# The Janson inequalities for general up-sets 

Oliver Riordan* and Lutz Warnke ${ }^{\dagger}$

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#### Abstract

Janson and Janson, Łuczak and Ruciński proved several inequalities for the lower tail of the distribution of the number of events that hold, when all the events are up-sets (increasing events) of a special form each event is the intersection of some subset of a single set of independent events (i.e., a principal up-set). We show that these inequalities in fact hold for arbitrary up-sets, by modifying existing proofs to use only positive correlation, avoiding the need to assume positive correlation conditioned on one of the events.


Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{I} \subseteq \mathcal{F}$ a collection of events with the following properties:

$$
\begin{equation*}
A, B \in \mathcal{I} \Longrightarrow \mathbb{P}(A \cap B) \geqslant \mathbb{P}(A) \mathbb{P}(B) \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
A, B \in \mathcal{I} \Longrightarrow A \cap B \in \mathcal{I} \text { and } A \cup B \in \mathcal{I} \tag{2}
\end{equation*}
$$

The standard and most important example is when $\Omega=\{0,1\}^{S}$ is a product probability space (with product measure), and $\mathcal{I}$ is the collection of all increasing events, i.e., events $A$ such that $\omega \in A$ and $\omega \leqslant \omega^{\prime}$ pointwise imply $\omega^{\prime} \in A$. (Of course, one can instead take all decreasing events.) Then (11) is simply Harris's Lemma [4] (also known as Kleitman's Lemma [9). There are other examples, such as increasing events in random cluster models with parameter $q \geqslant 1$; see [3].

Let $A_{1}, \ldots, A_{k} \in \mathcal{I}$, write $I_{i}$ for the indicator function of $A_{i}$, and set

$$
X=\sum_{i} I_{i}, \quad \mu=\mathbb{E}(X)=\sum_{i} \mathbb{P}\left(A_{i}\right)
$$

and

$$
\Delta=\sum_{i} \sum_{j \sim i} \mathbb{P}\left(A_{i} \cap A_{j}\right)
$$

where we write $i \sim j$ if $i \neq j$ and $A_{i}$ and $A_{j}$ are dependent. (Note that we sum over ordered pairs, and exclude the term $i=j$. These conventions are not universal!)

[^0]Theorem 1. Under the conditions above we have

$$
\begin{equation*}
\mathbb{P}(X=0) \leqslant \exp (-\mu+\Delta / 2), \tag{3}
\end{equation*}
$$

and, for any $0 \leqslant t \leqslant \mu$,

$$
\begin{equation*}
\mathbb{P}(X \leqslant \mu-t) \leqslant \exp \left(-\frac{\phi(-t / \mu) \mu^{2}}{\mu+\Delta}\right) \leqslant \exp \left(-\frac{t^{2}}{2(\mu+\Delta)}\right) \tag{4}
\end{equation*}
$$

where $\phi(x)=(1+x) \log (1+x)-x$ with $\phi(-1)=1$.
When the events $A_{i}$ are principal up-sets, i.e., events in a product space $\{0,1\}^{S}$ of the form $A_{i}=\left\{\omega: \omega_{x}=1\right.$ for all $\left.x \in \alpha_{i}\right\}$ for some $\alpha_{i} \subseteq S$, the first inequality in (4) is the well known Janson inequality (5). The second is also given in [5]; for other convenient weaker forms see [7]. Under the same assumptions, (3) was proved earlier by Janson, Luczak and Ruciński [6]. We shall prove Theorem 1 by modifying the proofs of these inequalities to avoid applying Harris's Lemma to the conditional measure $\mathbb{P}\left(\cdot \mid A_{i}\right)$.

Proof. We begin with a simple observation that, in the standard setting, follows from the equality conditions in Harris's Lemma. Indeed, we claim that, for each $i$,

$$
\begin{equation*}
A_{i} \text { is independent of the set } \mathcal{S}_{i}=\left\{A_{j}: j \neq i, j \nsim i\right\} . \tag{5}
\end{equation*}
$$

To see this, note first that if $A, B, C \in \mathcal{I}$ and $A$ is independent of $B$ and of $C$, then

$$
\begin{equation*}
\mathbb{P}(A \cap(B \cap C))+\mathbb{P}(A \cap(B \cup C))=\mathbb{P}(A \cap B)+\mathbb{P}(A \cap C)=\mathbb{P}(A)(\mathbb{P}(B)+\mathbb{P}(C)) \tag{6}
\end{equation*}
$$

Since, by (2), $B \cap C$ and $B \cup C$ are in $\mathcal{I}$, by (11) we have

$$
\mathbb{P}(A \cap(B \cap C)) \geqslant \mathbb{P}(A) \mathbb{P}(B \cap C) \text { and } \mathbb{P}(A \cap(B \cup C)) \geqslant \mathbb{P}(A) \mathbb{P}(B \cup C)
$$

The sum of these inequalities is the equality (6), so both inequalities are equalities, and in particular $A$ is independent of $B \cap C$. Since $A_{i}$ is independent of each $A_{j} \in \mathcal{S}_{i}$ by definition, it follows inductively that $A_{i}$ is independent of any intersection of events $A_{j} \in \mathcal{S}_{i}$, so $A_{i}$ is independent of the set $\mathcal{S}_{i}$ of events, as claimed.

To prove (3) we modify the argument given by Boppana and Spencer [2], following the presentation in (1). Following Boppana and Spencer, set $r_{i}=$ $\mathbb{P}\left(A_{i} \mid A_{1}^{\mathrm{c}} \cap \cdots \cap A_{i-1}^{\mathrm{c}}\right)$, so $\mathbb{P}(X=0)=\prod_{i=1}^{k}\left(1-r_{i}\right) \leqslant \exp \left(-\sum r_{i}\right)$. It suffices to show that for each $i$ we have

$$
\begin{equation*}
r_{i} \geqslant \mathbb{P}\left(A_{i}\right)-\sum_{j<i, j \sim i} \mathbb{P}\left(A_{i} \cap A_{j}\right), \tag{7}
\end{equation*}
$$

since the sum of the final expression is exactly $\mu-\Delta / 2$. Fix $i$. Still following [2], 1], set

$$
D=\bigcap_{j<i, j \sim i} A_{j}^{\mathrm{c}} \quad \text { and } \quad R=\bigcap_{j<i, j \nsim i} A_{j}^{\mathrm{c}},
$$

noting that, by (5), $A_{i}$ is independent of $R$. Then

$$
\begin{equation*}
r_{i}=\mathbb{P}\left(A_{i} \mid D \cap R\right)=\frac{\mathbb{P}\left(A_{i} \cap D \cap R\right)}{\mathbb{P}(D \cap R)} \geqslant \frac{\mathbb{P}\left(A_{i} \cap D \cap R\right)}{\mathbb{P}(R)}=\mathbb{P}\left(A_{i} \cap D \mid R\right) \tag{8}
\end{equation*}
$$

(We may assume that $\mathbb{P}(D \cap R)>0$, since otherwise $\mathbb{P}(X=0)=0$.) At this point the original argument involves writing the final probability as $\mathbb{P}\left(A_{i} \mid\right.$ $R) \mathbb{P}\left(D \mid A_{i} \cap R\right)$. Instead, we simply write

$$
\begin{equation*}
\mathbb{P}\left(A_{i} \cap D \mid R\right)=\mathbb{P}\left(A_{i} \mid R\right)-\mathbb{P}\left(A_{i} \cap D^{\mathrm{c}} \mid R\right) \tag{9}
\end{equation*}
$$

By (2), $D^{\mathrm{c}}=\bigcup_{j<i, j \sim i} A_{j} \in \mathcal{I}$, so $A_{i} \cap D^{\mathrm{c}} \in \mathcal{I}$. Since $R^{\mathrm{c}} \in \mathcal{I}$, using (11) and the union bound it follows that

$$
\begin{equation*}
\mathbb{P}\left(A_{i} \cap D^{\mathrm{c}} \mid R\right) \leqslant \mathbb{P}\left(A_{i} \cap D^{\mathrm{c}}\right)=\mathbb{P}\left(A_{i} \cap \bigcup_{j<i, j \sim i} A_{j}\right) \leqslant \sum_{j<i, j \sim i} \mathbb{P}\left(A_{i} \cap A_{j}\right) . \tag{10}
\end{equation*}
$$

Recalling that $A_{i}$ is independent of $R$, combining (8), (9) and (10) gives (7), completing the proof of (3).

Turning to (4), fix $1 \leqslant i \leqslant k$ and let

$$
Y_{i}=I_{i}+\sum_{j \sim i} I_{j} \quad \text { and } \quad Z_{i}=\sum_{j \neq i, j \nsim i} I_{j}
$$

so $X=Y_{i}+Z_{i}$, with $Z_{i}$ containing the terms independent of $I_{i}$ and $Y_{i}$ the others (including $I_{i}$ itself). In the proof of (4) given in [7], the only step in which anything is assumed about the events $A_{i}$ is (2.20) on page 32, where it is shown (in our notation) that for $s \geqslant 0$ and each $1 \leqslant i \leqslant k$,

$$
\begin{equation*}
\mathbb{E}\left(I_{i} e^{-s X}\right) \geqslant \mathbb{E}\left(I_{i} e^{-s Y_{i}}\right) \mathbb{E}\left(e^{-s X}\right) \tag{11}
\end{equation*}
$$

Proceeding much as in the proof of (7), note that

$$
\begin{equation*}
\frac{\mathbb{E}\left(I_{i} e^{-s X}\right)}{\mathbb{E}\left(e^{-s X}\right)}=\frac{\mathbb{E}\left(I_{i} e^{-s X}\right)}{\mathbb{E}\left(e^{-s Y_{i}} e^{-s Z_{i}}\right)} \geqslant \frac{\mathbb{E}\left(I_{i} e^{-s X}\right)}{\mathbb{E}\left(e^{-s Z_{i}}\right)} . \tag{12}
\end{equation*}
$$

Also,

$$
I_{i} e^{-s X}=I_{i} e^{-s Y_{i}} e^{-s Z_{i}}=I_{i} e^{-s Z_{i}}-I_{i} e^{-s Z_{i}}\left(1-e^{-s Y_{i}}\right)=I_{i} g-f g,
$$

where

$$
f=I_{i}\left(1-e^{-s Y_{i}}\right) \quad \text { and } \quad g=e^{-s Z_{i}}
$$

Now from (5), $I_{i}$ and $Z_{i}$ are independent, so $\mathbb{E}\left(I_{i} g\right)=\mathbb{E}\left(I_{i}\right) \mathbb{E}(g)$. We may write $f$ in the form $f=v_{0}+\sum_{j}\left(v_{j}-v_{j-1}\right) J_{j}$, where $0 \leqslant v_{0}<v_{1}<\cdots$ are the distinct values taken by $f$, and each $J_{j}$ is the indicator function of the event $B_{j}=\left\{f \geqslant v_{j}\right\}$. Note that any such $B_{j}$ may be expressed as $\bigcup_{\alpha \in \mathcal{J}} \bigcap_{i \in \alpha} A_{i}$ for some set $\mathcal{J}$ of subsets of $\{1,2, \ldots, k\}$, so (2) implies $B_{j} \in \mathcal{I}$. Writing $1-g$
in an analogous form, it follows from (11) that $\mathbb{E}(f(1-g)) \geqslant \mathbb{E}(f) \mathbb{E}(1-g)$, so $\mathbb{E}(f g) \leqslant \mathbb{E}(f) \mathbb{E}(g)$. Hence,

$$
\mathbb{E}\left(I_{i} e^{-s X}\right)=\mathbb{E}\left(I_{i} g-f g\right) \geqslant \mathbb{E}\left(I_{i}\right) \mathbb{E}(g)-\mathbb{E}(f) \mathbb{E}(g)
$$

Using (12) for the first step this gives

$$
\frac{\mathbb{E}\left(I_{i} e^{-s X}\right)}{\mathbb{E}\left(e^{-s X}\right)} \geqslant \frac{\mathbb{E}\left(I_{i} e^{-s X}\right)}{\mathbb{E}(g)} \geqslant \mathbb{E}\left(I_{i}\right)-\mathbb{E}(f)=\mathbb{E}\left(I_{i}-f\right)=\mathbb{E}\left(I_{i} e^{-s Y_{i}}\right)
$$

This is exactly (11), and the rest of the proof in [7] is unaltered.
Remark 1. We have stated two of the best-known and cleanest forms of the inequalities in Theorem 1. Let us note that other forms also hold in the present more general context. For example, the inequality (7) leads to the bound

$$
\begin{equation*}
\mathbb{P}(X=0) \leqslant \prod_{i=1}^{k}\left(1-\mathbb{P}\left(A_{i}\right)\right) \exp \left(\frac{1}{1-\varepsilon} \frac{\Delta}{2}\right) \tag{13}
\end{equation*}
$$

where $\varepsilon=\max _{i} \mathbb{P}\left(A_{i}\right)$. This bound was given by Boppana and Spencer [2] (for $\varepsilon=1 / 2)$; see also [1, 7].

Furthermore, (11) is the only step in the proof of Lemma 1 of [6] that requires any assumptions about the $A_{i}$. Hence this result, which is slightly stronger than (3), also holds in the present setting, giving (in our notation)

$$
\begin{equation*}
\log \mathbb{P}(X=0) \leqslant-\sum_{i} \mathbb{E}\left(\frac{I_{i}}{I_{i}+\sum_{j \sim i} I_{j}}\right) \tag{14}
\end{equation*}
$$

Remark 2. A key feature of the various Janson inequalities is that when $\Delta$ is small, then the bounds are close to best possible, since $\mathbb{P}(X=0) \geqslant \prod_{i}(1-$ $\left.\mathbb{P}\left(A_{i}\right)\right)$. We have not stressed this since it is well known that this lower bound applies to general up-sets $A_{i}$, by Harris's Lemma. Similarly, it applies whenever the $A_{i}$ are in some collection $\mathcal{I}$ of events satisfying (11) and (21).

Remark 3. The bounds in (4) can be extended to the weighted case $X=$ $\sum_{i} c_{i} I_{i}$ with positive $c_{i}$, studied, e.g., in [8]: we obtain

$$
\begin{equation*}
\mathbb{P}(X \leqslant \mu-t) \leqslant \exp \left(-\frac{\phi(-t / \mu) \mu^{2}}{\bar{\Delta}}\right) \leqslant \exp \left(-\frac{t^{2}}{2 \bar{\Delta}}\right) \tag{15}
\end{equation*}
$$

where $\mu=\mathbb{E}(X)$ and

$$
\bar{\Delta}=\sum_{i} c_{i}^{2} \mathbb{P}\left(A_{i}\right)+\sum_{i} \sum_{j \sim i} c_{i} c_{j} \mathbb{P}\left(A_{i} \cap A_{j}\right)=\sum_{i} \mathbb{E}\left(J_{i}^{2}\right)+\sum_{i} \sum_{j \sim i} \mathbb{E}\left(J_{i} J_{j}\right)
$$

for $J_{\ell}=c_{\ell} I_{\ell}$. (In applications, it may be convenient to note that $\sum_{i} \mathbb{E}\left(J_{i}^{2}\right) \leqslant C \mu$ when $\max _{i} c_{i} \leqslant C$.) The proof of (15) is a straightforward modification of that of Theorem 2.14 in [7]. The key inequality is again (11), now with $I_{\ell}$ replaced by $J_{\ell}$ in the definitions of $X, Y_{i}$ and $Z_{i}$. The proof above carries over since all $c_{i}$ are positive. Finally, in this setting (14) also holds, with $I_{\ell}$ replaced by $J_{\ell}$.

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[^0]:    *Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, UK. E-mail: riordan@maths.ox.ac.uk.
    ${ }^{\dagger}$ Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WB, UK. E-mail: L.Warnke@dpmms.cam.ac.uk.

