Sizes of the largest clusters for supercritical percolation on random recursive trees

Jean Bertoin

Sizes of the largest clusters for supercritical percolation on random recursive trees

Jean Bertoin*

Abstract

We consider Bernoulli bond-percolation on a random recursive tree of size $n \gg 1$, with supercritical parameter $p(n) = 1 - t/\ln n + o(1/\ln n)$ for some $t > 0$ fixed. We show that with high probability, the largest cluster has size close to $\mathrm{e}^{-t}n$ whereas the next largest clusters have size of order $n/\ln n$ only and are distributed according to some Poisson random measure.

Key words: Random recursive tree, supercritical bond-percolation.

1 Introduction

Consider Bernoulli bond-percolation on a large but finite connected graph. So each edge is removed with some fixed probability $1 - p \in (0, 1)$ and independently of the other edges, inducing a partition of the set of vertices into connected clusters. Loosely speaking, assume that p is supercritical, in the sense that with high probability, there exists a giant cluster, that is of size comparable to that of the entire graph. A natural problem in this setting is then to estimate the size of the next largest clusters; let us merely recall the known answers to this question in three important instances.

First, in the case of the complete graph with n vertices, a classical result due to Erdős and Rényi (see for instance [4]) shows that for $p(n) \sim c/n$ with $c > 1$ fixed, with high probability when $n \gg 1$, the largest cluster has size close to $\theta(c)n$ where $\theta(c)$ is the unique strictly positive solution to the equation $x + \mathrm{e}^{-cx} = 1$, while the second, third, etc. largest clusters have only size of order $\ln n$.

*Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland. Email: jean.bertoin@math.uzh.ch

Second, consider a box in the d-dimensional cubic lattice with $d \geq 2$, say $[1, \ell]^d$, so this box contains $n = \ell^d$ vertices. Let p be greater than the critical parameter p_c of Bernoulli bond-percolation on the entire lattice and denote by $\theta(p)$ the probability that the origin belongs to the infinite cluster. Then with high probability when $n \gg 1$, the size of the largest percolation cluster in the box is close to $\theta(p)n$, and the next largest clusters have size of order $(\ln n)^{d/(d-1)}$ as it can deduced for instance from results in Chapter 8 of [8].

Third, in the case of a uniform Cayley tree of size n (i.e. a tree picked uniformly at random amongst the n^{n-2} trees on a set of n labelled vertices), Pitman [18, 19] showed that for $1-p(n) \sim t/\sqrt{n}$ with a fixed $t > 0$, the sequence of the sizes of the clusters ranked in decreasing order and renormalized by a factor $1/n$ converges weakly as $n \to \infty$ to a random mass partition which can be described explicitly in terms of a conditioned Poisson measure. It is interesting to point out that in this situation, the number of giant components is unbounded as $n \to \infty$.

The main purpose of this work is to investigate analogously the case of large random recursive trees. More precisely, a tree on the set of vertices $\{0, 1, \ldots, n\}$ (note that the parameter n now corresponds to the number of edges) is called recursive if when rooted at 0, the sequence of vertices along any branch from the root to a leaf is increasing. Recursive trees are sometimes also referred to as increasing trees in the literature; they arise for instance in computer science as data structures, or as simple epidemic models; they can be used to represent permutations or the Bolthausen-Sznitman coalescent (see Goldschmidt and Martin [7]). There are $n!$ such recursive trees, and we pick one of them uniformly at random. Loosely speaking, our main result shows that for $1 - p(n) \sim t/\ln n$ with $t > 0$ fixed and $n \gg 1$, with high probability there is a unique giant cluster of size $\sim \mathrm{e}^{-t}n$ and the next largest clusters have size of order $n/\ln n$. A more precise description will be given in Theorem 1 below.

Our approach relies crucially on a coupling between the so-called cutting down of random recursive trees and a certain random walk, which is due to Iksanov and Möhle [6] and was used by these authors to investigate the asymptotic behavior of number of random cuts which are needed to isolate the root. More precisely, this coupling combined with well-known features of extreme value theory also provided useful estimates for the sizes of the largest tree-components that became disconnected from the root in the cutting down process. Essentially, all what is needed then is to remove some further edges to these components in order to observe the percolation configuration.

This work leaves open similar questions in the sub-critical regime. Typically, it would be interesting to estimate the size of the largest percolation clusters when $1 - p(n) \gg 1/\ln n$. Possibly our approach for tackling this problem in the super-critical case might still be of some use in sub-critical situations provided that $p(n)$ still remains sufficiently close to 1; however there will probably be several technical difficulties to overcome.

The plan of the rest of this paper is as follows. Section 2 introduces the cutting down procedure in discrete time and investigates consequences of the coupling of Iksanov and Möhle. Section 3 is devoted to the continuous-time version of the cutting down process and its limiting behavior when $n \to \infty$. Our main results for supercritical percolation on large random recursive trees are stated and proved in Section 4. Finally, in Section 5, we compare the effects of the same fragmentation procedures on random recursive trees and random Cayley trees.

2 Cutting down a tree and a chain of ordered forests

Consider a tree $T = (V, E)$, where as usual V is the set of vertices and E the set of edges. We denote by $n = |E|$ the number of edges, so $|V| = n + 1$. The parameter n will have of course a crucial role in this work although for the sake of conciseness it will be frequently (but not always) omitted from the notation. We assume that one of the vertices has been distinguished; it will be referred to as the root. A forest on V is a graph with no cycles, in other words the connected components are trees. The tree-component that contains the root is called the root-component, we denote it with a prime to distinguish it from the other tree-components. A forest is called ordered if its tree-components have been ranked, with the convention that the root-component is always ranked last. So a typical ordered forest has the form $(T_1, T_2, \ldots, T_j, T')$ where T' is the root-component and $T_1, \ldots, T_j$ are the tree-components that do not contain the root.

We will be interested in the following Markov chain $\mathcal{F} = (\mathcal{F}_k, k \geq 0)$ with values in the space of ordered forests, which can be traced back to the work of Meir and Moon [14], and will be referred to as the cutting-down chain. Let the chain start from $\mathcal{F}_0 = T$. We then remove an edge of T uniformly at random; this yields an ordered forest with two subtrees, $\mathcal{F}_1 = (T_1, T_1')$, where T_1' contains the root. The subtree T_1 will be left unchanged in the next steps, whereas we keep removing randomly edges from the root-component. That is, at the second step, we remove an edge from T_1' picked uniformly at random, provided of course that T_1' is not yet reduced to the root. This disconnects T_1' into two subtrees T_2 and T_2' where T_2' contains the root. Then $\mathcal{F}_2$ is given by the sequence (T_1, T_2, T_2'). We iterate in an obvious way, the chain reaching its terminal state after ζ steps when the root has been isolated, that is when all the edges adjacent to the root have been removed. The study of the absorption time ζ for various types of random trees has motivated a number of works in the literature, see in particular Meir and Moon [14, 15], Janson [12, 13], Panholzer [16, 17], Iksanov and Möhle [6], Drmota *et al.* [5], Holmgren [10, 11], etc.

We now further assume that the set of vertices V is endowed with a total order and that the root is the smallest vertex. Recall that a recursive tree on V is a tree such that the sequence of vertices from the root to any leaf is increasing. A random recursive tree $\mathcal{T}$ on V

refers to a recursive tree picked uniformly at random; it can be constructed by the following elementary algorithm. If we denote by $v_0, v_1, \ldots, v_n$ the ordered sequence of vertices, then for every $i = 1, \ldots, n$, we pick v'_i uniformly at random from $\{v_0, \ldots, v_{i-1}\}$ and independently of the v'_j for $j \neq i$, and declare that $\{(v'_i, v_i) : i = 1, \ldots n\}$ is the set of edges of $\mathcal{T}$.

From now on, we shall always assume that the initial state of the forest-valued chain $\mathcal{F}$ is given by a random recursive tree on V, i.e. $T = \mathcal{T}$. The following statement should be plain from induction using the construction of $\mathcal{T}$ described above; it is already implicit in several works in this area.

Lemma 1 *Fix $k \in \mathbb{N}$ and denote by $V_1, \ldots, V_{k\wedge\zeta}$, and $V'_{k\wedge\zeta}$ the subsets of vertices of the subtrees $T_1, \ldots, T_{k\wedge\zeta}$, and $T'_{k\wedge\zeta}$ of the ordered forest $\mathcal{F}_k$. Then conditionally on $V_1, \ldots, V_{k\wedge\zeta}$, and $V'_{k\wedge\zeta}$, the trees $T_1, \ldots, T_{k\wedge\zeta}$, and $T'_{k\wedge\zeta}$ are independent random recursive trees on their respective sets of vertices.*

The key quantitative ingredient for studying the distribution of $\mathcal{F}_k$ is the fact that the distribution of $|V_1|$, the number of vertices of the subtree T_1 that do not contain the root after the first edge removal, is known explicitly, see [15, 16]. Specifically, introduce first an integer-valued random variable ξ with distribution

$$\mathbb{P}(\xi = \ell) = \frac{1}{\ell(\ell+1)}\,, \qquad \ell \in \mathbb{N}\,.$$

Recall that $|V| = n + 1$; then one has

$$\mathbb{P}(|V_1| = \ell) = \mathbb{P}(\xi = \ell \mid \xi \leq n) = \frac{n+1}{n\ell(\ell+1)}\,, \qquad \ell = 1, \ldots, n\,. \tag{1}$$

Lemma 1 now incites us to introduce a random walk

$$S_j = \xi_1 + \ldots + \xi_j\,, \qquad j \in \mathbb{N}$$

where the ξ_i are i.i.d. copies of ξ, together with its first passage time

$$N(n) = \min\{j \geq 1 : S_j > n\}\,.$$

Following Iksanov and Möhle [6], one can couple the random walk S and the Markov chain $\mathcal{F}$ as follows. [In fact the coupling of Iksanov and Möhle is stronger than that described in Lemma 2 below; see Section 2 in [6] for details. The present weaker version will be sufficient for our purpose.]

Lemma 2 *One can construct on the same probability space a version of the random walk S and a version of the Markov chain $\mathcal{F}$ started from a random recursive tree with n edges such that:*

(i) *The absorption time ζ of $\mathcal{F}$ is bounded from below by $N(n)$,*

(ii) *For every $k < N(n)$, there is the identity*

$$(|V_1|, \ldots, |V_k|, |V_k'|) = (\xi_1, \ldots, \xi_k, n + 1 - S_k)\,,$$

where for $i = 1, \ldots, k$, $|V_i|$ denotes the number of vertices of the i-th tree component of $\mathcal{F}_k$, and $|V_k'|$ the number of vertices of the root-component of $\mathcal{F}_k$.

In this direction, we recall that the asymptotic behavior as $n \to \infty$ of the first passage time $N(n)$ has been determined in Proposition 2 of [6]: $n^{-1}(\ln n)^2 N(n) - \ln n - \ln\ln n$ converges in distribution to a completely asymmetric Cauchy variable. For the purpose of this work, it will be sufficient to record that

$$\lim_{n\to\infty} \frac{\ln n}{n} N(n) = 1 \qquad \text{in probability.} \tag{2}$$

Lemma 2 enables one to rephrase questions about the sizes of the largest components of $\mathcal{F}_k$ in terms of the random walk S and its increments, at least on the event $\{N(n) > k\}$. Observe that, according to (2), this event has a high probability as soon as $k \leq cn/\ln n$ for some fixed $c < 1$ and $n \gg 1$. In this direction, we will be mainly interested in the following easy feature.

For $k \in \mathbb{N}$, denote by

$$\xi_{1,k} \geq \xi_{2,k} \geq \ldots \geq \xi_{k,k}$$

the order statistics of the first k i.i.d. copies of ξ. Let also $\sigma(\cdot, k)$ be the random permutation of $\{1, \ldots, k\}$ such that $\xi_{i,k} = \xi_{\sigma(i,k)}$, that is $\sigma(i, k)$ should be thought of as the index of the i-th record in the k-sample.

Lemma 3 *Let $(k(n) : n \in \mathbb{N})$ be a random sequence of integers such that*

$$\lim_{n\to\infty} \frac{\ln n}{n} k(n) = c \qquad \textit{in probability}$$

for some fixed $c \in (0, 1)$. Then as $n \to \infty$, $n^{-1} S_{k(n)}$ converges in probability to c, and for every fixed $j \in \mathbb{N}$, the $(2j)$-tuple

$$\left(\frac{\ln n}{n} \xi_{1,k(n)}, \ldots, \frac{\ln n}{n} \xi_{j,k(n)}; \frac{\ln n}{n} \sigma(1, k(n)), \ldots, \frac{\ln n}{n} \sigma(j, k(n)) \right)$$

converges in distribution to

$$(a_1, \ldots, a_j; u_1, \ldots, u_j)$$

where $(a_1, u_1), (a_2, u_2), \ldots$ *is the sequence of the atoms of a Poisson random measure on* $(0, \infty) \times [0, c]$ *with intensity* $a^{-2}\mathrm{d}a\mathrm{d}u$*, ranked in the decreasing order of the first coordinate.*

Remark. Basic properties of Poisson random measures entail that the variables $u_1, \ldots, u_j$ are i.i.d. uniformly distributed on $[0, c]$, and independent of $a_1, \ldots, a_j$. Further $1/a_1, 1/a_2 - 1/a_1, \ldots, 1/a_j - 1/a_{j-1}$ is a sequence of j independent standard exponential variables. In particular a_j has the same distribution as $1/\gamma_j$, with γ_j a gamma variable with parameter $(j, 1)$.

Proof: It is easily verified using generating functions that if the sequence $k(n)$ is deterministic with $k(n) \sim cn/\ln n$, then the random walk S fulfills the following weak law of large numbers: $n^{-1}S_{k(n)}$ converges in probability to c. See the proof of Proposition 2 in [6] for a second order limit theorem in this vein.

Then consider the general case when $k(n)$ may be random. The assumptions that $n^{-1}k(n)\ln n$ converges in probability to a fixed $c > 0$ enables us to introduce two deterministic sequences of integers $k'(n)$ and $k''(n)$ such that $k'(n) \sim k''(n) \sim cn/\ln n$ and the probability of the event $\{k'(n) \leq k(n) \leq k''(n)\}$ tends to 1 as $n \to \infty$. On this event, we have by monotonicity

$$\frac{S_{k'(n)}}{n} \leq \frac{S_{k(n)}}{n} \leq \frac{S_{k''(n)}}{n},$$

from which we conclude that $n^{-1}S_{k(n)}$ converges to c in probability.

The second part of the statement about the largest increments of the random walk and their locations is immediate from extreme value theory, since the tail-distribution of ξ fulfills $\mathbb{P}(\xi > x) \sim 1/x$ as $x \to \infty$. See, for instance, Proposition 3.21 in [20]. □

We conclude this section with a technical consequence of Lemmas 2 and 3 which shall be useful later on. For every $j \geq 0$, let $|E'_j(n)|$ denote the number of edges of the root-component of $\mathcal{F}_j$; in particular $|E'_0(n)| = n$, $|E'_j(n)| = 0$ for $j \geq \zeta$, and $|E'_j(n)| = |V'_j| - 1 = n - S_j$ if $j < N(n)$.

Corollary 1 *Let* $(k(n) : n \in \mathbb{N})$ *be a sequence of random integers such that*

$$\lim_{n\to\infty} \frac{\ln n}{n} k(n) = c \qquad \textit{in probability}$$

for some fixed $c \in (0,1)$. Then

$$\lim_{n\to\infty} \sum_{j=0}^{k(n)-1} \frac{\ln n}{|E'_j(n)|} = -\ln(1-c) \quad \textit{and} \quad \lim_{n\to\infty} \sum_{j=0}^{k(n)-1} \left(\frac{\ln n}{|E'_j(n)|}\right)^2 = 0 \qquad \textit{in probability.}$$

Proof: Fix some integer $m \geq 1$ and for $i = 1, \ldots, m-1$, introduce sequences of integers $(k_i(n))_{n\in\mathbb{N}}$ such that $k_i(n) \sim (i/m)k(n)$. Then decompose

$$\sum_{j=0}^{k(n)-1} \frac{\ln n}{|E'_j(n)|} = \sum_{j=0}^{k_1(n)-1} \frac{\ln n}{|E'_j(n)|} + \sum_{j=k_1(n)}^{k_2(n)-1} \frac{\ln n}{|E'_j(n)|} + \cdots + \sum_{j=k_{m-1}(n)}^{k(n)-1} \frac{\ln n}{|E'_j(n)|}\,.$$

Next observe by monotonicity that for $i = 0, \ldots, m-1$ and $k_0(n) \equiv 0$,

$$\frac{k_{i+1}(n) - k_i(n)}{|E'_{k_{i+1}(n)}(n)|} \leq \sum_{j=k_i(n)}^{k_{i+1}(n)-1} \frac{1}{|E'_j(n)|} \leq \frac{k_{i+1}(n) - k_i(n)}{|E'_{k_i(n)}(n)|}\,.$$

On the one hand, combining Lemmas 2 and 3, we know that

$$|E'_{k_i(n)}(n)| = |V'_{k_i(n)}| - 1 \sim (1 - c(i/m))n \qquad \text{in probability,}$$

while on the other hand

$$k_{i+1}(n) - k_i(n) \sim m^{-1}cn/\ln n \qquad \text{in probability.}$$

Letting $m \to \infty$, we conclude from Riemann-sum approximation that

$$\lim_{n\to\infty} \sum_{j=0}^{k(n)-1} \frac{\ln n}{|E'_j(n)|} = c\int_0^1 \frac{\mathrm{d}s}{1-cs} = -\ln(1-c) \qquad \text{in probability.}$$

The second claim follows immediately since for every $j = 0, \ldots, k(n)-1$ there is the bound

$$\left(\frac{\ln n}{|E'_j(n)|}\right)^2 \leq \left(\frac{\ln n}{|E'_j(n)|}\right)\left(\frac{\ln n}{|E'_{k(n)}(n)|}\right)$$

and the second term of the product in the right-hand side tends to 0 in probability (again from Lemmas 2 and 3). □

3 Cutting down in continuous time

The purpose here is to translate the results of the preceding section in terms of a continuous time process $F = (F_s, s \geq 0)$ whose discrete time skeleton is the Markov chain $\mathcal{F}$, and which will be instrumental in the study of percolation on large random recursive trees. We shall often refer to F as the cutting-down process.

We start by attaching to each edge e of $\mathcal{T}$ an independent exponential variable with parameter $1/\ln n$, say $\varepsilon(e)$, which should be thought of as the time at which the edge e may be removed. However we do effectively remove this edge at that time if and only if e currently belongs to the root-component. That is, only the root-component splits as time passes, the other components appearing in the process being instantaneously frozen.

Rigorously, let $e_1, \ldots, e_n$ denote the edges of $\mathcal{T}$ listed in the increasing order of the attached exponential variables, i.e. such that $\varepsilon(e_1) < \varepsilon(e_2) < \ldots$. Then we declare that $F_s \equiv \mathcal{T}$ for all $0 \leq s < \varepsilon(e_1)$ and $F_s = (T_1, T_1')$ for $\varepsilon(e_1) \leq s < \varepsilon(e_2)$, where T_1 and T_1' are the two subtrees resulting from $\mathcal{T}$ after removing the edge e_1, with the convention that T_1' contains the root. Next, for $\varepsilon(e_2) \leq s < \varepsilon(e_3)$, if $e_2 \in T_1$ then $F_s = (T_1, T_1')$, whereas otherwise $F_s = (T_1, T_2, T_2')$ where T_2 and T_2' are the two subtrees resulting from T_1' after removing the edge e_2, with the convention that T_2' contains the root. We iterate in an obvious way.

Plainly, F is a continuous-time Markov chain whose discrete-time skeleton coincides with the forest-valued Markov chain $\mathcal{F}$ of the preceding section. More precisely, if $R_s(n)$ is the number of edges of the root-component which have been removed up to time s in the procedure described above, that is if $R_s(n) + 1$ is the number of tree-components of F_s, then there is the identity

$$F_s = \mathcal{F}_{R_s(n)}. \tag{3}$$

We stress that the processes F and $\mathcal{F}$ also depend on the parameter n, although this does not appear in the notation for the sake of simplicity.

Lemma 4 *We have for every fixed $t > 0$*

$$\lim_{n\to\infty} \sup_{0\leq s\leq t} \left| \frac{\ln n}{n} R_s(n) - (1 - \mathrm{e}^{-s}) \right| = 0 \qquad \textit{in probability.}$$

Proof: Recall that ζ denotes the absorption time of the chain $\mathcal{F}$, i.e. the number of edges that have been removed when the root is isolated. For every $k \leq \zeta$, define

$$\rho_k(n) = \inf\{s \geq 0 : R_s(n) = k\},$$

the first instant when k edges have been removed from the root-component in the process F.

Since in the continuous-time process, each edge of the root-component is removed with rate $1/\ln n$ independently of the other edges, we have

$$\rho_k(n) = \sum_{j=0}^{k-1} \frac{\ln n}{|E'_j(n)|} \varepsilon_j$$

where $|E'_j(n)|$ denotes the number of edges of the root-component T'_j of $\mathcal{F}_j$ and $(\varepsilon_j : j = 0, 1, \ldots)$ a sequence of i.i.d. standard exponential variables which is independent of the chain $\mathcal{F}$. We readily deduce from Corollary 1 that for $k = k(n) \sim cn/\ln n$ with $0 < c < 1$,

$$\lim_{n\to\infty} \rho_k(n) = -\ln(1-c) \qquad \text{in probability,}$$

and then by inversion that for every $s \geq 0$

$$\lim_{n\to\infty} \frac{\ln n}{n} R_s(n) = 1 - \mathrm{e}^{-s} \qquad \text{in probability.}$$

By the diagonal procedure, we may extract from an arbitrary increasing sequence of integers a subsequence, say $(n_\ell)_{\ell\in\mathbb{N}}$, such that with probability one,

$$\lim_{\ell\to\infty} \frac{\ln n_\ell}{n_\ell} R_s(n_\ell) = 1 - \mathrm{e}^{-s} \qquad \text{for all rational } s \geq 0.$$

As $s \to R_s(n_\ell)$ increases, the above convergence holds uniformly on $[0, t]$ for an arbitrarily fixed $t > 0$, i.e.

$$\lim_{\ell\to\infty} \sup_{0\leq s\leq t} \left| \frac{\ln n_\ell}{n_\ell} R_s(n_\ell) - (1 - \mathrm{e}^{-s}) \right| = 0 \qquad \text{a.s.}$$

and our claim follows. □

It will be convenient for our purpose to enrich the process F by keeping track of the time at which tree-components not containing the root have appeared. Specifically, if immediately before time $\varepsilon(e)$ the edge e belongs to the root-component, then removing e disconnects the root-component into say T and T' where T' contains the root, and we assign the mark $\varepsilon(e)$ to T.

Then denote by γ'_t the size of the root-component at time t, and by $\gamma_{1,t} \geq \gamma_{2,t} \geq \ldots$ the sequence ranked in the decreasing order of the sizes of the other components of F_t. Let also $\varepsilon_{i,t}$ be the mark attached to the i-th largest component of F_t that does not contain the root, i.e. the instant at which this component became disconnected from the root. We stress that all these quantities depend implicitly on the parameter n.

Corollary 2 *Consider a sequence of times $(t(n), n \in \mathbb{N})$ which converges to some $t > 0$. When*

$n \to \infty$, $n^{-1}\gamma'_{t(n)}$ converges towards e^{-t} in probability, and for every fixed $j \in \mathbb{N}$, the $(2j)$-tuple

$$\left(\frac{\ln n}{n}\gamma_{1,t(n)}, \ldots, \frac{\ln n}{n}\gamma_{j,t(n)}; \varepsilon_{1,t(n)}, \ldots, \varepsilon_{j,t(n)}\right)$$

converges in distribution towards

$$(a_1, \ldots, a_j; s_1, \ldots, s_j)$$

where $(a_1, s_1), (a_2, s_2), \ldots$ is the sequence of the atoms of a Poisson random measure on $(0, \infty) \times [0, t]$ with intensity $a^{-2}\mathrm{d}a \otimes \mathrm{e}^{-s}\mathrm{d}s$, ranked in the decreasing order of the first coordinate.

Proof: Recall that $|V_1|, |V_2|, \ldots$ denotes the sequence of the sizes of the tree-components that do not contain the root ranked in their order of appearance in the discrete-time Markov chain $\mathcal{F}$, where again for the sake of simplicity the parameter n is omitted from the notation. Also $|V'_k|$ is the size of the root-component of $\mathcal{F}_k$.

Now take $k(n) = R_{t(n)}(n)$, so according Lemma 4, $k(n) \sim (1 - \mathrm{e}^{-t})n/\ln n$ in probability. Note also from (2) that $k(n) < N(n)$ with high probability when $n \gg 1$. Introduce the random permutation $\sigma(\cdot, k(n))$ of $\{1, \ldots, k(n)\}$ which ranks the sequence $|V_1|, |V_2|, \ldots, |V_{k(n)}|$ in the decreasing order, i.e. such that

$$|V_{\sigma(1,k(n))}| \geq |V_{\sigma(2,k(n))}| \geq \ldots \geq |V_{\sigma(k(n),k(n))}|\,.$$

We see from (3) that there are the identities for $i = 1, \ldots, k(n)$

$$\gamma'_{t(n)} = |V'_{k(n)}|\ ,\ \gamma_{i,t(n)} = |V_{\sigma(i,k(n))}| \quad \text{and} \quad \varepsilon_{i,t(n)} = \rho_{\sigma(i,k(n))}(n)\,, \tag{4}$$

where $|V'_{k(n)}|$ is the size of the root-component in $\mathcal{F}_{k(n)}$ and $\ell \to \rho_\ell(n)$ denotes the generalized inverse of $s \to R_s(n)$.

It follows from Lemma 2, Lemma 3 and the estimate $k(n) \sim (1 - \mathrm{e}^{-t})n/\ln n$ in probability, that $n^{-1}|V'_{k(n)}|$ converges in probability to e^{-t} as $n \to \infty$. Further, for $j \in \mathbb{N}$ fixed, the $(2j)$-tuple

$$\left(\frac{\ln n}{n}|V_{\sigma(1,k(n))}|, \ldots, \frac{\ln n}{n}|V_{\sigma(j,k(n))}|; \frac{\ln n}{n}\sigma(1, k(n)), \ldots, \frac{\ln n}{n}\sigma(j, k(n))\right)$$

converges in distribution as $n \to \infty$ towards

$$(a_1, \ldots, a_j; u_1, \ldots, u_j)$$

where $(a_1, u_1), (a_2, u_2), \ldots$ is the sequence of the atoms of a Poisson measure on $(0, \infty) \times [0, 1 -$

$\mathrm{e}^{-t}]$ with intensity $a^{-2}\mathrm{d}a\mathrm{d}u$, ranked in the decreasing order of the first coordinate.

Recall from Lemma 4 that $\rho_\ell(n) \sim -\ln(1-u)$ whenever $\ell \sim un/\ln n$ and u remains bounded away from 1. Combining with (4), this establishes our claim since the image of the measure $a^{-2}\mathrm{d}a\mathrm{d}u$ on $(0,\infty)\times[0,1-\mathrm{e}^{-t}]$ by the map $(a,u)\to(a,-\ln(1-u))$ is $a^{-2}\mathrm{d}a\otimes\mathrm{e}^{-s}\mathrm{d}s$ on $(0,\infty)\times[0,t]$. □

4 Asymptotic sizes of the largest percolation clusters

In this section, we consider a Bernoulli bond-percolation with parameter $p(n)$ on $\mathcal{T}$, a random recursive tree on a set of $n+1$ ordered vertices. We shall use the name cluster to designate connected components induced by percolation, keeping the terminology tree-components for connected components arising from cutting down $\mathcal{T}$ as in Sections 2 and 3 (even though obviously percolation clusters are trees). Again for the sake of simplicity, the parameter n shall often be omitted from the notation.

We denote by C' the size of the cluster that contains the root and by $C_1, C_2, \ldots$ the sequence of the sizes of the remaining clusters ranked in the decreasing order. The purpose of this section is to establish the following weak limit theorem, which is the main result of this work.

Theorem 1 *Assume that*

$$1-p(n)\sim t/\ln n$$

where $t>0$ is fixed. Then as $n\to\infty$, $n^{-1}C'$ converges in probability towards e^{-t}, and for every fixed integer j,

$$\left(\frac{\ln n}{n}C_1,\ldots,\frac{\ln n}{n}C_j\right)$$

converges in distribution towards

$$(\mathtt{x}_1,\ldots,\mathtt{x}_j)$$

where $\mathtt{x}_1>\mathtt{x}_2>\ldots$ denotes the sequence of the atoms of a Poisson random measure on $(0,\infty)$ with intensity $t\mathrm{e}^{-t}x^{-2}\mathrm{d}x$.

Remarks. 1. Alternatively, the law of the limiting sequence can be specified as follows: $1/\mathtt{x}_1, 1/\mathtt{x}_2-1/\mathtt{x}_1,\ldots,1/\mathtt{x}_j-1/\mathtt{x}_{j-1}$ are i.i.d. exponential variables with parameter $t\mathrm{e}^{-t}$. In particular $\mathtt{x}_j$ has the same distribution as the inverse of a gamma variable with parameter $(j,t\mathrm{e}^{-t})$, $\mathtt{x}_j\sim t\mathrm{e}^{-t}/j$ as $j\to\infty$, and the m-th moment of $\mathtt{x}_j$ is finite iff $m<j$.

2. We stress that just as for supercritical percolation on a large complete graph or on a large d-dimensional cubic box, with high probability when $n\gg 1$, there is a unique giant component

whose size is essentially deterministic. However the next largest components only fail to be giant by a logarithmic factor, which contrasts sharply with the preceding examples.

The rest of this section is devoted to the proof of Theorem 1. Set $t(n) = -\ln n \times \ln p(n)$, so

$$p(n) = \exp(-t(n)/\ln n) \quad \text{and} \quad \lim_{n\to\infty} t(n) = t\,.$$

We will use a dynamical version of percolation which yields a natural coupling with the cutting down process F of Section 3. Recall that we attach to each edge e of $\mathcal{T}$ an independent exponential variable with parameter $1/\ln n$, say $\varepsilon(e)$. If we decide to remove e at time $\varepsilon(e)$, no matter whether e belongs or not to the root-cluster, then we observe at time $t(n)$ a Bernoulli bond-percolation on $\mathcal{T}$ with parameter $p(n)$.

The cutting down process F should be thought of as a dynamical percolation in which components that do not contain the root are instantaneously frozen. In particular, the root-cluster for percolation and the root-component for the continuous cutting down process coincide, i.e. $C' = \gamma'_{t(n)}$, and we conclude from Corollary 2 that indeed

$$\lim_{n\to\infty} n^{-1}C' = \mathrm{e}^{-t} \qquad \text{in probability,}$$

as stated in Theorem 1. We also stress that each percolation cluster is contained in some tree-component generated by the cutting down process up to time $t(n)$.

Let us sketch our approach to establishing the more interesting part of Theorem 1, namely that about the largest clusters that do not contain the root. Roughly speaking, we are led to de-freeze the tree-components that are disconnected from the root in the cutting down process up to time $t(n)$, that is to perform a further bond-percolation on the latter. More precisely, if $\tilde{T} = (\tilde{V}, \tilde{E})$ is such a tree-component which got disconnected from the root at time $s \in (0, t(n))$, then by the lack of memory of exponential variables, we should consider a Bernoulli bond percolation on $\tilde{T}$ with parameter $\exp(-(t(n) - s)/\ln n) = p(n)\exp(s/\ln n)$ to obtain the sub-family of percolation clusters of $\mathcal{T}$ at time $t(n)$ on $\tilde{V}$. Because we are interested in clusters of size of order $n/\ln n$, we may restrict our attention to tree-components of size of the same order, as smaller components will obviously not contribute. We shall see that if $|\tilde{V}| \asymp n/\ln n$, then with high probability when $n \gg 1$, percolation on $\tilde{T}$ with parameter $\exp(-(t(n)-s)/\ln n)$ produces a single giant cluster of size $\sim \mathrm{e}^{-(t-s)}|\tilde{V}|$ while the next clusters have size $O(n/\ln^2 n)$. Hence, only the giant percolation cluster on $\tilde{T}$ contributes to the family of clusters on $\mathcal{T}$ with size of order $n/\ln n$. More precisely, a tree-component $\tilde{T}$ of size $\sim an/\ln n$ born at time s yields a single cluster at time $t(n)$ of size $\sim \mathrm{e}^{-(t-s)}an/\ln n$. The asymptotic description of the tree-components in Corollary 2 then easily entails that of Theorem 1.

We now provide the technical details that justify rigorously the approach sketched above. We

first fix $\ell \in \mathbb{N}$ and for n sufficiently large, we consider the ℓ largest components of $F_{t(n)}$ that do not contain the root, say $\tilde{T}(1) = (\tilde{V}(1), \tilde{E}(1)), \ldots, \tilde{T}(\ell) = (\tilde{V}(\ell), \tilde{E}(\ell))$ in the decreasing order of their sizes, together with their birth-times, say $s(1), \ldots, s(\ell)$. We stress again that these quantities depend also on the parameter n, although for simplicity the latter does not appear in the notation. Note further that in the notation of Section 3, the size of $\tilde{T}(i)$ is $|\tilde{V}(i)| = \gamma_{i,t(n)}$ and $s(i) = \varepsilon_{i,t(n)}$.

We work conditionally on the variables $\tilde{V}(1), \ldots, \tilde{V}(\ell)$ and $s(1), \ldots, s(\ell)$, and recall from Lemma 1 that $\tilde{T}(1), \ldots, \tilde{T}(\ell)$ are then independent random recursive trees on their respective sets of vertices. As explained in the sketch of the proof above, we perform independent Bernoulli bond-percolations on each subtree $\tilde{T}(i)$ with parameter $\exp(-(t(n) - s(i)))$. For $i = 1, \ldots, \ell$, we denote by $\tilde{C}'_i$ the size of the root-cluster and by $\tilde{C}^*_i$ the size of the largest cluster that does not contain the root, where here of course root means the root of $\tilde{T}(i)$, that is the smallest vertex in $\tilde{V}(i)$.

Lemma 5 *In the notation above,*

$$\tilde{C}^*_i = O(n/\ln^2(n)) \qquad \text{in probability for every } i = 1, \ldots, \ell\,.$$

Further the ℓ-tuple

$$\left(\frac{\ln n}{n}\tilde{C}'_1, \ldots, \frac{\ln n}{n}\tilde{C}'_\ell\right)$$

converges in distribution towards

$$\left(\mathrm{e}^{-(t-s_1)}a_1, \ldots, \mathrm{e}^{-(t-s_\ell)}a_\ell\right)$$

where $(a_1, s_1), (a_2, s_2), \ldots$ is the sequence of the atoms of a Poisson random measure on $(0, \infty) \times [0, t]$ with intensity $a^{-2}\mathrm{d}a \otimes \mathrm{e}^{-s}\mathrm{d}s$, ranked in the decreasing order of the first coordinate.

Proof: We couple percolation on each random recursive tree $\tilde{\mathcal{T}}(i)$ with a cutting down process $\tilde{F}(i)$ in which edges of the root-component are removed at rate $1/\ln(|\tilde{V}(i)|-1)$ (remember that $|\tilde{V}(i)| - 1$ is the number of edges in $\tilde{\mathcal{T}}(i)$). If we set

$$r(i) = (t(n) - s(i))\frac{\ln(|\tilde{V}(i)| - 1)}{\ln n}\,, \tag{5}$$

then $\tilde{C}'_i$ coincides with $|\tilde{V}'_{r(i)}(i)|$, the size of the root-component of $\tilde{F}_{r(i)}(i)$, while $\tilde{C}^*_i$ is bounded from above by $|\tilde{V}^*_{r(i)}(i)|$, the size of the largest tree-component of $\tilde{F}_{r(i)}(i)$ that does not contain the root.

First note from Corollary 2 that $\ln(|\tilde{V}(i)| - 1) \sim \ln n$ in probability, and therefore $r(i)$ can

be bounded from above by $2t$ with high probability when $n \gg 1$. In that case, we have

$$\tilde{C}_i^* \leq |\tilde{V}^*_{r(i)}(i)| \leq |\tilde{V}^*_{2t}(i)|$$

and we conclude again from Corollary 2 that

$$\tilde{C}_i^* = O\left(\frac{|\tilde{V}(i)|-1}{\ln(|\tilde{V}(i)|-1)}\right) = O(n/\ln^2(n)) \qquad \text{in probability,}$$

establishing the first claim.

We then turn our attention to the second part of the statement. To ease the notation we shall only consider the one-parameter case $\ell = 1$, the argument for $\ell \geq 2$ being similar. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. We have for every $m = 1, \ldots, n$ and $s \geq 0$

$$\mathbb{E}(f(n^{-1}C_1') \mid |\tilde{V}(1)| = m, s(1) = s) = \mathbb{E}_m(f(n^{-1}|\tilde{V}_r'(1)|))$$

where $\mathbb{E}_m$ refers to the mathematical expectation starting from a random recursive tree of size m and $r = r(1)$ in (5) with $s(1) = s$ and $|\tilde{V}(1)| = m$. We know from the first part of the proof that $\ln(|\tilde{V}(1)|-1) \sim \ln n$ in probability, so from (5) we may focus on the regime $r \sim t - s$. On the other hand, according to Corollary 2, we know that if $m \sim an$ for some fixed $a \in (0,1)$, then $\mathbb{E}_m(f(n^{-1}|\tilde{V}_r'(1)|)) \sim f(a\mathrm{e}^{-r}) \sim f(a\mathrm{e}^{-(t-s)})$.

Still according to Corollary 2, the pair $(n^{-1}|\tilde{V}(1)|, s(1))$ converges in distribution as $n \to \infty$ towards (a_1, s_1), the atom with the largest the first coordinate of a Poisson random measure on $(0,\infty) \times [0,t]$ with intensity $a^{-2}\mathrm{d}a \otimes \mathrm{e}^{-s}\mathrm{d}s$. Since the function $(a,s) \to f(a\mathrm{e}^{-(t-s)})$ is uniformly continuous on $[0,1] \times [0,t]$, we conclude that

$$\lim_{n\to\infty} \mathbb{E}(f(n^{-1}C_1')) = \mathbb{E}(f(a_1\mathrm{e}^{-(t-s_1)}))$$

This completes the proof of Lemma 5. □

The second brick of the proof of Theorem 1 consists of verifying that, roughly speaking, for every fixed $j \in \mathbb{N}$, one can chose ℓ sufficiently large such that the probability that each of the j largest percolation clusters of $\mathcal{T}$ with parameter $p(n)$ can be found amongst the root-clusters of the independent percolations on the ℓ largest tree-components $\tilde{T}_1, \ldots, \tilde{T}_\ell$ with respective parameters $\exp(-(t(n) - s_i)) = p(n)\exp(s_i)$, remains as close to 1 as we wish as $n \to \infty$.

Rigorously, in the notation of Lemma 5, denote by

$$\tilde{C}'_{1,\ell} \geq \tilde{C}'_{2,\ell} \geq \ldots \geq \tilde{C}'_{\ell,\ell}$$

the rearrangement in decreasing order of the $\tilde{C}'_i$ for $i = 1, \ldots, \ell$.

Lemma 6 *For each fixed $j \in \mathbb{N}$, it holds that*

$$\lim_{\ell\to\infty} \liminf_{n\to\infty} \mathbb{P}\left(\tilde{C}'_{i,\ell} = C_i \text{ for every } i = 1, \ldots, j\right) = 1\,.$$

Proof: A Poisson random measure on $(0,\infty) \times [0,t]$ with intensity measure $a^{-2}\mathrm{d}a \otimes \mathrm{e}^{-s}\mathrm{d}s$ has infinitely many atoms, so in the notation of Lemma 5, we have $\min_{i=1,\ldots,j} \mathrm{e}^{-(t-s_i)}a_i > 0$ a.s. Since the size C_j of the j-th largest cluster cannot be smaller than $\min_{i=1,\ldots,j} \tilde{C}'_i$, where $\tilde{C}'_i$ denotes the size of the root-cluster in the percolation on the i-th subtree $\tilde{T}_i$ with parameter $\exp(-(t(n) - s_i)))$, we deduce from Lemma 5 and the portmanteau theorem that

$$\lim_{\alpha\to 0+} \limsup_{n\to\infty} \mathbb{P}(C_j < \alpha n/\ln n) = 0. \tag{6}$$

Roughly speaking, this means that for α sufficiently small, when $n \gg 1$ the size of the j-th largest cluster that does not contain the root is unlikely to be smaller than $\alpha n/\ln n$. On the other hand, in the notation of Corollary 2, a_ℓ decreases to 0 when $\ell \to \infty$ a.s. We deduce from Corollary 2 and again the portmanteau theorem that for each fixed $\alpha > 0$,

$$\lim_{\ell\to\infty} \lim_{n\to\infty} \mathbb{P}(\gamma_{\ell,t(n)} > \alpha n/\ln n) = 0\,,$$

where $\gamma_{\ell,t(n)} = |\tilde{V}(\ell)|$ is the size of the ℓ-th largest tree-component that does not contain the root generated by the cutting-down process F up to time $t(n)$.

Recall that the coupling of percolation with the cutting-down process entails that every percolation cluster is contained in some tree-component of $F_{t(n)}$. Combining the preceding two observations, we deduce that if we denote by $\Lambda_{i,\ell}(n)$ the event that the i-th largest cluster that does not contain the root is not included in one of the first ℓ largest tree-components generated by the cutting-down process up to time $t(n)$, then for each fixed i

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{P}(\Lambda_{i,\ell}(n)) = 0\,.$$

This means that if we chose ℓ sufficiently large, then the probability that each of the j largest clusters is included in one of the ℓ largest tree-components generated by the cutting-down process, remains as close to 1 as we wish when $n \to \infty$.

We now conclude from (6) and the first part of Lemma 5 that provided that ℓ is sufficiently large, the probability that each of the j largest percolation clusters coincides with one the ℓ root-clusters of the independent percolations on the tree-components $\tilde{T}_1, \ldots, \tilde{T}_\ell$ with respective parameters $\exp(-(t(n) - s_i))$, remains as close to 1 as we wish when $n \to \infty$. This entails our

claim. □

The final intermediate step we will need consists of the following elementary features.

Lemma 7 *Let* $(a_1, s_1), \ldots$ *be as in Lemma 5 and* $\mathtt{x}_1 \geq \mathtt{x}_2 \geq \ldots$ *the decreasing rearrangement of the infinite sequence* $(\mathrm{e}^{-(t-s_i)}a_i : i \in \mathbb{N})$.

(i) *The* $\mathtt{x}_i$ *form the sequence of the atoms of a Poisson random measure on* $(0, \infty)$ *with intensity* $t\mathrm{e}^{-t}x^{-2}\mathrm{d}x$*, listed in decreasing order.*

(ii) *For every* $\ell \in \mathbb{N}$*, let* $\mathtt{x}_{1,\ell} \geq \mathtt{x}_{2,\ell} \geq \ldots \geq \mathtt{x}_{\ell,\ell}$ *denote the decreasing rearrangement of the finite sequence* $(\mathrm{e}^{-(t-s_i)}a_i : i = 1, \ldots, \ell)$*. Then for each fixed* $j \in \mathbb{N}$*,*

$$\lim_{\ell\to\infty} \mathbb{P}\left(\mathtt{x}_{i,\ell} = \mathtt{x}_i \text{ for every } i = 1, \ldots, j\right) = 1\,.$$

Proof: The first part is immediate from the image property of Poisson random measures and the fact that the measure $t\mathrm{e}^{-t}x^{-2}\mathrm{d}x$ on $(0, \infty)$ arises in this setting as the image of the intensity measure $a^{-2}\mathrm{d}a \otimes \mathrm{e}^{-s}\mathrm{d}s$ on $(0, \infty) \times [0, t]$ by the map $(a, s) \to x = \mathrm{e}^{-(t-s)}a$.

The second part is obvious since $\lim_{i\to\infty} a_i = 0$ almost surely. □

We can now finish the proof of Theorem 1. Consider a continuous function $f : [0, \infty)^j \to [0, 1]$ and fix $\eta > 0$. According to Lemma 7(ii), we may chose ℓ sufficiently large so that

$$\mathbb{E}\left(f\left(\mathtt{x}_1, \ldots, \mathtt{x}_j\right)\right) \leq \mathbb{E}\left(f\left(\mathtt{x}_{1,\ell}, \ldots, \mathtt{x}_{j,\ell}\right)\right) + \eta\,.$$

On the other hand, again provided that ℓ is chosen sufficiently large, we get from Lemma 6 that there exists $n_\eta \in \mathbb{N}$ such that the upper bound

$$\mathbb{E}\left(f\left(\frac{\ln n}{n}C_1, \ldots, \frac{\ln n}{n}C_j\right)\right) \leq \mathbb{E}\left(f\left(\frac{\ln n}{n}\tilde{C}'_{1,\ell}, \ldots, \frac{\ln n}{n}\tilde{C}'_{j,\ell}\right)\right) + \eta$$

holds for all $n \geq n_\eta$. We now deduce from Lemma 5 and these two bounds that

$$\limsup_{n\to\infty} \mathbb{E}\left(f\left(\frac{\ln n}{n}C_1, \ldots, \frac{\ln n}{n}C_j\right)\right) \leq \mathbb{E}\left(f\left(\mathtt{x}_1, \ldots, \mathtt{x}_j\right)\right) + 2\eta\,.$$

Because η can be arbitrarily small and f replaced by $1 - f$, this establishes Theorem 1.

5 Comparing edge-deletion for Cayley and recursive trees

We conclude this work by commenting on similitudes and differences between random edge-deletion processes on random Cayley trees and on random recursive trees. For every integer

n, we denote now by $\mathcal{T}^{(\mathrm{r},n)}$ a random recursive tree on an ordered set of $n+1$ vertices, and by $\mathcal{T}^{(\mathrm{c},n)}$ a random Cayley tree on the same set of vertices. In other words, $\mathcal{T}^{(\mathrm{c},n)}$ is a tree picked uniformly at random amongst the $(n+1)^{n-1}$ trees on these $n+1$ vertices, relaxing the requirement that vertices on the branch from the root to a leaf should increase, and $\mathcal{T}^{(\mathrm{r},n)}$ can be viewed as a conditioned version of $\mathcal{T}^{(\mathrm{c},n)}$. We stress that, informally, the conditioning becomes singular as $n \to \infty$, and indeed the asymptotic behaviors of $\mathcal{T}^{(\mathrm{c},n)}$ and $\mathcal{T}^{(\mathrm{r},n)}$ are notoriously different. For instant, the typical height of $\mathcal{T}^{(\mathrm{c},n)}$ is of order $\sqrt{n}$ while that of $\mathcal{T}^{(\mathrm{r},n)}$ is only of order $\ln n$.

Both random Cayley trees and random recursive trees fulfill the following fundamental splitting property. If we remove one edge uniformly at random, thus disconnecting that tree into two subtrees, then conditionally on the partition of vertices that results, the structure of each component is again a random tree of the same type (Cayley or recursive), and further the two subtrees are independent. This splitting property is of course crucial when iterating the edge-deletion; I am not aware of other non-degenerate types of random trees fulfilling this property.

We shall now discuss the effects of two related algorithms on $\mathcal{T}^{(\mathrm{c},n)}$ and $\mathcal{T}^{(\mathrm{r},n)}$, the first corresponds to viewing iterated edge-deletion as a continuous-time fragmentation process, while the second takes the view point of so-called Markov branching trees.

5.1 Fragmentation chains and processes

So imagine first that given $\mathcal{T}^{(\mathrm{c},n)}$ or $\mathcal{T}^{(\mathrm{r},n)}$, we attach to each edge e an independent exponential variable $\varepsilon(e)$, and remove the edge e at time $\varepsilon(e)$. The parameter of $\varepsilon(e)$ will be specified later on, depending both on n and the nature of the random tree. We are interested in the processes $X^{(\mathrm{c},n)}$ and $X^{(\mathrm{r},n)}$ which record the sizes of the tree-components at time t, re-scaled by a factor $1/(n+1)$ and ranked in the decreasing order. These two processes take their values in the space of mass-partitions, i.e. of non-increasing sequences of nonnegative real numbers with sum at most 1. The splitting property entails that both are fragmentation chains in the sense of Definition 1.1 in [2].

We are concerned with the asymptotic behaviors of these processes when $n \to \infty$. In the Cayley case, Pitman [18, 19] noted that one should take the parameter of the exponential variable $\varepsilon(e)$ attached to a generic edge e to be equal to $1/\sqrt{n}$. Then the fragmentation chain $X^{(\mathrm{c},n)}$ has a weak limit $X^{(\mathrm{c})}$ as $n \to \infty$; more precisely $X^{(\mathrm{c})}$ is a self-similar fragmentation process with index $1/2$ which appears as the dual to the standard additive coalescent. See Aldous and Pitman [1] and Section 5.3.4 in [2]. In the recursive case, we have seen that the parameter of $\varepsilon(e)$ should be chosen as $1/\ln n$, and it follows readily from Theorem 1 that $X^{(\mathrm{r},n)}$ also converges as $n \to \infty$, but now in probability to a deterministic process given by

$X_t^{(\mathrm{r})} = (\mathrm{e}^{-t}, 0, \ldots)$. The latter should be viewed as a pure erosion, a somewhat degenerate instance of homogeneous fragmentation. See Chapter 3 in [2] for background.

5.2 Markov branching trees and fragmentation trees

A recent work of Haas and Miermont [9] also invites us to consider a variation of the random edge-deletion procedure in discrete time. Specifically, given $\mathcal{T}^{(\mathrm{c},n)}$ or $\mathcal{T}^{(\mathrm{r},n)}$, we consider the following algorithm. At the first step, we remove an edge uniformly at random, disconnecting the tree into two subtrees. We iterate independently with each subtree, i.e. at the second step, we remove an edge chosen uniformly at random in each of the two subtrees (at least provided that none is reduced to a single vertex), and so on until the tree has been completely destroyed.

Following Haas and Miermont, we represent this algorithm by a random rooted binary tree, where vertices of the binary tree corresponds to the connected components that arise in the procedure described above. More precisely, the root of this binary tree is given by the initial set V of $n+1$ vertices, and its leaves are the singletons $\{i\}$ for $i \in V$. Further, whenever the edge-removal for some component C produces two sub-components C_1 and C_2, respectively, then we create two edges connecting C and its two children, C_1 and C_2.

Let us denote by $T^{(\mathrm{c},n)}$ and $T^{(\mathrm{r},n)}$ the two random binary trees which result from this construction, where of course the exponent c refers to the case when one starts from the random Cayley tree $\mathcal{T}^{(\mathrm{c},n)}$ and the exponent r to the case when one starts from the random recursive tree $\mathcal{T}^{(\mathrm{r},n)}$. By the splitting property, both $T^{(\mathrm{c},n)}$ and $T^{(\mathrm{r},n)}$ are then Markov branching trees (see [9] for background). Since Haas and Miermont [9] have recently established a general weak limit theorem in the sense of Gromov-Hausdorff for sequences of Markov branching trees, it is therefore natural to consider the asymptotic behaviors of $T^{(\mathrm{c},n)}$ and $T^{(\mathrm{r},n)}$.

The asymptotic behavior of $T^{(\mathrm{c},n)}$ has been determined in [3], see the proof of Lemma 1 there. Roughly speaking, if we rescale the edge-lengths of $T^{(\mathrm{c},n)}$ by a factor $1/\sqrt{n}$, then the sequence of rescaled random trees converges weakly towards Aldous' Continuum Random Tree. We stress that this is a direct application of the limit theorem of Haas and Miermont.

On the other hand, the results of Haas and Miermont do *not* apply for the recursive case. More precisely their fundamental assumption, namely Equation (3) in [9], fails for $T^{(\mathrm{r},n)}$, as it can be seen from (1). Nonetheless, it is easy to deduce from the approach developed in the present work that if we rescale the edge-lengths of $T^{(\mathrm{r},n)}$ by a factor $n^{-1} \ln n$, then the sequence of rescaled random trees converges in probability to a degenerate deterministic real tree which can be identified as the unit interval $[0, 1]$. Details are left to the interested reader.

References

[1] Aldous, D.J. and Pitman, J. The standard additive coalescent. *Ann. Probab.* **26** (1998), 1703-1726

[2] Bertoin, J. *Random Fragmentation and Coagulation Processes.* Cambridge Studies in Advanced Mathematics, Cambridge, 2006

[3] Bertoin, J. Fires on trees. To appear in *Ann. Inst. H. Poincaré Probab. Stat.* Available at `http://www.imstat.org/aihp/accepted.html`

[4] Bollobás, B. *Random graphs*, Cambridge Studies in Advanced Mathematics, Cambridge, 2001

[5] Drmota, M., Iksanov, A., Möhle, M. and Rösler, U. A limiting distribution for the number of cuts needed to isolate the root of a random recursive tree, *Random Structures Algorithms* **34-3** (2009), 319–336

[6] Iksanov, A. and Möhle, M. A probabilistic proof of a weak limit law for the number of cuts needed to isolate the root of a random recursive tree, *Electron. Comm. Probab.* **12** (2007), 28–35

[7] Goldschmidt, C. and Martin, J. B. Random recursive trees and the Bolthausen-Sznitman coalescent, *Electron. J. Probab.* **10** (2005), 718–745

[8] Grimmett, G. *Percolation*, Springer-Verlag, Berlin, 1999

[9] Haas, B. and Miermont, G. Scaling limits of Markov branching trees, with applications to Galton-Watson and random unordered trees. To appear in *Ann. Probab.*, available at `http://www.imstat.org/aop/future_papers.htm`

[10] Holmgren, C. Random records and cuttings in binary search trees, *Combin. Probab. Comput.* **19-3** (2010), 391–424

[11] Holmgren, C. A weakly 1-stable distribution for the number of random records and cuttings in split trees, *Adv. in Appl. Probab.* **43-1** (2011), 151–177

[12] Janson, S. Random records and cuttings in complete binary trees, in : *Mathematics and computer science. III*, Trends Math., Birkhäuser, 2004, pp. 241–253

[13] Janson, S. Random cutting and records in deterministic and random trees, *Random Structures Algorithms* **29-2** (2006), 139-179

[14] Meir, A. and Moon, J. W. Cutting down random trees, *J. Austral. Math. Soc.* **11** (1970), 313–324

[15] Meir, A. and Moon, J. W. Cutting down recursive trees, *Mathematical Biosciences* **21** (1974), 173–181

[16] Panholzer, A. Destruction of recursive trees, in : *Mathematics and computer science. III*, Trends Math., Birkhäuser, 2004, pp. 267–280

[17] Panholzer, A. Cutting down very simple trees, *Quaest. Math.* **29-2** (2006), 211–227

[18] Pitman, J. Coalescent random forests, *J. Combin. Theory Ser. A* **85-2** (1999), 165–193

[19] Pitman, J. *Combinatorial stochastic processes*, Lectures from the 32nd Summer School on Probability Theory, Lecture Notes in Mathematics **1875**, Springer-Verlag, Berlin, 2006

[20] Resnick, S. I. *Extreme values, regular variation, and point processes.* A Series of the Applied Probability Trust. Springer-Verlag, New York, 1987