# The Marčenko-Pastur law for sparse random bipartite biregular graphs 

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#### Abstract

We prove that the empirical spectral distribution of a $\left(d_{L}, d_{R}\right)$-biregular, bipartite random graph, under certain conditions, converges to a symmetrization of the MarčenkoPastur distribution of random matrix theory. This convergence is not only global (on fixed-length intervals) but also local (on intervals of increasingly smaller length). Our method parallels the one used previously by Dumitriu and Pal (2012).


## 1 Motivation

In classical random matrix theory there are two basic types of symmetric ensembles: Wigner matrices and Wishart-like ones. There is a simple parallel between them, roughly expressed in the following way. Given a non-symmetric real matrix $G$, there are two natural ways to construct from it a symmetric matrix: if $G$ is square, one way is to consider its symmetric part $A=\frac{G+G^{T}}{2}$; the other way works for rectangular matrices, too, and consists of multiplying it by its transpose: $W=G G^{T}$. If one starts with a random matrix $G$ with i.i.d. entries of norm 0 and variance 1 , if $G$ is square, the first symmetrization yields a Wigner matrix; the second symmetrization yields a Wishart-like matrix (it is Wishart, more precisely central Wishart, if $G$ consists of standard normal variables; we call it Wishart-like otherwise).

The spectra and eigenvectors of Wigner and Wishart-like matrices have been shown to exhibit universal behavior: many of their eigenstatistics have limiting distributions which are independent of the entry distribution, modulo certain technical conditions (in addition to being mean 0 , variance 1). These results, along with successive weakenings of the technical conditions, are the subject of a recent set of breakthrough papers TV11, TV10, ESY09b, ESY09a, $\mathrm{EPR}^{+}$10, ERSY10, ERS ${ }^{+} 10$.

A natural question for the discrete probability community is whether this universal behavior extends to adjacency matrices of random graphs; we are specifically interested in the

[^0]case of random regular or semi-regular graphs. Such graphs are known to have very interesting properties: they are good expanders, some classes are quasi-Ramanujan, they have wide spectral gaps and as such they mix rapidly, and they are of interest in computer science and engineering and in coding theory.

Ordinary random regular graphs (or $d$-regular graphs), where every vertex has the same degree (which grows as a function of the number of vertices), have been recently investigated in DP12, DJPP13, Joh14, JP14, TVW13, BL13; we aim to extend some of the results to bipartite, biregular random graphs, where the two sets of vertices have the property that all vertices in the same set have the same degree (also growing with the total number of vertices).

The question of whether the spectra of random $d$-regular graphs have the same behavior as the spectra of Wigner matrices is non-trivial in nature, since the adjacency matrices of regular graphs have strong dependencies, namely, all rows and columns add to the same number - their common degree. As such, they are not Wigner; in fact, for $d$ fixed, McKay [McK81] showed that the scaled empirical distribution function (or ESD; defined below) of random $d$-regular graphs on $n$ vertices converges in probability (and almost surely, if the random graphs are defined on the same probability space) as $n \rightarrow \infty$ to the Kesten-McKay distribution, which has density

$$
f_{d}(x)= \begin{cases}\frac{d \sqrt{4(d-1)-x^{2}}}{2 \pi\left(d^{2}-x^{2}\right)} & \text { if }|x| \leq 2 \sqrt{d-1}, \\ 0 & \text { otherwise }\end{cases}
$$

This differs from the Wigner matrix case, where the scaled ESD converges in probability to the semicircle law, which has density

$$
f_{s}(x)= \begin{cases}\frac{\sqrt{4-x^{2}}}{2 \pi} & \text { if }|x| \leq 2 \\ 0 & \text { otherwise }\end{cases}
$$

When $d$ is allowed to grow with $n$, the scaled empirical spectral distribution of the random $d$-regular graph does converge in probability to the semicircle law (see [DP12] and [TV13]). Thus, even if for $d$ fixed they are rather different, the spectra of $d$-regular graphs with $d$ and $n$ growing to infinity are similar to the spectra of large Wigner matrices.

Motivated by a question asked by Babak Hassibi, we study here the spectra of bipartite, biregular random graphs. We find that their spectra have similar behavior to the spectra of Wishart-like matrices; at first glance, this may appear suprising, but further examination reveals linear algebraic reasons why this should be so. Our notation and main results are presented in Section 2. In Sections 3 and 4, we prove global and local convergence, respectively, of the ESD to its limiting measure. The appendix contains the proofs of some statements on the distribution of cycles in biregular bipartite graphs, used in Sections 3 and 4 to show that our graphs are locally well approximated by trees.

## 2 Preliminaries and statements of results

We assume that graphs do not have loops or parallel edges. The adjacency matrix of a graph $G$ is defined as

$$
A(i, j)=\left\{\begin{array}{lr}
1, & \text { if } i \sim j \\
0, & \text { otherwise }
\end{array}\right.
$$

Note that $A$ is symmetric, and therefore all of its eigenvalues are real.

Bipartite graphs are graphs composed of two sets $L$ and $R$ of vertices, with edges only between vertices from $L$ and vertices from $R$. With proper labeling, their adjacency matrices have the special form

$$
B=\left[\begin{array}{cc}
0 & X \\
X^{T} & 0
\end{array}\right]
$$

where the matrix $X$ is defined by the edges between the two classes of vertices. Note that if $L$ and $R$ have sizes $m$ and $n$, respectively, with $m \leq n$, then $X$ is an $m \times n$ matrix of 0 s and 1 s . It is a simple linear algebra result that the non-zero eigenvalues of $B$ come in pairs $(-\lambda, \lambda)$, with $\lambda \geq 0$ an eigenvalue of $X X^{T}$, and that $B$ has (at least) $n-m$ eigenvalues equal to 0 .

If $G$ is a bipartite graph with vertex classes $L$ and $R$, then we say it is $\left(d_{L}, d_{R}\right)$-biregular if all the vertices in $L$ have degree $d_{L}$, and all the vertices in $R$ have degree $d_{R}$. For simplicity, we will always assume that $d_{R} \geq d_{L}$ (and therefore that $|L| \geq|R|$ ).

By a random $\left(d_{L}, d_{R}\right)$-biregular bipartite graph with $(m+n)$ vertices we mean a graph selected uniformly from the space of all $\left(d_{L}, d_{R}\right)$-biregular bipartite graphs with $|L|=m$ and $|R|=n$.

We define the empirical spectral distribution or ESD of a symmetric $n \times n$ matrix $M$ to be the probability measure $\mu_{n}$ on the real numbers given by

$$
\mu_{n}=\frac{1}{n} \sum_{i=1} \delta_{\lambda_{i}}
$$

where $\delta_{x}$ is the point mass at $x$ and $\lambda_{1}, \ldots, \lambda_{n}$ are the eigenvalues of $M$. Note that if $M$ is random, $\mu$ is a random probability measure.

We say that a sequence $\mu_{1}, \mu_{2}, \ldots$ of random probability measures on the real numbers converges almost surely to a deterministic probability measure $\mu$ if as $n \rightarrow \infty$,

$$
\int f d \mu_{n} \rightarrow \int f d \mu \text { a.s. }
$$

for all bounded continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$. This is equivalent to the slightly different statement that with probability one, $\mu_{n}$ converges weakly to $\mu$.

Following combinatorialists' conventions, we will often refer to a random graph $G$ when we really mean a sequence of random graphs with an increasing number of vertices (depending on $m$ and $n)$. We assume that all of these random graphs are defined on the same probability space, but we make no assumptions about their joint distribution; all of our results hold for any arbitrary joint distribution, so long as the marginal distributions are as described. Many variables that we mention implicitly depend on $n$, and asymptotic expressions $O(\cdot)$ or $o(\cdot)$ reflect behavior as $m, n \rightarrow \infty$. We will occasionally use the notation $O_{A}(\cdot)$ to indicate that the constant in the big-O expression depends on some other constant $A$.

It is a standard result in random matrix theory that if $X$ is an $m \times n$ random matrix whose entries are i.i.d. real random variables with mean zero and variance one, and $m / n$ converges to a finite limit, then the ESD of $\frac{1}{n} X^{T} X$ converges to the Marčenko-Pastur law (see [BS10], for example). We show here an analogous result:

Theorem 1. Let $G$ be a random $\left(d_{L}, d_{R}\right)$-biregular bipartite graph on $m+n$ vertices, with
the following conditions on the growth of $d_{L}$ and $d_{R}$ :

$$
\begin{align*}
& \lim _{n \rightarrow \infty} d_{R}=\infty,  \tag{1}\\
& \text { for any fixed } \epsilon>0, \quad d_{R}=o\left(n^{\epsilon}\right) \text {, }  \tag{2}\\
& \frac{d_{R}}{d_{L}} \rightarrow y \geq 1 . \tag{3}
\end{align*}
$$

Let $A=\left(\begin{array}{cc}0 & X \\ X^{T} & 0\end{array}\right)$ be the adjacency matrix of $G$ (under proper labeling). Then as $n \rightarrow \infty$, the ESD of $\frac{1}{d_{R}} X^{T} X$ converges almost surely to the Marčenko-Pastur law with ratio $y^{-1}$. This distribution is supported on $\left[a^{2}, b^{2}\right]$ and is given on that interval by the density

$$
p(x)=\frac{y}{2 \pi x} \sqrt{\left(b^{2}-x\right)\left(x-a^{2}\right)},
$$

where $a=1-y^{-1 / 2}$ and $b=1+y^{-1 / 2}$.
Theorem 1 agrees with the results of [MS03, in which Mizuno and Sato derive the limiting distribution of the eigenvalues of a sequence of deterministic biregular graphs with girths growing to infinity. They do so using the Ihara zeta function, and they express their result in a different form from ours, which turns out to be equivalent. It should be noted that our result is much stronger than theirs; as we will see, most bipartite biregular graphs have small girth, but this does not affect the convergence of the ESD.

As mentioned above, the ESD of $d_{R}^{-1} X^{T} X$ is the distribution of the squares of the nontrivial eigenvalues of $d_{R}^{-1 / 2} A$. We can thus find the limiting distribution for the ESD of this matrix as well:
Corollary 2. The ESD of $d_{R}^{-1 / 2} A$ converges almost surely to the distribution $\mu$ supported on $[-b,-a] \cup[a, b]$ and given on that set by the density

$$
\begin{equation*}
\frac{2}{1+y} p\left(x^{2}\right)|x|=\frac{y}{(1+y) \pi|x|} \sqrt{\left(b^{2}-x^{2}\right)\left(x^{2}-a^{2}\right)}, \tag{4}
\end{equation*}
$$

along with a point mass of $\frac{y-1}{y+1}$ at 0 .
It is also known that when $d \rightarrow \infty$, the ESD of random $d$-regular graphs converges to the semicircle law on short scales (see DP12, [TVW13]). In Section [4 we prove this for the biregular case, under slightly different conditions on the growth of $d_{R}$ :

Theorem 3. Let $G$ be a random $\left(d_{L}, d_{R}\right)$-biregular bipartite graph on $m+n$ vertices satisfying (11) -(3), as well as the more stringent condition $d_{R}=\exp (o(1) \sqrt{\log n})$. Fix $\epsilon>0$. Let $A$ be the adjacency matrix of $G$ and $\mu_{n}$ be the ESD of $\left(d_{R}-1\right)^{-1 / 2} A$, and let $\mu$ be the limiting ESD of Corollary 圂. There exists a constant $C_{\epsilon}$ such that for all sufficiently large $n$ and $\delta>0$, for any interval $I \subseteq \mathbb{R}$ avoiding $[-\epsilon, \epsilon]$ and with length $|I| \geq \max (2 \eta, \eta /(-\delta \log \delta))$, it holds that

$$
\left|\mu_{n}(I)-\mu(I)\right|<\delta C_{\epsilon}|I|
$$

with probability $1-o(1 / n)$. The quantity $\eta$, which gives the minimum length of an interval $I$ that we consider, is given by the following series of definitions:

$$
\begin{aligned}
& a=\min \left(\frac{\log n}{9\left(\log d_{R}\right)^{2}}, d_{R}\right), \\
& r=e^{1 / a}, \\
& \eta=r^{1 / 2}-r^{-1 / 2} .
\end{aligned}
$$

This theorem is obscured by the technicalities in its statement, so we give some discussion of its meaning. Corollary 2 only gives information on $\mu_{n}(I)$ for fixed $|I|$. Theorem 3, on the other hand, allows $|I|$ to shrink as $\eta$ does. We note that $\eta \sim 1 / a$.

We restricted our intervals $I$ away from 0 to avoid complications caused by the point mass that $\mu$ has when $d_{R} / d_{L} \rightarrow y>1$. Since the support of $\mu$ except for this mass is bounded away from 0 , this restriction costs us nothing.

To prove our results, we use the moment method along with Stieltjes transforms, combined with a careful examination of the local structure of the bipartite, biregular graph. Along the way, we need to adapt some of the results proved by McKay, Wormald, and Wysocka MWW04 for random $d$-regular graphs to our $\left(d_{L}, d_{R}\right)$ biregular random graphs. We give these results below. The method used follows [MWW04 very closely, which is why we relegate the proofs to the appendix.

Let $G$ be a random $\left(d_{L}, d_{R}\right)$-biregular bipartite graph on $m+n$ vertices. Assume $d_{L} \leq d_{R}$, and let $\alpha=d_{R} / d_{L}$. As always, all of these variables depend on $n$, and any expressions $O(\cdot)$ or $o(\cdot)$ reflect behavior as $n \rightarrow \infty$. We assume that $\alpha$ converges to a finite value as $n \rightarrow \infty$ (this assumption is also necessary in the case of Wishart matrices). Here and throughout the paper, "cycle" always refers to a simple cycle, with no repeated vertices. Let $X_{r}$ denote the number of cycles of length $2 r$ in $G$. (Note that as $G$ is bipartite, its cycles all have even length.)

Proposition 4. Let

$$
\mu_{r}=\frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r}}{2 r}
$$

If $d_{R}=o(n), r=O(\log n)$, and $r d_{R}=o(n)$, then

$$
\begin{aligned}
\mathbf{E}\left[X_{r}\right] & =\mu_{r}\left(1+O\left(\frac{r\left(r+d_{R}\right)}{n}\right)\right) \\
\operatorname{Var}\left[X_{r}\right] & =\mu_{r}\left(1+O\left(\frac{d_{R}^{2 r}\left(r \alpha^{2 r-1}+\alpha^{-r} d_{R}\right)}{n}\right)\right)
\end{aligned}
$$

In Section 3, we use this proposition to show that with high probability, our biregular bipartite graph $G$ is locally well approximated by a tree. This allows us to approximate the traces of the adjacency matrix of $G$, thus computing the moments of its ESD. Finally, we refine these results in Section 4 to prove local convergence on vanishing-length intervals. To prove this theorem, we give estimates on the rate of convergence of the Stieltjes transform of the ESD, the same approach used in DP12 and TV11.

## 3 Global convergence to the Marčenko-Pastur law

To find the limiting ESD of a biregular bipartite graph $G$, we will first show that in a sense that we will make precise, most neighborhoods in $G$ have no cycles and are trees. This will allow us to estimate the traces of the adjacency matrix of $G$, and we will find the limit of these as $G$ grows with a combinatorial argument.

For this entire section, let $G$ be a random $\left(d_{L}, d_{R}\right)$-biregular bipartite graph on $n+m$ vertices, and assume that conditions (1)-(3) on the growth of $d_{L}$ and $d_{R}$ hold. As before, let $\alpha=\frac{d_{R}}{d_{L}}$.

We make precise the property of $G$ being locally a tree in the following lemma:

Lemma 5. Let $r$ be fixed, and let $\tau$ be the set of vertices in $G$ whose $r$-neighborhoods contain no cycles. Then, if $d_{R}$ satisfies (11) and (2),

$$
\mathbf{P}\left[1-\frac{|\tau|}{n+m}>n^{-1 / 4}\right]=o\left(n^{-5 / 4}\right)
$$

Proof. This is the same statement proven in [DP12] for regular graphs, and using Proposition 4 we can prove it in the same way. If a vertex is not in $\tau$, then for some $s \leq r$ there exists a $2 s$-cycle within $r-s$ of the vertex. The size of all $(r-s)$-neighborhoods of $2 s$-cycles hence serves as a bound on the number of "bad" vertices. For any given $2 s$-cycle, the size of its $(r-s)$-neighborhood is bounded by $2 s\left(d_{R}-1\right)^{r-s}$. If we define

$$
N_{r}^{*}=\sum_{s=2}^{r} 2 s\left(d_{R}-1\right)^{r-s} X_{s},
$$

then this gives us the bound $n+m-|\tau| \leq N_{r}^{*}$.
Now we compute $\mathbf{E}\left[N_{r}^{*}\right]$ and $\operatorname{Var}\left[N_{r}^{*}\right]$. Using our expression for $\mathbf{E}\left[X_{r}\right]$ from Proposition 4 ,

$$
\begin{aligned}
\mathbf{E}\left[N_{r}^{*}\right] & =\sum_{s=2}^{r} 2 s\left(d_{R}-1\right)^{r-s}\left(d_{L}-1\right)^{s}\left(d_{R}-1\right)^{s} O(1) \\
& =\left(d_{R}-1\right)^{r} \sum_{s=2}^{r} O\left(\left(d_{L}-1\right)^{s}\right) \\
& =O\left(\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r}\right)=O\left(d_{R}^{2 r}\right)
\end{aligned}
$$

To compute the variance, first notice that (2) implies that $\operatorname{Var}\left[X_{s}\right]=\mu_{s}(1+o(1))$. By Cauchy-Schwarz,

$$
\begin{aligned}
\operatorname{Var}\left[N_{r}^{*}\right] & \leq r \sum_{s=2}^{r} 4 s^{2}\left(d_{R}-1\right)^{2 r-2 s} \operatorname{Var}\left[X_{s}\right] \\
& \leq r \sum_{s=2}^{r} 4 s^{2}\left(d_{R}-1\right)^{2 r-2 s} \mu_{s}(1+o(1)) \\
& =r\left(d_{R}-1\right)^{2 r} \sum_{s=2}^{r} s \frac{\left(d_{L}-1\right)^{s}\left(d_{R}-1\right)^{s}}{\left(d_{R}-1\right)^{2 s}}(1+o(1)) \\
& \leq r\left(d_{R}-1\right)^{2 r}(1+o(1)) \sum_{s=2}^{r} s \\
& =O\left(d_{R}^{2 r}\right)
\end{aligned}
$$

The rest of the lemma follows from Markov's inequality:

$$
\begin{aligned}
\mathbf{P}\left[1-\frac{|\tau|}{n+m}>n^{-1 / 4}\right] & =\mathbf{P}\left[n+m-|\tau|>(1+\alpha) n^{3 / 4}\right] \\
& \leq \mathbf{P}\left[N_{r}^{*}>(1+\alpha) n^{3 / 4}\right] \\
& \leq \frac{\operatorname{Var}\left[N_{r}^{*}\right]+\mathbf{E}\left[N_{r}^{*}\right]^{2}}{(1+\alpha)^{2} n^{3 / 2}} \\
& =O\left(\frac{d_{R}^{4 r}}{n^{3 / 2}}\right)=o\left(n^{-5 / 4}\right)
\end{aligned}
$$

This result shows that there are few "bad" vertices. It easily follows that this is true within the left and the right vertex classes of $G$ as well.

Corollary 6. Let $\tau_{L}$ and $\tau_{R}$ be the number of vertices in the left and right classes of $G$, respectively, with acyclic r-neighborhoods. Then

$$
\begin{aligned}
& \mathbf{P}\left[\frac{m-\left|\tau_{L}\right|}{n+m}>n^{-1 / 4}\right]=o\left(n^{-5 / 4}\right), \\
& \mathbf{P}\left[\frac{n-\left|\tau_{R}\right|}{n+m}>n^{-1 / 4}\right]=o\left(n^{-5 / 4}\right) .
\end{aligned}
$$

Proof. Note that $1-\frac{|\tau|}{n+m}=\frac{m-\left|\tau_{L}\right|}{n+m}+\frac{n-\left|\tau_{R}\right|}{n+m}$, so $1-\frac{|\tau|}{n+m}>c$ whenever $\frac{m-\left|\tau_{L}\right|}{n+m}>c$ or $\frac{n-\left|\tau_{R}\right|}{n+m}>c$.

Let $\beta_{k}\left(r, \sigma^{2}\right)$ be the $k$ th moment of the Marčenko-Pastur law with ratio $r$ and scaling factor $\sigma^{2}$ as defined in BS10.
Proposition 7. Let $A$ be the adjacency matrix of $G$, and let $\mu_{n}$ be the ESD of $d_{R}^{-1 / 2} A$. Recalling that $y=\lim _{n \rightarrow \infty} \alpha$,

$$
\begin{aligned}
& \int x^{2 k+1} d \mu_{n}(x) \rightarrow 0 \text { a.s. } \\
& \int x^{2 k} d \mu_{n}(x) \rightarrow \frac{2}{1+y} \beta_{k}\left(y^{-1}, 1\right) \text { a.s. }
\end{aligned}
$$

as $n \rightarrow \infty$.
Proof. Consider the infinite $\left(d_{L}, d_{R}\right)$-biregular tree. Let $B_{r}$ denote the number of closed walks of length $r$ on this tree, starting from some fixed vertex of degree $d_{L}$, and let $C_{r}$ denote the number of closed walks of length $r$ starting from a vertex of degree $d_{R}$. Note that as $d_{L}$ and $d_{R}$ depend on $n$, so do $B_{r}$ and $C_{r}$.

First, we formulate the $r$ th moment of $\mu_{n}$ in terms of $B_{r}$ and $C_{r}$.

$$
\int x^{r} d \mu_{n}(x)=\frac{d_{R}^{-r / 2}}{n+m} \sum_{v \in V(G)} A^{r}(v, v)
$$

The quantity $A^{r}(v, v)$ is the number of closed walks of length $r$ from $v$ in $G$. With the same definitions of $\tau, \tau_{L}$, and $\tau_{R}$ as in Lemma 5 and Corollary 6, this is equal to $B_{r}$ when $v \in \tau_{L}$ and $C_{r}$ when $v \in \tau_{R}$. For $v \notin \tau$, we can use the bound $A^{r}(v, v) \leq d_{R}^{r}$. Hence we can bound the $r$ th moment of $\mu_{n}$ by

$$
\begin{aligned}
& \frac{d_{R}^{-r / 2}}{n+m}\left(\left|\tau_{L}\right| B_{r}+\left|\tau_{R}\right| C_{r}\right) \leq \int x^{r} d \mu_{n}(x) \\
& \quad \leq \frac{d_{R}^{-r / 2}}{n+m}\left(m B_{r}+n C_{r}+(n+m-|\tau|) d_{R}^{r}\right)
\end{aligned}
$$

Define

$$
\begin{aligned}
a_{n} & =d_{R}^{-r / 2}\left(\left(\frac{m}{n+m}-n^{-1 / 4}\right) B_{r}+\left(\frac{n}{n+m}-n^{-1 / 4}\right) C_{r}\right), \\
b_{n} & =d_{R}^{-r / 2}\left(\frac{m B_{r}+n C_{r}}{n+m}+\frac{n^{-1 / 4}}{n+m} d_{R}^{r}\right) .
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathbf{P}\left[a_{n} \leq \int x^{r} d \mu_{n}(x)\right] & \geq \mathbf{P}\left[\frac{m-\left|\tau_{L}\right|}{n+m} \leq n^{-1 / 4} \text { and } \frac{n-\left|\tau_{R}\right|}{n+m} \leq n^{-1 / 4}\right] \\
& =1-o\left(n^{-5 / 4}\right)
\end{aligned}
$$

and in the same way, $\mathbf{P}\left[\int x^{r} d \mu_{n}(x) \leq b_{n}\right] \geq 1-o\left(n^{-5 / 4}\right)$. By the Borel-Cantelli lemma, it holds almost surely that $a_{n} \leq \int x^{r} d \mu_{n}(x) \leq b_{n}$ for all but finitely many $n$. If we show that $a_{n}$ and $b_{n}$ converge to a common limit (which we will do next), it will follow that $\int x^{r} d \mu_{n}(x)$ converges to this limit almost surely.

To find the limits of $a_{n}$ and $b_{n}$ as $n \rightarrow \infty$, we first note that $n^{-1 / 4} d_{R}^{r} \rightarrow 0$ by (2). Since $B_{r}, C_{r} \leq d_{R}^{r}$, this also implies that $n^{-1 / 4} B_{r} \rightarrow 0$ and $n^{-1 / 4} C_{r} \rightarrow 0$. So, it suffices to show that

$$
\begin{align*}
& \frac{d_{R}^{-(2 k+1) / 2}}{n+m}\left(m B_{2 k+1}+n C_{2 k+1}\right) \rightarrow 0  \tag{5}\\
& \frac{d_{R}^{-k}}{n+m}\left(m B_{2 k}+n C_{2 k}\right) \rightarrow \frac{2}{1+\alpha} \beta_{k}\left(y^{-1}, 1\right) \tag{6}
\end{align*}
$$

Equation (5) is trivial, since $B_{2 k+1}=C_{2 k+1}=0$. To prove (6), we introduce the Narayana numbers (see [Sta99, p. 237]), defined as

$$
N(k, a)=\frac{1}{a+1}\binom{k}{a}\binom{k-1}{a} .
$$

The moments of the Marčenko-Pastur distribution can be given in terms of these numbers [BS10, Lemma 3.1]:

$$
\begin{equation*}
\beta_{k}\left(y^{-1}, 1\right)=\sum_{r=0}^{k-1} y^{-r} N(k, r) \tag{7}
\end{equation*}
$$

We will give a combinatorial argument to relate the closed walks on the tree to the Narayana numbers. We mention that another approach to proving (6) is to calculate $B_{r}$ and $C_{r}$ using the spectral density of the infinite $\left(d_{L}, d_{R}\right)$-biregular tree, as calculated in [GM88, (5.7)].

A Motzkin path of length $2 k$ is a lattice path that starts at $(0,0)$, ends at $(2 k, 0)$, and stays above the $x$-axis; each step can be a rise ( $\nearrow$ ), a fall $(\searrow$ ), or a level step $(\rightarrow)$. An alternating Motzkin path is a Motzkin path that rises only on even steps and that falls only on odd steps. See Figure 1 for an example of the five alternating Motzkin paths of length 6.

The alternating Motzkin paths have the following connection to the Narayana numbers:
Lemma 8 (Lemma 6.1.7, Dum03). The number of alternating Motzkin paths of length $2 k$ with exactly a rises is $N(k, a)$.

We relate the Narayana numbers to the walks on a tree by the following two lemmas. A ballot sequence of length $2 k$ is a sequence $x_{1}, \ldots, x_{2 k}$ of 1 's and -1 's such that all partial sums $x_{1}+\cdots+x_{j}$ are nonnegative.

Lemma 9. The number of ballot sequences of length $2 k$ with a 1's at even locations and $k-a$ 1 's at odd locations is $N(k, a)$.


Figure 1: The alternating Motzkin paths of length 6.


```
+1
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Figure 2: An alternating Motzkin path and its corresponding ballot sequence.

Proof. We give a bijection between alternating Motzkin paths and ballot sequences. Encode the alternating Motzkin path as $p_{1}, \ldots, p_{2 k}$, where $p_{i}$ is 1,0 , or -1 depending on whether the $i$ th step is rising, level, or falling. Define a sequence by $x_{i}=2 p_{i}+(-1)^{i-1}$. We will confirm that this is a ballot sequence: each $x_{i}$ is either 1 or -1 ; for any $j$,

$$
x_{1}+\cdots+x_{i}=2\left(p_{1}+\cdots+p_{i}\right)+\sum_{i=1}^{j}(-1)^{i-1}
$$

and both of these terms are nonnegative; and $x_{1}+\cdots+x_{2 k}=2\left(p_{1}+\cdots+p_{2 k}\right)=0$. So, $x_{1}, \ldots, x_{2 k}$ is a ballot sequence.

To map back from ballot sequences to alternating Motzkin paths, we let $p_{i}=\left(x_{i}-\right.$ $\left.(-1)^{i-1}\right) / 2$. For any even $j$,

$$
p_{1}+\cdots+p_{j}=\frac{1}{2}\left(x_{1}+\cdots+x_{j}\right) \geq 0
$$

For any odd $j$,

$$
p_{1}+\cdots+p_{j}=\frac{1}{2}\left(x_{1}+\cdots+x_{j}-1\right) .
$$

Since $x_{1}+\cdots+x_{j} \geq 1$ when $j$ is odd, this expression is also nonnegative. So, our path stays above the $x$-axis. The other properties of being an alternating Motzkin path are easy to check.

This bijection takes alternating Motzkin paths with $a$ rises to ballot sequences with $a$ 1's at even locations, so the lemma follows from Lemma [8,

## Lemma 10.

$$
\begin{align*}
B_{2 k} & =\sum_{a=0}^{k-1}\left(d_{R}-1\right)^{a} \widetilde{d}_{L}^{k-a} N(k, a),  \tag{8}\\
C_{2 k} & =\sum_{a=0}^{k-1}\left(d_{L}-1\right)^{a} \widetilde{d}_{R}^{k-a} N(k, a), \tag{9}
\end{align*}
$$

for some $\widetilde{d}_{L}$ and $\widetilde{d}_{R}$ satisfying $d_{L}-1 \leq \widetilde{d}_{L} \leq d_{L}$ and $d_{R}-1 \leq \widetilde{d}_{R} \leq d_{R}$.
Proof. Fix some vertex $v$ in the $\left(d_{L}, d_{R}\right)$-biregular tree with degree $d_{L}$ to serve as the root. We will enumerate the closed walks of length $2 k$ starting at $v$. To any such walk we can associate a ballot sequence of length $2 k$, given by putting a 1 at every step of the walk going away from $v$ and a -1 at every step returning toward $v$. We will count the number of closed walks associated to each ballot sequence.

Fix some ballot sequence, and suppose we are constructing a closed walk associated with it. For every 1 in the ballot sequence, our walk must go outward from the root. If the 1 is at an even location, we have $d_{R}-1$ choices for where to move; if it is at an odd location, we have either $d_{L}$ or $d_{L}-1$, depending on whether we are moving from $v$ or from some other vertex. For every -1 in the ballot sequence, our walk must move backward towards the root, and there is no choice to be made. So, given a ballot sequence with a rises on even steps, the number of closed walks from $v$ associated with that ballot sequence is between $\left(d_{R}-1\right)^{a}\left(d_{L}-1\right)^{k-a}$ and $\left(d_{R}-1\right)^{a} d_{L}^{k-a}$. Using Lemma 9 to count the number of ballot sequences with $r$ rises on even steps, we obtain (8). The same proof starting with a vertex $v$ with degree $d_{R}$ gives us (9).

Now we finish the proof of Proposition 7 by computing the limit of

$$
\frac{d_{R}^{-k}}{n+m}\left(m B_{2 k}+n C_{2 k}\right)
$$

as $n \rightarrow \infty$. Recall that all of these variables depend on $n$ except for $k$, which is fixed. By Lemma 10, we can rewrite the above expression as

$$
\frac{m}{n+m} \sum_{r=0}^{k-1} \frac{\left(d_{R}-1\right)^{r} d_{R}^{-k}}{\widetilde{d}_{L}^{r-k}} N(k, r)+\frac{n}{n+m} \sum_{r=0}^{k-1} \frac{\left(d_{L}-1\right)^{r}}{\widetilde{d}_{R}^{r-k} d_{R}^{k}} N(k, r) .
$$

Replacing $m /(n+m)$ and $n /(n+m)$ by $\alpha /(1+\alpha)$ and $1 /(1+\alpha)$ respectively, and taking the limit as $n \rightarrow \infty$ yields

$$
\frac{y}{1+y} \sum_{r=0}^{k-1} y^{r-k} N(k, r)+\frac{1}{1+y} \sum_{r=0}^{k-1} y^{-r} N(k, r) .
$$

Note that this is where we used (1). We can simplify this expression to

$$
\frac{2}{1+y} \sum_{r=0}^{k-1} y^{-r} N(k, r)
$$

This is exactly $\frac{2}{1+y} \beta_{k}\left(y^{-1}, 1\right)$ as given in (7).


Figure 3: This figure depicts the limiting density of the ESD of a random $\left(d_{L}, d_{R}\right)$-biregular bipartite graph for two different values of $\alpha=\frac{d_{R}}{d_{L}}$. The spike at 0 denotes a point mass. The continuous part of the density is given by (4), and the point mass has size $\frac{\alpha-1}{\alpha+1}$. Each of the left and right spikes are scaled down copies of the Marĉenko-Pastur distribution under the transformation $x \mapsto \sqrt{x}$. When $\alpha=1$, the density reduces to that of the semicircle law.

Proof of Theorem [1. Let $\nu_{n}$ be the ESD of $d_{R}^{-1} X^{T} X$. As described in the introduction, the eigenvalues of $d_{R}^{-1 / 2} A$ consist of $\pm \sigma_{i}$ for the singular values $\sigma_{1}, \ldots, \sigma_{n}$ of $d_{R}^{-1 / 2} X$, along with $m-n 0$ 's. It follows that

$$
\begin{aligned}
\int x^{k} d \nu_{n} & =\frac{m+n}{2 n} \int x^{2 k} d \mu_{n} \\
& =\frac{\alpha+1}{2} \int x^{2 k} d \mu_{n} .
\end{aligned}
$$

It follows from Proposition 7 and the convergence of $\alpha$ to $y$ that $\int x^{k} d \nu_{n} \rightarrow \beta_{k}\left(y^{-1}, 1\right)$ a.s. as $n \rightarrow \infty$. Since the moments of $\nu_{n}$ converge almost surely to the moments of the MarčenkoPastur distribution, which is supported on a compact interval, $\nu_{n}$ converges almost surely to this distribution.

To wrap things up, we compute the density of the limiting distribution of $\mu_{n}$.
Proof of Corollary 圆. We only need to show that the moments of the measure given by the density $(2 /(1+y)) p\left(x^{2}\right)|x|$ agree with the limits of the moments of $\mu_{n}$ found in Proposition 7 The odd moments of this measure are 0 by symmetry, and the even moments are easily computed by integrating and substituting $u=x^{2}$.

See Figure 3 for a picture of the limiting distribution for a few values of $\alpha$.

## 4 Convergence on short scales

To show convergence of the ESD on short scales, we will use the method of [DP12, Section 3]. The basic idea is to use the local approximation of our graph as a tree to estimate the graph's Stieltjes transform.

First, we will define some terms and sketch the proof. The Stieltjes transform of a probability measure $\mu$ is the function $s(z)=\int(z-x)^{-1} d \mu(x)$ defined on the complex upper half-plane. The Stieltjes transform of the ESD of an $n \times n$ Hermitian matrix $A$ is then $s(z)=\frac{1}{n} \operatorname{tr} R(z)$, where $R(z)=(A-z I)^{-1}$ is the resolvent of $A$. If one can show that the Stieltjes tranform of the ESD converges, standard arguments from random matrix theory allow one to show that the ESD itself converges, with quantitative estimates on the Stieltjes transform translating into quantitative estimates on the ESD.

To simplify language, we will refer to the resolvent of a graph instead of the resolvent of the adjacency matrix of the graph. Similarly, we use the Stieltjes transform of a graph to mean the Stieltjes transform of the ESD of the adjacency matrix of the graph. Our goal is to show that the Stieltjes transform of a random biregular bipartite graph is close to its limit. We break the proof into the following steps:

1. (Section 4.2) Compute the resolvent matrix of a biregular tree of a given depth $\zeta$.
2. (Section 4.3) Let $v$ be a vertex of a deterministic biregular graph $G$ with no cycles in its $(\zeta+1)$-neighborhood. Show that the $(v, v)$ entry of the resolvent of $G$ is close to the (root, root) entry of the resolvent of a biregular tree of depth $\zeta$.
3. (Section 4.4) Show that nearly all the vertices of a random biregular graph have a large acyclic neighborhood, and use this fact to transfer the estimates of the earlier parts to random graphs, giving us an estimate of the Stieltjes transform.

We finish by invoking a standard argument from TV11, Lemma 64] to deduce Theorem 3 from the estimate on the Stieltjes transform.

Our task is made slightly more difficult by the need to consider two different $\left(d_{L}, d_{R}\right)$ biregular trees: one in which the root has degree $d_{L}$, and one in which the root has degree $d_{R}$.

### 4.1 Preliminaries

We will use the following well-known formula for the inverse of a block matrix.
Proposition 11. Let $A$ and $D$ be $n \times n$ matrices of size $n \times n$ and $m \times m$, respectively, and let $B$ be $n \times m$. Let

$$
M=\left[\begin{array}{cc}
A & B \\
B^{T} & D
\end{array}\right]
$$

Then

$$
M^{-1}=\left[\begin{array}{cc}
A^{-1}+A^{-1} B F^{-1} B^{T} A^{-1} & -A^{-1} B F^{-1} \\
-F^{-1} B^{T} A^{-1} & F^{-1}
\end{array}\right], \quad F=D-B^{T} A^{-1} B
$$

In this section of the paper, we will define the ratio $\alpha$ by $\alpha=\left(d_{R}-1\right) /\left(d_{L}-1\right)$ rather than $d_{R} / d_{L}$. Let $U_{n}(z)$ be the Chebyshev polynomial of the second kind of degree $n$. We define the following shifted Chebyshev polynomial,

$$
q_{n}(z)=\alpha^{-n / 2} U_{n}\left(\frac{\sqrt{\alpha}\left(z^{2}-\alpha^{-1}-1\right)}{2}\right)
$$

This family of polynomials satisfies the recurrence

$$
\begin{align*}
q_{-1}(z) & =0 \\
q_{0}(z) & =1 \\
q_{n}(z) & =\left(z^{2}-\alpha^{-1}-1\right) q_{n-1}(z)-\alpha^{-1} q_{n-2}(z), \quad n \geq 1 \tag{10}
\end{align*}
$$

which follows by applying the recurrence $U_{n}(z)=2 z U_{n-1}(z)-U_{n-2}(z)$.

### 4.2 Resolvents of trees

Our aim is to calculate the resolvents of biregular trees. We start, however, by considering trees in which each vertex has either $d_{L}-1$ or $d_{R}-1$ children; this means that every vertex has degree $d_{L}$ or $d_{R}$ except for the root, which has degree $d_{L}-1$ or $d_{R}-1$.

We define $T_{L}(\zeta)$ to be the tree with depth $\zeta$ where the root has $d_{L}-1$ children, its children each have $d_{R}-1$ children, their children each have $d_{L}-1$ children, and so on. The tree $T_{L}(0)$ is a single vertex. We define $T_{R}(\zeta)$ similarly, but with the root having $d_{R}-1$ children. To determine the adjacency matrices of these trees, we place an ordering on the vertices as follows: If $\zeta=0$, then there is only one vertex and hence one possible labeling. For $\zeta>0$, we will define an ordering inductively. Choose a subtree of the root and list of all its vertices in the order already determined for $\zeta-1$. Then, do this with the remaining subtrees of the root. Finally, put the root last.

Let $H_{L}$ and $H_{R}$ be the adjacency matrices of $T_{L}(\zeta)$ and $T_{R}(\zeta)$, respectively. We define

$$
\begin{aligned}
& \varphi_{L}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} H_{L}-z\right)_{\text {root,root }}^{-1} \\
& \psi_{L}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} H_{L}-z\right)_{\text {root,leaf }}^{-1}
\end{aligned}
$$

We note that $\psi_{L}(\zeta)$ is independent of the particular leaf chosen. We will make use of the recursive structure of the trees to calculate these values.

Lemma 12. (a)

$$
\begin{array}{ll}
\varphi_{L}(2 \zeta)=-\frac{q_{\zeta}(z)+\alpha^{-1} q_{\zeta-1}(z)}{z q_{\zeta}(z)}, & \varphi_{L}(2 \zeta+1)=-\frac{z q_{\zeta}(z)}{q_{\zeta+1}(z)+q_{\zeta}(z)}, \\
\varphi_{R}(2 \zeta)=-\frac{q_{\zeta}(z)+q_{\zeta-1}(z)}{z q_{\zeta}(z)}, & \varphi_{R}(2 \zeta+1)=-\frac{z q_{\zeta}(z)}{q_{\zeta+1}(z)+\alpha^{-1} q_{\zeta}(z)},
\end{array}
$$

(b)

$$
\begin{array}{rlrl}
\psi_{L}(2 \zeta) & =\psi_{R}(2 \zeta)=-\frac{\left(d_{R}-1\right)^{-\zeta}}{z q_{\zeta}(z)} \\
\psi_{L}(2 \zeta+1) & =-\frac{\left(d_{R}-1\right)^{-\zeta-1 / 2}}{q_{\zeta+1}(z)+q_{\zeta}(z)}, & \psi_{R}(2 \zeta+1)=-\frac{\left(d_{R}-1\right)^{-\zeta-1 / 2}}{q_{\zeta+1}(z)+\alpha^{-1} q_{\zeta}(z)}
\end{array}
$$

Proof. We will start by showing that $\varphi_{L}(\zeta)$ and $\varphi_{R}(\zeta)$ satisfy the recurrences

$$
\begin{align*}
& \varphi_{L}(\zeta)=-\left(z+\alpha^{-1} \varphi_{R}(\zeta-1)\right)^{-1}  \tag{11}\\
& \varphi_{R}(\zeta)=-\left(z+\varphi_{L}(\zeta-1)\right)^{-1} \tag{12}
\end{align*}
$$

Consider the tree $T_{L}$ of depth $\zeta$ and let $H_{1}, \ldots, H_{d_{L}-1}$ denote the adjacency matrices of the subtrees of the root. Using the given ordering for the vertices, we have
$\frac{1}{\sqrt{d_{R}-1}} H_{L}(\zeta)-z=\left[\begin{array}{ccccc}\frac{1}{\sqrt{d_{R}-1}} H_{R}(\zeta-1)-z & & & & \\ & \frac{1}{\sqrt{d_{R}-1}} H_{R}(\zeta-1)-z & & & \\ & & \ddots & & \\ & u^{T} & & \frac{1}{\sqrt{d_{R}-1}} H_{R}(\zeta-1)-z & \\ & & & & -z\end{array}\right]$
where $u$ is a column vector representing the children of the root. This vector is $\left(d_{R}-1\right)^{-1 / 2}$ in the root of each of the subtrees of the root and 0 elsewhere. Using Proposition 11 and thinking of the $-z$ in the bottom right corner as a $1 \times 1$ block, we find

$$
\varphi_{L}(\zeta)=\left(-z-\frac{d_{L}-1}{d_{R}-1} \varphi_{R}(\zeta-1)\right)^{-1}
$$

which is (11). The proof for (12) is the same.
Unwinding these recurrences and noting that $\varphi_{L}(0)=\varphi_{R}(0)=-z^{-1}$, we have the following continued fraction representation of $\varphi_{L}(\zeta)$ :

$$
\varphi_{L}(\zeta)=-\frac{1}{z-\frac{\alpha^{-1}}{z-\frac{1}{z-\frac{\cdots}{z-z^{-1}}}}}
$$

Using standard formulas for the evaluation of continued fractions (see [LW08), we find that $\varphi_{L}(\zeta)=\frac{A_{\zeta}}{B_{\zeta}}$, where

$$
\begin{array}{ll}
A_{2 \zeta}=z A_{2 \zeta-1}-A_{2 \zeta-2}, & A_{2 \zeta+1}=z A_{2 \zeta}-\alpha^{-1} A_{2 \zeta-1} \\
B_{2 \zeta}=z B_{2 \zeta-1}-B_{2 \zeta-2}, & B_{2 \zeta+1}=z B_{2 \zeta}-\alpha^{-1} B_{2 \zeta-1}
\end{array}
$$

with the initial conditions

$$
\begin{array}{ll}
A_{0}=-1, & A_{1}=-z, \\
B_{0}=z, & B_{1}=z^{2}-\alpha^{-1}
\end{array}
$$

We can iterate these recurrences as follows:

$$
\begin{aligned}
A_{2 \zeta} & =z\left(z A_{2 \zeta-2}-\alpha^{-1} A_{2 \zeta-3}\right)-A_{2 \zeta-2} \\
& =z\left(z A_{2 \zeta-2}-\frac{\alpha^{-1}}{z}\left(A_{2 \zeta-2}+A_{2 \zeta-4}\right)\right)-A_{2 \zeta-2} \\
& =\left(z^{2}-\alpha^{-1}-1\right) A_{2 \zeta-2}-\alpha^{-1} A_{2 \zeta-4} .
\end{aligned}
$$

Applying this procedure to the $A_{2 \zeta+1}$ and to the $B_{2 \zeta}$ and $B_{2 \zeta+1}$ cases give the same result, yielding

$$
\begin{aligned}
& A_{\zeta}=\left(z^{2}-\alpha^{-1}-1\right) A_{\zeta-1}-\alpha^{-1} A_{\zeta-2} \\
& B_{\zeta}=\left(z^{2}-\alpha^{-1}-1\right) B_{\zeta-1}-\alpha^{-1} B_{\zeta-2}
\end{aligned}
$$

It is easily checked using (10) that

$$
\begin{array}{ll}
A_{2 \zeta}=-\left(q_{\zeta}(z)+\alpha^{-1} q_{\zeta-1}(z)\right), & A_{2 \zeta+1}=-z q_{\zeta}(z) \\
B_{2 \zeta}=z q_{\zeta}(z), & B_{2 \zeta+1}=q_{\zeta+1}(z)+q_{\zeta}(z)
\end{array}
$$

From these expressions and (12), it is straightforward to derive the expressions for $\varphi_{R}(2 \zeta)$ and $\varphi_{R}(2 \zeta+1)$.

To compute $\psi_{L}(2 \zeta)$ and $\psi_{R}(2 \zeta)$, we will first show

$$
\begin{align*}
& \psi_{L}(\zeta)=-\left(d_{R}-1\right)^{-1 / 2} \varphi_{L}(\zeta) \psi_{R}(\zeta-1)  \tag{13}\\
& \psi_{R}(\zeta)=-\left(d_{R}-1\right)^{-1 / 2} \varphi_{R}(\zeta) \psi_{L}(\zeta-1) \tag{14}
\end{align*}
$$

Using Proposition [11, if $A$ is the minor of $\left(d_{R}-1\right)^{-1 / 2} H_{L}-z$ consisting of all but the last row and column, we have

$$
\begin{aligned}
\psi_{L}(\zeta) & =\left(\frac{1}{\sqrt{d_{R}-1}} H_{L}-z\right)_{1, \text { root }}^{-1} \\
& =-\varphi_{L}(\zeta)\left(A^{-1} u\right)_{1} \\
& =-\left(d_{R}-1\right)^{-1 / 2} \varphi_{L}(\zeta) \psi_{R}(\zeta-1)
\end{aligned}
$$

proving (13). Equation (14) is derived in the same way. By combining these, we get $\psi_{L}(\zeta)=$ $\left(d_{R}-1\right)^{-1} \varphi_{L}(\zeta) \varphi_{R}(\zeta-1) \psi_{L}(\zeta-2)$, whence

$$
\begin{aligned}
\psi_{L}(2 \zeta) & =\psi_{L}(0)\left(d_{R}-1\right)^{-\zeta} \prod_{j=1}^{\zeta} \varphi_{L}(2 j) \varphi_{R}(2 j-1) \\
& =-z^{-1}\left(d_{R}-1\right)^{-\zeta} \prod_{j=1}^{\zeta} \frac{q_{j-1}(z)}{q_{j}(z)} \\
& =-\frac{\left(d_{R}-1\right)^{-\zeta}}{z q_{\zeta}(z)}
\end{aligned}
$$

The proof of the expression for $\psi_{R}(2 \zeta)$ is identical, and the expressions for $\psi_{L}(2 \zeta+1)$ and $\psi_{R}(2 \zeta+1)$ follow immediately by applying (13) and (14).

We now turn from these almost regular trees to the real thing. Let $\widetilde{T}_{L}(\zeta)$ be the $\left(d_{L}, d_{R}\right)$ biregular tree of depth $\zeta$ whose root has degree $d_{L}$, and let $\widetilde{T}_{R}(\zeta)$ be the $\left(d_{L}, d_{R}\right)$-biregular tree of depth $\zeta$ whose root has degree $d_{R}$. Let $\widetilde{H}_{L}$ and $\widetilde{H}_{R}$ be the adjacency matrices of $\widetilde{T}_{L}(\zeta)$ and $\widetilde{T}_{R}(\zeta)$, respectively, with vertices ordered as with $H_{L}$ and $H_{R}$. We define

$$
\begin{array}{ll}
\widetilde{\varphi}_{L}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} \widetilde{H}_{L}-z\right)_{\text {root,root }}^{-1}, & \widetilde{\varphi}_{R}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} \widetilde{H}_{R}-z\right)_{\text {root,root }}^{-1}, \\
\widetilde{\psi}_{L}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} \widetilde{H}_{L}-z\right)_{\text {root,leaf }}^{-1}, & \widetilde{\psi}_{R}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} \widetilde{H}_{R}-z\right)_{\text {root,leaf }}^{-1} .
\end{array}
$$

## Lemma 13.

$$
\begin{array}{ll}
\widetilde{\varphi}_{L}(2 \zeta) & =-\frac{q_{\zeta}(z)+\alpha^{-1} q_{\zeta-1}(z)}{z\left(q_{\zeta}(z)-\frac{1}{d_{R}-1} q_{\zeta-1}(z)\right)}, \\
\widetilde{\psi}_{L}(2 \zeta) & =\widetilde{\psi}_{R}(2 \zeta)=-\frac{\left(d_{R}-1\right)^{-\zeta}}{z\left(q_{\zeta}(z)-\frac{1}{d_{R}-1} q_{\zeta-1}(z)\right)} .
\end{array}
$$

Proof. Because the root of $\widetilde{H_{L}}$ has $d_{L}$ children instead of $d_{L}-1$, the methods of Lemma 12 give

$$
\widetilde{\varphi}_{L}(\zeta)=\left(-z-\frac{d_{L}}{d_{R}-1} \varphi_{R}(\zeta-1)\right)^{-1}
$$

Substituting in the value for $\varphi_{R}(\zeta-1)$ from Lemma 12 yields the desired expression. The expression for $\widetilde{\varphi}_{R}(2 \zeta)$ is derived in the same way.

The same procedure shows that

$$
\tilde{\psi}_{L}(2 \zeta)=-\left(d_{R}-1\right)^{-1 / 2} \widetilde{\varphi}_{L}(2 \zeta) \psi_{R}(2 \zeta-1),
$$

and substituting the value from Lemma 12 yields the desired expression.
We will now bound the rate of convergence of some of these functions to their limits as $\zeta \rightarrow \infty$. First, define the complex function $F(z)=z+\sqrt{z^{2}-1}$, with branch cut $[0, \infty)$ for the square root. Let $w(z)=F\left(\frac{1}{2} \sqrt{\alpha}\left(z^{2}-\alpha^{-1}-1\right)\right)$ and $r(z)=|w(z)|$. We will refer to $w(z)$ and $r(z)$ as simply $w$ and $r$. Note that $r>1$ for all $\Im(z)>0$. Using a well-known expression for the Chebyshev function $U_{n}(z)$ (see (MH03), we can expand $q_{\zeta}(z)$ as

$$
\begin{equation*}
q_{\zeta}(z)=\alpha^{-\zeta / 2} \frac{w^{\zeta+1}-w^{-\zeta-1}}{w-w^{-1}} \tag{15}
\end{equation*}
$$

As we will see, the limits of $\varphi_{L}(\zeta)$ and $\varphi_{R}(\zeta)$ as $\zeta \rightarrow \infty$ are given by the following functions, defined on the upper half-plane:

$$
s_{L}(z)=-\frac{1}{z}-\frac{\alpha^{-1 / 2} w^{-1}}{z}, \quad \quad s_{R}(z)=-\frac{1}{z}-\frac{\alpha^{1 / 2} w^{-1}}{z},
$$

with branch cut $[0, \infty)$ for the square root. We define

$$
s(z)=\frac{\alpha s_{L}(z)+s_{R}(z)}{1+\alpha}=\frac{\alpha}{1+\alpha}\left(-z+\sqrt{\left(z-\frac{1-\alpha}{\alpha z}\right)^{2}-4},\right)
$$

again with branch cut $[0, \infty)$ for the square root, and we note that $s(z)$ is the Stieltjes transform of the limiting ESD $\mu$ of Corollary 2,

We bound the convergence in terms of $r=r(z)$ and $\zeta$ :

## Lemma 14.

$$
\begin{aligned}
\left|\varphi_{L}(2 \zeta)-s_{L}(z)\right| & \leq \frac{2 \alpha^{-1 / 2} r^{-2 \zeta}}{|z|\left(1-r^{-2 \zeta-2}\right)} \\
\left|\varphi_{R}(2 \zeta)-s_{R}(z)\right| & \leq \frac{2 \alpha^{1 / 2} r^{-2 \zeta}}{|z|\left(1-r^{-2 \zeta-2}\right)} \\
\left|\psi_{L}(2 \zeta)\right|,\left|\psi_{R}(2 \zeta)\right| & \leq \frac{2 \alpha^{\zeta / 2}\left(d_{R}-1\right)^{-\zeta} r^{-\zeta}}{|z|\left(1-r^{-2 \zeta-1}\right)}
\end{aligned}
$$

Proof. Applying (15) to the formula for $\varphi_{L}(2 \zeta)$,

$$
\begin{aligned}
\left|\varphi_{L}(2 \zeta)-s_{L}(z)\right| & =\frac{\alpha^{-1 / 2}}{|z|}\left|w^{-1}-\frac{w^{\zeta}-w^{-\zeta}}{w^{\zeta+1}-w^{-\zeta-1}}\right| \\
& =\frac{\alpha^{-1 / 2}}{|z|}\left|\frac{w^{-2 \zeta-1}\left(1-w^{-2}\right)}{1-w^{-2 \zeta-2}}\right| \\
& \leq \frac{2 \alpha^{-1 / 2} r^{-2 \zeta-1}}{|z|\left(1-r^{-2 \zeta-2}\right)} .
\end{aligned}
$$

The exact same procedure establishes the corresponding inequality for $\left|\varphi_{R}(2 \zeta)-s_{R}(2 \zeta)\right|$. We can similarly compute

$$
\begin{aligned}
\left|\psi_{L}(2 \zeta)\right|=\left|\psi_{R}(2 \zeta)\right| & =\left|\frac{\left(d_{R}-1\right)^{-\zeta}\left(w-w^{-1}\right)}{z\left(w^{\zeta+1}-w^{-\zeta-1}\right)}\right| \\
& \leq \frac{2 \alpha^{\zeta / 2}\left(d_{R}-1\right)^{-\zeta} r^{-\zeta}}{|z|\left(1-r^{-2 \zeta-2}\right)} .
\end{aligned}
$$

### 4.3 From trees to deterministic graphs

We now move from trees to graphs with large, acyclic neighborhoods. Let $G$ be a deterministic $\left(d_{L}, d_{R}\right)$-biregular graph that has a root vertex with an acyclic $(\zeta+1)$-neighborhood. Let $A$ be the adjacency matrix of this graph. Our goal is to show that the resolvent of $A$ is well approximated by the resolvent of $\widetilde{H}_{L}$ or $\widetilde{H}_{R}$. If the root of $G$ has degree $d_{L}$, then we consider the error term

$$
E_{L}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{\text {root, ,root }}^{-1}-\widetilde{\varphi}_{L}(\zeta)
$$

and if the root of $G$ has degree $d_{R}$, then we consider the error term

$$
E_{R}(\zeta)=\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{\text {root,root }}^{-1}-\widetilde{\varphi}_{R}(\zeta)
$$

Lemma 15. We can bound $E_{L}$ and $E_{R}$ by

$$
\begin{aligned}
& \left|E_{L}(\zeta)\right| \leq \frac{\left|\widetilde{\psi}_{L}(\zeta)\right|^{2} d_{L}\left(d_{R}-1\right)^{\lceil\zeta / 2\rceil}\left(d_{L}-1\right)^{\lfloor\zeta / 2\rfloor}}{\left(d_{R}-1\right) \Im(z)} \\
& \left|E_{R}(\zeta)\right| \leq \frac{\left|\widetilde{\psi}_{R}(\zeta)\right|^{2} d_{R}\left(d_{L}-1\right)^{\lceil\zeta / 2\rceil}\left(d_{R}-1\right)^{\lfloor\zeta / 2\rfloor}}{\left(d_{R}-1\right) \Im(z)}
\end{aligned}
$$

Proof. Let $C_{L}$ and $C_{R}$ denote the number of vertices of distance $\zeta+1$ from the root in the biregular trees $\widetilde{T}_{L}$ and $\widetilde{T}_{R}$, respectively. We claim that

$$
\begin{align*}
\left|E_{L}(\zeta)\right| & \leq \frac{\left|\tilde{\psi}_{L}(\zeta)\right|^{2} C_{L}}{\left(d_{R}-1\right) \Im(z)}  \tag{16}\\
\left|E_{R}(\zeta)\right| & \leq \frac{\left|\widetilde{\psi}_{R}(\zeta)\right|^{2} C_{R}}{\left(d_{R}-1\right) \Im(z)} \tag{17}
\end{align*}
$$

These statements can be proven exactly as in [DP12, Lemma 10]; we will only give a sketch here. The gist of the argument is to partition the vertices of $G$ into two parts, those at distance $\zeta$ or less from the root and those at distance greater than $\zeta$. This decomposes $A$ into blocks, one of which is simply $\widetilde{H}_{L}$ or $\widetilde{H}_{R}$. An application of Proposition 11 and some calculations then prove (16) and (17). The rest is simply a calculation of $C_{L}$ and $C_{R}$.

Now, we take a sequence of graphs as above and let $\zeta$ grow to infinity.
Lemma 16. Let $G$ be a sequence of deterministic graphs, each with a root with an acyclic $2 \zeta+1$ neighborhood. Suppose $d_{R} \rightarrow \infty$ and $\zeta \rightarrow \infty$. Fix $\epsilon>0$, and let $z$ be a sequence with $|\operatorname{Re}(z)|>\epsilon$ and $|\Im(z)| \geq 1 / d_{R}$. Suppose that $r^{-2 \zeta}=o\left(1 / d_{R}^{2}\right)$. Then either

$$
\left|\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{\text {root, root }}^{-1}-s_{L}(z)\right|=O_{\epsilon}\left(1 / d_{R}\right)
$$

or

$$
\left|\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{\text {root, root }}^{-1}-s_{R}(z)\right|=O_{\epsilon}\left(1 / d_{R}\right),
$$

depending on whether the root of $G$ has degree $d_{L}$ or $d_{R}$. We use $O_{\epsilon}(\cdot)$ to indicate that the constant in the expression depends on $\epsilon$.
Proof. Consider the case where the root of $G$ has degree $d_{L}$. We will proceed in two steps, bounding first $\left|\widetilde{\varphi}_{L}(2 \zeta)-s_{L}(z)\right|$ and then $E_{L}(2 \zeta)$.

We define the quantity

$$
\begin{aligned}
\beta & =\frac{q_{2 \zeta}(z)}{q_{2 \zeta}(z)-\left(d_{R}-1\right)^{-1} q_{2 \zeta-1}(z)} \\
& =\frac{1}{1-\left(d_{R}-1\right)^{-1} \frac{w^{-1}\left(1-w^{-2 \zeta}\right.}{1-w^{-2 \zeta-2}}}
\end{aligned}
$$

This is of interest because $\widetilde{\varphi}_{L}(2 \zeta)=\beta \varphi_{L}(2 \zeta)$. With the assumptions of this lemma, one can calculate directly that $|\beta|=1+O\left(1 / d_{R}\right)$. By these assumptions and Lemma 14, | $\varphi_{L}(2 \zeta)-$ $s_{L}(z) \mid=o_{\epsilon}\left(1 / d_{R}\right)$. Since $s_{L}(z)$ is bounded for $|z|>\epsilon$, this also implies that $\varphi_{L}(2 \zeta)=O_{\epsilon}(1)$. Thus

$$
\begin{aligned}
\left|\widetilde{\varphi}_{L}(2 \zeta)-s_{L}(z)\right| & =\left|\beta \varphi_{L}(2 \zeta)-s_{L}(z)\right| \\
& \leq\left|\varphi_{L}(2 \zeta)-s_{L}(z)\right|+\left|(\beta-1) \varphi_{L}(2 \zeta)\right| \\
& \leq o_{\epsilon}\left(1 / d_{R}\right)+O_{\epsilon}\left(1 / d_{R}\right)=O_{\epsilon}\left(1 / d_{R}\right) .
\end{aligned}
$$

Since $\widetilde{\psi}_{L}(2 \zeta)=\beta \psi_{L}(2 \zeta)$, by Lemma 14 and our bound on $\beta$,

$$
\left|\widetilde{\psi}_{L}(2 \zeta)\right|=\alpha^{\zeta / 2}\left(d_{R}-1\right)^{-\zeta} o_{\epsilon}\left(1 / d_{R}\right)
$$

Combining this with our bound on $E_{L}(2 \zeta)$ from Lemma 15 gives

$$
\left|E_{L}(2 \zeta)\right| \leq \frac{o_{\epsilon}\left(1 / d_{R}^{2}\right)}{\Im(z)}=o_{\epsilon}\left(1 / d_{R}\right)
$$

These two bounds prove the lemma. The case when the root of $G$ has degree $d_{R}$ is the same.

### 4.4 From deterministic graphs to random graphs

The main actors of this section will be sequences $s, \zeta$, and $\eta$. We will choose $s$ and $z$ in such a way that $s \geq r(z)$, and $\zeta$ will represent the size of an acyclic neighborhood in the graph. We will choose $\eta$ so that we can control the Stieltjes transform of our graph on the set $U=\{z: \Im(z) \geq \eta\}$. The following lemma gives us the relation between $s$ and $\eta$ :

Lemma 17. Let $r=r(z)$, fix some $s>1$, and let $\eta=s^{1 / 2}-s^{-1 / 2}$. If $\Im(z) \geq \eta$, then $r \geq s$.
Proof. First we prove this when $\alpha=1$. Consider the set $E_{s}=\{z:|F(z)|<s\}$. This set is the interior of an ellipse whose foci are -1 and 1 and whose radii are $\frac{1}{2}\left(s+s^{-1}\right)$ and $\frac{1}{2}\left(s-s^{-1}\right)$ (see [MH03, p. 14]). It suffices to show that if $\Im(z) \geq \eta$, then $\frac{1}{2} z^{2}-1$ lies outside of $E_{s}$. To this end, we note that the transformation $z \mapsto \frac{1}{2} z^{2}-1$ takes the region given by $\Im(z) \geq \eta$ to the region bounded on the right by the parabola $P=\left\{\frac{1}{2}\left(t^{2}-\eta^{2}\right)-1+\eta t i: t \in \mathbb{R}\right\}$. This can be checked to touch $E_{s}$ at $-\frac{1}{2}\left(s+s^{-1}\right)$ and otherwise to lie to the left of it. This proves the lemma when $\alpha=1$.

To extend this to the case where $\alpha>1$, we consider the image of $E_{s}$ under the map $z \mapsto\left(\alpha^{-1 / 2}\left(2 z+\alpha^{1 / 2}+\alpha^{-1 / 2}\right)\right)^{1 / 2}$, which is the inverse of $\frac{1}{2} \sqrt{\alpha}\left(z^{2}-\alpha^{-1}-1\right)$. Our argument for $\alpha=1$ establishes that in this case, the maximum imaginary part of this set is $\eta$. It is straightforward to check that for any $z$, the quantity $\Im\left(\alpha^{-1 / 2}\left(2 z+\alpha^{1 / 2}+\alpha^{-1 / 2}\right)\right)^{1 / 2}$ decreases as $\alpha$ increases, which establishes the lemma.

For the remainder of this section, let $G$ be a random biregular bipartite graph on $m+n$ vertices satisfying (1)-(3) as well as the condition $d_{R}=\exp (o(1) \sqrt{\log n})$. Let $A$ be the adjacency matrix of $G$. We define the sequences

$$
\begin{aligned}
& a=\min \left(\frac{\log n}{9\left(\log d_{R}\right)^{2}}, d_{R}\right), \\
& s=e^{1 / a}, \\
& \zeta=\frac{\log n}{8 \log d_{R}}-1, \\
& \eta=s^{1 / 2}-s^{-1 / 2} .
\end{aligned}
$$

We now show that sufficiently many vertices of $G$ have tree-like neighborhoods.
Lemma 18. Let $J$ be the set of vertices in $G$ whose $2 \zeta$-neighborhoods are acyclic. Then

$$
\mathbf{P}\left[1-\frac{|J|}{n+m} \geq \frac{\eta}{d_{R}}\right]=o(1 / n) .
$$

Proof. This is nearly the same as Lemma [5. We may assume $\zeta$ is an integer by replacing it with $\lfloor\zeta\rfloor$. We define

$$
N^{*}=\sum_{i=2}^{2 \zeta} 2 i\left(d_{R}-1\right)^{2 \zeta-i} X_{i}
$$

recalling that $X_{i}$ is the random variable denoting the number of $2 i$-cycles in $G$. We have the bound $n+m-|J| \leq N^{*}$. Now we apply Proposition 4 to calculate

$$
\begin{aligned}
\mathbf{E}\left[N^{*}\right] & =\sum_{i=2}^{2 \zeta} 2 i\left(d_{R}-1\right)^{2 \zeta-i} \mu_{i}\left(1+O\left(\frac{i\left(i+d_{R}\right)}{n}\right)\right) \\
& =O\left(d_{R}^{4 \zeta}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Var}\left[N^{*}\right] & \leq 2 \zeta \sum_{i=2}^{2 \zeta} 4 i^{2}\left(d_{R}-1\right)^{4 \zeta-2 i} \mu_{i}\left(1+O\left(\frac{d_{R}^{2 i}\left(i \alpha^{2 i-1}+\alpha^{-i} d_{R}\right)}{n}\right)\right) \\
& \leq 2 \zeta \sum_{i=2}^{\zeta} 2 i\left(d_{R}-1\right)^{4 \zeta}\left(1+O\left(\frac{n^{4 c}\left(\zeta \alpha^{4 \zeta-1}+d_{R}\right)}{n}\right)\right) \\
& =O\left(\zeta^{3} d_{R}^{4 \zeta}\right)
\end{aligned}
$$

By Markov's inequality,

$$
\begin{aligned}
\mathbf{P}\left[1-\frac{|J|}{n+m} \geq \frac{\eta}{d_{R}}\right] & \leq \mathbf{P}\left[N^{*} \geq \frac{(1+\alpha) n \eta}{d_{R}}\right] \\
& \leq \frac{O\left(d_{R}^{8 \zeta+2}+\zeta^{3} d_{R}^{4 \zeta+2}\right)}{n^{2} \eta^{2}} \\
& \leq O\left(n^{-1} d_{R}^{-4}+n^{-3 / 2} \zeta^{3}\right)=o(1 / n)
\end{aligned}
$$

Let $J_{L}$ and $J_{R}$ denote the sets of vertices with acyclic $2 \zeta$-neighborhoods in the left and right vertex classes, respectively. As in Corollary 6, it is immediate that

$$
\mathbf{P}\left[\frac{m-\left|J_{L}\right|}{n+m} \geq \frac{\eta}{d_{R}}\right]=o(1 / n)
$$

and

$$
\mathbf{P}\left[\frac{n-\left|J_{R}\right|}{n+m} \geq \frac{\eta}{d_{R}}\right]=o(1 / n) .
$$

It is also straightforward to see that this lemma holds when we require the vertices to have acyclic $2 \zeta+1$ neighborhoods rather than just $2 \zeta$ neighborhoods.

We can now apply all of these results to the task of bounding the rate of convergence of the Stieltjes transform:

Theorem 19. Fix some $\epsilon$ and let $U$ denote the set of complex numbers

$$
U=\{z \in \mathbb{C}: \operatorname{Re}(z) \geq \epsilon, \Im(z) \geq \eta\} .
$$

Let $s_{n}(z)$ denote the Stieltjes transform of $\left(d_{R}-1\right)^{-1} A$. Then for sufficiently large $C_{\epsilon}$,

$$
\mathbf{P}\left[\sup _{z \in U}\left|s_{n}(z)-s(z)\right|>C_{\epsilon} / d_{R}\right]=o(1 / n) .
$$

Proof. Let $J$ denote the vertices with acyclic $2 \zeta+1$ neighborhoods. We condition on the event that $\left(m-\left|J_{L}\right|\right) /(m+n)<\eta / d_{R}$ and $\left(n-\left|J_{R}\right| /(m+n)<\eta / d_{R}\right.$, which by the discussion following Lemma 18 holds with probability $1-o(1 / n)$. We call this event $\Omega$. We compute $s_{n}(z)$ for $z \in U$, breaking up the vertices into $J_{L}, J_{R}$, and the remaining vertices:

$$
\begin{aligned}
s_{n}(z)= & \frac{1}{m+n} \sum_{v \in J_{L}}\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{v, v}^{-1}+\frac{1}{m+n} \sum_{v \in J_{R}}\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{v, v}^{-1} \\
& +\frac{1}{n+m} \sum_{v \notin J}\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{v, v}^{-1} .
\end{aligned}
$$

We begin with the third term. Applying the bound $\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{v, v}^{-1} \leq \eta^{-1}$, we have

$$
\begin{equation*}
\left|\frac{1}{n+m} \sum_{v \notin J}\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{v, v}^{-1}\right| \leq \frac{m+n-|J|}{(m+n) \eta}<\frac{1}{d_{R}} \tag{18}
\end{equation*}
$$

on the event $\Omega$.
Since every vertex in $J$ has an acyclic $2 \zeta+1$ neighborhood, we will apply Lemma 16 to estimate the first two terms. First, we confirm that the conditions of the lemma hold. By expanding $\eta$ as a power series, we deduce the bound $\eta \geq 1 / a \geq 1 / d_{R}$. We can calculate

$$
\begin{aligned}
s^{-2 \zeta} & =\exp \left(-\frac{2}{a}\left(\frac{\log n}{8 \log d_{R}}-1\right)\right) \\
& \leq \exp \left(-\frac{9}{4} \log d_{R}+o(1)\right)=o\left(1 / d_{R}^{2}\right) .
\end{aligned}
$$

By Lemma 17, it follows from $\Im(z) \geq \eta$ that $r(z) \geq s$. Hence for any sequence $z \in U$, we have $r(z)^{-2 \zeta} \leq s^{-2 \zeta}=o\left(1 / d_{R}^{2}\right)$. Thus the conditions of Lemma 16 hold, and so for all $v \in J_{L}$,

$$
\left|\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{v, v}^{-1}-s_{L}(z)\right|=O_{\epsilon}\left(1 / d_{R}\right),
$$

and for all $v \in J_{R}$,

$$
\left|\left(\left(d_{R}-1\right)^{-1 / 2} A-z\right)_{v, v}^{-1}-s_{R}(z)\right|=O_{\epsilon}\left(1 / d_{R}\right) .
$$

Combining these estimates with (18),

$$
\begin{aligned}
s_{n}(z) & =\frac{\left|J_{L}\right|}{m+n} s_{L}(z)+\frac{\left|J_{R}\right|}{m+n} s_{R}(z)+O_{\epsilon}\left(1 / d_{R}\right) \\
& =\frac{m}{m+n} s_{L}(z)+\frac{n}{m+n} s_{R}(z)+O_{\epsilon}\left(1 / d_{R}\right) \\
& =s(z)+O_{\epsilon}\left(1 / d_{R}\right)
\end{aligned}
$$

on the event $\Omega$, which proves the theorem.
We now restate and prove our local convergence law.
Theorem 3. Fix $\epsilon>0$. Let $\mu_{n}$ be the ESD of $\left(d_{R}-1\right)^{-1 / 2} A$, and let $\mu$ be the limiting ESD defined in Corollary 园. There exists a constant $C_{\epsilon}$ such that for all sufficiently large $n$ and $\delta>0$, for any interval $I \subseteq \mathbb{R}$ avoiding $[-\epsilon, \epsilon]$ and with length $|I| \geq \max (2 \eta, \eta /(-\delta \log \delta))$, it holds that

$$
\left|\mu_{n}(I)-\mu(I)\right|<\delta C_{\epsilon}|I|
$$

with probability $1-o(1 / n)$.
Proof. This theorem follows from the arguments of [TV11, Lemma 64], which we will sketch. Define

$$
F(y)=\frac{1}{\pi} \int_{I} \frac{\eta}{\eta^{2}+(y-x)^{2}} d x
$$

This function $F(y)$ approximates the indicator function on the interval $I$, and the following statements hold:

$$
\begin{aligned}
\int F(y) d \mu(y) & =\mu(I)+O_{\epsilon}\left(\eta \log \frac{|I|}{\eta}\right) \\
\int F(y) d \mu_{n}(y) & =\mu_{n}(I)+O_{\epsilon}\left(\eta \log \frac{|I|}{\eta}\right)
\end{aligned}
$$

The proofs of these statements in [TV11, Lemma 64] have $\mu$ as the semicircle law, but they apply just as well to our limiting measure $\mu$; the only thing necessary for the proof to go through is that $\mu$ has a bounded density outside of the interval $[-\epsilon, \epsilon]$. On the event $\Omega$ of the previous theorem,

$$
\begin{aligned}
\left|\int F(y) d \mu(y)-\int F(y) d \mu_{n}(y)\right| & =\frac{1}{\pi}\left|\int_{I}\left(\Im(s(x+\eta i))-\Im\left(s_{n}(x+\eta i)\right)\right) d x\right| \\
& \leq \frac{C_{\epsilon}|I|}{\pi d_{R}}
\end{aligned}
$$

As observed in [TV11], it follows from the condition $|I| \geq \eta /(-\delta \log \delta)$ that $\eta \log \frac{|I|}{\eta}=O(\delta|I|)$. Since $d_{R} \rightarrow \infty$, for $n$ sufficiently large,

$$
\left|\mu(I)-\mu_{n}(I)\right| \leq C_{\epsilon} \delta|I|
$$

for some constant $C_{\epsilon}$ (not necessarily the same one as before) on the event $\Omega$.

## Appendix

Our goal here is to prove Proposition 4. We mention that it is possible to prove much more than this. The main result of [MWW04] is that the distribution of short cycles in a random regular graph is approximately Poisson, and this result holds for biregular bipartite graphs as well, with suitable modifications of the proofs.

We will use a theorem from McK81 that gives us the probability that $G$ contains some subgraph $L \subseteq K_{m, n}$. For any $v \in K_{m, n}$, let $g_{v}$ and $l_{v}$ denote the the degree of $v$ considered as a vertex of $G$ and of $L$, respectively. Let $l_{\max }$ be the largest value of $l_{i}$. Consider $L$ to be a collection of edges, so that $|L|$ is the number of edges of $L$. The notation $[x]_{a}$ denotes the falling factorial, $x(x-1) \cdots(x-a+1)$.

Proposition 20. Let $L \subseteq K_{m, n}$.
(a) If $|L|+2 d_{R}\left(d_{R}+l_{\max }-2\right) \leq n d_{R}-1$, then

$$
\mathbf{P}[L \subseteq G] \leq \frac{\prod\left[g_{i}\right]_{l_{i}}}{\left[n d_{R}-4 d_{R}^{2}-1\right]_{|L|}}
$$

(b) If $n d_{R}-2 d_{R}\left(d_{R}+l_{\max }-1\right)-1-|L| \geq d_{R} l_{\max }$, then

$$
\begin{aligned}
& \mathbf{P}[L \subseteq G] \geq \frac{\prod\left[g_{i}\right]_{i}}{\left[n d_{R}-1\right]_{|L|}} \\
\times & {\left[\left(1-\frac{d_{R} l_{\max }}{n d_{R}-|L|-2 d_{R}\left(d_{R}+l_{\max }-1\right)-1}\right) /\left(1+\frac{d_{R}^{2}}{n d_{R}-2 d_{R}\left(d_{R}+l_{\max }-2\right)-1-|L|(e-1) / e}\right)\right]^{|L|} }
\end{aligned}
$$

Proof. This is an application of [McK81, Theorem 3.5], with the set $H$ from that theorem equal to $\varnothing$.

We are most interested in when $L$ is a cycle with $2 r$ edges, in which case the above theorem reduces to the following:

Corollary 21. Let $L \subseteq K_{m, n}$ be a cycle with $2 r$ edges whose presence in $G$ we wish to test.
(a) If $2 r+2 d_{R}^{2} \leq n d_{R}-1$, then

$$
\mathbf{P}[L \subseteq G] \leq \frac{d_{L}^{r}\left(d_{L}-1\right)^{r} d_{R}^{r}\left(d_{R}-1\right)^{r}}{\left[n d_{R}-2 d_{R}^{2}-1\right]_{2 r}}
$$

(b) If $n d_{R}-2 d_{R}\left(d_{R}+1\right)-1-2 r \geq 2 d_{R}$, then

$$
\begin{aligned}
& \mathbf{P}[L \subseteq G] \geq \frac{d_{L}^{r}\left(d_{L}-1\right)^{r} d_{R}^{r}\left(d_{R}-1\right)^{r}}{\left[n d_{R}-1\right]_{2 r}} \\
& \quad \times\left[\left(1-\frac{2 d_{R}}{n d_{R}-2 d_{R}\left(d_{R}+1\right)-1-2 r}\right) /\left(1+\frac{d_{R}^{2}}{n d_{R}-2 d_{R}^{2}-1-2 r(e-1) / e}\right)\right]^{2 r} .
\end{aligned}
$$

Proof of Proposition 园. Let $L \subseteq K_{m, n}$ be a cycle with $2 r$ edges. We start by showing that

$$
\begin{equation*}
\mathbf{P}[L \subseteq G]=\frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(1+O\left(\frac{r d_{R}}{n}+\frac{r^{2}}{n d_{R}}\right)\right) . \tag{19}
\end{equation*}
$$

Since $d_{R}=o(n)$, all of the conditions for Proposition 20 and Corollary 21 apply. By Corollary 21面,

$$
\begin{align*}
\mathbf{P}[L \subseteq G] & \leq \frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r} d_{R}^{2 r}}{\left(n d_{R}-2 d_{R}^{2}-2 r\right)^{2 r}} \\
& =\frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(\frac{n d_{R}}{n d_{R}-2 d_{R}^{2}-2 r}\right)^{2 r} \\
& =\frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(1+\frac{2 d_{R}^{2}+2 r}{n d_{R}-2 d_{R}^{2}-2 r}\right)^{2 r} \\
& =\frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(1+O\left(\frac{r d_{R}}{n}+\frac{r^{2}}{n d_{R}}\right)\right) . \tag{20}
\end{align*}
$$

The last line follows from the fact that if $x>-1$,

$$
(1+x)^{r} \leq e^{r x}=1+O(r x)
$$

as $r x \rightarrow 0$.

From Corollary 211b,

$$
\begin{aligned}
\mathbf{P}[L \subseteq G] \geq & \frac{d_{L}^{r}\left(d_{L}-1\right)^{r} d_{R}^{r}\left(d_{R}-1\right)^{r}}{\left(n d_{R}\right)^{2 r}} \\
& \times\left[\left(1-\frac{2 d_{R}}{n d_{R}-2 d_{R}\left(d_{R}+1\right)-1-2 r}\right) /\left(1+\frac{d_{R}^{2}}{n d_{R}-2 d_{R}^{2}-1-2 r(e-1) / e}\right)\right]^{2 r} \\
= & \frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(\frac{1+O(1 / n)}{1+O\left(d_{R} / n\right)}\right)^{2 r} \\
= & \frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(1+O\left(\frac{d_{R}}{n}\right)\right)^{2 r} \\
= & \frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(1+O\left(\frac{r d_{R}}{n}\right)\right) .
\end{aligned}
$$

These two inequalities prove (19).
The number of cycles of length $2 r$ in $K_{m, n}$ is $[m]_{r}[n]_{r} / 2 r$. Using $[n]_{r}=n^{r}\left(1+O\left(r^{2} / n\right)\right)$, which can be proven by showing inductively that $[n]_{r} \geq n^{r}\left(1-r^{2} / 2 n\right)$, we calculate the expected number of such cycles:

$$
\begin{align*}
\mathbf{E}\left[X_{r}\right] & =\frac{[m]_{r}[n]_{r}}{2 r} \frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(1+O\left(\frac{r d_{R}}{n}+\frac{r^{2}}{n d_{R}}\right)\right) \\
& =\frac{(n \alpha)^{r} n^{r}\left(1+O\left(r^{2} / n\right)\right)}{2 r} \frac{\left(d_{L}-1\right)^{r}\left(d_{R}-1\right)^{r} \alpha^{-r}}{n^{2 r}}\left(1+O\left(\frac{r d_{R}}{n}+\frac{r^{2}}{n d_{R}}\right)\right) \\
& =\mu_{r}\left(1+O\left(\frac{r\left(r+d_{R}\right)}{n}\right)\right) . \tag{21}
\end{align*}
$$

Let $\mathcal{C}$ be the set of $2 r$-cycles in $K_{m, n}$. We will calculate $\operatorname{Var}\left[X_{r}\right]$ using the equation

$$
\mathbf{E}\left[X_{r}^{2}\right]=\sum_{C_{1} \in \mathcal{C}} \sum_{C_{2} \in \mathcal{C}} \mathbf{P}\left[C_{1} \cup C_{2} \subseteq G\right] .
$$

We break up $\mathcal{C} \times \mathcal{C}$ to help calculate this sum.

$$
\begin{aligned}
& \mathcal{C}_{1}=\left\{\left(C_{1}, C_{2}\right) \in \mathcal{C} \times \mathcal{C}: C_{1} \cap C_{2}=\varnothing\right\}, \\
& \mathcal{C}_{2}=\left\{\left(C_{1}, C_{2}\right) \in \mathcal{C} \times \mathcal{C}: C_{1} \cap C_{2} \neq \varnothing, \text { but } C_{1} \neq C_{2}\right\}, \\
& \mathcal{C}_{3}=\left\{\left(C_{1}, C_{2}\right) \in \mathcal{C} \times \mathcal{C}: C_{1}=C_{2}\right\} .
\end{aligned}
$$

We are considering cycles as collections of edges, so pairs of cycles that share vertices but not edges belong in $\mathcal{C}_{1}$ rather than $\mathcal{C}_{2}$.

For $\left(C_{1}, C_{2}\right) \in \mathcal{C}_{1}$, Proposition 20自 and a calculation identical to the one in (20) show that

$$
\mathbf{P}\left[C_{1} \cup C_{2} \subseteq G\right] \leq \frac{\left(d_{L}-1\right)^{2 r}\left(d_{R}-1\right)^{2 r} \alpha^{-2 r}}{n^{4 r}}\left(1+O\left(\frac{r d_{R}}{n}+\frac{r^{2}}{n d_{R}}\right)\right)
$$

Bounding $\left|\mathcal{C}_{1}\right|$ by $|\mathcal{C}|=\left([m]_{r}[n]_{r} / 2 r\right)^{2}$ and repeating the calculation in (21) gives

$$
\begin{equation*}
\sum_{\left(C_{1}, C_{2}\right) \in \mathcal{C}_{1}} \mathbf{P}\left[C_{1} \cup C_{2} \subseteq G\right] \leq \mu_{r}^{2}\left(1+O\left(\frac{r\left(r+d_{R}\right)}{n}\right)\right) \tag{22}
\end{equation*}
$$

For a lower bound on this sum, we note that by Proposition 20b, for any $2 r$-cycles $C_{1}$ and $C_{2}$ that share no vertices,

$$
\begin{aligned}
\mathbf{P}\left[C_{1} \cup C_{2} \subseteq G\right] & \geq \frac{d_{L}^{2 r}\left(d_{L}-1\right)^{2 r} d_{R}^{2 r}\left(d_{R}-1\right)^{2 r}}{\left(n d_{R}\right)^{4 r}}\left(\frac{1+O(1 / n)}{1+O\left(d_{R} / n\right)}\right)^{4 r} \\
& =\frac{\left(d_{L}-1\right)^{2 r}\left(d_{R}-1\right)^{2 r} \alpha^{-2 r}}{n^{4 r}}\left(1+O\left(\frac{r d_{R}}{n}\right)\right) .
\end{aligned}
$$

Summing this over the $\left([n]_{r}[m]_{r} / 2 r\right)\left([n-r]_{r}[m-r]_{r} / 2 r\right)$ such pairs of $2 r$-cycles provides the bound

$$
\begin{aligned}
\sum_{\left(C_{1}, C_{2}\right) \in \mathcal{C}_{1}} \mathbf{P}\left[C_{1} \cup C_{2} \subseteq G\right] & \geq \frac{[n]_{r}[m]_{r}[n-r]_{r}[m-r]_{r} \alpha^{-2 r}}{n^{4 r}} \mu_{r}^{2}\left(1+O\left(\frac{r d_{R}}{n}\right)\right) \\
& =\frac{n^{r} m^{r}(n-r)^{r}(m-r)^{r}\left(1+O\left(r^{2} / n\right)\right) \alpha^{-2 r}}{n^{4 r}} \mu_{r}^{2}\left(1+O\left(\frac{r d_{R}}{n}\right)\right) \\
& =\left(\left(1-\frac{r}{n}\right)\left(1-\frac{r}{\alpha n}\right)\right)^{r} \mu_{r}^{2}\left(1+O\left(\frac{r^{2}}{n}\right)\right)\left(1+O\left(\frac{r d_{R}}{n}\right)\right) \\
& =\mu_{r}^{2}\left(1+O\left(\frac{r\left(r+d_{R}\right)}{n}\right)\right) .
\end{aligned}
$$

The sum over $\mathcal{C}_{3}$ is

$$
\begin{equation*}
\sum_{\left(C_{1}, C_{2}\right) \in \mathcal{C}_{3}} \mathbf{P}\left[C_{1} \cup C_{2} \subseteq G\right]=\mathbf{E}\left[X_{r}\right]=\mu_{r}\left(1+O\left(\frac{r\left(r+d_{R}\right)}{n}\right)\right) . \tag{23}
\end{equation*}
$$

To estimate the sum over $\mathcal{C}_{2}$, we bound the number of isomorphism types of a graph $H=C_{1} \cup C_{2}$ for $\left(C_{1}, C_{2}\right) \in \mathcal{C}_{2}$. Let $H^{\prime}$ be the graph $\left(V\left(C_{1}\right) \cap V\left(C_{2}\right), E\left(C_{1}\right) \cap E\left(C_{2}\right)\right)$. Say that $H^{\prime}$ has $p$ components and $j$ edges. As $H^{\prime}$ is a forest, it has $p+j$ vertices, so $H$ has $4 r-p-j$ vertices. We also note that $H$ has $4 r-j$ edges.

Let $C_{1}=a_{1} a_{2} \cdots a_{2 r}$ and $C_{2}=b_{1} b_{2} \cdots b_{2 r}$, with $a_{1}=b_{1}$. Let $A_{1}, \ldots, A_{p}$ be the components of $H^{\prime}$ ordered as the appear in $C_{1}$. We can encode the isomorphism type of $H$ in the following four sequences:

- $s_{i}$ is the number of vertices in $A_{i}$
- $t_{i}$ is the smallest $j$ such that $a_{j} \in A_{i}$
- $u_{i}$ is the smallest $j$ such that $b_{j} \in A_{i}$
- $v_{i}$ specifies whether $A_{i}$ is oriented the same way in $C_{1}$ as in $C_{2}$ (if $A_{i}$ is a single vertex, consider it to be oriented the same)

For example, the following diagram is the union of two cycles of length 8, the first colored black and the second colored gray.


The intersection graph $H^{\prime}$ has three components, $A_{1}=a_{1}=b_{1}, A_{2}=a_{3} a_{4}=b_{2} b_{3}$, and $A_{3}=a_{6} a_{7} a_{8}=b_{6} b_{7} b_{8}$. The four sequences for this graph are

$$
\begin{aligned}
& \mathbf{s}: 1,2,3 \\
& \mathbf{t}: 1,3,6 \\
& \mathbf{u}: 1,2,6 \\
& \mathbf{v}: \text { yes, yes, no }
\end{aligned}
$$

To illustrate that the isomorphism class of $H$ is encoded in these sequences, We will demonstrate how to recover it in this example. Start by drawing a cycle and labeling its vertices $a_{1}, \ldots, a_{8}$. From $\mathbf{s}$ and $\mathbf{t}$, we can deduce that $A_{1}=a_{1}, A_{2}=a_{3} a_{4}$, and $A_{3}=a_{6} a_{7} a_{8}$. From $\mathbf{u}$ and $\mathbf{v}$, we deduce that $b_{1}=a_{1}, b_{2}=a_{3}, b_{3}=a_{4}, b_{6}=a_{8}, b_{7}=a_{7}$, and $b_{8}=a_{6}$. Since $b_{4}$ and $b_{5}$ are unaccounted for, we conclude that they are not contained in $a_{1} \cdots a_{8}$. Once we add edges to connect $b_{1}$ to $b_{2}, b_{2}$ to $b_{3}$, and so on, we have recreated $H$ up to isomorphism.

Now, we consider the number of possible isomorphism classes of some $H=C_{1} \cup C_{2}$. The sequence $\mathbf{s}$ is a composition of $p+j$ into exactly $p$ parts, so there are $\binom{p+j-1}{p-1}$ possibilities. We know that $t_{1}=1$, and we know that $t_{2}, \ldots, t_{p}$ are ordered and are distinct, so there are at most $\binom{2 r-1}{p-1}$ choices for $\mathbf{t}$. We know that $u_{1}=1$ and that $u_{2}, \ldots, u_{p}$ are distinct but not necessarily in any order, so there are at most $\binom{2 r-1}{p-1}(p-1)$ ! choices for $\mathbf{u}$. For $\mathbf{v}$, there are $2^{p-1}$ choices. For a fixed choice of of $p$ and $j$, the number of possible isomorphism classes of $H$ is hence bounded by

$$
\binom{p+j-1}{p-1}\binom{2 r-1}{p-1}^{2}(p-1)!2^{p-1} .
$$

By replacing $p+j-1$ and $2 r-1$ by $2 r$, we bound this quantity by

$$
\binom{2 r}{p-1}^{3}(p-1)!2^{p-1} \leq \frac{\left(16 r^{3}\right)^{p-1}}{(p-1)!^{2}}
$$

Suppose an isomorphism type has $a$ vertices from the left vertex class and $b$ from the right vertex class (with $a+b=4 r-p-j$ ). Then this isomorphism type can be realized in at most $[n]_{a}[m]_{b}+[n]_{b}[m]_{a} \leq 2 \alpha^{4 r-p-j} n^{4 r-p-j}$ ways. By Proposition [20甸, the probability of any one of these realizations being a subgraph of $G$ is bounded by

$$
\frac{d_{L}^{(4 r-j) / 2}\left(d_{L}-1\right)^{(4 r-j) / 2} d_{R}^{(4 r-j) / 2}\left(d_{R}-1\right)^{(4 r-j) / 2}}{\left[n d_{R}-4 d_{R}^{2}-1\right]_{4 r-j}}=\frac{\left(d_{L}-1\right)^{(4 r-j) / 2}\left(d_{R}-1\right)^{(4 r-j) / 2}}{\alpha^{(4 r-j) / 2} n^{4 r-j}} O(1) .
$$

All together, we have the bound

$$
\begin{align*}
\sum_{\left(C_{1}, C_{2}\right) \in \mathcal{C}_{3}} \mathbf{P}\left[C_{1} \cup C_{2} \subseteq G\right] & \leq \sum_{p=1}^{2 r} \sum_{j=1}^{2 r} \frac{\left(16 r^{3}\right)^{p-1}}{(p-1)!^{2}} \alpha^{4 r-p-j} n^{4 r-p-j} \frac{\left(d_{L}-1\right)^{(4 r-j) / 2}\left(d_{R}-1\right)^{(4 r-j) / 2}}{\alpha^{(4 r-j) / 2} n^{4 r-j}} O(1) \\
& =\sum_{p=1}^{2 r} \sum_{j=1}^{2 r} \frac{\left(16 r^{3}\right)^{p-1}}{(p-1)!^{2}} \frac{\left(d_{L}-1\right)^{(4 r-j) / 2}\left(d_{R}-1\right)^{(4 r-j) / 2}}{\alpha^{p-2 r-j / 2} n^{p}} O(1) \\
& =O\left(\frac{\alpha^{(4 r-1) / 2}\left(d_{L}-1\right)^{(4 r-1) / 2}\left(d_{R}-1\right)^{(4 r-1) / 2}}{n}\right) . \tag{24}
\end{align*}
$$

Combining this with (22) and (23) shows

$$
\begin{aligned}
\operatorname{Var}\left[X_{r}\right]= & \mu_{r}^{2}\left(1+O\left(\frac{r\left(r+d_{R}\right)}{n}\right)\right)+\mu_{r}\left(1+O\left(\frac{r\left(r+d_{R}\right)}{n}\right)\right) \\
& -\mathbf{E}\left[X_{r}\right]^{2}+O\left(\frac{\alpha^{(4 r-1) / 2}\left(d_{L}-1\right)^{(4 r-1) / 2}\left(d_{R}-1\right)^{(4 r-1) / 2}}{n}\right) \\
= & \mu_{r}+\mu_{r}^{2}\left(O\left(\frac{r\left(r+d_{R}\right)}{n}\right)+O\left(\frac{r\left(r+d_{R}\right)}{\mu_{r} n}\right)+O\left(\frac{r^{2} \alpha^{(4 r-1) / 2}}{n\left(d_{L}-1\right)^{1 / 2}\left(d_{R}-1\right)^{1 / 2}}\right)\right) \\
= & \mu_{r}+\mu_{r}^{2} O\left(\frac{r\left(\alpha^{(4 r-1) / 2} r+d_{R}\right)}{n}\right) .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\operatorname{Var}\left[X_{r}\right] & =\mu_{r}\left(1+\mu_{r} O\left(\frac{r\left(\alpha^{(4 r-1) / 2} r+d_{R}\right)}{n}\right)\right) \\
& =\mu_{r}\left(1+O\left(\frac{d_{R}^{2 r}\left(r \alpha^{2 r-1}+\alpha^{-r} d_{R}\right)}{n}\right)\right) .
\end{aligned}
$$

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