

Asymptotic enumeration of sparse 2-connected graphs

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Abstract

We determine an asymptotic formula for the number of labelled 2-connected (simple) graphs on n vertices and m edges, provided that $m - n \rightarrow \infty$ and $m = O(n \log n)$ as $n \rightarrow \infty$. This is the entire range of m not covered by previous results. The proof involves determining properties of the core and kernel of random graphs with minimum degree at least 2. The case of 2-edge-connectedness is treated similarly. We also obtain formulae for the number of 2-connected graphs with given degree sequence for most ('typical') sequences. Our main result solves a problem of Wright from 1983.

1 Introduction

Counting graphs with a given property is a fundamental and often difficult problem. G.E. Uhlenbeck, in the Gibbs Lecture at an American Mathematical Society meeting in 1950, cited the enumeration of 2-connected graphs as one of the unsolved problems in statistical mechanics. In the ensuing years, ways were found to efficiently calculate the number of such graphs with a given number of vertices, or vertices and edges (see Harary and Palmer [HP73] for example). However, no very simple formula was found, which brings up the question of asymptotic formulae. In the two-parameter case, there are some ranges of the parameters for which such a formula is unknown. This is the subject of the present paper.

Call a (simple) graph on the vertex set $[n] = \{1, \dots, n\}$ with m edges an (n, m) -graph. (Thus, we are concerned with labelled graphs.) A well-studied problem is to count (n, m) -graphs with minimum degree at least some fixed number, k . Korshunov [Kor94], and Bender, Canfield and McKay [BMC97] provided asymptotic formulae for the case $k = 1$, that is, graphs with no isolated

*This author's research was primarily carried out while at the University of California at San Diego under an NSERC postdoctoral award.

†Supported by the Canada Research Chairs Program and NSERC.

vertices. The case $k = 2$, which is relevant for 2-connected graphs, was studied by Wright [Wri78] and others such as Ravelomanana and Thimonier [RT04]. Later, Pittel and Wormald [PW03] found asymptotic formulae for the case $k \geq 2$.

A number of authors have addressed the problem of counting connected (n, m) -graphs. After results by various authors for various ranges of m with various degrees of approximation, Bender, Canfield and McKay [BMC90] provided an asymptotic formula for the number whenever $m - n \rightarrow \infty$ as $n \rightarrow \infty$. They obtained this formula by studying a differential equation related to a recurrence relation for the number of connected graphs. Pittel and Wormald [PW05] provided a somewhat simpler proof for this formula, with an improved error term for some ranges of m .

A natural next step would be to count k -connected (n, m) -graphs. This problem turns out to be essentially already solved for $k \geq 3$. Łuczak [Luc92] showed that a random graph with given degree sequence, all degrees between 3 and d , a.a.s. has connectivity equal to minimum degree. As observed in the introduction of [PW03], this implies that, for $m = O(n \log n)$, a random (n, m) -graph with minimum degree $k \geq 3$ is a.a.s. k -connected. (To deduce this, one needs to know that such a random (n, m) -graph has no large degree vertices, which can be deduced from the results of [PW03], or alternatively by a more direct argument if m/n is bounded.) Thus, using the above-mentioned result from [PW03], one immediately obtains an asymptotic formula for k -connected (n, m) -graphs. However, this argument does not apply for 2-connected graphs.

In this article, we derive an asymptotic formula for the number of 2-connected (n, m) -graphs when $m - n \rightarrow \infty$ with $m = O(n \log n)$. Above this range, for any fixed k , it is well known that almost all graphs are k -connected. This follows by the classic result of Erdős and Rényi [ER61], that for fixed $k \geq 0$ and $m = m(n) = \frac{1}{2}n(\log n + k \log \log n + x + o(1))$,

$$\mathbb{P}(\mathcal{G}(n, m) \text{ is } k\text{-connected}) \rightarrow 1 - e^{-e^{-x}/k!},$$

where $\mathcal{G}(n, m)$ denotes an (n, m) -graph chosen uniformly at random. Wright [Wri83] found an asymptotic formula for the case $m - n = o(\sqrt{n})$ with $m - n \rightarrow \infty$, and it was noted that the problem of finding a formula for $m - n$ growing faster than \sqrt{n} seems difficult. We solve this problem here.

Regarding exact enumeration, Temperley [Tem59] proved a recurrence relation for the number of 2-connected (n, m) -graphs. His proof uses calculus to deduce the recurrence formula from a well known differential equation for the generating functions of the number of connected graphs and 2-connected graphs; see [HP73] (pp. 10, 11). For a combinatorial proof, see [WW79]. Wright [Wri78] also found an exact formula for the number 2-connected $(n, n + k)$ -graphs with fixed k . It is possible that following an approach close to the one in [BMC90], one could obtain an asymptotic formula for the 2-connected (n, m) -graphs. However, we believe that this would be very difficult since the proof in [BMC90] is not simple and the recurrence relation for 2-connected graphs is more complicated than the one for connected graphs.

The k -core of a graph G is a maximal subgraph of G with minimum degree at least k . We extend this definition and simply call a graph a k -core if it has minimum degree at least k .

Since 2-connected graphs must have minimum degree at least 2 (if they have more than two vertices), we work with 2-cores. Our approach uses some of the basic ideas in [PW05] where the (sub-)problem being addressed was asymptotic enumeration of connected 2-cores with a given number of edges and vertices. One possible plan can be described as follows, and could potentially be of use for any graph property. First, compute the probability that a random graph with given degree sequence is 2-connected, where the degree sequence is chosen randomly, the degrees being independent truncated Poisson random variables, conditioned on the sum being $2m$. (Truncated means conditioning on the value being at least 2.) Next, using the results in [PW03] we can try to deduce that the same probability of connectedness holds for a random 2-core. In that paper, it is essentially shown, starting with the basic enumeration results of McKay [McK85],

that the degree sequence of a random (n, m) -graph which is a 2-core is strongly related to a sequence of independent truncated Poisson random variables, conditioned on the sum being $2m$ (see [PW03, Equation (13)] for example). Note that, in the first step, we do not need to estimate the probability for some degree sequences (e.g., if the maximum degree is too high) as long as we show that they have very low probability of occurring as the degree sequence of a random (n, m) -graph with minimum degree at least 2.

This plan works quite well provided $m/n \rightarrow \infty$, in which case we use another result of Łuczak [Luc92] to show that the probability of 2-connectedness tends to 1 in the Poisson-based model. For this we need to use the pairing model, a common model used for analysing random graphs with given degrees. However, if $m/n \rightarrow 1$, a random 2-core tends to have many isolated cycles, so the probability of 2-connectedness tends to 0 and then the plan is difficult to carry out. For such m , and, for convenience, whenever m is bounded, we use a construction in [PW05] called the kernel configuration model, a modification of the pairing model. This is a probability space enabling direct analysis of the 2-cores that have no isolated cycles, and the above plan is readily adapted to using this model.

The models mentioned above are explained in Section 3.

Combining the results obtained for degree sequences we obtain asymptotic formulae for the number of 2-connected (n, m) -graphs for the following three cases: $m/n \rightarrow 1$, m/n bounded away from 1 and $m/n \rightarrow \infty$. We then combine all three cases into a single formula (Theorem 1). The pieces of the proof of this are finally gathered together in Section 7. The final section adapts the method to counting 2-edge-connected graphs.

2 Main results

We assume that $m > n$. Let $T(n, m)$ denote the number of labelled 2-connected (simple) graphs with n vertices and m edges. We may assume that the vertex set is $[n]$.

In preparation for the statement of our results we define the odd falling factorial $(2m-1)!! := (2m-1)(2m-3)\cdots 1$, and the average degree $c := 2m/n$.

Define $g : \mathbb{R}_{++} \rightarrow \mathbb{R}$ by $g(\lambda) := \lambda(e^\lambda - 1)/(e^\lambda - 1 - \lambda)$. Then g is an increasing function with $g(\lambda) \rightarrow 2$ as $\lambda \rightarrow 0$. Since $c > 2$, we may let λ_c be the (unique) positive root of

$$g(\lambda) = \frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = c,$$

and we set

$$\bar{\eta}_c := \frac{\lambda_c e^{\lambda_c}}{e^{\lambda_c} - 1} \quad \text{and} \quad p_c := \frac{\lambda_c^2}{2(e^{\lambda_c} - 1 - \lambda_c)}.$$

Our main result is the following.

Theorem 1. *Suppose $m = O(n \log n)$ and $m - n \rightarrow \infty$. Then*

$$T(n, m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n c (1 + \bar{\eta}_c - c)}} \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right).$$

To prove this, we first obtain asymptotic formulae for $T(n, m)$ for the following three cases: $c \rightarrow 2$, bounded $c > 2$, and $c \rightarrow \infty$.

Theorem 2. *Suppose $m = O(n \log n)$ and $r := 2m - 2n \rightarrow \infty$. Then*

(a) *if $c \rightarrow 2$,*

$$T(n, m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n (c-2)}} \cdot \frac{\sqrt{3r}}{e\sqrt{2m}}.$$

(b) if $c = O(1)$ and $c > C_0$ for a constant $C_0 > 2$ for n large enough,

$$T(n, m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n c (1 + \bar{\eta}_c - c)}} \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right).$$

(c) if $c \rightarrow \infty$,

$$T(n, m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n c}} \exp\left(-\frac{\bar{\eta}_c}{2} - \frac{\bar{\eta}_c^2}{4}\right).$$

For each case, we prove that the formula obtained is asymptotically equivalent to the formula in Theorem 1. Then we show how to combine these to obtain Theorem 1.

As we have already mentioned, Wright [Wri83] proved an asymptotic formula for the case $k := m - n = o(\sqrt{n})$ with $k \rightarrow \infty$. His formula is

$$T(n, m) = a \sqrt{6\pi n}^{n+3k-1/2} e^{2k-n} (18k^2)^{-k} (1 + O(k^{-1}) + O(k^2/n)),$$

where a is a constant. Wright gave a method of estimating a , and computed it to be $0.058549831 \dots$. Voblyř [Vob87] determined that a is $1/(2e\pi)$. To compare Wright's formula to our own, we compute $\lambda_c = 3r/n - \frac{3}{2}(r/n)^2 + \frac{6}{5}(r/n)^3 + O((r/n)^4)$ for $k = o(n^{2/3})$, and then from Theorem 2(c) it is easy to obtain the following.

Corollary 3. *Suppose that $k = m - n = o(n^{2/3})$ and $k \rightarrow \infty$. Then*

$$T(n, m) \sim \frac{\sqrt{3}}{e\sqrt{2\pi}} n^{n+3k-1/2} e^{2k-n+3k^2/(2n)} (18k^2)^{-k}.$$

One can also check easily that for $m \approx n \log n$, our formula is asymptotic to the total number of (n, m) -graphs, in accordance with the result of Erdős and Rényi mentioned above.

The proof of each case in Theorem 2 follows the same strategy. First we study the general “typical” degree sequences of each case, computing the (asymptotic) probability that a graph with a given degree sequence is 2-connected. With this, we obtain an asymptotic formula for the number of 2-connected graphs with degree sequence \mathbf{d} . For all typical sequences, we obtain the same probability (within a uniform error). This allows us to sum over degree sequences obtaining an asymptotic formula for $T(n, m)$ in each case.

We use $\mathcal{D}(n, m)$ to represent the set of degree sequences $\mathbf{d} := (d_1, \dots, d_n)$ such that $\sum_{i=1}^n d_i = 2m$ and $d_i \geq 2$ for all $i \in [n]$. For $\mathbf{d} \in \mathcal{D}(n, m)$ define $\eta(\mathbf{d}) := \frac{1}{2m} \sum d_j(d_j - 1)$ and let $T(\mathbf{d})$ represent the number of 2-connected graphs with degree sequence \mathbf{d} . For every integer j , let $D_j = D_j(\mathbf{d})$ denote $|\{i: d_i = j\}|$.

Define $\mathbf{Y} = (Y_1, \dots, Y_n)$ to be a vector of independent *truncated Poisson* random variables $Y \sim \text{Po}(2, \lambda_c)$ defined by

$$\mathbb{P}(Y = j) = \frac{\lambda_c^j}{j!(e^{\lambda_c} - 1 - \lambda_c)}$$

for $j = 2, 3, \dots$

Theorem 4. *Suppose $m = O(n \log n)$ and $r := 2m - 2n \rightarrow \infty$. Then*

(a) *Suppose further that $c \rightarrow 2$. Let $\psi(n) = r^{1-\varepsilon}$ for some $\varepsilon \in (0, 1/4)$. If $\mathbf{d} \in \mathcal{D}(n, m)$ satisfies*

- (i) $|D_2 - \mathbb{E}(D_2(\mathbf{Y}))| \leq \psi(n),$
- (ii) $|D_3 - \mathbb{E}(D_3(\mathbf{Y}))| \leq \psi(n),$
- (iii) $|\sum_i \binom{d_i}{2} - \mathbb{E}(\sum_i \binom{Y_i}{2})| \leq \psi(n),$

(iv) $d_i \leq 8 \log(n - D_2(\mathbf{d}))$ for every i ,

then

$$T(\mathbf{d}) \sim \frac{\sqrt{3r}}{e\sqrt{2m}} \cdot \frac{(2m-1)!!}{\prod_{j=1}^n d_j!}.$$

(b) Suppose further that $c = O(1)$ and $c = c(n) = 2m/n > C_0$ for a constant $C_0 > 2$ for n large enough. Let $\psi(n) = 1/n^\varepsilon$ for some $\varepsilon \in (0, 1/4)$. If $\mathbf{d} \in \mathcal{D}(n, m)$ satisfies

(i) $d_i \leq 6 \log n$ for every i ,

(ii) $|\eta(\mathbf{d}) - \bar{\eta}_c| \leq \psi(n)$ and

(iii) $|D_2/n - p_c| \leq \psi(n)$,

then

$$T(\mathbf{d}) \sim \frac{(2m-1)!!}{\prod_{j=1}^n d_j!} \sqrt{\frac{c-2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right).$$

(c) If $c \rightarrow \infty$ and $\mathbf{d} \in \mathcal{D}(n, m)$ satisfies $\max d_i \leq n^\epsilon$ for some $\epsilon \in (0, 0.01)$ then

$$T(\mathbf{d}) \sim \frac{(2m-1)!!}{\prod_{j=1}^n d_j!} \exp\left(-\frac{\eta(\mathbf{d})}{2} - \frac{\eta(\mathbf{d})^2}{4}\right).$$

3 Background and preliminary results

3.1 Models for graphs of given degree sequence

The *pairing model* or *configuration model* is a standard theoretical tool for studying graphs of a given degree sequence. For $\mathbf{d} \in \mathcal{D}(n, m)$ a random perfect matching is placed on a set of $2m$ points which are grouped into n cells of size d_1, d_2, \dots, d_n . This random pairing naturally corresponds in an obvious way to a random pseudograph (possibly containing loops or parallel edges) of degree sequence \mathbf{d} in which each cell becomes a vertex.

Let $U(\mathbf{d})$ denote the probability the random pairing model is simple, and $U'(\mathbf{d})$ the probability that a random pairing is both 2-connected and simple. It is well known that the number of pairings corresponding to a given (simple) graph is $\prod_{j=1}^n d_j!$, thus

$$T(\mathbf{d}) = \frac{(2m-1)!!}{\prod_{j=1}^n d_j!} U'(\mathbf{d}). \quad (1)$$

Let $C(n, m)$ denote the number of labelled (simple) graphs with n vertices and m edges (with vertex set $[n]$) with minimum degree at least 2 and let

$$Q(n, m) := \sum_{\mathbf{d} \in \mathcal{D}(n, m)} \prod_{j=1}^n \frac{1}{d_j!}.$$

Recall the definition of the sequence $\mathbf{Y} = (Y_1, \dots, Y_n)$ of independent truncated Poisson random variables with parameter $(2, \lambda_c)$. Let Σ denote the event that $\sum_i Y_i = 2m$. Then [PW03, Eq. (13)] states that

$$C(n, m) = (2m-1)!! Q(n, m) \mathbb{E}(U(\mathbf{Y}) | \Sigma). \quad (2)$$

This was obtained by summing the analogue of (1) for $U(\mathbf{d})$ over $\mathbf{d} \in \mathcal{D}(n, m)$. Applying the same argument to (1), one easily obtains

$$T(n, m) = (2m-1)!! Q(n, m) \mathbb{E}(U'(\mathbf{Y}) | \Sigma). \quad (3)$$

This distribution \mathbf{Y} has been studied by several authors. Facts about it will be introduced as they are needed. The probability of the event Σ has been estimated quite precisely in terms of the variance of Y . (See Lemma 2 and Theorem 4(a) of [PW03].)

We will use the estimate in Theorem 4(a) of [PW03]:

$$\mathbb{P}(\Sigma) = \frac{1 + O(r^{-1})}{\sqrt{2\pi nc(1 + \bar{\eta}_c - c)}} = \Omega(1/\sqrt{r}), \quad (4)$$

where $r := 2m - 2n$ under the conditions that $r = O(n \log n)$ and $r \rightarrow \infty$. One of the reasons for which (3) holds is that $Q(n, m)$ can be rewritten as $(e^{\lambda_c} - 1 - \lambda_c)^n \lambda_c^{-2m} \mathbb{P}(\Sigma)$, and so (4) gives

$$Q(n, m) = \frac{(e^{\lambda_c} - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi nc(1 + \bar{\eta}_c - c)}} (1 + O(r^{-1})). \quad (5)$$

When the degrees are all at least 2, the *kernel configuration model* of Pittel and Wormald [PW05] can provide some advantages. Before describing the model we need some definitions. The *2-core* of a graph is its maximal subgraph of minimum degree at least 2. The *pre-kernel* of a graph is obtained from the 2-core by throwing away any components which are simply cycles. The *kernel* of a graph is obtained from the pre-kernel by replacing each maximal path of degree-2 vertices by a single edge. We say that a pseudograph is a pre-kernel (respectively, a kernel) if it is the pre-kernel (respectively, kernel) of some graph. Now we are ready to describe the kernel configuration model for a degree sequence $\mathbf{d} \in \mathcal{D}(n, m)$.

For each i with $d_i \geq 3$ create a set S_i of d_i points. Choose, uniformly at random, a perfect matching on the union of these sets of points. Assign the remaining numbers $\{i : d_i = 2\}$ to the edges of the perfect matching and, for each edge, choose a linear order for these numbers. The assignment and the linear ordering are chosen uniformly at random. The pairing and assignment (with linear orderings) are the configuration. A pseudograph G is constructed by collapsing each set S_i to a vertex (producing a kernel K) and placing the degree-2 vertices on the edges of the kernel according to the assignment and linear orderings.

It is not hard to show (see Corollary 2 in [PW05]) that each pre-kernel can be produced by the same number of configurations, and

$$T(\mathbf{d}) = \frac{(2m' - 1)!!(m - 1)! \mathbb{P}(\mathbf{2cs}(\mathbf{d}))}{(m' - 1)! \prod_{i \in R(\mathbf{d})} d_i!},$$

where $R = R(\mathbf{d}) := \{i \in [n] : d_i \geq 3\}$, $m' = m'(\mathbf{d}) := \frac{1}{2} \sum_{i \in R} d_i$, and $\mathbf{2cs}(\mathbf{d})$ is the event that the pre-kernel produced by the kernel configuration model is 2-connected and simple. For later use, let $n' = n'(\mathbf{d}) := |R(\mathbf{d})| = \sum_{j \geq 3} D_j(\mathbf{d})$. By Stirling's formula,

$$T(\mathbf{d}) = \frac{(2m - 1)!! \sqrt{m'/m} \mathbb{P}(\mathbf{2cs})}{\prod_{i=1}^n d_i!} (1 - O(1/m')). \quad (6)$$

In [PW05, (5.3)], a similar expression for the event of being connected and simple was summed over $\mathbf{d} \in \mathcal{D}(n, m)$. We can use the same argument, using (4) and (5), and defining

$$w(\mathbf{d}) := \mathbb{P}(\mathbf{2cs}(\mathbf{d})) \sqrt{m'}, \quad (7)$$

to get

$$\begin{aligned} T(n, m) &= (1 - O(1/m')) (2m - 1)!! Q(n, m) \sqrt{m^{-1}} \mathbb{E}(w(\mathbf{Y})|\Sigma). \\ &= (1 - O(1/r)) (2m - 1)!! \frac{(e^{\lambda_c} - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi nc(1 + \bar{\eta}_c - c)}} \sqrt{m^{-1}} \mathbb{E}(w(\mathbf{Y})|\Sigma). \end{aligned} \quad (8)$$

3.2 Relation between vertex and edge connectivity

In this section we investigate some properties of 2-connected graphs which may be of independent interest. We show that, asymptotically almost surely, a random kernel is 2-connected if and only if it is 2-edge-connected. (An event is said to occur asymptotically almost surely (a.a.s.) if its probability is $1 - o(1)$.) In this article, for convenience, we allow 2-connected pseudographs and 2-edge-connected pseudographs to have loops. In particular, a cut-vertex of a pseudograph is a vertex whose removal (along with all incident edges) increases the number of components, and a graph is 2-connected if it has no cutvertices and at least three vertices.

Proposition 5. *Let $\mathbf{d} \in \mathcal{D}(n, m)$ satisfying $n \geq 3$ and $3 \leq \delta = d_1 \leq \dots \leq d_n = \Delta \leq n^{0.04}$. Let K be the kernel of the random pseudograph produced by the pairing model using degree sequence \mathbf{d} . A.a.s., K is 2-connected iff it is 2-edge-connected.*

Proof. Let K be the random kernel produced by the pairing model using degree sequence \mathbf{d} satisfying $n \geq 3$ and $3 \leq \delta = d_1 \leq \dots \leq d_n = \Delta \leq n^{0.04}$. By closely following Łuczak's proofs of properties of (simple) graphs with given degree sequence in [Łuc92, Section 12.3]), it is straightforward to prove the following lemmas.

Lemma 6. *A.a.s., no subgraph of K with s vertices, $2 \leq s \leq n^{0.4}$, has more than $1.2s$ edges.*

Lemma 7. *A.a.s., each subset of K with s vertices, $n^{0.3} \leq s \leq \lceil n/2 \rceil$ has more than δ neighbours.*

Suppose that v is a cut-vertex in K not in a bridge. Then v decomposes K into components W_1 and W_2 with $|W_1| \leq |W_2|$. Note that v sends at least 2 edges to W_1 and at least 2 edges to W_2 . (Otherwise v would be in a bridge).

Suppose that $|W_1| = 1$. Then the number of edges induced by $W_1 \cup \{v\}$ is at least 3 (since $\delta \geq 3$) which is $\frac{3}{2}|W_1 \cup \{v\}|$. On the other hand, if $|W_1| \geq 2$, the number of edges induced by $W_1 \cup \{v\}$ is at least $(3|W_1| + 2)/2 \geq 1.25|W_1 \cup \{v\}|$. For $|W_1 \cup \{v\}| \leq n^{0.4}$, we conclude that such v a.a.s. does not exist, by Lemma 6. Otherwise, $|W_1| \geq n^{0.3}$ and such v a.a.s. does not exist by Lemma 7.

So a.a.s. K has a bridge if it has a cut-vertex. The converse is deterministically true for pseudographs with at least three vertices, and the proposition follows. \square

Note that Lemmas 6 and 7 actually imply that a.a.s. there are no cut-sets of cardinality from 2 to $\delta - 1$ inclusive.

4 The case $c \rightarrow 2$

In this case, we can directly implement the plan presented in the introduction: we examine the probability that a random n -vertex graph is 2-connected when its vertex degrees are chosen as independent truncated Poissons random variables, conditioned on the sum being $2m$. We do this for typical degree sequences and then transfer this result to a random (n, m) -graph with minimum degree 2.

Recall the definition of the sequence \mathbf{Y} of independent truncated Poisson random variables and the associated event Σ used in (8).

Define $\mu_2 = \mathbb{E}(D_2(\mathbf{Y}))$, $\mu_3 = \mathbb{E}(D_3(\mathbf{Y}))$ and $\mu = \mathbb{E}(\sum_{i=1}^n \binom{Y_i}{2})$. We need to know the asymptotic behaviour of these expected values.

Lemma 8. *We have $\mu_2 = n - r + o(r)$, $\mu_3 = r + o(r)$ and $\mu = n + 2r + o(r)$.*

The proof of this lemma is straightforward and depends only on properties of \mathbf{Y} that follow easily from facts established by Pittel and Wormald [PW03, PW05]. The proof is presented at the end of the section.

We now define a set of “typical” degree sequences. Let $\psi(n) : \mathbb{N} \rightarrow \mathbb{R}_+$ be any function such that $\psi(n) = o(r)$. (We will specify a particular such function later.) Recall the definition $n'(\mathbf{d}) = \sum_{j \geq 3} D_j$. Let

$$\tilde{\mathcal{D}}(\psi) := \left\{ \mathbf{d} \in \mathcal{D}(n, m) : |D_2(\mathbf{d}) - \mu_2| \leq \psi(n); |D_3(\mathbf{d}) - \mu_3| \leq \psi(n); \right. \\ \left. \left| \sum_{i=1}^n \binom{d_i}{2} - \mu \right| \leq \psi(n); \max_i d_i \leq 8 \log n'(\mathbf{d}) \right\}.$$

and define $\tilde{\mathcal{D}}^c(\psi) := \mathcal{D}(n, m) \setminus \tilde{\mathcal{D}}(\psi)$.

Let $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. We want to compute the probability of $\mathbf{2cs}(\mathbf{d})$ as in (7). It is easy to see that this is the same as the event that G is simple and K is 2-connected and loopless (but permitting K to have multiple edges). Let B denote the event that G is simple and K is 2-edge-connected and has no loops. The maximum degree in K is at most $8 \log(n') < (n')^{0.04}$ so we may use Proposition 5 to deduce $\mathbb{P}(B) = \mathbb{P}(\mathbf{2cs}(\mathbf{d})) + o(1)$.

We have that $\max d_i \leq 6 \log n$ and, by Lemma 8 and the definition of $\tilde{\mathcal{D}}(\psi)$,

$$\sum_{i \in R(\mathbf{d})} \binom{d_i}{2} = \sum_{i=1}^n \binom{d_i}{2} - D_2(\mathbf{d}) = n + 2r + o(r) - (n - r + o(r)) = 3r + o(r) < 4r,$$

for large n , which are sufficient conditions (by Lemma 5 in [PW05]) for a random kernel configuration to be a.a.s. simple. Thus, the probability of G being simple is $1 + o(1)$. (Actually, in [PW05] it is shown to be $1 - O(r/n + 1/r)$.)

For a random pairing having a given degree sequence of minimum degree at least 3, the probability of being 2-edge-connectivity was investigated by Łuczak in [Luc92]. He shows (in his Lemma 12.1(iii)) that this probability approaches $\exp(-\frac{3}{2}D_3/m')$ provided D_3/m' approaches a positive constant. Using Lemma 8 for μ_2 ,

$$m'(\mathbf{d}) = m - D_2(\mathbf{d}) = m - \mu_2 + O(\psi) = m - \mu_2 + o(r) = \frac{3r + o(r)}{2}.$$

Applying this to K , we have

$$\frac{D_3(\mathbf{d})}{m'} = \frac{r + o(r)}{(3/2)r + o(r)} \sim \frac{2}{3}$$

so the probability that K is 2-edge-connected goes to $1/e$. Note that K being 2-edge-connected implies that there are no loops on vertices of degree 3 in K . The expected number of loops in K on vertices of degree at least 4 is

$$\sum_{i: d_i \geq 4} \binom{d_i}{2} \frac{1}{2m' - 1} = \left(\sum_{i=1}^n \binom{d_i}{2} - D_2 - 3D_3 \right) \frac{1}{3r + o(r)} = o(1)$$

by Lemma 8, so a.a.s. no such loops exist. We conclude that $\mathbb{P}(B) \sim 1/e$, and thus

$$\mathbb{P}(\mathbf{2cs}(\mathbf{d})) \sim \frac{1}{e}. \tag{9}$$

These two results together with (6) give Theorem 4(a) and, in particular recalling $w(\mathbf{d}) = \mathbb{P}(\mathbf{2cs}(\mathbf{d}))\sqrt{m'}$ from (7),

$$w(\mathbf{d}) \sim \frac{1}{e} \sqrt{\frac{3r}{2}}.$$

Finally, we will show Theorem 2(a). Let $\psi(n) = r^{1-\varepsilon}$ for some $\varepsilon \in (0, 1/4)$. We will show

$$\mathbb{P}(\tilde{\mathcal{D}}(\psi)|\Sigma) = 1 + O(\sqrt{r}/n) + O(r^{2\varepsilon-1/2}). \quad (10)$$

Then using the formula for $w(\mathbf{d})$ shown above for any $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$, we have

$$\begin{aligned} \mathbb{E}(w(\mathbf{Y})|\Sigma) &= \mathbb{E}(w(\mathbf{Y})|\tilde{\mathcal{D}}(\psi))\mathbb{P}(\tilde{\mathcal{D}}(\psi)|\Sigma) + \mathbb{E}(w(\mathbf{Y})|\tilde{\mathcal{D}}^c(\psi))\mathbb{P}(\tilde{\mathcal{D}}^c(\psi)|\Sigma) \\ &= \mathbb{E}(w(\mathbf{Y})|\tilde{\mathcal{D}}(\psi))(1 - O(\sqrt{r}/n) - O(r^{2\varepsilon-1/2})) + o(\sqrt{r}) \\ &\sim \frac{1}{e} \sqrt{\frac{3r}{2}}, \end{aligned}$$

which combined with (8), and using

$$c(1 + \bar{\eta}_c - c) \sim c - 2 \quad (11)$$

for $c \rightarrow 2$ (see [PW03](20)), gives the conclusion of Theorem 2(a).

So it suffices to prove (10). Let p_i denote the probability that a variable with distribution $\text{Po}(2, \lambda_c)$ has value i . Recall that $\eta(\mathbf{d}) = (\sum_{i=1}^n d_i(d_i - 1))/(\sum_{i=1}^n d_i)$. First we will study the first three conditions in the definition of $\tilde{\mathcal{D}}(\psi)$. Let F be the event that \mathbf{Y} fails to satisfy any of the three conditions. Using Chebyshev's inequality, together with $p_2(1 - p_2)n = O(r)$ and $p_3(1 - p_3)n \leq p_3n = O(r)$ by straightforward calculations, we have that

$$\mathbb{P}(|D_2(\mathbf{Y}) - \mu_2| \geq \psi(n)) \leq \frac{p_2(1 - p_2)n}{\psi(n)^2} = O\left(\frac{r}{\psi(n)^2}\right)$$

and

$$\mathbb{P}(|D_3(\mathbf{Y}) - \mu_3| \geq \psi(n)) \leq \frac{p_3(1 - p_3)n}{\psi(n)^2} = O\left(\frac{r}{\psi(n)^2}\right).$$

There is a concentration result for μ shown in [PW03, p. 262] which may be expressed as

$$\mathbb{P}\left(\left|\sum_{i=1}^n \binom{Y_i}{2} - \mu\right| \geq \psi(n)\right) = O\left(\frac{\lambda_c m^2}{n\psi(n)^2}\right).$$

In [PW03], Lemma 1(a) states

$$\lambda_c = 3(c - 2) + O((c - 2)^2) = 3r/n + O(r^2/n^2). \quad (12)$$

Thus,

$$\frac{\lambda_c m^2}{n} = \frac{3rm^2}{n^2} + O\left(\frac{r^2 m^2}{n^3}\right) = O(r).$$

Hence

$$\mathbb{P}\left(\left|\sum_{i=1}^n \binom{Y_i}{2} - \mu\right| \geq \psi(n)\right) = O\left(\frac{r}{\psi(n)^2}\right).$$

This implies that $\mathbb{P}(F) = O(r/\psi(n)^2)$. Using (4) we get

$$\mathbb{P}(F|\Sigma) \leq \frac{\mathbb{P}(F)}{\mathbb{P}(\Sigma)} = O(\sqrt{r})O\left(\frac{r}{\psi(n)^2}\right) = O\left(\frac{r^{3/2}}{\psi(n)^2}\right).$$

Now consider the last condition in the definition of $\tilde{\mathcal{D}}(\psi)$: $\max_i d_i \leq 8 \log n'(\mathbf{d})$. If the first condition in the definition of $\tilde{\mathcal{D}}(\psi)$ holds, then, using Lemma 8, we have $D_2(\mathbf{d}) = n - r + \phi(n)$

for some function $\phi(n) = o(r)$ and so $n'(\mathbf{d}) = r - \phi(n)$. Let F' denote the event that the first condition holds but the last condition fails. Thus, $\mathbb{P}(F') \leq \mathbb{P}(\max_i Y_i \geq 8 \log(r - \phi(n)))$. For $r \leq \sqrt{n}$, it is easy to see that $\mathbb{E}(D_j(\mathbf{Y})) = O(r^{j-2}/n^{j-3})$ for every $j \geq 4$. Thus, using Markov's inequality and the union bound, one can prove that

$$\mathbb{P}(D_j(\mathbf{Y}) \geq 1 \text{ for some } j \geq 8) \leq n \cdot O(1/n^2) = O(1/n).$$

For $r > \sqrt{n}$, it is easy to bound the tail probability of Y_i (see (3.17) of [PW05] for example) as

$$\mathbb{P}(Y_i \geq 8 \log(r - \phi(n))) = O(\exp(-4 \log(r - \phi(n)))) = O(\exp(-4 \log r)) = O\left(\frac{1}{n^2}\right).$$

Thus, $\mathbb{P}(\max_i Y_i > 8 \log(r - \phi(n))) = O(1/n)$. Since $\mathbb{P}(\Sigma) = \Omega(1/\sqrt{r})$, we conclude that

$$\mathbb{P}(F'|\Sigma) \leq O(\sqrt{r})O(1/n) = O(\sqrt{r}/n).$$

Hence

$$\mathbb{P}(\tilde{D}(\psi)|\Sigma) \geq 1 - \mathbb{P}(F|\Sigma) - \mathbb{P}(F'|\Sigma) = 1 + O\left(\frac{r\sqrt{r}}{\psi(n)^2}\right) + O(\sqrt{r}/n) = 1 + O(r^{2\varepsilon-1/2}) + O(\sqrt{r}/n),$$

and we proved (10).

Proof of Lemma 8. Let $r(\mathbf{Y}) := \sum_{i=1}^n Y_i - 2n$ and $n'(\mathbf{Y}) := n - D_2(\mathbf{Y})$. Note that $r(\mathbf{Y})$ may not coincide with r because we are not conditioning on Σ . But

$$\mathbb{E}(r(\mathbf{Y})) = \mathbb{E}\left(\sum_{i=1}^n Y_i\right) - 2n = \sum_{i=1}^n \frac{2m}{n} - 2n = 2m - 2n = r. \quad (13)$$

Note that

$$\sum_{i=1}^n Y_i = \sum_{i \in R(\mathbf{Y})} Y_i + 2D_2(\mathbf{Y}) \geq \sum_{i \in R(\mathbf{Y})} 3 + 2n - 2n'(\mathbf{Y}) = 3n'(\mathbf{Y}) + 2n - 2n'(\mathbf{Y}) = n'(\mathbf{Y}) + 2n.$$

Hence, $n'(\mathbf{Y}) \leq r(\mathbf{Y})$.

Thus,

$$D_2(\mathbf{Y}) = n - n'(\mathbf{Y}) \geq n - r(\mathbf{Y}). \quad (14)$$

and so, by (13),

$$\mathbb{E}(D_2(\mathbf{Y})) \geq n - r. \quad (15)$$

Moreover, $D_2(\mathbf{Y}) \leq n - D_3(\mathbf{Y})$, which implies that

$$\mathbb{E}(D_2(\mathbf{Y})) \leq n - \mathbb{E}(D_3(\mathbf{Y})). \quad (16)$$

Since $n - r = n + o(n)$ and $n - D_3(\mathbf{Y}) \leq n$, we conclude that $\mathbb{E}(D_2(\mathbf{Y})) = n + o(n)$. Using (12),

$$\begin{aligned} \mathbb{E}(D_3(\mathbf{Y})) &= \frac{\lambda_c^3}{3!(e^{\lambda_c} - 1 - \lambda_c)} n = \frac{\lambda_c}{3} \mathbb{E}(D_2(\mathbf{Y})) = \left(\frac{r}{n} + O\left(\frac{r^2}{n^2}\right)\right)(n + o(n)) \\ &= r + o(r) + O(r^2/n) = r + o(r). \end{aligned}$$

By (16), $\mathbb{E}(D_2(\mathbf{Y})) \leq n - \mathbb{E}(D_3(\mathbf{Y})) = n - r + o(r)$. So by (15), we conclude that $\mathbb{E}(D_2(\mathbf{Y})) = n - r + o(r)$.

In [PW05], the line after (5.6) (with error term corrected to $O(r^2/n)$) states $\mathbb{E}\left(\sum_{i=1}^n \binom{Y_i}{2}\right) = n + 2r + O(r^2/n) = n + 2r + o(r)$. \square

5 The case c bounded away from 2, and bounded

Let $\psi : \mathbb{N} \rightarrow \mathbb{R}$ be a function such that $\psi(n) = o(1)$. Let

$$\tilde{\mathcal{D}}(\psi) := \left\{ \mathbf{d} \in \mathcal{D}(n, m) : d_i \leq 6 \log n \ \forall i; \ |\eta(\mathbf{d}) - \bar{\eta}_c| \leq \psi(n); \ |D_2(\mathbf{d}) - p_c n| \leq n\psi(n) \right\}$$

Let $\tilde{\mathcal{D}}^c(\psi) := \{\mathbf{d} \in \mathbb{N}^n : d_i \geq 2 \ \forall i; \mathbf{d} \notin \tilde{\mathcal{D}}(\psi)\}$. (Note that if $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$ then $\sum d_i = 2m$ but we do not have this constraint for $\tilde{\mathcal{D}}^c(\psi)$.)

Let $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. We use the kernel configuration model to investigate the graphs with no isolated cycles and with degree sequences in $\tilde{\mathcal{D}}(\psi)$. According to the general plan in the introduction, we will then see that the probability such graphs are 2-connected is concentrated around a given value when the degree sequence consists of independent truncated Poissons, and show how this probability then carries over to random graphs with a given number of edges.

Let \mathbf{d}' be the restriction of \mathbf{d} to the coordinates with value at least 3, and let G be obtained using the kernel configuration model with degree sequence \mathbf{d} . Recall $n' = |\{i : d_i \geq 3\}|$.

Let P be the random perfect matching placed on a set S with $\sum_{i=1}^{n'} d'_i$ points grouped in cells of size $d'_1, d'_2, \dots, d'_{n'}$. Let K be the kernel (obtained by contracting the cells of P). Let v_i denote the vertex with degree d'_i in K . Let M denote the number of edges in K .

We want to compute the probability that G is 2-connected and simple. Let B be the event that G is simple and that K is 2-edge-connected and has no loops. Since $n' = (1 - p_c)n + o(n) = \Theta(n)$, we have $\max_i d_i \leq 6 \log n \leq (n')^{0.04}$, and so Proposition 5 says that, conditioning on B , K is a.s. 2-connected. If K is 2-connected and loopless, it is easy to show that G is also 2-connected. In other words,

$$\mathbb{P}(\mathbf{2cs}|B) = (1 + o(1)).$$

Note that $\mathbf{2cs} \subseteq B$ for $n > 2$.

Let A denote the event that G has no multiple edges and K has no loops. Łuczak has shown (see Lemma 12.1(ii) in [Luc92]) that in a random pseudograph with given degree sequence, with the distribution of pairing model, having minimum degree at least 3, a.s. all 2-edge-connected components, except at most one, are loops at vertices of degree 3. Hence, $\mathbb{P}(A \setminus B) = o(1)$. Since $B \subseteq A$, we deduce $\mathbb{P}(A) = \mathbb{P}(B) + o(1)$.

We will show that

$$\mathbb{P}(A) \sim p_a := \exp(-c/2 - \lambda_c^2/4). \quad (17)$$

Hence,

$$\mathbb{P}(\mathbf{2cs}) = \mathbb{P}(\mathbf{2cs}|B)\mathbb{P}(B) = (1 + o(1))\mathbb{P}(A) \sim p_a. \quad (18)$$

Note that $D_2(\mathbf{d}) = p_c n + nO(\psi(n)) \sim p_c n$ and $\eta(\mathbf{d}) = \bar{\eta}_c + O(\psi(n)) \sim \bar{\eta}_c$. Thus

$$\sqrt{\frac{m'(\mathbf{d})}{m}} = \sqrt{\frac{m - D_2(\mathbf{d})}{m}} = \sqrt{\frac{(c/2)n - p_c n + o(n)}{(c/2)n}} \sim \sqrt{\frac{c - 2p_c}{c}}$$

since $c > 2$ and $p_c \leq 1$. Using this fact together with (18),

$$\mathbb{P}(\mathbf{2cs}(\mathbf{d}))\sqrt{m'} \sim \sqrt{m}\sqrt{\frac{c - 2p_c}{c}}p_a, \quad (19)$$

which together with (6) proves Theorem 4(b).

So in order to prove Theorem 4(b), it suffices to prove (17). The proof is presented in Section 5.1.

We now prove Theorem 2(b). First we show that

$$\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c(\psi)) = O(n^{-1}\psi(n)^{-2}) \quad \text{and} \quad \mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c(\psi)|\Sigma) = O(n^{-1/2}\psi(n)^{-2}). \quad (20)$$

We will use some properties of \mathbf{Y} developed by Pittel and Wormald [PW03]. Equation (27) in [PW03] states that $\mathbb{P}(Y \geq j_0) = O(\exp(-j_0/2))$ provided $j_0 > 2e\lambda_c$, where $Y \sim \text{Po}(2, \lambda_c)$. Lemma 1(b) in the same paper assures $\lambda_c \leq 2m/n$, which is $O(1)$ in the present case, allowing us to choose $j_0 = 6 \log n$, apply the union bound, and conclude

$$\mathbb{P}(\max_i Y_i > 6 \log n) = O\left(\frac{1}{n^2}\right).$$

Note that $D_2(\mathbf{Y})$ has binomial distribution with probability p_c . Using Chebyshev's inequality,

$$\mathbb{P}(|D_2(\mathbf{Y}) - p_c n| \geq n\psi(n)) \leq \frac{p_c(1-p_c)n}{n^2\psi(n)^2} = O\left(\frac{1}{n\psi(n)^2}\right)$$

since $0 \leq p_c \leq 1$

Pittel and Wormald also show (see [PW03, p. 262]),

$$\mathbb{P}(|\eta(\mathbf{Y}) - \bar{\eta}_c| \geq \alpha) = O\left(\frac{\lambda_c}{n\alpha^2}\right).$$

Since $\lambda_c \leq c = O(1)$,

$$\mathbb{P}(|\eta(\mathbf{Y}) - \bar{\eta}_c| \geq \psi(n)) = O\left(\frac{\lambda_c}{n\psi(n)^2}\right) = O\left(\frac{1}{n\psi(n)^2}\right).$$

Hence,

$$\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c) = O\left(\frac{1}{n\psi(n)^2}\right).$$

Since $r := 2m - 2n = \Theta(n)$, (4) implies that $\mathbb{P}(\Sigma) = \Omega(1/\sqrt{n})$. Conditioning on Σ , we have

$$\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c | \Sigma) \leq \frac{\mathbb{P}(\mathbf{Y} \in \tilde{\mathcal{D}}^c)}{\mathbb{P}(\Sigma)} = O\left(\frac{n^{1/2}}{n\psi(n)^2}\right) = O\left(\frac{1}{n^{1/2}\psi(n)^2}\right).$$

This proves (20).

Let $\psi(n) = n^{-\varepsilon}$ for some constant $\varepsilon \in (0, 1/4)$. Using (19) and (20),

$$\mathbb{E}(w(\mathbf{Y}) | \Sigma) = \mathbb{E}(w(\mathbf{Y}) | \tilde{\mathcal{D}}(\psi))\mathbb{P}(\tilde{\mathcal{D}}(\psi) | \Sigma) + \mathbb{E}(w(\mathbf{Y}) | \Sigma \cap \tilde{\mathcal{D}}^c(\psi))\mathbb{P}(\tilde{\mathcal{D}}^c(\psi) | \Sigma).$$

Note that $w(\mathbf{Y}) \leq \sqrt{m'}$ since $\mathbb{P}(\mathbf{2cs}) \leq 1$. By (20), we have that $\mathbb{P}(\tilde{\mathcal{D}}^c(\psi) | \Sigma) = O(1/n^{1/2-2\varepsilon})$. So $\mathbb{E}(w(\mathbf{Y}) | \Sigma \cap \tilde{\mathcal{D}}^c)\mathbb{P}(\tilde{\mathcal{D}}^c | \Sigma) = O(\sqrt{m'}/n^{1/2-2\varepsilon})$. Hence,

$$\begin{aligned} \mathbb{E}(w(\mathbf{Y}) | \Sigma) &= \mathbb{E}(w(\mathbf{Y}) | \Sigma \cap \tilde{\mathcal{D}}(\psi))(1 - O(1/n^{1/2-2\varepsilon})) + O(\sqrt{m'}/n^{1/2-2\varepsilon}) \\ &= \sqrt{m'} \sqrt{\frac{c-2p_c}{c}} p_a (1 + o(1)) (1 - O(1/n^{1/2-2\varepsilon})) + O(\sqrt{m'}/n^{1/2-2\varepsilon}) \\ &= \sqrt{m'} \sqrt{\frac{c-2p_c}{c}} \exp(-c/2 - \lambda_c^2/4) (1 + o(1)), \end{aligned}$$

which together with (8) proves Theorem 2(b).

5.1 Showing $\mathbb{P}(A) \sim p_a$

Here we show (17). Recall that $\mathbf{d} \in \tilde{\mathcal{D}}(\psi)$. Let e_1, \dots, e_ℓ denote the possible loops in K . For every $1 \leq i \leq \ell$, let X_i be the indicator variable for $e_i \in E(K)$. Let $X = \sum_{i=1}^\ell X_i$. Let f_1, \dots, f_t denote the possible double edges in K (here we do not include double loops). For every $1 \leq j \leq t$, let Y_j be the indicator variable for $f_j \subseteq E(G)$. Let $Y = \sum_{j=1}^t Y_j$.

Using the method of moments, we will show that $X + Y \xrightarrow{d} \text{Po}(c/2 + \lambda_c^2/4)$, which gives (17). For this we need to show, for every positive integer k , that

$$\mathbb{E}([X + Y]_k) = \left(\frac{c}{2} + \frac{\lambda_c^2}{4}\right)^k + o(1).$$

Considering the first moment, note that for every $1 \leq i \leq \ell$, we have that $\mathbb{P}(X_i = 1) \sim 1/(2M)$. For the double edges, we need to know the probability that a given set of edges of the kernel is not assigned any vertices of degree 2 in the kernel configuration model. Let

$$\delta = \left(\frac{c - 2p_c}{c}\right)^2 = \left(\frac{\lambda_c}{c}\right)^2. \quad (21)$$

For any fixed q and any set of edges $\{e_1, \dots, e_q\}$ in K , the probability that none of these kernel edges is assigned a vertex of degree 2 (and hence become edges of G) can be estimated as follows.

$$\begin{aligned} \mathbb{P}(\{e_1, \dots, e_q\} \subseteq E(G) | \{e_1, \dots, e_q\} \subseteq E(K)) &= \prod_{i=0}^{D_2-1} \left(1 - \frac{q}{M+i}\right) \sim \exp\left(-q \sum_{i=0}^{D_2-1} \frac{1}{M+i}\right) \\ &\sim \left(\frac{M-1}{M+D_2-1}\right)^q \sim \left(\frac{cn/2 - p_cn}{cn/2}\right)^q = \delta^{q/2}. \end{aligned} \quad (22)$$

Thus, for every $1 \leq j \leq t$, we have that

$$\mathbb{P}(Y_j = 1) = \mathbb{P}(f_j \subseteq E(K)) \cdot \mathbb{P}(f_j \subseteq E(G) | f_j \subseteq E(K)) \sim \frac{\delta}{(2M)^2}.$$

Hence,

$$\begin{aligned} \mathbb{E}(X + Y) &= \mathbb{E}(X) + \mathbb{E}(Y) \sim \ell \cdot \frac{1}{2M} + t \cdot \frac{\delta}{(2M)^2} \\ &= \frac{\sum_{i=1}^{n'} \binom{d'_i}{2}}{2M} + \frac{\delta}{(2M)^2} \sum_{\substack{(i,j) \\ i \neq j}} \binom{d'_i}{2} \binom{d'_j}{2}. \end{aligned}$$

We will use the following lemma, which is proved in the end of the section.

Lemma 9. *Let q be a fixed positive integer. For $d \in \tilde{\mathcal{D}}(\psi)$,*

$$\sum_{(i_1, \dots, i_q)} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} \sim \left(\frac{c}{2}\right)^q,$$

where the sum is over all $(i_1, \dots, i_q) \in [n']^q$ where $i_j \neq i_{j'}$ for all $j \neq j'$.

Thus,

$$\mathbb{E}(X + Y) \sim \frac{c}{2} + \frac{\lambda_c^2}{4}.$$

It only remains to examine the higher moments, and show that

$$\mathbb{E}([X + Y]_k) = \sum_{k_1+k_2=k} \binom{k}{k_1} \sum_{y \in I(k_1, k_2)} \mathbb{P}(W(y) = 1)$$

for $y \in I(k_1, k_2)$, where $I(k_1, k_2)$ is the set of tuples $y \in (\{e_1, \dots, e_\ell\})^{k_1} \times (\{f_1, \dots, f_t\})^{k_2}$ such that $y_i \neq y_j$ for $i \neq j$ and $\bigcup_{i=1}^k \{y_i\}$ induces a matching on the set of points S , and $W(y)$ is the indicator variable for the event that $X_i = 1$ for every $e_i \in \{y_1, \dots, y_k\}$ and $Y_j = 1$ for every $f_j \in \{y_1, \dots, y_k\}$.

Let $I'(k_1, k_2)$ be the set of tuples $y \in I(k_1, k_2)$ such that, in the graph induced by $\bigcup_{i=1}^k \{y_i\}$ in K , the degree of every vertex is either 0 or 2. (This is the non-overlapping case.) Let $I''(k_1, k_2) := I(k_1, k_2) \setminus I'(k_1, k_2)$.

For $y \in I''(k_1, k_2)$, it is easy to see that the graph induced by $\bigcup_{i=1}^k \{y_i\}$ in K has more edges than vertices. For any fixed multigraph H with more edges than vertices, the expected number of copies of H in K can be bounded as follows. There are at most $(n')^{|V(H)|}$ ways of assigning the vertices of H to vertices of K . If we assign a vertex with degree d in H to a vertex v in K , then there are at most Δ^d ways of choosing the points inside v to be the points of the vertex in H . So there are at most $(n')^{|V(H)|} \Delta^{2|E(H)|} = O((n')^{|V(H)|} (\log n)^{2|E(H)|})$ possible copies of H in K . The probability that a set of $|E(H)|$ edges in K is $O(M^{-|E(H)|})$. Thus, the expected number of copies of H in K is at most

$$O\left(\frac{(n')^{|V(H)|} (\log n)^{2|E(H)|}}{M^{|E(H)|}}\right) = O\left(\frac{(n')^{|V(H)|} (\log n)^{2|E(H)|}}{(n')^{|V(H)|+1}}\right) = o(1).$$

From this, since k is fixed, we deduce that

$$\sum_{k_1+k_2=k} \binom{k}{k_1} \sum_{y \in I''(k_1, k_2)} \mathbb{P}(W(y) = 1) = o(1).$$

For $I'(k_1, k_2)$, using (22) and Lemma 9,

$$\begin{aligned} \sum_{y \in I'(k_1, k_2)} \mathbb{P}(W(y) = 1) &= \sum_{y \in I'(k_1, k_2)} \frac{\delta^{k_2}}{(2M)^{k_1+2k_2}} = |I'(k_1, k_2)| \frac{1}{(2M)^{k_1+2k_2}} \delta^{k_2} \\ &= \sum_{(v_1, \dots, v_{k_1+2k_2})} \prod_{i=1}^{k_1+2k_2} \binom{d'_{v_i}}{2} \frac{1}{(2M)^{k_1+2k_2}} \cdot \delta^{k_2} \\ &\sim \left(\frac{c}{2}\right)^{k_1+2k_2} \delta^{k_2}, \end{aligned}$$

where $v_i \neq v_j$ in $(v_1, \dots, v_{k_1+2k_2})$ for every $i \neq j$.

Thus,

$$\begin{aligned} \mathbb{E}([X + Y]_k) &= o(1) + \sum_{k_1+k_2=k} \binom{k}{k_1} \sum_{y \in I'(k_1, k_2)} \mathbb{P}(W(y) = 1) \\ &= \sum_{k_1+k_2=k} \binom{k}{k_1} \left(\frac{c}{2}\right)^{k_1+2k_2} \delta^{k_2} + o(1) \\ &= \left(\frac{c}{2} + \frac{\lambda_c^2}{4}\right)^k + o(1), \end{aligned}$$

as required to establish (17).

Proof of Lemma 9. For every $q \geq 1$, let

$$L_q := \{(i_1, \dots, i_q) : 1 \leq i_j \leq n' \ \forall j\};$$

$$L_q^\neq := \{(i_1, \dots, i_q) \in L_q : i_j \neq i_{j'} \ \forall j \neq j'\}$$

and

$$L_q^\equiv := \{(i_1, \dots, i_q) \in L_q : i_j = i_{j'} \text{ for some } j \neq j'\}.$$

We have

$$\frac{\sum_{i=1}^{n'} d'_i (d'_i - 1)}{\sum_{i=1}^{n'} d'_i} = \frac{\sum_{i=1}^n d_i (d_i - 1) - 2D_2}{\sum_{i=1}^n d_i - 2D_2} \sim \frac{\bar{\eta}_c c n - 2p_c n}{c n - 2p_c n} = \frac{\bar{\eta}_c c - 2p_c}{c - 2p_c} = c.$$

So, for every $q \geq 1$,

$$\sum_{(i_1, \dots, i_q) \in L_q} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} = \left(\frac{\sum_i \binom{d'_i}{2}}{2M} \right)^q \sim \left(\frac{c}{2} \right)^q = \Theta(1). \quad (23)$$

For $q \geq 2$, we have that

$$\begin{aligned} \sum_{(i_1, \dots, i_q) \in L_q^\neq} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} &\leq q! \cdot \sum_{(i_1, \dots, i_{q-1}) \in L_{q-1}} \binom{d'_{i_1}}{2} \prod_{j=1}^{q-1} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} \\ &\leq q! \frac{\Delta^2}{4M} \sum_{(i_1, \dots, i_{q-1}) \in L_{q-1}} \prod_{j=1}^{q-1} \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^{q-1}} \\ &\sim q! \frac{\Delta^2}{4M} \left(\frac{c}{2} \right)^{q-1} = o(1). \end{aligned} \quad (24)$$

Note that for $q = 1$, we have $L_q = L_q^\neq$ and we are done by (23). So suppose $q \geq 2$. Then L_q is the disjoint union of L_q^\neq and L_q^\equiv . Thus, using (23) and (24),

$$\begin{aligned} \sum_{(i_1, \dots, i_q) \in L_q} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} &= \sum_{(i_1, \dots, i_q) \in L_q^\neq} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} + \sum_{(i_1, \dots, i_q) \in L_q^\equiv} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} \\ &= \sum_{(i_1, \dots, i_q) \in L_q} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} - \sum_{(i_1, \dots, i_q) \in L_q^\equiv} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} \\ &= \sum_{(i_1, \dots, i_q) \in L_q} \prod_{j=1}^q \binom{d'_{i_j}}{2} \cdot \frac{1}{(2M)^q} + o(1) \sim \left(\frac{c}{2} \right)^q. \end{aligned} \quad \square$$

6 The case $c \rightarrow \infty$

Recall that n and m are such that $m = O(n \log n)$, $m > n$ and $m/n \rightarrow \infty$. The set $\mathcal{D}(n, m)$ contains all degree sequences \mathbf{d} such that $\sum_{i=1}^n d_i = 2m$ and $d_i \geq 2$ for all $i \in [n]$.

Recall that $U(\mathbf{d})$ is the probability of obtaining a simple graph using the pairing model with degree sequence \mathbf{d} , and $U'(\mathbf{d})$ is defined similarly, for the event that it is additionally 2-connected.

Let $0 < \varepsilon < 0.01$ be a constant, and let

$$\tilde{\mathcal{D}} := \{\mathbf{d} \in \mathcal{D}(n, m) : \max d_i \leq n^\varepsilon\} \quad \text{and} \quad \tilde{\mathcal{D}}^c := \mathcal{D}(n, m) \setminus \tilde{\mathcal{D}}.$$

By [Luc92, Theorem 12.2(iii)],

$$U(\mathbf{d}) \sim U'(\mathbf{d}). \quad (25)$$

when \mathbf{d} is in $\mathcal{D}(n, m)$ and satisfies $D_2(\mathbf{d})/m \rightarrow 0$ and $\max_i d_i \leq n^{0.01}$. The condition on D_2 is satisfied by all \mathbf{d} of concern when n is large since $D_2(\mathbf{d}) \leq n$ and $c \rightarrow \infty$. Thus (25) holds for any sequence $\mathbf{d}(n)$ with $\mathbf{d} \in \tilde{\mathcal{D}}$ and $m/n \rightarrow \infty$ where $m = \frac{1}{2} \sum_{i=1}^n d_i$.

It is known [McK85] that

$$U(\mathbf{d}) \sim \exp(-\eta(\mathbf{d})/2 - \eta(\mathbf{d})^2/4). \quad (26)$$

This result, together with (1), proves Theorem 4(c).

If all degree sequences were in $\tilde{\mathcal{D}}$, we could immediately deduce Theorem 2(c) from (25). So it remains to show that the other degree sequences have no effect asymptotically. We need to randomize \mathbf{d} with the distribution of the vector \mathbf{Y} of independent truncated Poissons as defined in Section 3.1, we have

$$\begin{aligned} \mathbb{E}(U'(\mathbf{Y})|\Sigma) &= \mathbb{E}(U'(\mathbf{Y})|\tilde{\mathcal{D}})\mathbb{P}(\tilde{\mathcal{D}}|\Sigma) + \mathbb{E}(U'(\mathbf{Y})|\tilde{\mathcal{D}}^c)\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma) \\ &= \mathbb{E}(U(\mathbf{Y})|\tilde{\mathcal{D}})(1 + o(1))\mathbb{P}(\tilde{\mathcal{D}}|\Sigma) + O(\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma)) \end{aligned} \quad (27)$$

by (25). Properties of \mathbf{Y} were investigated by Pittel and Wormald, and in particular [PW03, Eq. (27)] implies for any $\beta > 0$

$$\mathbb{P}(\max_j Y_j \geq m^\beta) \leq \exp(-n^\alpha)$$

for some fixed $\alpha(\beta)$. This shows that $\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma) = O(\exp(-n^\alpha))$ for some fixed positive α . Also, [PW03, Theorem 4(b) and (21)] give

$$\mathbb{E}(\exp(-\eta(\mathbf{Y})/2 - \eta(\mathbf{Y})^2/4)|\Sigma) \geq \exp(-O(\log^2 n)).$$

Using (26) and the bound on $\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma)$, we may now deduce that the first term in (27) dominates the second, and thus

$$\mathbb{E}(U'(\mathbf{Y})|\Sigma) \sim \mathbb{E}(U(\mathbf{Y})|\tilde{\mathcal{D}}).$$

Similarly,

$$\mathbb{E}(U(\mathbf{Y})|\Sigma) = \mathbb{E}(U(\mathbf{Y})|\tilde{\mathcal{D}})\mathbb{P}(\tilde{\mathcal{D}}|\Sigma) + O(\mathbb{P}(\tilde{\mathcal{D}}^c|\Sigma)) \sim \mathbb{E}(U(\mathbf{Y})|\tilde{\mathcal{D}})$$

and so

$$\mathbb{E}(U'(\mathbf{Y})|\Sigma) \sim \mathbb{E}(U(\mathbf{Y})|\Sigma). \quad (28)$$

By Theorem 3 ([PW03]) and equation (13) ([PW03]),

$$C(n, m) \sim (2m-1)!!Q(n, m)\mathbb{E}(U(\mathbf{Y})|\Sigma) \sim (2m-1)!!Q(n, m)\exp(-\bar{\eta}_c/2 - \bar{\eta}_c^2/4).$$

So by (3) and (5),

$$T(n, m) \sim C(n, m) \sim (2m-1)!! \frac{(e^{\lambda_c} - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n c(1 + \bar{\eta}_c - c)}} \exp(-\bar{\eta}_c/2 - \bar{\eta}_c^2/4). \quad (29)$$

Since $c \rightarrow \infty$, we have that $\lambda_c \sim c$ (see Lemma 1(c) from [PW03]). This implies that $\bar{\eta}_c = \lambda_c e^{\lambda_c} / (e^{\lambda_c} - 1) \sim c$. This fact together with (29) implies Theorem 2(c).

7 Proof of Theorem 1

Note that we have already proved Theorem 2. If we prove that in each of the three cases in Theorem 2

$$T(n, m) \sim (2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n c (1 + \bar{\eta}_c - c)}} \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right)$$

then the subsubsequence principle easily implies Theorem 1. (See [JLR00] (Section 1.2) for the subsubsequence principle.)

It suffices to show

$$\sqrt{\frac{3r}{2m}} \frac{1}{e} \sim \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right), \text{ when } c \rightarrow 2$$

and

$$\exp\left(-\frac{\bar{\eta}_c}{2} - \frac{\bar{\eta}_c^2}{4}\right) \sim \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right), \text{ when } c \rightarrow \infty.$$

(See (11) and (29).)

So suppose $c \rightarrow 2$. Using Lemma 1 from [PW03], $\lambda_c = 3(c-2) + O((c-2)^2) = o(1)$. Thus,

$$\exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right) \sim \exp\left(-\frac{c}{2}\right) \sim \frac{1}{e}.$$

By series expansion, $p_c = 1 - \frac{1}{3} \cdot (3(c-2)) + O((c-2)^2) = 3 - c + O\left(\frac{r^2}{n^2}\right)$. Using $c = 2m/n = 2 + r/n$,

$$\sqrt{\frac{c - 2p_c}{c}} = \sqrt{\frac{c - 6 + 2c + O(r^2/n^2)}{c}} = \sqrt{\frac{3(2 + r/n) - 6 + O(r^2/n^2)}{2 + r/n}} \sim \sqrt{\frac{3r}{2m}}.$$

Now suppose $c \rightarrow \infty$. In this case $\lambda_c \sim c$ (see Lemma 1(c) from [PW03]). From the definition of λ_c we have $c = \lambda_c + O(\lambda_c^2 e^{-\lambda_c})$. Also,

$$\bar{\eta}_c = \lambda_c \cdot \frac{e^{\lambda_c}}{e^{\lambda_c} - 1} = \lambda_c + O(\lambda_c e^{-\lambda_c}) \quad \text{and} \quad p_c = \frac{\lambda_c^2}{2(e^{\lambda_c} - 1 - \lambda_c)} \rightarrow 0.$$

This implies

$$\sqrt{\frac{c - 2p_c}{c}} \sim 1 \quad \text{and} \quad \exp\left(-\frac{\bar{\eta}_c}{2} - \frac{\bar{\eta}_c^2}{4}\right) \sim \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4}\right).$$

We now have Theorem 1.

8 Enumeration of k -edge-connected graphs

In the introduction we observed that for $k \geq 3$ and for m under consideration, almost all k -cores on n vertices and m edges are k -connected, so it follows that almost all are also k -edge-connected. This settles the enumeration of k -edge-connected (n, m) -graphs for fixed $k \geq 3$. When $k = 2$ we have the following result.

Theorem 10. *Suppose $m = O(n \log n)$ and $m - n \rightarrow \infty$. Then the number of 2-edge-connected (n, m) -graphs is asymptotic to*

$$(2m-1)!! \frac{(\exp(\lambda_c) - 1 - \lambda_c)^n}{\lambda_c^{2m} \sqrt{2\pi n c (1 + \bar{\eta}_c - c)}} \sqrt{\frac{c - 2p_c}{c}} \exp\left(-\frac{c}{2} - \frac{\lambda_c^2}{4} + \frac{\lambda_c^3}{2(e^{\lambda_c} - 1)^2}\right).$$

Proof. The result is established by adapting the methods used for 2-connected graphs, so we omit unimportant details.

For $c \rightarrow \infty$ we have shown that almost all (n, m) -graphs are 2-connected, hence the asymptotic formula for the 2-connected graphs also holds for the 2-edge-connected ones.

For $c \rightarrow 2$ our proof actually showed that the probability of a 2-edge-connected simple graph in the kernel configuration model is asymptotic to the probability of a 2-connected simple graph. So, once again, the asymptotic formula for the 2-connected graphs also holds for the 2-edge-connected graphs.

When c is bounded away from 2 and bounded, the situation is more interesting. For 2-connectivity, the key computation used the method of moments to deduce the Poisson distribution of the number of loops in the kernel plus the number of double edges in the pseudograph. (See Section 5.1.) Note that, using [Luc92, Lemma 12.1(ii)] as we did in Section 5, the graph G will a.a.s. be 2-edge-connected if it has no loops or multiple edges and no cycle in the kernel on a vertex of degree 3. Thus, in the present case we must study the random variable $X + Y + Z$, where X counts loops on vertices of degree 3 in the kernel, Y counts double edges in the kernel which are assigned no vertices of degree 2, and Z counts loops in the kernel at vertices of degree at least 4 which are not assigned at least 2 degree-2 vertices. Analogous arguments establish the Poisson distribution of $X + Y + Z$. We discuss only the computation of the first moment here.

There are three ways to attach a loop to each of the kernel's D_3 vertices of degree 3. Analogous to the condition on $D_2(\mathbf{d}) - p_c n$ in the definition of $\tilde{\mathcal{D}}(\psi)$ in Section 5, we can assume for the crucial computations that $D_3 \sim p_3 n$, where p_3 is the probability that a truncated Poisson $\text{Po}(2, \lambda_c)$ takes the value 3. Each possible loop occurs with probability $1/(2M)$, giving $\mathbb{E}(X) = D_3/(2M) \sim c/2 - \lambda_c/2$.

From Section 5.1 we have $\mathbb{E}(Y) \sim \lambda_c^2/4$. To compute $\mathbb{E}(Z)$ we must first estimate the probability that a given kernel edge is not assigned at least two degree-2 vertices. The number of assignments of the D_2 degree-2 vertices to the M kernel edges is the rising factorial $[M]^{D_2}$. Either the given kernel edge is assigned no vertices, which has probability

$$\frac{[M-1]^{D_2}}{[M]^{D_2}} = \frac{M-1}{m-1} \sim \sqrt{\delta},$$

or the edge is assigned exactly one vertex, which has probability

$$D_2 \frac{[M-1]^{D_2-1}}{[M]^{D_2}} = D_2 \frac{M-1}{(m-2)(m-1)} \sim (1 - \sqrt{\delta})\sqrt{\delta}$$

since the degree-2 vertex may be chosen in D_2 ways. The sum of these two probabilities is $2\sqrt{\delta} - \delta$. The number of ways to attach a loop among the vertices of degree at least 4 is $\sum_{i=1}^{n'} \binom{d'_i}{2} - 3D_3$. Each occurs with probability $1/(2M)$. Using Lemma 9 we have

$$\mathbb{E}(Z) = \frac{\sum_{i=1}^{n'} \binom{d'_i}{2} - 3D_3}{2M} (2\sqrt{\delta} - \delta) \sim \left(\frac{c}{2} - \frac{3D_3}{2M} \right) (2\sqrt{\delta} - \delta) \sim \frac{\lambda_c}{2} - \frac{\lambda_c^3}{2(e^{\lambda_c} - 1)^2}.$$

The probability that G is 2-edge-connected and simple is thus

$$\exp \left(-\frac{c}{2} - \frac{\lambda_c^2}{4} + \frac{\lambda_c^3}{2(e^{\lambda_c} - 1)^2} \right),$$

and the formula for the number of 2-edge-connected graphs follows as in the 2-connected case. This concludes the proof of the theorem. \square

Note: The alert reader will notice that an alternative way to derive this result would be to take Łuczak’s corollary at the end of Section 12.5 in [Luc92], which gives the probability of 2-edge-connectedness of graphs with a given degree sequence, and then use our argument to extend this to graphs with minimum degree 2. The resulting formula agrees with ours if one corrects the formulae in Theorem 12.4 of his paper, and its Corollary, to let $D_3/M' \rightarrow c$ (not D_3/M) in his notation. (We believe the source of this problem is in the first displayed equation in the proof of [Luc92, Theorem 12.4]. The correct definition of c appears just after this equation.)

Acknowledgement This work is based on the foundational work of Boris Pittel with the third author on counting graphs with given minimum degree, and we gratefully acknowledge discussions with Boris on the initial stages of the present work.

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