $(\frac{r-2}{r-1} + \delta) \cdot e(G)$ edges. Hence $\mathcal{R}^{(r)}(n, p_n)$ is a.a.s. δ-Turánnical for $G(n, Cq_n)$. □

To prove the 1-statement of Theorem 11 it now suffices to check that the conditions of Theorem 11 guarantee that $\mathcal{R}^{(r)}(n, p)$ satisfies the conditions of Corollary 26. We will make use of the Chernoff bound for a binomial random variable X (see, e.g., [11, Theorem 2.1])

$$\mathbb{P}\big(X \ge (1 + \gamma)\mathbb{E}X\big) \le \exp(-\gamma^2 \mathbb{E}X/3)\,, \qquad \text{for } \gamma \le 1/2\,. \tag{20}$$

The last tool we shall need for our proof of Theorem 11 is a counterpart of Theorem 14 for random graphs due to Kohayakawa, Rödl and Schacht.

Theorem 27 (Kohayakawa, Rödl & Schacht [13])**.** *Given any $r \in \mathbb{N}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any sequence of probabilities $(q_n)_{n \in \mathbb{N}}$ with $\liminf_n q_n > 0$ the following is a.a.s. true for the random graph $G =$*

$G(n,q_n)$. If $G' \subseteq G$ is a graph with at least $(1+\varepsilon)\frac{r-2}{r-1}e(G)$ edges, then there are at least $\delta q_n^{r(r-1)/2} n^r$ copies of K_r in G'.

Kohayakawa, Rödl and Schacht prove their result for a wider range of probabilities, allowing q_n's to decrease roughly at the speed $n^{-\frac{1}{r-1}}$. However we do not need this stronger result. Actually, in our setting when $\liminf_n q_n > 0$, Theorem 27 has a relatively simple proof using Szemerédi's Regularity Lemma. Let us remark that Theorem 27 was one of the early contributions to the Kohayakawa-Łuczak-Rödl conjecture.

Proof of Theorem 11. Given r and $\varepsilon \in (0, 1/(r-2))$, set $\delta' := \varepsilon$ and $\varepsilon' := \delta'/8^{\binom{r}{2}}$. Let $\zeta > 0$ be the constant provided by Theorem 14 for r and ε'. Now set

$$K' := r^{2r+5} 2^{r^2+3} \tag{21}$$

and let C' be the constant returned by Corollary 26 for input r, K', δ' and ζ. Let δ^* be given by Theorem 27 for input parameters ε and r. Set

$$c := \tfrac{1}{16}\left(\tfrac{1}{r-1} - \varepsilon\tfrac{r-2}{r-1}\right) \qquad \text{and} \qquad C := \max\left\{\tfrac{8}{\zeta}, C'^{(r+1)(r-2)/2}, \tfrac{2}{\delta^*}\right\}. \tag{22}$$

The constants c and C from (22) define the thresholds for the 0-statement and 1-statement of Theorem 11. Let $\mathbf{p} = (p_n)_{n\in\mathbb{N}}$ and $\mathbf{q} = (q_n)_{n\in\mathbb{N}}$ be given. We let $\mathcal{T}_n$ denote the event that $\mathcal{R}^{(r)}(n,p_n)$ is ε-Turánnical for $G(n,q_n)$.

First we prove the 0-*statement.* Since adding hyperedges to a sequence of hypergraphs does not destroy their property of being a.a.s. ε-Turánnical for $G(n,q_n)$, we can assume that

$$p_n = c\big(nq_n^{(r+1)/2}\big)^{2-r} \quad \text{and hence} \quad q_n = c'\left(np_n^{1/(r-2)}\right)^{-2/(r+1)}, \tag{23}$$

where $c' := c^{2/((r+1)(r-2))}$. In particular, since $1 \geq p_n$, we have that

$$q_n \geq c' n^{-2/(r+1)} \tag{24}$$

Recall that we are dealing with two random objects, the random graph $G(n,q_n)$ and the random hypergraph $\mathcal{R}^{(r)}(n,p_n)$. In the following argumentation we shall first perform the random experiment for $G(n,q_n)$ and then the one for $\mathcal{R}^{(r)}(n,p_n)$.

Let us first expose the graph $G(n,q_n)$. The Chernoff bound (16) implies that the probability that $G(n,q_n)$ has less than $q_n n^2/4$ edges tends to zero. Moreover, the random variable X counting copies of K_r in $G(n,q_n)$ has expectation $\binom{n}{r}q_n^{r(r-1)/2}$ and variance $\mathcal{O}\big(n^r q_n^{r(r-1)/2}\big)$ (see for example Lemma 3.5 of [11]). Hence, applying Chebyshev's inequality and observing that $n^r q_n^{r(r-1)/2} \to \infty$ by (24), we obtain that $\mathbb{P}\big[X \geq 2\binom{n}{r}q_n^{r(r-1)/2}\big] = o(1)$.

Since a.a.s. $G(n,q_n)$ has at least $q_n n^2/4$ edges and

$$X < 2\tbinom{n}{r}q_n^{r(r-1)/2}, \tag{25}$$

from now we assume these two events occur. We next expose the hypergraph $\mathcal{R}^{(r)}(n,p_n)$. Let Y be the random variable counting the hyperedges of $\mathcal{R}^{(r)}(n,p_n)$ which induce copies of K_r in $G=G(n,q_n)$. Observe that Y has distribution $\mathrm{Bin}\big(X,p_n\big)$. From the Chernoff bound (20) and from (25) we infer that a.a.s. Y does not exceed $4\binom{n}{r}q_n^{r(r-1)/2}p_n$. Because

$$\binom{n}{r}q_n^{\binom{r}{2}}p_n=\binom{n}{r}q_n^{(r-2)(r+1)/2}p_nq_n\overset{(23)}{=}\binom{n}{r}cn^{2-r}q_n\,, \tag{26}$$

we thus a.a.s. have

$$\begin{aligned}Y&\le 4\tbinom{n}{r}q_n^{\binom{r}{2}}p_n\overset{(26)}{=}4\tbinom{n}{r}cn^{2-r}q_n\le 4cq_nn^2\\ &\overset{(22)}{=}\left(\frac{1}{r-1}-\varepsilon\frac{r-2}{r-1}\right)\frac{q_nn^2}{4}\le\left(\frac{1}{r-1}-\varepsilon\frac{r-2}{r-1}\right)e(G)\,.\end{aligned}$$

Hence, a.a.s. $\mathcal{R}_n=\mathcal{R}^{(r)}(n,p_n)$ does not detect some subgraph G' of G which is obtained by deleting at most $(\frac{1}{r-1}-\varepsilon\frac{r-2}{r-1})e(G)$ edges from G. In particular, $e(G')\ge(1+\varepsilon)\frac{r-2}{r-1}e(G)$, which finishes the proof of the 0-statement.

We now turn to the 1-*statement*. Again, by monotonicity, we can assume that

$$p_n=C\big(nq_n^{(r+1)/2}\big)^{2-r}\quad\text{and hence}\quad q_n=C_q\big(np_n^{1/(r-2)}\big)^{-2/(r+1)}\,, \tag{27}$$

where $C_q:=C^{2/((r+1)(r-2))}\overset{(22)}{\ge}C'$. Since $p_n\le 1$ and $q_n\le 1$ we have that

$$q_n\ge C_qn^{-2/(r+1)}\qquad\text{and}\qquad p_n\ge Cn^{2-r}\,. \tag{28}$$

We can assume (by taking subsequences if it is necessary) that either $\liminf_n q_n>0$, or $q_n=o(1)$. In the former case we mimic our proof of Theorem 6 while in the latter case we apply Corollary 26.

Let us first prove the 1-statement when $\liminf_n q_n>0$. We repeat the proof strategy of the 1-statement of Theorem 6. Suppose that G' is an arbitrary graph on the vertex set $[n]$ with at least $\delta^*q_n^{r(r-1)/2}n^r$ copies of K_r. The probability that G' is not detected by $\mathcal{R}^{(r)}(n,p_n)$ is at most

$$(1-p_n)^{\delta^*q_n^{r(r-1)/2}n^r}\,.$$

Suppose now that a random graph $G=G(n,q_n)$ is given. We can assume that G has at most q_nn^2 edges as this property is a.a.s. satisfied. Consequently, G contains at most $2^{q_nn^2}$ subgraphs G' on the same vertex set. By Theorem 27 we a.a.s. have that each such subgraph with at least $(1+\varepsilon)\frac{r-2}{r-1}e(G)$ edges contains at least $\delta^*q_n^{r(r-1)/2}n^r$ copies of K_r. Therefore, the union bound over all such graphs G' gives that

$$\begin{aligned}\mathbb{P}\Big[\mathcal{R}^{(r)}(n,p_n)\text{ is not }\varepsilon\text{-Turánnical for }G(n,q_n)\Big]&\le 2^{q_nn^2}\cdot(1-p_n)^{\delta^*q_n^{\binom{r}{2}}n^r}\\ &\le\exp\left(q_nn^2-p_n\delta^*q_n^{\binom{r}{2}}n^r\right)\overset{(27)}{=}\exp\big(q_nn^2-Cn^2q_n\delta^*\big)\overset{(22)}{\to}0\,,\end{aligned}$$

and the statement follows in this case.

Let us now focus on the 1-statement in the case $q_n = o(1)$. The claim will follow from Corollary 26 (with parameters r, K', δ', ζ, C') applied to the sequences of probabilities $\mathbf{p}$ and $\mathbf{q}' = (q'_n)_{n\in\mathbb{N}} := \mathbf{q}/C'$, together with the following claim.

Claim 28. We have that

(a) for every L a.a.s. $(q'_n)^{r(r-1)/2} \cdot e\big(\mathcal{R}^{(r)}(n,p_n)\big) > L$,
(b) $\mathcal{R}^{(r)}(n,p_n)$ is a.a.s. $\big(\frac{r-1}{r-2}, \varepsilon', \zeta\big)$-dense, and
(c) $\mathcal{R}^{(r)}(n,p_n)$ is a.a.s. $(K', \mathbf{q}')$-bounded.

Proof of Claim 28. We first verify (a). We have

$$\mathbb{E}\Big(e\big(\mathcal{R}^{(r)}(n,p_n)\big)\Big) = p_n\binom{n}{r}\,,$$

which tends to infinity by (28). Consequently, the Chernoff bound (16) guarantees that a.a.s. $\mathcal{R}^{(r)}(n,p_n)$ has at least $p_n\binom{n}{r}/2$ hyperedges. Now we have

$$\frac{(q'_n)^{\binom{r}{2}}p_n\binom{n}{r}}{2} \overset{(27)}{=} \frac{(q'_n)^{\frac{r^2-r}{2}}Cn^{2-r}q_n^{(r+1)(2-r)/2}\binom{n}{r}}{2} = \Omega\left(q_n n^{2-r}\binom{n}{r}\right)\,,$$

and by (28) this tends to infinity.

Now we verify (b). Given an n-vertex graph H with $e(H) \geq \big(\frac{r-2}{r-1}+\varepsilon'\big)\binom{n}{2}$, by Theorem 14, H contains at least ζn^r copies of K_r. It follows that the expected number of hyperedges of $\mathcal{R}_n = \mathcal{R}^{(r)}(n,p_n)$ which induce copies of K_r in H is at least $\zeta n^r p_n$. By the Chernoff bound (16), the probability that less than $\zeta n^r p_n/2$ copies of K_r in H are induced by hyperedges of $\mathcal{R}_n$ is at most

$$\exp\left(-\frac{\zeta n^r p_n}{8}\right) \overset{(28)}{\leq} \exp\left(-\frac{C\zeta n^2}{8}\right) \overset{(22)}{=} o(2^{-n^2})\,.$$

Applying the union bound (on at most $2^{\binom{n}{2}}$ graphs H) we conclude that the probability that there exists any n-vertex graph H with at least $\big(\frac{r-1}{r-2}+\varepsilon'\big)\binom{n}{2}$ edges and less than $3\zeta\binom{n}{r}p_n/2 \leq \zeta n^r p_n/2$ copies of K_r on hyperedges of $\mathcal{R}_n$ tends to zero as n tends to infinity. Furthermore, applying the Chernoff bound (20) in conjunction with (28), the probability that $\mathcal{R}^{(r)}(n,p)$ has more than $3p_n\binom{n}{r}/2$ hyperedges tends to zero as n tends to infinity. It follows that for $\mathcal{R}_n$ a.a.s. every n-vertex graph H with more than $\big(\frac{r-2}{r-1}+\varepsilon'\big)\binom{n}{2}$ edges has at least $\zeta e(\mathcal{R}_n)$ copies of K_r on hyperedges of $\mathcal{R}_n$. Therefore, $\mathcal{R}^{(r)}(n,p_n)$ is a.a.s. $\big(\frac{r-2}{r-1}, \varepsilon', \zeta\big)$-dense.

Now we prove (c). We need to show that $\mathcal{R}_n = \mathcal{R}^{(r)}(n,p_n)$ a.a.s. has the property that for each $1 \leq i \leq \binom{r}{2}-1$ and each $\tilde{q} \geq q'_n$, we have

$$\mu_i(\mathcal{R}_n, \tilde{q}) \leq K'\tilde{q}^{2i}\frac{e(\mathcal{R}_n)^2}{n^2}\,. \tag{29}$$

We will show that (29) holds for all $1 \le i \le \binom{r}{2} - 1$ and $\tilde{q} \ge q'_n$ provided that $\mathcal{R}_n$ obeys a simple bound (inequality (31) below); this bound will turns out to hold a.a.s. for our random hypergraph.

Given a hypergraph $\mathcal{R}_n$ and two distinct vertices u and v, let F_1 and F_2 be two hyperedges containing u and v and intersecting in a set A of j vertices. Then the probability $P_{i,j}$ that both F_1 and F_2 contain at least i edges of the random graph $G = G(n, \tilde{q})$, not counting uv, can be bounded as follows. We use the random variables $X_A := |E(G[A]) \setminus uv|$, $X_{F_1} := e(G[F_1 \setminus A]) + e(G[F_1 \setminus A, A])$, and $X_{F_2} := e(G[F_2 \setminus A]) + e(G[F_2 \setminus A, A])$. Then

$$\begin{aligned} P_{i,j} &\le \sum_{k=0}^{\binom{j}{2}-1} \mathbb{P}(X_A = k)\mathbb{P}(X_{F_1} \ge i-k)\mathbb{P}(X_{F_2} \ge i-k) \\ &\le \sum_{k=0}^{\binom{j}{2}-1} \binom{\binom{j}{2}-1}{k} \tilde{q}^k \left(\binom{\binom{r}{2} - \binom{j}{2}}{i-k} \tilde{q}^{i-k} \right)^2 \\ &\le 2^{\binom{j}{2}-1+2(\binom{r}{2}-\binom{j}{2})} \sum_{k=0}^{\binom{j}{2}-1} \tilde{q}^{2i-k} \le j^2 2^{r^2} \cdot \tilde{q}^{2i+1-\binom{j}{2}} \,. \end{aligned} \tag{30}$$

Let $N(j)$ count the number of pairs of hyperedges in $\mathcal{R}_n$ intersecting in exactly j vertices. Then we have

$$\begin{aligned} \mu_i(\mathcal{R}_n, \tilde{q}) &= \mathbb{E}\left[\sum_{\substack{u,v \\ u \ne v}} \deg_i^2(u, v, G(n, \tilde{q})) \right] = \sum_{\substack{u,v \\ u \ne v}} \sum_{\substack{F_1 \in \mathcal{E}(\mathcal{R}_n) \\ F_1 \ni u,v}} \sum_{\substack{F_2 \in \mathcal{E}(\mathcal{R}_n) \\ F_2 \ni u,v}} P_{i,|F_1 \cap F_2|} \\ &= \sum_{j=2}^{r} N(j) j(j-1) P_{i,j} \overset{(30)}{\le} r^4 2^{r^2} \sum_{j=2}^{r} N(j) \tilde{q}^{2i+1-\binom{j}{2}} \,. \end{aligned}$$

It follows that $\mathcal{R}_n$ satisfies (29) if we have, for each $2 \le j \le r$ and $\tilde{q} \ge q'_n$,

$$r^5 2^{r^2} \cdot N(j) \cdot \tilde{q}^{1-\binom{j}{2}} \le K' \frac{e(\mathcal{R}_n)^2}{n^2} \,. \tag{31}$$

Since $j \ge 2$ we have $1 - \binom{j}{2} \le 0$. Therefore, the left-hand side of (31) is non-increasing in $\tilde{q}$. The right-hand side of (31) does not depend upon $\tilde{q}$. It follows that we need only verify that a.a.s. $\mathcal{R}_n = \mathcal{R}^{(r)}(n, p_n)$ satisfies (31) for each $2 \le j \le r$, with $\tilde{q} = q'_n$. We have that a.a.s. $e(\mathcal{R}^{(r)}(n, p_n)) \ge p_n \binom{n}{r}/2 \ge p_n n^r/(2r^r)$, by the Chernoff bound (16). So it is enough to show that a.a.s. for each $2 \le j \le r$ we have

$$N(j) \le \frac{K'}{r^5 2^{r^2}} (q'_n)^{\frac{(j-2)(j+1)}{2}} \frac{p_n^2 n^{2r-2}}{4r^{2r}} \overset{(21)}{=} 2(q'_n)^{\frac{(j-2)(j+1)}{2}} p_n^2 n^{2r-2} \,. \tag{32}$$

To show that (32) holds, we first consider the case $j = r$. Observe that $N(r)$ is simply the number of hyperedges in $\mathcal{R}^{(r)}(n, p_n)$, and is therefore (by the Chernoff bound (20)) a.a.s. at most $2p_n\binom{n}{r} \le 2p_n n^r$. Substituting $q_n' \ge \left(np_n^{1/(r-2)}\right)^{-2/(r+1)}$ into the right-hand side of (32) (for $j = r$), we have

$$2(q_n')^{\frac{(r-2)(r+1)}{2}} p_n^2 n^{2r-2} \ge 2\left(np_n^{\frac{1}{r-2}}\right)^{2-r} p_n^2 n^{2r-2} = 2p_n n^r \ .$$

Therefore (32) holds for $j = r$.

Suppose now that $2 \le j \le r-1$. Then we have

$$\mathbb{E}(N(j)) = \binom{n}{r}\binom{r}{j}\binom{n-r}{r-j} p_n^2 = \mathcal{O}(n^{2r-j}p_n^2)\,.$$

We have by (28) that $q_n' = \Omega\left(n^{-\frac{2}{r+1}}\right) = \omega\left(n^{-\frac{2}{j+1}}\right)$ for each $2 \le j \le r-1$. Consequently,

$$\mathbb{E}(N(j)) = \mathcal{O}(n^{2r-j}p_n^2) = \mathcal{O}(n^{2-j}p_n^2 n^{2r-2}) = o\left((q_n')^{\frac{(j-2)(j+1)}{2}} p_n^2 n^{2r-2}\right).$$

By Markov's inequality, (32) holds a.a.s. for every $2 \le j \le r-1$. This completes the proof that $\mathcal{R}^{(r)}(n, p_n)$ is a.a.s. $(K', \mathbf{q}')$-bounded. □

It follows that a.a.s. $\mathcal{R}^{(r)}(n, p_n)$ satisfies the conditions to apply Corollary 26, that is, a.a.s. $\mathcal{R}^{(r)}(n, p_n)$ is ε-Turánnical for $G(n, q_n)$. □

7. Sharp thresholds

In this section we use Friedgut's [8] condition for sharp thresholds to prove that the threshold we obtained in Theorem 7 is sharp. For a background on threshold phenomena we refer the reader to [8]. We show the following result.

Theorem 29. *For every integer $r \ge 3$ there are $c, C > 0$ and a sequence of numbers $(c_n \in (c, C))_{n\in\mathbb{N}}$ such that for every $\gamma > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathcal{R}^{(r)}\left(n, (c_n - \gamma)n^{3-r}\right) \text{ is Turánnical}\right) = 0 \quad \text{and,}$$
$$\lim_{n\to\infty} \mathbb{P}\left(\mathcal{R}^{(r)}\left(n, (c_n + \gamma)n^{3-r}\right) \text{ is Turánnical}\right) = 1\ .$$

As usual it is reasonable to conjecture that the sequence (c_n) in this theorem converges, and as usual in the field we are not able to prove this.

Before we can state Friedgut's result we need to introduce some notation. Given two hypergraphs $\mathcal{G}$ and $\mathcal{M}$ with $|V(\mathcal{G})| \ge |V(\mathcal{M})|$ we write $\mathcal{G} \cup \mathcal{M}^*$ for the random hypergraph obtained from the following random experiment. Let ϕ be a (uniformly chosen) random injection from $V(\mathcal{M})$ to $V(\mathcal{G})$ and for each hyperedge F of $\mathcal{M}$ add the hyperedge $\phi(F)$ to $\mathcal{G}$ (without creating multiple hyperedges). A family of r-uniform hypergraphs is called a *hypergraph property* if it is closed under isomorphism and under adding hyperedges.

Friedgut formulates his result for graphs. Here, we use the corresponding hypergraph result, specialised to our situation; see also [7] for a discussion of this result and for extensions to other combinatorial structures.

Theorem 30 (Friedgut [8, Theorem 2.4])**.** *Suppose that Theorem 29 does not hold for some $r \geq 3$. Then there exists a sequence $p = p_n$, $\tau > 0$, a fixed r-uniform hypergraph $\mathcal{M}$ with*

$$\mathbb{P}\big(\mathcal{M} \subseteq \mathcal{R}^{(r)}(n,p)\big) > \tau \ , \tag{33}$$

and $\alpha > 0$ with

$$\alpha < \mathbb{P}\big(\mathcal{R}^{(r)}(n,p) \text{ is Turánnical}\big) < 1 - 3\alpha \ , \tag{34}$$

and a constant $\varepsilon > 0$ such that, for every hypergraph property $\mathcal{P}$ which satisfies that $\mathcal{R}^{(r)}(n,p)$ is a.a.s. in $\mathcal{P}$, the following holds. There exists an infinite set $Z \subseteq \mathbb{N}$ and for each $n \in Z$ a hypergraph $\mathcal{G}_n \in \mathcal{P}$ such that

$$\mathbb{P}\big(\mathcal{G}_n \cup \mathcal{M}^* \text{ is Turánnical}\big) > 1 - \alpha \ , \tag{35}$$

$$\mathbb{P}\big(\mathcal{G}_n \cup \mathcal{R}^{(r)}(n,\varepsilon p) \text{ is Turánnical}\big) < 1 - 2\alpha \ . \tag{36}$$

With this result at hand, we can now give a proof of Theorem 29. It turns out that we do not need to utilise Theorem 30 in its full strength; in particular we shall not use assertion (33).

Proof of Theorem 29. Suppose that Theorem 29 does not hold for some $r \geq 3$. Let p_n, the r-uniform hypergraph $\mathcal{M}$, and $\alpha > 0$ be given by Theorem 30. In particular, by (34) we have that $\alpha < 1/4$. It follows from (34) and from Theorem 7 that

$$cn^{3-r} \leq p \leq Cn^{3-r} \ ,$$

for some absolute constants $c, C > 0$. Let $\beta := \frac{1}{2e(\mathcal{M})}$ and let $\mathcal{P}$ be the family of n-vertex hypergraphs which detect every n-vertex graph F with at least $\beta\binom{n}{r}$ r-cliques. It follows from the proof of Theorem 6 that a.a.s. $\mathcal{R}^{(r)}(n,p) \in \mathcal{P}$.

Let now $Z \subseteq \mathbb{N}$ and $(\mathcal{G}_n)_{n\in Z}$ be given by Theorem 30. We will derive a contradiction using just a single hypergraph $\mathcal{G}_n$, $n \in Z$. Indeed, from (36) we see that $\mathcal{G}_n$ itself cannot be Turánnical. Let W be a graph which witnesses this, i.e., W is an n-vertex graph with more than $t_r(n)$ edges which is not detected by $\mathcal{G}_n$. By the definition of $\mathcal{P}$ and since $\mathcal{G}_n \in \mathcal{P}$, the graph W contains less than $\beta\binom{n}{r}$ r-cliques. If $\mathcal{G}_n \cup \mathcal{M}^*$ is Turánnical then at least one hyperedge of $\mathcal{M}$ must be placed on an r-clique of W. Therefore we have

$$\mathbb{P}\big(\mathcal{G}_n \cup \mathcal{M}^* \text{ is Turánnical}\big) \leq e(\mathcal{M})\beta < \frac{1}{2} \ ,$$

which contradicts (35). □

8. Random restrictions

Traditional extremal combinatorics deals with questions in the following framework. Given a combinatorial structure $\mathcal{S}$ (such as the edge set of the complete graph K_n, or the set $2^{[n]}$ of subsets of $[n]$) and a monotone increasing parameter $f\colon 2^{\mathcal{S}} \to \mathbb{N}$ (such as the minimum degree of $H \subseteq K_n$, or the number of sets in the set family $H \subseteq 2^{[n]}$), we ask:

> What is the maximum possible value $f(H)$ for $H \subseteq \mathcal{S}$ satisfying a set of restrictions $\mathcal{R}$?

Often the restrictions $\mathcal{R}$ are simply *all* substructures of $\mathcal{S}$ of a certain type. For example, in the setting of Turán's theorem every r-tuple of vertices forbids a clique; in that of Sperner's theorem [20], every pair of sets $A \subseteq B \subseteq [n]$ is forbidden to be in the set family $H \subseteq 2^{[n]}$.

In this framework there are two places where randomness may come into play. Firstly, one could choose $\mathcal{S}$ to be a random structure (and thus H be a substructure of a random structure). A famous example of this type of randomness is the Kohayakawa-Łuczak-Rödl conjecture concerning a version of Turán's theorem for random graphs (see [12]) mentioned already in the introduction. Versions of the famous Erdős-Ko-Rado theorem for random hypergraphs as studied by Balogh, Bohman, and Mubayi [1] form another example.

Secondly, the restriction set can be relaxed to a random subset of all possible restrictions $\mathcal{R}$. This is exemplified in Theorems 6 and 7 in the context of Turán's theorem. Moreover, the two types of randomness can be combined, as shown in Theorem 11.

Obviously, similar randomised versions can be formulated for many other problems. Probably the closest one to the present paper would be a variant of the Erdős-Stone theorem about the extremal number of H-free graphs with random restrictions. While the statement and the proof of Theorem 6 translates *mutatis mutandis* to that setting when $\chi(H) \geq 3$, obtaining either a proof for $\chi(H) = 2$ or an analogue of Theorem 7 seem to be significantly harder. We conclude by mentioning two additional problems which seem interesting for further research.

Ramsey theory. Graph Ramsey theory deals with estimating the parameter $R(H)$, which is the smallest number n such that any two-colouring of edges of the complete graph K_n contains a monochromatic copy of H.

In a randomised version of this problem of the first type mentioned above, we colour the edges of the random graph $G(n, q)$ instead of K_n and search for a monochromatic copy of H in such a colouring. The threshold for this problem was determined by Rödl and Ruciński [15] (see also Friedgut, Rödl and Schacht [9] and Conlon and Gowers [3] for some recent progress).

Concerning the second approach for randomisation mentioned above, we suggest considering the following problem. Given n and a probability p, let $\mathcal{R}(n, p)$ be a set of copies of H in K_n obtained by picking H-copies independently at random with probability p from the set of all copies of H in K_n. What is the threshold $p = p_n$ such that a.a.s. $\mathcal{R} = \mathcal{R}(n, p)$ has the property that for every two-edge-colouring of K_n, there is a monochromatic copy of H contained in $\mathcal{R}$?

VC-dimension. The celebrated Sauer-Shelah Lemma [16, 18] states that if $\mathcal{A}$ is a family of subsets of $[n]$ with $|\mathcal{A}| > \binom{n}{0} + \ldots + \binom{n}{k-1}$ then there is a set $X \subseteq [n]$ of size k which is *shattered* by $\mathcal{A}$, i.e., for every $Y \subseteq X$, there is $A \in \mathcal{A}$ such that $Y = X \cap A$.

A randomised variant of this Lemma of the first type mentioned above would generate a random family $\mathcal{X} = \binom{[n]}{k}_p$ of k-sets in $[n]$, each k-set being present in this family independently with probability $p = p_n$. The question is then: How large must $|\mathcal{A}|$ be in order to guarantee a shattered k-set $X \in \mathcal{X}$?

A randomised version of the second type, instead, would randomise the concept of a shattering in the Sauer-Shelah Lemma. More precisely, a p-*shattering* does not require every subset $Y \subseteq X$ to be represented as $X \cap A$ for some $A \in \mathcal{A}$, but only for each $X \subseteq [n]$ of size k a family of subsets Y which are selected randomly and independently from 2^X with probability p. The question then is: Given $0 < c \leq 1$, what is the threshold $p = p_n$ such that a.a.s. there exists a set family with $c\left(\binom{n}{0} + \ldots + \binom{n}{k-1}\right)$ members which does not even p-shatter any k-set in $[n]$?

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