The Janson inequalities for general up-sets

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Abstract

Janson and Janson, Luczak and Ruciński proved several inequalities for the lower tail of the distribution of the number of events that hold, when all the events are up-sets (increasing events) of a special form – each event is the intersection of some subset of a single set of independent events (i.e., a principal up-set). We show that these inequalities in fact hold for arbitrary up-sets, by modifying existing proofs to use only positive correlation, avoiding the need to assume positive correlation conditioned on one of the events.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\mathcal{I} \subseteq \mathcal{F}$ a collection of events with the following properties:

$$A, B \in \mathcal{I} \implies \mathbb{P}(A \cap B) \ge \mathbb{P}(A)\mathbb{P}(B) \tag{1}$$

and

$$A, B \in \mathcal{I} \implies A \cap B \in \mathcal{I} \text{ and } A \cup B \in \mathcal{I}.$$
 (2)

The standard and most important example is when $\Omega = \{0, 1\}^S$ is a product probability space (with product measure), and \mathcal{I} is the collection of all increasing events, i.e., events A such that $\omega \in A$ and $\omega \leq \omega'$ pointwise imply $\omega' \in A$. (Of course, one can instead take all decreasing events.) Then (1) is simply Harris's Lemma [4] (also known as Kleitman's Lemma [9]). There are other examples, such as increasing events in random cluster models with parameter $q \ge 1$; see [3].

Let $A_1, \ldots, A_k \in \mathcal{I}$, write I_i for the indicator function of A_i , and set

$$X = \sum_{i} I_i, \qquad \mu = \mathbb{E}(X) = \sum_{i} \mathbb{P}(A_i)$$

and

$$\Delta = \sum_{i} \sum_{j \sim i} \mathbb{P}(A_i \cap A_j)$$

where we write $i \sim j$ if $i \neq j$ and A_i and A_j are dependent. (Note that we sum over *ordered* pairs, and exclude the term i = j. These conventions are not universal!)

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Theorem 1. Under the conditions above we have

$$\mathbb{P}(X=0) \leqslant \exp(-\mu + \Delta/2), \tag{3}$$

and, for any $0 \leq t \leq \mu$,

$$\mathbb{P}(X \leqslant \mu - t) \leqslant \exp\left(-\frac{\phi(-t/\mu)\mu^2}{\mu + \Delta}\right) \leqslant \exp\left(-\frac{t^2}{2(\mu + \Delta)}\right),\tag{4}$$

where $\phi(x) = (1+x)\log(1+x) - x$ with $\phi(-1) = 1$.

When the events A_i are principal up-sets, i.e., events in a product space $\{0,1\}^S$ of the form $A_i = \{\omega : \omega_x = 1 \text{ for all } x \in \alpha_i\}$ for some $\alpha_i \subseteq S$, the first inequality in (4) is the well known Janson inequality [5]. The second is also given in [5]; for other convenient weaker forms see [7]. Under the same assumptions, (3) was proved earlier by Janson, Luczak and Ruciński [6]. We shall prove Theorem 1 by modifying the proofs of these inequalities to avoid applying Harris's Lemma to the conditional measure $\mathbb{P}(\cdot | A_i)$.

Proof. We begin with a simple observation that, in the standard setting, follows from the equality conditions in Harris's Lemma. Indeed, we claim that, for each i,

$$A_i$$
 is independent of the set $S_i = \{A_j : j \neq i, j \not\sim i\}.$ (5)

To see this, note first that if $A, B, C \in \mathcal{I}$ and A is independent of B and of C, then

 $\mathbb{P}(A \cap (B \cap C)) + \mathbb{P}(A \cap (B \cup C)) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) = \mathbb{P}(A)(\mathbb{P}(B) + \mathbb{P}(C)).$ (6)

Since, by (2), $B \cap C$ and $B \cup C$ are in \mathcal{I} , by (1) we have

$$\mathbb{P}(A \cap (B \cap C)) \ge \mathbb{P}(A)\mathbb{P}(B \cap C) \text{ and } \mathbb{P}(A \cap (B \cup C)) \ge \mathbb{P}(A)\mathbb{P}(B \cup C).$$

The sum of these inequalities is the equality (6), so both inequalities are equalities, and in particular A is independent of $B \cap C$. Since A_i is independent of each $A_j \in S_i$ by definition, it follows inductively that A_i is independent of any intersection of events $A_j \in S_i$, so A_i is independent of the set S_i of events, as claimed.

To prove (3) we modify the argument given by Boppana and Spencer [2], following the presentation in [1]. Following Boppana and Spencer, set $r_i = \mathbb{P}(A_i \mid A_1^c \cap \cdots \cap A_{i-1}^c)$, so $\mathbb{P}(X = 0) = \prod_{i=1}^k (1 - r_i) \leq \exp(-\sum r_i)$. It suffices to show that for each *i* we have

$$r_i \ge \mathbb{P}(A_i) - \sum_{j < i, \, j \sim i} \mathbb{P}(A_i \cap A_j),\tag{7}$$

since the sum of the final expression is exactly $\mu - \Delta/2$. Fix *i*. Still following [2, 1], set

$$D = \bigcap_{j < i, j \sim i} A_j^c \quad \text{and} \quad R = \bigcap_{j < i, j \not\sim i} A_j^c,$$

noting that, by (5), A_i is independent of R. Then

$$r_i = \mathbb{P}(A_i \mid D \cap R) = \frac{\mathbb{P}(A_i \cap D \cap R)}{\mathbb{P}(D \cap R)} \ge \frac{\mathbb{P}(A_i \cap D \cap R)}{\mathbb{P}(R)} = \mathbb{P}(A_i \cap D \mid R).$$
(8)

(We may assume that $\mathbb{P}(D \cap R) > 0$, since otherwise $\mathbb{P}(X = 0) = 0$.) At this point the original argument involves writing the final probability as $\mathbb{P}(A_i \mid R)\mathbb{P}(D \mid A_i \cap R)$. Instead, we simply write

$$\mathbb{P}(A_i \cap D \mid R) = \mathbb{P}(A_i \mid R) - \mathbb{P}(A_i \cap D^c \mid R).$$
(9)

By (2), $D^{c} = \bigcup_{j < i, j \sim i} A_{j} \in \mathcal{I}$, so $A_{i} \cap D^{c} \in \mathcal{I}$. Since $R^{c} \in \mathcal{I}$, using (1) and the union bound it follows that

$$\mathbb{P}(A_i \cap D^{c} \mid R) \leq \mathbb{P}(A_i \cap D^{c}) = \mathbb{P}\left(A_i \cap \bigcup_{j < i, j \sim i} A_j\right) \leq \sum_{j < i, j \sim i} \mathbb{P}(A_i \cap A_j).$$
(10)

Recalling that A_i is independent of R, combining (8), (9) and (10) gives (7), completing the proof of (3).

Turning to (4), fix $1 \leq i \leq k$ and let

$$Y_i = I_i + \sum_{j \sim i} I_j$$
 and $Z_i = \sum_{j \neq i, \ j \not\sim i} I_j$

so $X = Y_i + Z_i$, with Z_i containing the terms independent of I_i and Y_i the others (including I_i itself). In the proof of (4) given in [7], the only step in which anything is assumed about the events A_i is (2.20) on page 32, where it is shown (in our notation) that for $s \ge 0$ and each $1 \le i \le k$,

$$\mathbb{E}(I_i e^{-sX}) \ge \mathbb{E}(I_i e^{-sY_i}) \mathbb{E}(e^{-sX}).$$
(11)

Proceeding much as in the proof of (7), note that

$$\frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(e^{-sX})} = \frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(e^{-sY_i} e^{-sZ_i})} \ge \frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(e^{-sZ_i})}.$$
(12)

Also,

$$I_i e^{-sX} = I_i e^{-sY_i} e^{-sZ_i} = I_i e^{-sZ_i} - I_i e^{-sZ_i} (1 - e^{-sY_i}) = I_i g - fg,$$

where

$$f = I_i(1 - e^{-sY_i})$$
 and $g = e^{-sZ_i}$.

Now from (5), I_i and Z_i are independent, so $\mathbb{E}(I_ig) = \mathbb{E}(I_i)\mathbb{E}(g)$. We may write f in the form $f = v_0 + \sum_j (v_j - v_{j-1})J_j$, where $0 \leq v_0 < v_1 < \cdots$ are the distinct values taken by f, and each J_j is the indicator function of the event $B_j = \{f \geq v_j\}$. Note that any such B_j may be expressed as $\bigcup_{\alpha \in \mathcal{J}} \bigcap_{i \in \alpha} A_i$ for some set \mathcal{J} of subsets of $\{1, 2, \ldots, k\}$, so (2) implies $B_j \in \mathcal{I}$. Writing 1 - g

in an analogous form, it follows from (1) that $\mathbb{E}(f(1-g)) \ge \mathbb{E}(f)\mathbb{E}(1-g)$, so $\mathbb{E}(fg) \le \mathbb{E}(f)\mathbb{E}(g)$. Hence,

$$\mathbb{E}(I_i e^{-sX}) = \mathbb{E}(I_i g - fg) \ge \mathbb{E}(I_i)\mathbb{E}(g) - \mathbb{E}(f)\mathbb{E}(g).$$

Using (12) for the first step this gives

$$\frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(e^{-sX})} \ge \frac{\mathbb{E}(I_i e^{-sX})}{\mathbb{E}(g)} \ge \mathbb{E}(I_i) - \mathbb{E}(f) = \mathbb{E}(I_i - f) = \mathbb{E}(I_i e^{-sY_i}).$$

This is exactly (11), and the rest of the proof in [7] is unaltered.

Remark 1. We have stated two of the best-known and cleanest forms of the inequalities in Theorem 1. Let us note that other forms also hold in the present more general context. For example, the inequality (7) leads to the bound

$$\mathbb{P}(X=0) \leqslant \prod_{i=1}^{k} (1 - \mathbb{P}(A_i)) \exp\left(\frac{1}{1-\varepsilon}\frac{\Delta}{2}\right),$$
(13)

where $\varepsilon = \max_i \mathbb{P}(A_i)$. This bound was given by Boppana and Spencer [2] (for $\varepsilon = 1/2$); see also [1, 7].

Furthermore, (11) is the only step in the proof of Lemma 1 of [6] that requires any assumptions about the A_i . Hence this result, which is slightly stronger than (3), also holds in the present setting, giving (in our notation)

$$\log \mathbb{P}(X=0) \leqslant -\sum_{i} \mathbb{E}\left(\frac{I_{i}}{I_{i} + \sum_{j \sim i} I_{j}}\right).$$
(14)

Remark 2. A key feature of the various Janson inequalities is that when Δ is small, then the bounds are close to best possible, since $\mathbb{P}(X = 0) \ge \prod_i (1 - \mathbb{P}(A_i))$. We have not stressed this since it is well known that this lower bound applies to general up-sets A_i , by Harris's Lemma. Similarly, it applies whenever the A_i are in some collection \mathcal{I} of events satisfying (1) and (2).

Remark 3. The bounds in (4) can be extended to the weighted case $X = \sum_i c_i I_i$ with positive c_i , studied, e.g., in [8]: we obtain

$$\mathbb{P}(X \leqslant \mu - t) \leqslant \exp\left(-\frac{\phi(-t/\mu)\mu^2}{\overline{\Delta}}\right) \leqslant \exp\left(-\frac{t^2}{2\overline{\Delta}}\right),\tag{15}$$

where $\mu = \mathbb{E}(X)$ and

$$\overline{\Delta} = \sum_{i} c_i^2 \mathbb{P}(A_i) + \sum_{i} \sum_{j \sim i} c_i c_j \mathbb{P}(A_i \cap A_j) = \sum_{i} \mathbb{E}(J_i^2) + \sum_{i} \sum_{j \sim i} \mathbb{E}(J_i J_j),$$

for $J_{\ell} = c_{\ell}I_{\ell}$. (In applications, it may be convenient to note that $\sum_{i} \mathbb{E}(J_{i}^{2}) \leq C\mu$ when $\max_{i} c_{i} \leq C$.) The proof of (15) is a straightforward modification of that of Theorem 2.14 in [7]. The key inequality is again (11), now with I_{ℓ} replaced by J_{ℓ} in the definitions of X, Y_{i} and Z_{i} . The proof above carries over since all c_{i} are positive. Finally, in this setting (14) also holds, with I_{ℓ} replaced by J_{ℓ} .

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