Random geometric subdivisions

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Abstract

We study several models of random geometric subdivisions arising from the model of Diaconis and Miclo (2011). In particular, we show that the limiting shape of an indefinite subdivision of a quadrilateral is a.s. a parallelogram. We also show that the geometric subdivisions of a triangle by angle bisectors converge (only weakly) to a non-atomic distribution, and that the geometric subdivisions of a triangle by choosing random points on its sides converges to a "flat" triangle, similarly to the result of Diaconis and Miclo (2011).

Keywords: barycentric subdivision, geometric probability, Markov chain, iterated random functions.
AMS subject classification: 60J05, 60D05.

1 Motivation

The aim of this paper is to consider several models involving random subdivision of geometrical objects. Markov chains involving geometry and polygons have been studied quite widely in the literature, see e.g. [3] and references therein; however, the main motivation of this paper comes from the paper by

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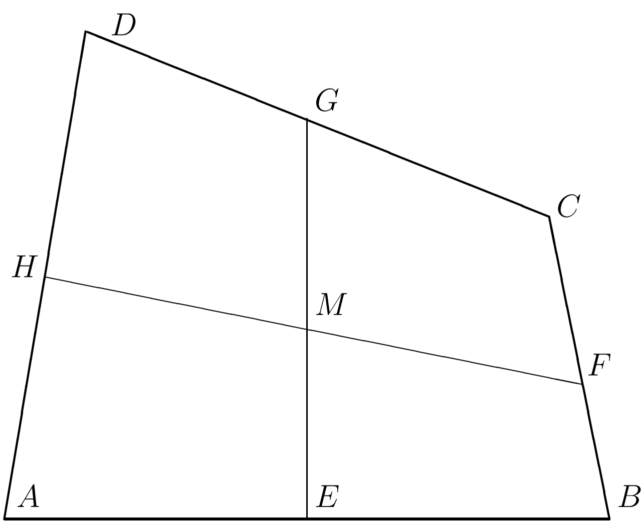
Diaconis and Miclo [2], who considered the following model, earlier also studied by Hough [5]. A non-degenerate triangle is divided by its three medians into 6 smaller triangles, and one of these new triangles is chosen with equal probability, and (optionally) re-scaled, thus becoming the "new" triangle. This procedure is repeated indefinitely. It was shown that in some sense the limiting triangle will be "flat", that is the largest angle will converge to π.

In the current paper we consider several generalizations of the above model. In Section 2 we consider subdivisions of a quadrilateral by the lines connecting the middle points on the opposite sides; in Section 3 we consider subdivision of a triangle using angle bisectors. Finally, in Section 4 we consider a sequence of triangles obtained by randomly choosing a point on each of the sides and letting them be vertices of the new triangle.

2 Random subdivision of quadrilateral



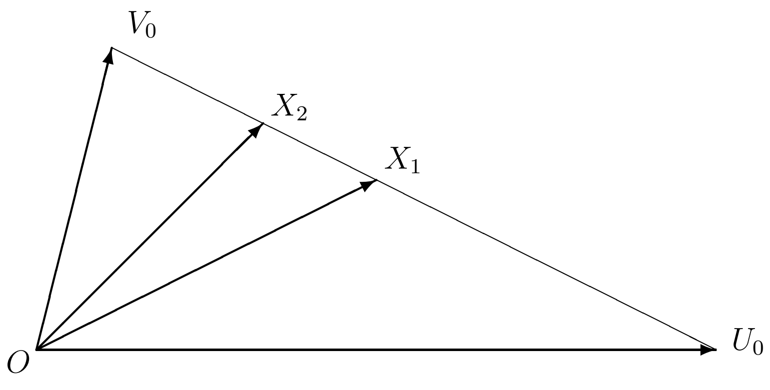
We are given a convex quadrilateral $ABCD$. Let E, F, G, H be the middle points of the sides AB, BC, CD, and DA respectively. Let M be the point of intersection of segments EG and FH. Now we replace $ABCD$ by one of the following four quadrilaterals $AEMH$, $EBFM$, $MFCG$ and $HMGD$ with equal probabilities. Suppose that we repeat this procedure indefinitely. What is the limiting *shape* of the quadrilateral obtained as the limit of this procedure?

Theorem 1 *Under the procedure described above, the limiting shape of the quadrilateral will be a parallelogram, and the rate of convergence is geometric. (Note that the shape of parallelogram is "invariant" for the procedure).*

Proof. Observe that $\vec{HF} = \frac{1}{2}(\vec{AB} + \vec{DC})$, and $\vec{HM} = \vec{MF} = \frac{1}{2}\vec{HF}$ where $\vec{x}$ denotes the vector x. Also suppose that when we replace the original quadrilateral by one of the four smaller ones, we rescale the smaller one twice thus making it bigger; this will not affect the shape. Let $\vec{u}_0 = \vec{AB}$ and $\vec{v}_0 = \vec{DC}$ be the vectors corresponding to the "horizontal" sides of $ABCD$, and $\vec{u}_1$, $\vec{v}_1$ be the corresponding vectors of the new quadrilateral obtained by subdivision. The crucial observation is that

$$\{\vec{u}_1, \vec{v}_1\} = \begin{cases} \left\{\vec{u}_0, \frac{1}{2}\left(\vec{u}_0 + \vec{v}_0\right)\right\} & \text{with probability } 1/2; \\ \left\{\vec{v}_0, \frac{1}{2}\left(\vec{u}_0 + \vec{v}_0\right)\right\} & \text{with probability } 1/2. \end{cases}$$

Similarly we can define $\{\vec{u}_n, \vec{v}_n\}$, $n = 2, 3, \ldots$.



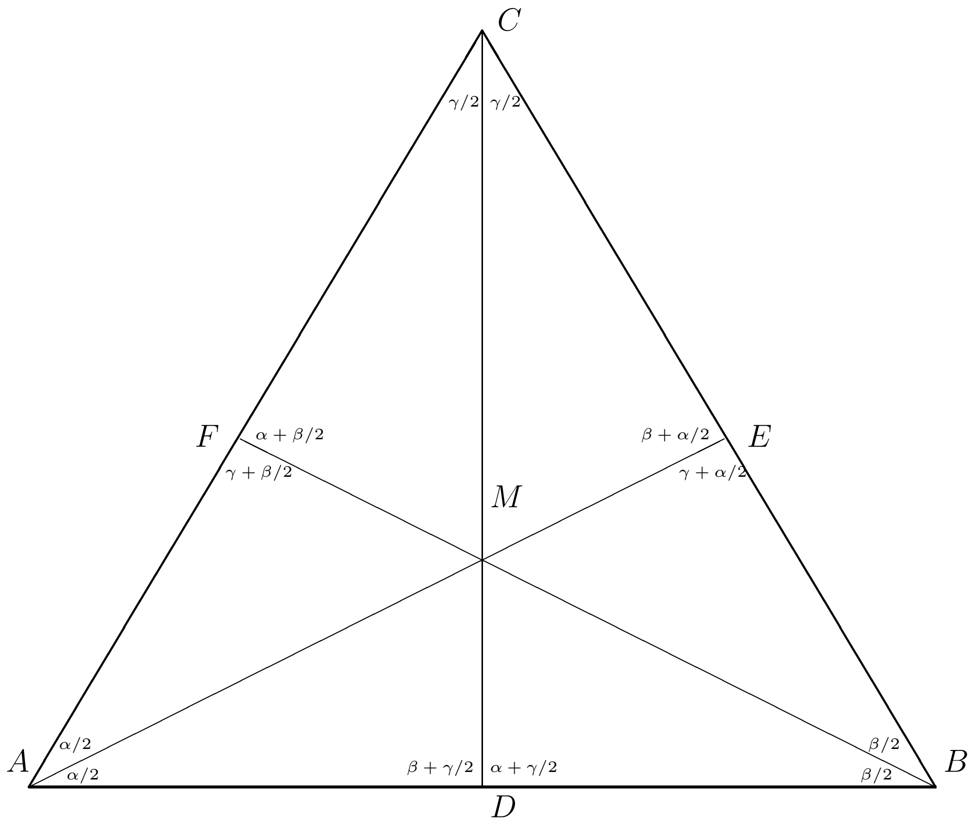
Let us place all vectors $\vec{u}_n, \vec{v}_n$ at the origin O and let U_n and V_n be the corresponding endpoints of these vectors. Let $X_n = U_n$ if $U_n \notin \{U_{n-1}, V_{n-1}\}$ and $X_n = V_n$ if $V_n \notin \{U_{n-1}, V_{n-1}\}$ (exactly one of these two statements must be true). Then we see that all points X_n lie on the segment U_0V_0. Moreover,

X_1 lies exactly in the middle of U_0V_0, X_2 with equal probabilities splits U_0X_1 or V_0X_1 in the middle, etc. As a result, we see that X_n a.s. converges to a point X_∞ which is uniformly distributed on the segment U_0V_0. Taking into account that $|U_nV_n| = 2^{-n}|U_0V_0|$, we obtain a deterministic speed of convergence of $\vec{u}_n$ and $\vec{v}_n$ towards $\vec{u}_\infty := \vec{OX_\infty}$:

$$\frac{|U_0V_0|}{2^{n+1}} \leq \max\{|\vec{u}_n - \vec{u}_\infty|, |\vec{v}_n - \vec{u}_\infty|\} \leq \frac{|U_0V_0|}{2^n}.$$

The analogous statement holds also for the "vertical" sides corresponding to AD and BC, thus yielding the desired convergence towards a parallelogram. ■

3 Random subdivision of triangle with angle bisectors



Unlike the median-subdivision model considered in [2], suppose that we subdivide the triangle ABC by the three angle bisectors which intersect the sides AB, BC, CA at points E, F, G respectively, and let M be the point of intersections of all angle bisectors, the centre of the inscribed circle. The replacement procedure now states that the triangle ABC is replaced with probabilities $1/6$ by one of the following triangles: ADM, DBM, BEM, ECM, CFM, FAM. As before, the object of interest is the *shape* of the limiting triangle obtained by indefinite repetition of the above replacement procedure.

It turns out to be convenient to work with the angles of the triangle. If the original triangle has the angles (α, β, γ), $\alpha + \beta + \gamma = \pi$, then the new triangle will have one of the following six sets of angles:

$$\begin{aligned}
&\left(\frac{\alpha}{2}, \gamma + \frac{\beta}{2}, \frac{\alpha+\beta}{2}\right), \ \left(\frac{\alpha}{2}, \beta + \frac{\gamma}{2}, \frac{\alpha+\gamma}{2}\right), \\
&\left(\frac{\beta}{2}, \alpha + \frac{\gamma}{2}, \frac{\beta+\gamma}{2}\right), \ \left(\frac{\beta}{2}, \gamma + \frac{\alpha}{2}, \frac{\alpha+\beta}{2}\right), \\
&\left(\frac{\gamma}{2}, \beta + \frac{\alpha}{2}, \frac{\alpha+\gamma}{2}\right), \ \left(\frac{\gamma}{2}, \alpha + \frac{\beta}{2}, \frac{\beta+\gamma}{2}\right).
\end{aligned} \tag{3.1}$$

Obviously, we cannot expect convergence almost surely for this procedure (since e.g. any angle can be halved on the next step with probability $1/3$).

Observe that we can generate the sequence of triangles by *always* choosing the left-bottom triangle, that is by using the mapping $(\alpha, \beta, \gamma) \to \left(\frac{\alpha}{2}, \gamma + \frac{\beta}{2}, \frac{\alpha+\beta}{2}\right)$, and then performing a random permutation of the set of three newly obtained angles. Formally, let $(\alpha_n, \beta_n, \gamma_n)$ denote the set of the angles of the n-th triangle, then

$$\begin{aligned}
(\alpha_{n+1}, \beta_{n+1}, \gamma_{n+1}) &= \sigma_n \left(\frac{\alpha_n}{2}, \gamma_n + \frac{\beta_n}{2}, \frac{\alpha_n + \beta_n}{2}\right) \\
&= \sigma_n \left(\frac{\alpha_n}{2}, \pi - \alpha_n - \frac{\beta_n}{2}, \frac{\alpha_n + \beta_n}{2}\right)
\end{aligned} \tag{3.2}$$

where $\sigma_n = \left\{\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}, \sigma^{(4)}, \sigma^{(5)}, \sigma^{(6)}\right\}$ is a random permutation of the set of three elements; $\sigma_n\big((a,b,c)\big)$ takes one of the six possible values

$$\{(a,b,c),\ (b,c,a),\ (c,a,b),\ (c,b,a),\ (b,a,c),\ (a,c,b)\}$$

with equal probability, and σ_n's are i.i.d. The advantage of this method is that the distribution of the set of the angles of the n-th triangle is completely symmetric with respect to exchanges of its components, even though the components are not independent. Unlike the case of barycentric subdivision studied in [2], only the weak limit exists in our case.

Theorem 2 *The sequence $(\alpha_n, \beta_n, \gamma_n)$ converges in distribution to some limit.*

Proof. Let $S = \{(\alpha, \beta, \gamma) : \ \alpha, \beta, \gamma \geq 0, \ \alpha+\beta+\gamma = \pi\}$ be the 2-simplex with the standard Euclidean distance. Let $f_i(u)$, $u \in S$, $i = 1, \ldots, 6$ be the set of functions given by (3.1). Let $u = (\alpha, \beta, \gamma)$ and $v = (\alpha + x, \beta + y, \gamma + z)$, so that $x + y + z = 0$. Observe that all f_i are Lipschitz and that

$$\begin{aligned}\sum_{i=1}^{2} \log \frac{|f_i(u) - f_i(v)|}{|u - v|} &= \log \sqrt{\frac{[x^2 + y^2 + [y + z]^2 + xz][x^2 + z^2 + [y + z]^2 + xy]}{4(x^2 + y^2 + z^2)^2}} \\ &= \frac{1}{2} \log \frac{(y^2 + 3z^2 + 3yz)(3y^2 + z^2 + 3yz)}{16(y^2 + z^2 + yz)^2} \\ &\leq \frac{1}{2} \log \frac{(3y^2 + 3z^2 + 3yz)(3y^2 + 3z^2 + 3yz)}{16(y^2 + z^2 + yz)^2} = \log \frac{3}{4} < 0.\end{aligned}$$

The identical bound holds for $i = 3, 4$ and $i = 5, 6$. Therefore, if Z_u denotes a set of the angles obtained from u by random subdivision, we have

$$\mathbb{E} \left[\log \frac{|Z_u - Z_v|}{|u - v|} \right] = \sum_{i=1}^{6} \frac{1}{6} \log \frac{|f_i(u) - f_i(v)|}{|u - v|} \leq \log \frac{\sqrt{3}}{2} < 0.$$

Therefore, by Theorem 1 from [1], see also the proof of Lemma 5.1 in [2], the mapping $u \to Z_u$ is ergodic, that is there is a (unique) probability measure ν on S such that for any starting configuration $u = (\alpha, \beta, \gamma) \in S$ we have $Z_u^{(n)} \to \nu$ weakly. (Here $Z^{(n)}$ stands for the superposition of n i.i.d. mappings Z.) ∎

Now let us try to get a handle on the distribution of the limiting triangle. For the purpose of simplicity, and without loss of generality, assume from now on that $\alpha_n + \beta_n + \gamma_n \equiv 1$ (as opposed to π).

We already know that (α_n, β_n) converges to some pair $(\bar\alpha, \bar\beta)$ in distribution. Since $\bar\alpha$ is bounded and $\bar\beta$ and $\bar\gamma := 1 - \bar\alpha - \bar\beta$ have the same distribution as $\bar\alpha$, and $\bar\alpha\bar\beta$, $\bar\beta\bar\gamma$, $\bar\gamma\bar\alpha$ all have the same distribution as well (from the symmetry), we conclude

$$\begin{aligned} \mathbb{E}\,[\bar\alpha + \bar\beta + \bar\gamma] = 1 &\Longrightarrow \mathbb{E}\,\bar\alpha = \mathbb{E}\,\bar\beta = \mathbb{E}\,\bar\gamma = \frac{1}{3}, \\ \mathbb{E}\,[\bar\alpha + \bar\beta + \bar\gamma]^2 = 1 &\Longrightarrow 3\mathbb{E}\,\bar\alpha^2 + 6\mathbb{E}\,[\bar\alpha\bar\beta] = 1 \end{aligned}$$

Also from (3.2) we have

$$\mathbb{E}\,\bar\alpha^2 = \frac{1}{3}\left[\frac{1}{4}\,\mathbb{E}\,\bar\alpha^2 + \mathbb{E}\,\left[\bar\gamma + \frac{\bar\beta}{2}\right]^2 + \frac{1}{4}\,\mathbb{E}\,(\bar\alpha + \bar\beta)^2\right].$$

This in turn yields

$$\mathbb{E}\,\bar\alpha^2 = \frac{1}{7},\ \mathbb{E}\,[\bar\alpha\bar\beta] = \frac{2}{21},\ \mathbb{V}\mathsf{ar}\,(\bar\alpha) = \frac{2}{63},\ Cov(\bar\alpha, \bar\beta) = -\frac{1}{63}$$

which sheds some light on the distribution of $\bar\alpha$ and the dependence between $\bar\alpha$ and $\bar\beta$.

A more interesting and subtle statement about the joint distribution of $\bar\alpha, \bar\beta, \bar\gamma$ is the following

Theorem 3 *Let c_1, c_2, c_3 be some real numbers, not all of which are 0. Then distribution of the random variable $c_1\bar\alpha + c_2\bar\beta + c_3\bar\gamma$ does not have atoms.*

In fact, we conjecture that the distribution of ν is continuous on the simplex, and so are the marginal distributions, e.g. the distribution of $\bar\alpha$. Numerical simulations suggest that the pdf of a randomly chosen angle of a limiting triangle looks like the one shown on Figure 1, which is obviously quite non-trivial.

Due to the symmetry of the triple $(\bar\alpha, \bar\beta, \bar\gamma)$ with respect to permutations and the fact that $\bar\alpha + \bar\beta + \bar\gamma = 1$, Theorem 3 follows immediately from the next statement.

Lemma 1 *For any $c \in \mathbb{R}$ and $x \in \mathbb{R}$, $\mathbb{P}(\bar\alpha + c\bar\beta = x) = 0$.*

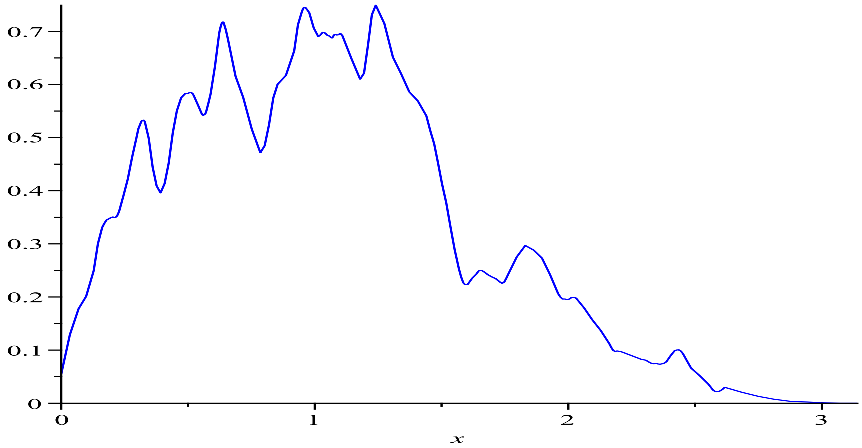


Figure 1: Density of a randomly chosen angle.

Proof. Equation (3.2) together with the weak convergence of $(\alpha_n, \beta_n, \gamma_n)$ yield

$$\begin{aligned}\mathbb{P}(\bar{\alpha} + c\bar{\beta} = x) = \frac{1}{6}\Bigg\{&\mathbb{P}\left(\frac{\bar{\alpha}}{2} + c\left[1 - \bar{\alpha} - \frac{\bar{\beta}}{2}\right] = x\right) + \mathbb{P}\left(c\frac{\bar{\alpha}}{2} + \left[1 - \bar{\alpha} - \frac{\bar{\beta}}{2}\right] = x\right)\\ &+ \mathbb{P}\left(\frac{\bar{\alpha}}{2} + c\frac{\bar{\alpha} + \bar{\beta}}{2} = x\right) + \mathbb{P}\left(c\frac{\bar{\alpha}}{2} + \frac{\bar{\alpha} + \bar{\beta}}{2} = x\right)\\ &+ \mathbb{P}\left(\left[1 - \bar{\alpha} - \frac{\bar{\beta}}{2}\right] + c\frac{\bar{\alpha} + \bar{\beta}}{2} = x\right)\\ &+ \mathbb{P}\left(c\left[1 - \bar{\alpha} - \frac{\bar{\beta}}{2}\right] + \frac{\bar{\alpha} + \bar{\beta}}{2} = x\right)\Bigg\}. \end{aligned} \tag{3.3}$$

In particular,

$$\mathbb{P}(\bar{\alpha} = x) = \frac{1}{3}\left\{\mathbb{P}(\bar{\alpha}/2 = x) + \mathbb{P}(1 - \bar{\alpha} - \bar{\beta}/2 = x) + \mathbb{P}([\bar{\alpha} + \bar{\beta}]/2 = x)\right\} \tag{3.4}$$

and if we set $x = 1$, then

$$\begin{aligned}\mathbb{P}(\bar{\alpha} = 1) &= \frac{1}{3}\left\{\mathbb{P}(\bar{\alpha}/2 = 1) + \mathbb{P}(1 - \bar{\alpha} - \bar{\beta}/2 = 1) + \mathbb{P}([1 - \bar{\gamma}]/2 = 1)\right\}\\ &= \frac{1}{3}\left\{0 + \mathbb{P}(\bar{\alpha} = 0, \bar{\beta} = 0) + 0\right\} = \frac{1}{3}\mathbb{P}(\bar{\gamma} = 1) = \frac{1}{3}\mathbb{P}(\bar{\alpha} = 1),\end{aligned}$$

yielding

$$\mathbb{P}(\bar{\alpha} = 1) = 0. \tag{3.5}$$

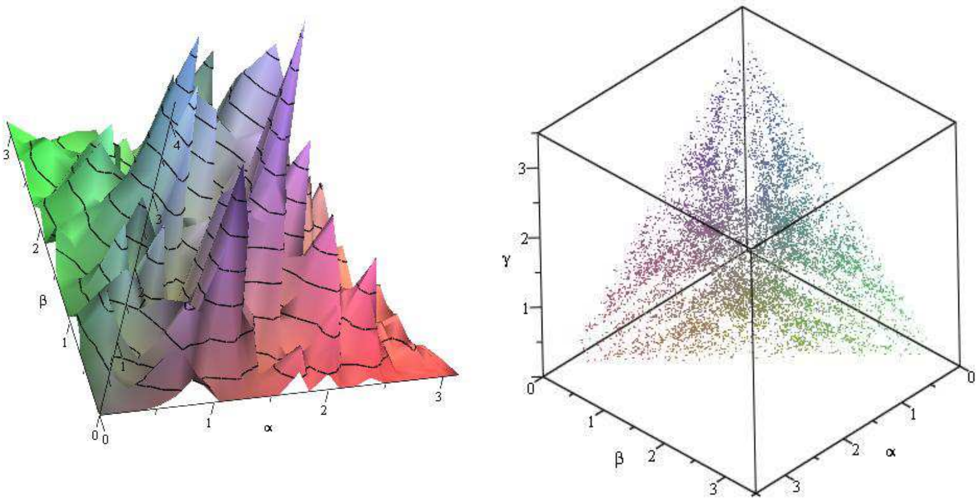


Figure 2: Sample density of ν on simplex S.

Rewriting (3.3) for $c \notin \{-1, 1/2, 2\}$ we have

$$\begin{aligned}\mathbb{P}(\bar{\alpha} + c\bar{\beta} = x) = \frac{1}{6}\Big\{&\mathbb{P}\left(\bar{\alpha} + \frac{c}{2c-1}\bar{\beta} = ?\right) + \mathbb{P}\left(\bar{\alpha} + \frac{1}{2-c}\bar{\beta} = ?\right) \\ &+ \mathbb{P}\left(\bar{\alpha} + \frac{c}{1+c}\bar{\beta} = ?\right) + \mathbb{P}\left(\bar{\alpha} + \frac{1}{1+c}\bar{\beta} = ?\right) \\ &+\mathbb{P}\left(\bar{\alpha} + \frac{c-1}{c-2}\bar{\beta} = ?\right) + \mathbb{P}\left(\bar{\alpha} + \frac{1-c}{1-2c}\bar{\beta} = ?\right)\Big\}\end{aligned} \tag{3.6}$$

where the question marks stand for some non-random numbers. We can make sense of the expression above also for $c \in \{-1, 1/2, 2\}$ if we replace the fractions equal to infinity by zeros, due to the symmetry between $\bar{\alpha}$ and $\bar{\beta}$.

For each c, let $x^*(c)$ be such that $\mathbb{P}(\bar{\alpha}+c\bar{\beta} = x^*(c)) = p^*(c) \equiv \max_{x\in\mathbb{R}} \mathbb{P}(\bar{\alpha}+c\bar{\beta} = x)$; clearly such $x^*(c)$ must exist. Suppose that the statement of the lemma does not hold; then $p^* = \sup_{c\in\mathbb{R}} p^*(c) > 0$. From (3.6) and the symmetry between $\bar{\alpha}$ and $\bar{\beta}$ it follows that

$$\begin{aligned}p^*(c) = \mathbb{P}\left(\bar{\alpha} + c\bar{\beta} = x^*(c)\right) &\le \frac{1}{6}\left\{p^* + p^* + p^* + p^*(c+1) + p^* + p^*\right\} \\ &= \frac{1}{6}\left\{5p^* + p^*(c+1)\right\},\end{aligned}$$

hence

$$p^*(c+1) = p^*(c) - 6\varepsilon \text{ whenever } p^*(c) > p^* - \varepsilon.$$

Fix a very small $\varepsilon' > 0$ and let c_0 be such that $p^*(c_0) > p^* - \varepsilon'$. Then for $c_i = c_0 + i$, $i = 1, 2, \ldots$, we have $p^*(c_i) > p^* - 6^i\varepsilon'$.

Let $A(c) = \{\omega : \ \bar{\alpha} + c\bar{\beta} = x^*(c)\}$ and $Z_N = 1_{A(c_0)} + 1_{A(c_1)} + \cdots + 1_{A(c_{N-1})}$. On one hand,

$$\mathbb{E}\, Z_N = \sum_{i=0}^{N-1} \mathbb{P}(A(c_i)) > Np^* - 0.2 \cdot 6^N \varepsilon'$$

since $\mathbb{P}(A(c_i)) = p^*(c_i) > p^* - 6^i\varepsilon'$. On the other hand, for $1 \le n \le N$,

$$\mathbb{E}\, Z_N = \sum_{i=1}^{N} \mathbb{P}(Z_N \ge i) \le n - 1 + \mathbb{P}(Z_N \ge n),$$

thus

$$\mathbb{P}(Z_N \ge n) > Np^* - (n-1) - 0.2 \cdot 6^N \varepsilon'.$$

Suppose that

$$Np^* > (n-1) + 0.2 \cdot 6^N \varepsilon'. \tag{3.7}$$

Then there is a subset of $A(c_0), \ldots, A(c_{N-1})$ containing n distinct elements, say $A(c_{i_1}), \ldots, A(c_{i_n})$ for which the probability of $\cap_{m=1}^n A(c_{i_m})$ is strictly positive. We are going to make use of

Claim 1 *Suppose that $z_1, z_2, \ldots, z_n$, $n \ge 3$, is a collection of n distinct real numbers. Let $B = \bigcap_{i=1}^n A(z_i)$. If $\mathbb{P}(B) > 0$ then the sets $A(z_i) \setminus B$, $i = 1, 2, \ldots, n$, are disjoint and as a result*

$$\mathbb{P}\left(\bigcup_{i=1}^n A(z_i)\right) = \mathbb{P}(B) + \sum_{i=1}^n \mathbb{P}(A(z_i) \setminus B) = \sum_{i=1}^n \mathbb{P}(A(z_i)) - (n-1)\mathbb{P}(B).$$

Proof of the claim. Since $\mathbb{P}(B) > 0$, the system of equations

$$\begin{cases} \bar{\alpha} + z_1\bar{\beta} &= x^*(z_1) \\ \bar{\alpha} + z_2\bar{\beta} &= x^*(z_2) \\ & \dots \\ \bar{\alpha} + z_n\bar{\beta} &= x^*(z_n) \end{cases}$$

must have a solution in $(\bar{\alpha}, \bar{\beta})$. This yields that for all distinct $i, j \in \{1, \dots, n\}$ we have

$$\bar{\beta} = \frac{x^*(z_i) - x^*(z_j)}{z_i - z_j} \tag{3.8}$$

and thus for any three distinct i, j, k

$$\frac{x^*(z_i) - x^*(z_j)}{z_i - z_j} = \frac{x^*(z_i) - x^*(z_k)}{z_i - z_k}$$

which in turn leads to $\omega \in A(z_i) \cap A(z_j) \Rightarrow \omega \in A(z_k)$ for all $k \Rightarrow \omega \in B$. Now the second statement of the claim is trivial. ■

From Claim 1 it follows that for $B = \cap_{m=1}^n A(c_{i_m})$

$$\mathbb{P}(B) \geq \frac{-1 + \sum_{m=1}^n \mathbb{P}(A(c_{i_m}))}{n-1} > p^* - 6^N \varepsilon' - \frac{1}{n-1}.$$

Let $\varepsilon'' = 6^N \varepsilon' + \frac{1}{n-1}$. From (3.8) we have that if we set $y = \frac{x^*(c_{i_1}) - x^*(c_{i_2})}{c_{i_1} - c_{i_2}}$ then $\mathbb{P}(\bar{\alpha} = y) = \mathbb{P}(\bar{\beta} = y) \geq \mathbb{P}(B) \geq p^* - \varepsilon''$. At the same time, from (3.4) it follows

$$\begin{aligned} p^* - \varepsilon'' \leq \mathbb{P}(\bar{\alpha} = y) &= \frac{1}{3}\left\{\mathbb{P}\left(\bar{\alpha} = 2y\right) + \mathbb{P}\left(\bar{\alpha} + \bar{\beta}/2 = 2(1-y)\right) + \mathbb{P}\left(\bar{\alpha} + \bar{\beta} = 2y\right)\right\} \\ &= \frac{1}{3}\left\{\mathbb{P}\left(\bar{\alpha} = 2y\right) + \mathbb{P}\left(\bar{\alpha} + \bar{\beta}/2 = 2(1-y)\right) + \mathbb{P}\left(\bar{\alpha} = 1 - 2y\right)\right\}. \end{aligned}$$

Since each of the expressions on the RHS does not exceed p^*, we have $\min\{\mathbb{P}(\bar{\alpha} = 2y), \mathbb{P}(\bar{\alpha} = 1 - 2y)\} \geq p^* - 3\varepsilon''$. W.l.o.g. assume that $2y \geq 1/2$. Reiterating (3.4) and recalling (3.5) (so that $\mathbb{P}(\bar{\alpha} \geq 1) = \mathbb{P}(\bar{\alpha} = 1) = 0$) we obtain

$$\begin{aligned} p^* - 3\varepsilon'' &\leq \mathbb{P}(\bar{\alpha} = 2y) \\ &= \frac{1}{3}\left\{\mathbb{P}\left(\bar{\alpha} = 4y\right) + \mathbb{P}\left(\bar{\alpha} + \frac{\bar{\beta}}{2} = 2(1-2y)\right) + \mathbb{P}\left(\bar{\alpha} = 1 - 4y\right)\right\} \\ &\leq \frac{1}{3}\left\{0 + \mathbb{P}\left(\bar{\alpha} + \bar{\beta}/2 = 2(1-2y)\right) + 0\right\} \leq \frac{p^*}{3} \end{aligned}$$

leading to a contradiction, provided $\varepsilon'' < 2p^*/9$. To finish the proof, we have to demonstrate that there exist n, N, ε' instantaneously satisfying this condition as well as (3.7). Indeed, fix an n so large that $n-1>9/p^*$. Next, let N be larger than $(n-1)/p^*$. Finally, set

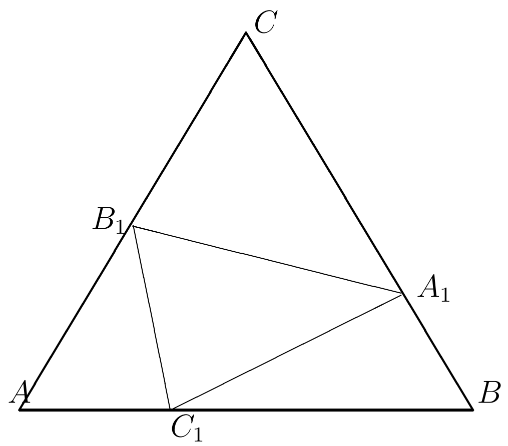
$$\varepsilon' = \frac{\min\{Np^* - (n-1), p^*/9\}}{6^N} > 0.$$

It is easy to see that (3.7) is fulfilled and moreover

$$\varepsilon'' = 6^N \varepsilon' + \frac{1}{n-1} < \frac{p^*}{9} + \frac{p^*}{9} = \frac{2p^*}{9}.$$

■

4 Random subtriangle of triangle



Now on each side of the triangle we randomly (independently and uniformly) choose points A_1, B_1, C_1. The new triangle is now formed by these three points ($A_1B_1C_1$). Repeat this procedure indefinitely. What is the limit of the shape?

Let $a_0 = |BC|$, $b_0 = |CA|$, and $c_0 = |AB|$. Also let $\xi_a = |BA_1|/|BC|$, $\xi_b = |CB_1|/|CA|$, and $\xi_c = |AC_1|/|AB|$. According to our assumption, ξ_i, $i = a, b, c$, are independent uniform $[0,1]$ random variables. Also let $a_1 = |B_1C_1|$, $b_1 = |C_1A_1|$, and $c_1 = |A_1B_1|$ be the sides of the new obtained triangle.

Theorem 4 *The limiting triangle shape is a.s. flat (i.e., maximum angle converges to π). Moreover, for any $c < 1 + \frac{\log 4}{3} - \frac{\pi^2}{9} = 0.365\dots$*

$$y_n \le e^{-cn} \text{ for all sufficiently large } n$$

where y_n is the ratio of the height of the triangle corresponding to its largest side and the length of this side.

Similarly to [2], let us rescale the triangle such that the largest side's length (say, AB) is 1, and fit this side on the coordinate plane such that $A = (0,0)$, $B = (1,0)$. Also suppose that C lies in the upper plane and $|AC| \ge |BC|$. Let $C = (x,y)$, then $x \in [1/2, 1]$, $y \ge 0$, and the pair (x,y) completely characterizes the shape of the triangle ABC. We also have

$$A_1 = (1-(1-x)\xi_a, y\xi_a)\,,\ B_1 = (x(1-\xi_b), y(1-\xi_b))\,,\ C_1 = (\xi_c, 0)\,.$$

The new side lengths will then satisfy

$$\begin{aligned} a_1^2 &= [x(1-\xi_b) - \xi_c]^2 + [y(1-\xi_b)]^2 \\ b_1^2 &= [(1-(1-x)\xi_a) - \xi_c]^2 + [y\xi_a]^2 \\ c_1^2 &= [(1-(1-x)\xi_a) - x(1-\xi_b)]^2 + [y(1-\xi_a-\xi_b)]^2 \end{aligned}$$

and according to the standard formulas, the area of the triangle with vertex coordinates A_1, B_1, and C_1 equals

$$\Delta := \frac{y}{2}\left[\xi_a\xi_b\xi_c + (1-\xi_a)(1-\xi_b)(1-\xi_c)\right]$$

Thus the new value of y is now

$$y_1 = \frac{2\Delta}{\max\{a_1,b_1,c_1\}^2} = y \cdot \frac{\xi_a\xi_b\xi_c + (1-\xi_a)(1-\xi_b)(1-\xi_c)}{\max\{a_1^2,b_1^2,c_1^2\}}$$

and also

$$x_1 = \begin{cases} \frac{a_1^2+b_1^2-c_1^2}{2\max\{a_1,b_1\}^2} & \text{if } c_1 = \min\{a_1,b_1,c_1\}; \\ \frac{a_1^2+c_1^2-b_1^2}{2\max\{a_1,c_1\}^2} & \text{if } b_1 = \min\{a_1,b_1,c_1\}; \\ \frac{b_1^2+c_1^2-a_1^2}{2\max\{b_1,c_1\}^2} & \text{if } a_1 = \min\{a_1,b_1,c_1\}. \end{cases}$$

which can be summarized as

$$\begin{aligned} y_1 &= y \cdot r, \\ x_1 &= \frac{a_1^2 + b_1^2 + c_1^2 - 2\min\{a_1^2, b_1^2, c_1^2\}}{2\max\{a_1^2, b_1^2, c_1^2\}} \end{aligned}$$

where

$$r = r(x,y) = \frac{R}{\max\{a_1^2, b_1^2, c_1^2\}}$$

and

$$R = R(\xi_a, \xi_b, \xi_c) = \xi_a\xi_b\xi_c + (1-\xi_a)(1-\xi_b)(1-\xi_c).$$

Observe that the denominator of $r(x,y)$ is increasing in y; hence $r(x,y) \le r(x,0)$.

From now on let us assume $y = 0$. For $y = 0$, we have that points A_1, B_1, C_1 all lie on the horizontal axes with coordinates $\mu = 1 - (1-x)\xi_a$, $\nu = x(1-\xi_b)$ and ξ_c respectively; we always have $0 \le \nu \le \mu \le 1$ since $\nu \in [0,x]$ and $\mu \in [x,1]$. Consequently,

$$\begin{aligned} S(x;\xi_a,\xi_b,\xi_c) &:= \max\{a_1, b_1, c_1\} \\ &= \max\{\mu - \xi_c, \mu - \nu, \xi_c - \nu\} = \begin{cases} \mu - \xi_c & \text{if } \xi_c < \nu; \\ \mu - \nu & \text{if } \nu \le \xi_c \le \mu; \\ \xi_c - \nu & \text{if } \xi_c > \mu. \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}\left[r(x,0) \mid \xi_a, \xi_b\right] &= \int_0^1 \frac{R(\xi_a,\xi_b,\xi_c)}{[\max\{a_1,b_1,c_1\}]^2}\,\mathrm{d}\xi_c = I_1 + I_2 + I_3 \\ &= \int_0^\nu \frac{R(\xi_a,\xi_b,\xi_c)}{(\mu-\xi_c)^2}\,\mathrm{d}\xi_c + \int_\nu^\mu \frac{R(\xi_a,\xi_b,\xi_c)}{(\mu-\nu)^2}\,\mathrm{d}\xi_c + \int_\mu^1 \frac{R(\xi_a,\xi_b,\xi_c)}{(\xi_c-\nu)^2}\,\mathrm{d}\xi_c. \end{aligned}$$

Easy algebra gives

$$\begin{aligned} I_1 &= (\xi_a + \xi_b - 1)\ln\left(\frac{\xi_b x + (1-x)(1-\xi_a)}{1 - \xi_a + \xi_a x}\right) + \frac{(1-\xi_b)\xi_a x}{1-\xi_a+\xi_a x}, \\ I_2 &= \frac{\xi_a + 1 - \xi_b}{2}, \\ I_3 &= (\xi_a + \xi_b - 1)\ln\left(\frac{1 - x + \xi_b x}{\xi_b x + (1-x)(1-\xi_a)}\right) + \frac{\xi_a(1-\xi_b)(1-x)}{1-x+\xi_b x}, \end{aligned}$$

hence

$$\mathbb{E}\left[r(x,0)\,|\,\xi_a,\xi_b\right] = (\xi_a+\xi_b-1)\ln\left(\frac{1-x+\xi_b x}{1-\xi_a+\xi_a x}\right) + \frac{\xi_a+1-\xi_b}{2} - \frac{\xi_a(1-\xi_b)(x^2(\xi_a+\xi_b-1)+\xi_a(1-2x)-1)}{(1-\xi_a+\xi_a x)(1-x+\xi_b x)}.$$

As a result,

$$\begin{aligned}\mathbb{E}\left[r(x,0)\,|\,\xi_a\right] &= \int_0^1 \mathbb{E}\left[r(x,0)\,|\,\xi_a,\xi_b\right]\mathrm{d}\xi_b \\ &= \frac{1}{2x^2(1-\xi_a+\xi_a x)}\left[x(x+1-\xi_a(1-x)(2x+3-\xi_a(2-x)))\right. \\ &\quad \left. + (1-2\xi_a)(1-\xi_a+\xi_a x)((1-x^2)\log(1-x)+x^2\log(1-\xi_a+\xi_a x))\right]\end{aligned}$$

and thus

$$\mathbb{E}\left[r(x,y)\right] \le \mathbb{E}\left[r(x,0)\right] = \int_0^1 \mathbb{E}\left[r(x,0)\,|\,\xi_a\right]\mathrm{d}\xi_a = 1.$$

Let (x_0,y_0), (x_1,y_1), (x_2,y_2), $\ldots$ be the sequence of coordinates corresponding to the sequence of subdivided triangles. Assume $y_0>0$, then with probability 1 we have $y_n>0$ for all n. Let $\mathcal{F}_n$ be the sigma-field generated by $\{(x_i,y_i), i=0,1,\ldots,n\}$. We have just established that

$$\mathbb{E}\left[y_{n+1}\,|\,\mathcal{F}_n\right] = y_n\mathbb{E}\left[r(x_n,y_n)\right] \le y_n.$$

Thus $y_n\in[0,1]$ is a supermartingale which must converge a.s. to some limit y_∞. Let $E=\{\xi_a<0.1,\ \xi_b>0.9\}$, then $\mathbb{P}(E)=0.01>0$. On the event E we have $c_1\ge 0.8$, hence

$$r(x,y) = \frac{\xi_a\cdot[\xi_b\xi_c]+(1-\xi_b)\cdot[(1-\xi_a)(1-\xi_c)]}{\max\{a_1,b_1,c_1\}^2} \le \frac{2\cdot 0.1}{c_1^2} < \frac{1}{3}.$$

Consequently, $\mathbb{P}(y_{n+1}<y_n/3\,|\,\mathcal{F}_n)>0.01$ which implies $y_\infty=0$ a.s.

Note that

$$\mathbb{E}\left[\log R\left(\xi_a,\xi_b,\xi_c\right)\right] = \int_0^1\int_0^1\int_0^1 \log(R)\,\mathrm{d}\xi_c\,\mathrm{d}\xi_b\,\mathrm{d}\xi_a = \frac{\pi^2}{9}-\frac{8}{3} = -1.57\ldots$$

and

$$\mathbb{E}\left[\log S(x;\xi_a,\xi_b,\xi_c)\right] = \int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b \left[\int_0^\nu \log(\mu-\xi_c)\,\mathrm{d}\xi_c + \int_\nu^\mu \log(\mu-\nu)\,\mathrm{d}\xi_c \right.$$
$$\left. + \int_\mu^1 \log(\xi_c-\nu)\,\mathrm{d}\xi_c\right] = -\frac{5}{6} - \frac{x^3\log(x)+(1-x)^3\log(x)}{3x(1-x)}$$
$$\geq \frac{\log 4 - 5}{6} = -0.602\cdots =: -\kappa \tag{4.9}$$

Observe that

$$\begin{aligned}\log y_n - \log y_{n-1} &= \log r(x_{n-1},y_{n-1}) \leq \log r(x_n,0)\\ &= \log R\left(\xi_a^{(n)},\xi_b^{(n)},\xi_c^{(n)}\right) - 2\log S\left(x_{n-1};\xi_a^{(n)},\xi_b^{(n)},\xi_c^{(n)}\right)\\ &= \rho_n + 2\sigma_n\end{aligned}$$

where $\rho_n = \log R\left(\xi_a^{(n)},\xi_b^{(n)},\xi_c^{(n)}\right)$ are i.i.d. with mean $\frac{\pi^2}{9}-\frac{8}{3}$ and $\sigma_n = -\log S\left(x_{n-1};\xi_a^{(n)},\xi_b^{(n)},\xi_c^{(n)}\right)$ are some $\mathcal{F}_n-$adapted random variables respectively. Since $0\leq S(\cdot)\leq 1$, we have $\sigma_n\geq 0$, and also $\mathbb{E}\left(\sigma_n \,|\, \mathcal{F}_{n-1}\right)\leq\kappa$ due to (4.9). Additionally, we have for any positive z

$$\begin{aligned}\mathbb{P}(\sigma_n\geq z \,|\, \mathcal{F}_{n-1}) &= \mathbb{P}\left(S\left(x_{n-1};\xi_a^{(n)},\xi_b^{(n)},\xi_c^{(n)}\right)\leq e^{-z}\,|\,\mathcal{F}_{n-1}\right)\\ &\leq \mathbb{P}(\mu-\nu\leq e^{-z}\,|\,\mathcal{F}_{n-1}) = \mathbb{P}(x_{n-1}\xi_b+(1-x_{n-1})(1-\xi_a)\leq e^{-z}\,|\,\mathcal{F}_{n-1})\\ &\leq \mathbb{P}(1_{\{x_{n-1}>1/2\}}x_{n-1}\xi_b + 1_{\{x_{n-1}\leq 1/2\}}(1-x_{n-1})(1-\xi_a)]\leq e^{-z}\,|\,\mathcal{F}_{n-1})\\ &\leq \mathbb{P}\left(\frac{\xi_b}{2}\leq e^{-z}\,|\,\mathcal{F}_{n-1}\right) = 2e^{-z}\end{aligned}$$

since $1-\xi_a$ has the same uniform $(0,1)$ distribution as ξ_b.

The following statement is related to exponential inequalities involving martingales; however since we did not find it in the form we needed, we present its short proof later.

Lemma 2 *Let $\mathcal{F}_n$ be an increasing family of $\sigma-$fields, and σ_n be $\mathcal{F}_n$-adapted random variables, possibly unbounded. Suppose that*

$$\mathbb{E}\left[\sigma_n \,|\, \mathcal{F}_{n-1}\right]\leq\kappa$$

and for some $\bar{z} \geq 0$ *and* $c > 0$ *we have*

$$\mathbb{P}(|\sigma_n| \geq z \,|\, \mathcal{F}_{n-1}] \leq e^{-cz} \text{ for all } z \geq \bar{z}.$$

Then

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n \sigma_i}{n} \leq \kappa \text{ a.s.}$$

Now from Lemma 2 and the strong law of large numbers it follows that

$$\limsup_{n\to\infty} \frac{\log y_n}{n} \leq \limsup_{n\to\infty} \frac{\sum_{i=1}^n \rho_n}{n} + 2 \limsup_{n\to\infty} \frac{\sum_{i=1}^n \sigma_n}{n}$$
$$\leq \mathbb{E}\,\rho_n + 2\kappa = \frac{\pi^2}{9} - 1 - \frac{\log 4}{3} = -0.365\ldots$$

This established the result of Theorem 4.

Theorem 5 x_n *converges to Uniform* $[1/2, 1]$ *distribution.*

Proof. Since $y_n \to 0$ a.s., and x_{n+1} is continuous in y_n near 0, it will suffice to show that, given $y_n = 0$, x_{n+1} has $U[1/2, 1]$ distribution. This will follow, in turn, from the following statement: put μ, ν, ξ_c in an increasing order, and denote the resulting values $0 \leq x^{(1)} < x^{(2)} < x^{(3)} \leq 1$, then $\chi := \frac{x^{(2)} - x^{(1)}}{x^{(3)} - x^{(1)}}$ has $U[0, 1]$ distribution. Indeed, for any $z \in (0, 1)$ we have

$$\mathbb{P}(\chi \leq z) = \mathbb{P}(\chi \leq z,\ \xi_c < \nu) + \mathbb{P}(\chi \leq z,\ \nu \leq \xi_c \leq \mu) + \mathbb{P}(\chi \leq z,\ \xi_c > \mu)$$
$$= (I) + (II) + (III).$$

We have

$$(I) = \int_0^1 \mathrm{d}\xi_b \int_0^1 \mathrm{d}\xi_a \int_0^\nu 1_{\left\{\frac{\xi_a x - \xi_c}{1-\xi_b(1-x)-\xi_c} < z\right\}} \mathrm{d}\xi_c = \begin{cases} \frac{(3z - xz^2 - zx - x)x}{6z(1-x)} & \text{if } x < z; \\ \frac{(3x - zx^2 - xz - z)z}{6x(1-z)} & \text{if } x \geq z, \end{cases}$$

$$(II) = \int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b \int_\nu^\mu 1_{\left\{\frac{\xi_c - \xi_a x}{1-\xi_b(1-x)-\xi_a x} < z\right\}} \mathrm{d}\xi_c = z/2,$$

$$(III) = \int_0^1 \mathrm{d}\xi_a \int_0^1 \mathrm{d}\xi_b \int_\mu^1 1_{\left\{\frac{1-\xi_b(1-x)-\xi_a x}{\xi_c - \xi_a x} < z\right\}} \mathrm{d}\xi_c$$
$$= \begin{cases} \frac{3z^2 + z^2x^2 - 3z^2x + zx^2 - 3zx + x^2}{6z(1-x)} & \text{if } x < z; \\ \frac{(1-x)^2 z^2}{6(1-z)x} & \text{if } x \geq z. \end{cases}$$

As a result, $\mathbb{P}(\chi \leq z) = (I) + (II) + (III) = z$ and thus χ has a uniform $[0,1]$ distribution. ■

4.1 Proof of Lemma 2

Proof. The result follows from the Borel-Cantelli lemma, once we establish that for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$\mathbb{P}\left(\sum_{i=1}^{n} \sigma_i \geq (\kappa + \varepsilon)n\right) \leq e^{-\delta n}. \tag{4.10}$$

First, by Markov inequality, for any $\lambda > 0$

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{n} \sigma_i \geq (\kappa + \varepsilon)n\right) &= \mathbb{P}\left(e^{\lambda \sum_{i=1}^{n} \sigma_i} \geq e^{\lambda n(\kappa+\varepsilon)}\right) \leq e^{-\lambda n(\kappa+\varepsilon)}\mathbb{E}\left[e^{\lambda \sum_{i=1}^{n} \sigma_i}\right] \\
&= e^{-\lambda n(\kappa+\varepsilon)}\mathbb{E}\,\mathbb{E}\left[e^{\lambda \sum_{i=1}^{n} \sigma_i} \,|\, \mathcal{F}_{n-1}\right] \\
&= e^{-\lambda n(\kappa+\varepsilon)}\mathbb{E}\left[e^{\lambda\left(\sum_{i=1}^{n-1} \sigma_i\right)}\mathbb{E}\left[e^{\lambda\sigma_n} \,|\, \mathcal{F}_{n-1}\right]\right].
\end{aligned}$$

Now to show (4.10) by induction, it suffices to demonstrate that there are $\lambda > 0$ and $\delta > 0$ such that $\mathbb{E}\left[e^{\lambda\sigma_n} \,|\, \mathcal{F}_{n-1}\right] < e^{\lambda(\kappa+\varepsilon)-\delta}$ for all n.

Indeed, (see e.g. [4], Lemma 5.7 in Section 1)

$$\begin{aligned}
\mathbb{E}\left[\frac{\lambda^k |\sigma_n|^k}{k!} \,|\, \mathcal{F}_{n-1}\right] &= \frac{\lambda^k}{k!}\int_0^\infty k z^{k-1}\mathbb{P}(|\sigma_n| > z \,|\, \mathcal{F}_{n-1})\,\mathrm{d}z \\
&\leq \frac{\lambda^k}{k!}\left[\bar{z}^k + \int_{\bar{z}}^\infty k z^{k-1} e^{-cz}\,\mathrm{d}z\right] \\
&\leq \frac{\lambda^k}{k!}\left[\bar{z}^k + k! c^{-k}\right] = \frac{(\lambda\bar{z})^k}{k!} + \left(\frac{\lambda}{c}\right)^k,
\end{aligned}$$

therefore, assuming $\lambda < \min(c,1)$, by Dominated convergence theorem we have

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda\sigma_n} \mid \mathcal{F}_{n-1}\right] &= 1 + \lambda\mathbb{E}\left[\sigma_n \mid \mathcal{F}_{n-1}\right] + \sum_{k=2}^{\infty} \frac{\lambda^k \mathbb{E}\left[\sigma_n^k \mid \mathcal{F}_{n-1}\right]}{k!} \\
&\leq 1 + \lambda\kappa + \sum_{k=2}^{\infty} \mathbb{E}\left[\frac{\lambda^k |\sigma_n|^k}{k!} \mid \mathcal{F}_{n-1}\right] \qquad (4.11) \\
&\leq 1 + \lambda\kappa + \sum_{k=2}^{\infty} \left(\frac{\lambda}{c}\right)^k + \sum_{k=2}^{\infty} \frac{(\lambda\bar{z})^k}{k!} \\
&\leq 1 + \lambda\kappa + \frac{\lambda^2}{c(c-\lambda)} + \lambda^2[e^{\bar{z}} - 1 - \bar{z}]
\end{aligned}$$

Now by choosing $\lambda > 0$ sufficiently small, we can make (4.11) smaller than $e^{\lambda(\kappa+\varepsilon)-\delta} > 1 + \lambda\kappa + (\lambda\varepsilon - \delta)$ for some $\delta > 0$, thus finishing the proof. ■

References

[1] Barnsley, M.; Elton, J. A new class of Markov processes for image encoding. *Adv. in Appl. Probab.* 20 (1988), no. 1, 14-32.

[2] Diaconis, P.; Miclo., L. On barycentric subdivision. *Combinatorics, Probability and Computing* 20 (2011), 213–237.

[3] Ding, J.; Hitt, L.R.; Zhang, X-M. Markov chains and dynamic geometry of polygons. *Linear Algebra and its Applications* 367 (2003), 255–270 .

[4] Durrett, R. (1996) *Probability: Theory and Examples.* (2nd. ed.) Duxbury Press, Belmont, California.

[5] Hough, B. Tessellation of a triangle by repeated barycentric subdivision. *Electron. Commun. Probab.* 14 (2009), 270-277.