ON THE THRESHOLD FOR THE MAKER-BREAKER H-GAME

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Abstract. We study the Maker-Breaker H-game played on the edge set of the random graph $G_{n,p}$. In this game two players, Maker and Breaker, alternately claim unclaimed edges of $G_{n,p}$, until all the edges are claimed. Maker wins if he claims all the edges of a copy of a fixed graph H; Breaker wins otherwise. In this paper we show that, with the exception of trees and triangles, the threshold for an H-game is given by the threshold of the corresponding Ramsey property of $G_{n,p}$ with respect to the graph H.

Keywords. Positional games; random graphs; Maker-Breaker

1. Introduction

Combinatorial games are games like Tic-Tac-Toe or Chess in which each player has perfect information and players move sequentially. Outcomes of such games can thus, at least in principle, be predicted by enumerating all possible ways in which the game may evolve. But, of course, such complete enumerations usually exceed available computing powers, which keeps these games interesting to study.

In this paper we take a look at a special class of combinatorial games, the so-called Maker-Breaker positional games. Given a finite set X and a family $\mathcal{E}$ of subsets of X, two players, Maker and Breaker, alternate in claiming unclaimed elements of X until all the elements are claimed. Unless explicitly stated otherwise, Maker starts the game. Maker wins if he claims all elements of a set from $\mathcal{E}$, and Breaker wins otherwise. The set X is referred to as the *board*, and the elements of $\mathcal{E}$ as the *winning sets*.

Given a (large) graph G and a (small) graph H, the *H-game* on G is played on the board $E(G)$ and the winning sets are the edge sets of all copies of H appearing in G as subgraphs. So, Maker and Breaker alternately claim unclaimed edges of the graph G until all the edges are claimed. Maker wins if he claims all the edges of a copy of H, otherwise Breaker wins.

Positional games played on edges of random graphs were first introduced and studied in [14]. Here we look at the H-game played on the random graph $G_{n,p}$, where H is a fixed graph. More precisely, we aim at determining a threshold function $p_0 = p_0(n, H)$ such that

$$\lim_{n\to\infty} \Pr[G_{n,p} \text{ is Maker's win in the } H\text{-game}] = \begin{cases} 1, & p \gg p_0(n,H), \\ 0, & p \ll p_0(n,H). \end{cases}$$

For the case that H is a clique such thresholds were recently obtained by Müller and Stojaković [7]. There is an easy intuitive argument for the location of such a threshold: if the random graph $G_{n,p}$ is so sparse that w.h.p. it only contains few scattered copies of H then this should be a Breaker's win. If on the other hand

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the graph contains many copies of H that heavily overlap then this should make Maker's task easier. As it turns out, the same intuition can also be applied to the threshold for the Ramsey property of $G_{n,p}$, thus one should expect that the two are related. We formalize this as follows.

For graphs G and H we denote by $G \to (H)^e_2$ the property that every edge-coloring of G with 2 colors contains a copy of H with all edges having the same color. For a graph $G = (V, E)$ on at least three vertices, we let $d_2(G) := (|E| - 1)/(|V| - 2)$ and denote by $m_2(G)$ the so-called *2-density*, defined as $m_2(G) = \max_{J \subseteq G, v_J \geq 3} d_2(J)$. If $m_2(G) = d_2(G)$, we say that G is *2-balanced*, and if in addition $m_2(G) > d_2(J)$ for every subgraph $J \subset G$ with $v_J \geq 3$, we say that G is *strictly 2-balanced*.

The Ramsey property of random graphs $G_{n,p}$ is well understood, as the following theorem shows, cf. also [9] for a short proof.

Theorem 1 (Rödl, Ruciński [10, 11, 12])**.** *Let H be a graph that is not a forest of stars or paths of length 3. Then there exist constants $c, C > 0$ such that*

$$\lim_{n\to\infty} \Pr[G_{n,p} \to (H)^e_2] = \begin{cases} 1, & \text{if } p \geq Cn^{-1/m_2(H)}, \\ 0, & \text{if } p \leq cn^{-1/m_2(H)}. \end{cases}$$

Note that $p = n^{-1/m_2(F)}$ is the density where we expect that every edge is contained in roughly a constant number of copies of H. Thus, if c is very small, the of copies of H will be scattered. If on the other hand C is big then these copies overlap so heavily that every coloring has to induce at least one monochromatic copy of H.

In this paper, we show that this intuition indeed provides the correct answer for most graphs H.

Theorem 2. *Let H be a graph for which there exists $H' \subseteq H$ such that $d_2(H') = m_2(H)$, H' is strictly 2-balanced and it is not a tree or a triangle. Then there exist constants $c, C > 0$ such that*

$$\lim_{n\to\infty} \Pr[G_{n,p} \text{ is Maker's win in the } H\text{-game}] = \begin{cases} 1, & p \geq Cn^{-1/m_2(H)}, \\ 0, & p \leq cn^{-1/m_2(H)}. \end{cases}$$

Next, we take a look at the graphs H that are not covered by Theorem 2. For $H = K_3$ we have $m_2(K_3) = 2$. Nevertheless, the threshold for the K_3-game is $n^{-5/9}$, cf. [14]. The reason turns out to be that K_5 minus an edge is a Maker's win (which can be easily checked by hand) – and this graph appears in $G_{n,p}$ w.h.p. whenever $p \gg n^{-5/9}$.

For graphs H that contain a triangle, various things can happen. If their 2-density is above two, then they are covered by the above theorem. If $m_2(H) = 2$ and H contains a subgraph with 2-density exactly two that does not contain a triangle, then this case is also covered by the above theorem. Otherwise, the threshold can be placed almost arbitrarily between $n^{-5/9}$ and $n^{-1/2}$ while the 2-density of H remains at 2, as our next theorem confirms. In particular, we show that there exists a class of graphs for which the threshold is not determined by the 2-densest subgraph.

For a graph H, we denote by H_P the graph obtained by adding a path of length 3 between a vertex of a K_3 and an arbitrary vertex of H, see Figure 1.

Theorem 3. *Let H be a graph which satisfies the conditions of Theorem 2. Then for $t = \min\{\frac{5}{9}, 1/m_2(H)\}$ we have*

$$\lim_{n\to\infty} \Pr[G_{n,p} \text{ is Maker's win in the } H_P\text{-game}] = \begin{cases} 1, & p \gg n^{-t}, \\ 0, & p \ll n^{-t}. \end{cases}$$

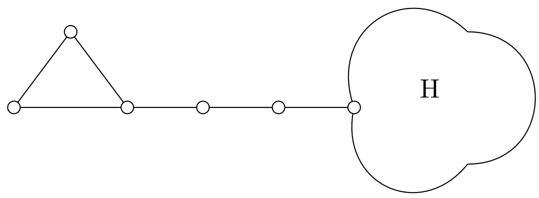

FIGURE 1. Graph H_P

Our paper is structured as follows. In the next section we collect some preliminaries. Then, in Sections 3-5 we prove Theorem 2, while in Section 6 we prove Theorem 3.

2. Preliminaries

In this section we collect some known properties about positional games, graph decompositions and random graphs. We follow the standard notation. In particular, for a graph G and a subset $A \subseteq V(G)$, we denote with $N_G(A)$ the neighborhood of A in $V(G) \setminus A$, i.e.

$$N_G(A) := \{v \in V(G) \setminus A \mid \exists a \in A \text{ such that } \{v, a\} \in E(G)\}.$$

If the graph G is clear from the context, we omit it in the subscript. Furthermore, for a graph G we use v_G and e_G to denote the number of vertices and edges of G, respectively.

2.1. Positional games.

For a Maker-Breaker game with the board X and the winning sets $\mathcal{E}$, the hypergraph $(X, \mathcal{E})$ is referred to as the *hypergraph of the game.* The following is a classical result in the theory of positional games.

Theorem 4 (Erdős-Selfridge criterion [3])**.** *Let $(X, \mathcal{E})$ be a hypergraph. Then, if Breaker has the first move in the game,*

$$\sum_{A \in \mathcal{E}} 2^{-|A|} < 1 \tag{1}$$

is a sufficient condition for Breaker's win in the game $(X, \mathcal{E})$.

To see why this condition is sufficient, consider the following strategy for Breaker: choose $x \in X$ such that $\sum_{A \in \mathcal{E}; x \in A} 2^{-|A|}$ is maximal, and denote with $\mathcal{E}'$ the set of hyperedges which does not contain x. Then Maker's move will result in a vertex $y \in X$ such that $\sum_{A \in \mathcal{E}'; y \in A} 2^{-|A|} \leq \sum_{A \in \mathcal{E}; x \in A} 2^{-|A|}$. Observe that all edges $A \in \mathcal{E}$ with $x \in A$ essentially disappear from the game, while the size of all edges $A \in \mathcal{E}'$ with $y \in A$ just shrink by one. The choice of x thus implies that the condition of the theorem remains valid and the theorem thus follows by induction.

The following result guarantees that the first player cannot claim a cycle in the game played on the union of two disjoint forests.

Theorem 5 ([4])**.** *Let $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ be two edge disjoint forests on the same vertex set V. Then if two players alternately claim unclaimed edges from $E_1 \cup E_2$, the second player can enforce that the edges of the first player span a forest.*

Finally, the following result determines the threshold for the K_3-game.

Theorem 6 ([14])**.** *Consider the K_3-game (i.e. the* triangle *game) played on the edge set of $G_{n,p}$. Then*

$$\lim_{n\to\infty} \Pr[G_{n,p} \text{ is Maker's win in the } K_3\text{-game}] = \begin{cases} 1, & p \gg n^{-5/9}, \\ 0, & p \ll n^{-5/9}. \end{cases}$$

2.2. **Graph decompositions.**

Theorem 7 (Nash-Williams' arboricity theorem [8])**.** *Any graph G can be decomposed into $\lceil ar(G) \rceil$ edge-disjoint forests, where*

$$ar(G) = \max_{G' \subseteq G} \frac{e(G')}{v(G') - 1}.$$

The next lemma follows immediately from Hall's theorem. For convenience of the reader we add its short proof.

Lemma 8. *The edges of any graph G can be oriented such that the maximal out-degree is at most $\lceil m(G) \rceil$, where*

$$m(G) = \max_{G' \subseteq G} \frac{e(G')}{v(G')}.$$

Proof. Let $k := \lceil m(G) \rceil$. We construct a bipartite graph $\hat{G}$ as follows. One vertex class consists of all edges of G (class P_e) and the other of k copies of each vertex of G (class P_v). Furthermore, we add an edge between edge e and a vertex v if and only if v is an endpoint of e in G. It follows immediately from the definition of $m(G)$ and the construction of $\hat{G}$ that $\hat{G}$ satisfies Hall's condition with respect to the class P_e. Thus, $\hat{G}$ contains a matching M that covers the set P_e. Orient an edge $e = \{v, u\}$ of G towards u if $\{e, v\}$ belongs to M (for some copy of v in P_v). Since each vertex appears only k times in P_v, we deduce from the construction that the out-degree of each vertex is bounded by k. Since M covers P_e, this process describes the orientation of every edge. □

2.3. **Hypergraph containers.** For the proof of the 1-statement of Theorem 2, we need the following consequence of the container theorems of Balogh, Morris, and Samotij [1] and Saxton and Thomason [13]. The following theorem for all graphs H is from [13]. A similar statement is obtained in [1] for all 2-balanced graphs H.

Definition 9. For a given set S, let $\mathcal{T}_{k,s}(S)$ be the family of k-tuples of subsets defined as follows,

$$\mathcal{T}_{k,s}(S) := \left\{ (S_1, \dots, S_k) \;\middle|\; S_i \subseteq S \text{ for } 1 \le i \le k \text{ and } \left| \bigcup_{i=1}^{k} S_i \right| \le s \right\}.$$

Theorem 10 ([13], Theorem 1.3)**.** *For any graph H there exist constants $n_0, s \in \mathbb{N}$ and $\delta < 1$ such that the following is true. For every $n \ge n_0$ there exists $t = t(n)$, pairwise distinct tuples $T_1, \dots, T_t \in \mathcal{T}_{s, sn^{2-1/m_2(H)}}(E(K_n))$ and sets $C_1, \dots, C_t \subseteq E(K_n)$, such that*

(a) each C_i contains at most $(1-\delta)\binom{n}{2}$ edges,
(b) for every H-free graph G on n vertices there exists $1 \le i \le t$ such that $T_i \subseteq E(G) \subseteq C_i$. (Here $T_i \subseteq E(G)$ means that all sets contained in T_i are subsets of $E(G)$.)

2.4. **Random graphs.**

Theorem 11 (Markov's Inequality)**.** *Let X be a non-negative random variable. For all $t > 0$ we have $\Pr[X \ge t] \le \frac{\mathbb{E}[X]}{t}$.*

Theorem 12 (Chernoff's Inequality)**.** *Let $X_1, \dots, X_n$ be independent Bernoulli distributed random variables with* $\Pr[X_i = 1] = p$ *and* $\Pr[X_i = 0] = 1 - p$*. Then for $X = \sum_{i=1}^n X_i$ we have*

$$\Pr[X \leq (1-\delta)\mathbb{E}[X]] \leq e^{-\mathbb{E}[X]\delta^2/2}, \quad \textit{for any} \quad 0 < \delta \leq 1.$$

The following is a standard result from the random graph theory. We include its simple proof for convenience of the reader.

Lemma 13. *Let α, c, L be positive constants and assume $p \leq cn^{-1/\alpha}$. Then w.h.p. every subgraph G' of $G_{n,p}$ on at most L vertices has density $m(G') \leq \alpha$.*

Proof. Observe that there exist only constantly many different graphs on L vertices. Let H be one such graph, and choose $\hat{H} \subseteq H$ such that $m(H) = e_{\hat{H}}/v_{\hat{H}}$. Then the expected number of $\hat{H}$-copies in $G_{n,p}$ is bounded by $n^{v_{\hat{H}}}p^{e_{\hat{H}}}$. Observe that for $p = cn^{-1/\alpha}$ we have $n^{v_{\hat{H}}}p^{e_{\hat{H}}} = o(1)$ whenever $m(H) = e_{\hat{H}}/v_{\hat{H}} > \alpha$. It thus follows from Markov's inequality that for $p \leq cn^{-1/\alpha}$ w.h.p. there is no $\hat{H}$-copy, and hence no H-copy in $G_{n,p}$. Therefore, it follows from the union bound that w.h.p. every subgraph G' of $G_{n,p}$ of size $v_{G'} \leq L$ satisfies $m(G') \leq \alpha$. □

Finally, in Section 6 we use the following lemma that follows from a standard application of Chernoff's inequality.

Lemma 14. *Let $p \gg \log n/n$ and $\varepsilon > 0$ be any constant. Then a graph $G := G_{n,p}$ satisfies w.h.p. the following property: for any subset $X \subseteq V(G)$ of size at most $1/p$ we have*

$$|N(X)| \geq (1-\varepsilon)|X|np,$$

3. Proof of the 1-statement of Theorem 2

Since we assume that Maker starts the game, the 1-statement of Theorem 2 follows directly from Theorem 1 and the strategy stealing argument. This argument can be easily augmented even for the case when Breaker starts, as the first move of Breaker typically cannot ruin the Ramsey property of the ground graph.

However, we would like to prove a strengthened version of part (i) of Theorem 2, namely that a *resilience*-type result also holds. In the proof we make use of the hypergraph containers, a new tool that seems to have potential for applications in positional games. A simplified version of this general approach was first utilized under a different name in [5], where the following observation has been put to good use – if there are two hypergraphs $\mathcal{H}_1 = (X, \mathcal{E}_1)$ and $\mathcal{H}_2 = (X, \mathcal{E}_2)$ such that every cover (set of vertices that intersects every hyperedge) of $\mathcal{H}_1$ is also a cover of $\mathcal{H}_2$, then a Breaker's win in the game played on $\mathcal{H}_1$ implies a Breaker's win on $\mathcal{H}_2$.

We note that the following theorem can alternatively be proved using the approach of derandomized Maker's strategy from [2], which is also well-suited for resilience-type results.

Theorem 15. *Let H be any graph. Then there exist constants $C > 0$ and $\gamma > 0$ such that $G := G_{n,p}$ with probability $1 - e^{-\Theta(n^2p)}$ satisfies the following: there exists a winning strategy for Maker in the H-game played on $E(G) \setminus R$, for any $R \subseteq E(G)$ with $|R| \leq \gamma \cdot n^2 p$, provided that $p \geq Cn^{-1/m_2(H)}$.*

Proof. Our proof is based on ideas of the proof from [9] of the 1-statement of Theorem 1. Note, however, that here we need to be much more careful: for the proof of Theorem 1 one has to show that *every* coloring contains a monochromatic copy of H in *some* color. Here we have to argue that we can find a *strategy* for Maker that ensures that he gets a monochromatic copy in *his* color. We achieve this by using the hypergraph game resp. Theorem 4.

Let δ and s be as given by Theorem 10 when applied on the graph H. We prove the theorem for $\gamma = \delta/16$ and C to be chosen later.

Let $G := G_{n,p}$, and consider some subset $R \subseteq E(G)$ with $|R| \leq \gamma \cdot n^2 p$. Observe that if Maker loses in the H-game on $E(G) \setminus R$, then by Theorem 10 there exists $1 \leq i \leq t$ such that $T_i \subseteq E_M \subseteq C_i$, where E_M is the set of Maker's edges.

Let us consider an auxiliary game played on the hypergraph $\mathcal{H} = (E(G) \setminus R, \mathcal{E})$ with the vertex set being the edge set of $G \setminus R$ and the edge set

$$\mathcal{E} = \{(E(K_n) \setminus C_i) \cap (G \setminus R) \; : \; T_i \subseteq G \setminus R\}.$$

In this game Breaker wins if he claims at least one edge from each set $(E(K_n) \setminus C_i) \cap (G \setminus R)$. Note that, by the previous observation, in case of Breaker's win the edge set of Breaker cannot be H-free. We can thus conclude that Maker has a winning strategy in the H-game if he has a winning strategy (as Breaker) in the auxiliary game. In the light of Theorem 4 it remains to check that the hypergraph $(E(G) \setminus R, \mathcal{E})$ satisfies condition (1).

First we show that all hyperedges typically have size at least $\delta n^2 p/16$. It follows from Theorem 10 that $|E(K_n) \setminus C_i| \geq \delta\binom{n}{2} \geq \delta n^2/4$, for every $1 \leq i \leq t(n)$, and thus from Chernoff's inequality we have

$$\Pr[|(E(K_n) \setminus C_i) \cap G| < \delta \cdot n^2 p/8] < e^{-\delta \cdot n^2 p/32}. \tag{2}$$

Let $\mathcal{B}$ be the event that there exists a hyperedge which has less than $\delta n^2 p/8$ vertices "before" the removal of R, i.e.

$$\mathcal{B} = \exists \, T_i \subseteq G \setminus R \; : \; |(E(K_n) \setminus C_i) \cap G| < \delta n^2 p/8.$$

Then

$$\Pr[\mathcal{B}] \leq \sum_{i=1}^{t(n)} \Pr[T_i \subseteq G \; \wedge \; |(E(K_n) \setminus C_i) \cap G| < \delta n^2 p/8].$$

As $T_i \subseteq C_i$, the two events are independent and we deduce

$$\begin{aligned}\Pr[\mathcal{B}] &\leq \sum_{i=1}^{t(n)} \Pr[T_i \subseteq G] \cdot \Pr[|(E(K_n) \setminus C_i) \cap G| < \delta n^2 p/8] \\ &\overset{(2)}{\leq} e^{-\delta n^2 p/32} \cdot \sum_{i=1}^{t(n)} p^{|T_i^+|},\end{aligned}$$

where T_i^+ is the union of all sets of the s-tuple T_i. Routine calculations (see [9] for details) imply that for any fixed $\varepsilon > 0$, by choosing C sufficiently large (with respect to s and ε), we have

$$\sum_{i=1}^{t(n)} p^{|T_i^+|} \leq 2^{\varepsilon n^2 p/2}. \tag{3}$$

Therefore, for a suitable chosen ε (with respect to δ), we have $\Pr[\mathcal{B}] < e^{-\Theta(n^2 p)}$. It now easily follows that

$$\Pr[\exists \, A \in \mathcal{E} \; : \; |A| < \delta \cdot n^2 p/16] = e^{-\Theta(n^2 p)},$$

regardless of the choice of R (recall that we set $\gamma = \delta/16$). Finally, observe that for the expected number of edges we have

$$\mathbb{E}[|\mathcal{E}|] \leq \sum_{i=1}^{t(n)} \Pr[T_i \subseteq G] = \sum_{i=1}^{t(n)} p^{|T_i^+|} \overset{(3)}{\leq} 2^{\varepsilon n^2 p/2}.$$

By Markov's inequality, we get

$$\Pr[|\mathcal{E}| \geq 2^{\varepsilon n^2 p}] \leq 2^{-\varepsilon n^2 p/2}.$$

Thus, with probability $1 - o(1)$, G is such that

$$\sum_{A \in \mathcal{E}} 2^{-|A|} \leq 2^{-\delta n^2 p/32 + \varepsilon n^2 p} < 1$$

for $\varepsilon > 0$ small enough. Therefore, by Theorem 4, Breaker has a winning strategy in the auxiliary game, hence by the previous discussion Maker has a winning strategy in the H-game played on $E(G) \setminus R$. □

4. Criteria for Breaker's win in an H-game

In this section we collect some graph properties that suffice for characterizing the graph as a Breaker's win in an H-game. These will be used later in the proof of the 0-statement of Theorem 2.

The following two criterions are fairly general and thus may be of independent interest.

Proposition 16. *Let G and H be graphs such that*

$$\left\lceil \frac{ar(G)}{2} \right\rceil < ar(H),$$

then Breaker can win the H-game played on the edge set of G, even if Maker starts.

Proof. Let $k := \left\lceil \frac{ar(G)}{2} \right\rceil$, and let $F_0, \ldots, F_{2k-1}$ be the edge-disjoint decomposition of G into forests which exists by Theorem 7. Assume Breaker uses the strategy from Theorem 5 for every pair of forests F_{2i} and F_{2i+1}, $0 \leq i < k$. Then Theorem 5 implies that Maker's edges can be partitioned into k forests. Any subset S of the vertex set can thus contain at most $k(|S|-1)$ Maker's edges. That is, the arboricity value for Maker's edges is at most k and, as $ar(H) > k$ by assumption, Maker's graph cannot contain H. □

Proposition 17. *Let G and H be graphs such that*

$$\left\lceil \frac{m(G)}{2} \right\rceil < m(H),$$

then Breaker can win the H-game played on the edge set of G, even if Maker starts.

Proof. Let us fix any orientation of the edges of G such that each vertex has out-degree at most $\lceil m(G) \rceil$. Such an orientation exists by Lemma 8. Now by a simple pairing strategy, it follows that Breaker can claim half of the outgoing edges of each vertex. In other words, the out-degree of each vertex, with respect to Maker's edges, is at most $\left\lceil \frac{\lceil m(G) \rceil}{2} \right\rceil = \left\lceil \frac{m(G)}{2} \right\rceil$. Therefore, by the condition of the proposition, the density of each subgraph of Maker's graph is less than $m(H)$, and thus it cannot contain H as a subgraph. □

With these two basic criteria at hand we can now prove the main theorem of this section.

Theorem 18. *Let G and H be graphs such that $m(G) \leq m_2(H)$ and H is strictly 2-balanced with at least 4 vertices. Then Breaker has a winning strategy for the H-game on the edge set of G.*

Proof. Let $m_2(H) = k + x$, for some $k \in \mathbb{N}$ and $0 \le x < 1$. We first handle the case when $0 \le x < 1/2$.

Since H is strictly 2-balanced we have

$$m_2(H) = \frac{e_H - 1}{v_H - 2} > \frac{e_H - \delta(H) - 1}{v_H - 3},$$

which easily implies $m_2(H) < \delta(H)$. For the sake of contradiction, let G be the smallest graph such that Maker has a winning strategy. We first deduce that then $\delta(G) \ge 2(\delta(H) - 1) + 1$. Assuming otherwise, let v be a vertex of degree at most $2(\delta(H) - 1)$. Then Breaker has the following winning strategy: whenever Maker claims an edge incident to v, Breaker does the same (if possible). If on the other hand Maker claims an edge from $G - \{v\}$, then Breaker follows his winning strategy for $G - \{v\}$ (which exists by choice of G). Then, clearly, Maker cannot build a copy of H in $G - \{v\}$. Further, the degree of v in the Maker's graph is at most $\delta(H) - 1$, thus it cannot be part of an H-copy either. Therefore, we have

$$m(G) \ge \frac{\sum_{v \in G} \deg(v)}{2n} \ge \delta(H) - 1/2.$$

It now follows from $m_2(H) < \delta(H)$ that $\delta(H) \ge k + 1$ and thus $m(G) \ge k + 1/2$, which is a contradiction to $m(G) \le m_2(H) < k + 1/2$.

From now on we can thus assume that $x \ge 1/2$. Next, we consider the case that $k \ge 3$. Observe that for every graph H with at least 4 vertices we have $\frac{3}{4}v_H^2 - v_H > \binom{v_H}{2} \ge e_H$, and thus

$$\frac{e_H}{v_H} + 3/2 > \frac{e_H - 1}{v_H - 2}. \tag{4}$$

Therefore $m(H) > m_2(H) - 3/2 \ge k - 1$, and so we have

$$\lceil m(G)/2 \rceil \le \lceil (k+1)/2 \rceil \overset{(k \ge 3)}{\le} k - 1 < m(H).$$

Breaker's win now follows from Proposition 17.

If H is not very dense, then a better estimate than the one in (4) can be made. In particular, $e_H < v_H^2/4$ implies that $\frac{e_H}{v_H} + 1/2 > \frac{e_H - 1}{v_H - 2}$. Since we also assumed that $x \ge 1/2$, this implies $m(H) > m_2(H) - 1/2 \ge k$. Similarly as before we have

$$\lceil m(G)/2 \rceil \le \lceil (k+1)/2 \rceil \le k < m(H),$$

and Breaker's win again follows from Proposition 17.

To summarize, so far we have shown that Breaker has a winning strategy for the H-game on graph G if one of the following holds,

(a) $0 \le x < 1/2$,
(b) $k \ge 3$, or
(c) $e_H < v_H^2/4$.

Let us consider a graph H which does not satisfy any of the above properties. Then $e_H \ge \lceil v_H^2/4 \rceil$ and thus

$$m_2(H) = \frac{e_H - 1}{v_H - 2} \ge \frac{\lceil v_H^2/4 \rceil - 1}{v_H - 2} \ge 2$$

for $v_H \ge 5$, and since H does not satisfy (a) and (b) we have $2.5 \le m_2(H) < 3$. Furthermore, it is easy to check that $ar(G) \le m(G) + 1/2$, and thus $ar(G) \le m(G) + 1/2 \le m_2(H) + 1/2 < 4$. On the other hand, from $m_2(H) \ge 2.5$ we have $e_H \ge \frac{5}{2}v_H - 4$, and thus $e_H > 2v_H - 2$ for $v_H \ge 5$, which implies $ar(H) > 2$. It follows now from $\lceil ar(G)/2 \rceil \le 2 < ar(H)$ and Proposition 16 that Breaker has a winning strategy in this case.

Finally, checking all graphs on 4 vertices we see that the only strictly 2-balanced graphs are K_4 and C_4. The case $H = K_4$ is covered by Lemma 2.1 in [7]. For

$H = C_4$ we have $ar(H) = 4/3$ and $ar(G) \leq m(G) + 1/2 \leq 2$, thus Proposition 16 implies that Breaker has a winning strategy also in this case. □

5. Proof of the 0-statement of Theorem 2

We need to show that with high probability Breaker has a strategy such that, when played on the random graph $G_{n,p}$ with $p = cn^{-1/m_2(H)}$, for $0 < c = c(H) < 1$ small enough, Maker's edges do not span an H-copy. Observe that we may assume, without loss of generality, that H is strictly 2-balanced. If not, replace H by a minimal subgraph H' with the same 2-density. Clearly, if Breaker has a strategy for winning the H'-game on $G_{n,p}$, then the same strategy prevents Maker from obtaining an H-copy.

Let us first give an intuition behind the Breaker's strategy. Observe that the expected number of copies of H on any given edge is bounded by

$$v_H^2 \cdot n^{v_H-2} \cdot p^{e_H-1} = v_H^2 \cdot c^{e_H-1}.$$

That is, for $0 < c < 1$ small enough we expect that the copies of H are scattered 'loosely' and that we even have many edges that are not contained in any copy of H. Clearly, whether such edges are claimed by Maker or Breaker is irrelevant for the outcome of the game. Assume now we find a copy of H that contains two edges which are not contained in any other copy of H. Then Breaker can easily ensure that this H-copy will never be claimed by Maker: fix two such edges arbitrarily and as soon as Maker claims the first of these edges, claim the other edge. Clearly, in this way this specific H-copy will never be a Maker's copy. We formalize these ideas as follows.

Definition 19. We call an edge *free* if it does not belong to any copy of H, *open* if it is contained in exactly one copy of H and *closed* otherwise. Furthermore, we call a copy of H *unproblematic* if it contains at least two open edges. Otherwise we call the copy *problematic*.

Preprocessing. Before starting the game, Breaker preprocesses the graph $G := G_{n,p}$ to obtain a subgraph $\hat{G}$ (with some special properties that we exhibit below) and a sequence of pairwise disjoint sets of edges $S_1, \ldots, S_k$ of cardinality two each:

$i := 0$; $k = 0$;
$G_i := G$;
while there exists an unproblematic copy $\hat{H}$ of H in G_i
  $k \leftarrow k + 1$;
  let $S_k \leftarrow \{$ two open edges (chosen arbitrarily) of $\hat{H}$ $\}$;
  $i \leftarrow i + 1$;
  $G_i \leftarrow G_{i-1} - \{$ all open edges of $\hat{H}$ $\}$;
while there exists a free edge $e \in G_i$
  $i \leftarrow i + 1$;
  $G_i \leftarrow G_{i-1} - e$;
$\hat{G} \leftarrow G_i$

Note that within this algorithm open, free and closed are always defined with respect to the current graph G_i.

Strategy. Assuming that Breaker has a winning strategy for the H-game when played on $\hat{G}$, the winning strategy for the whole graph G is defined as follows:

if Maker claims an edge from $\hat{G}$
  claim an edge from $\hat{G}$ according to the winning strategy for $\hat{G}$;
else if Maker claims an edge from a set S_j for some $1 \leq j \leq k$
  claim the other edge from the set S_j;
else

take an arbitrary edge.

We first show that this strategy extends a winning strategy for $\hat{G}$ to a winning strategy for the whole graph.

Claim 20. *Assuming that Breaker has a winning strategy for the H-game on $\hat{G}$, Breaker claims at least one edge from every copy of H in G.*

Proof. First, consider an H-copy $\hat{H}$ which is contained in $\hat{G}$. Since Breaker is playing according to the winning strategy on $\hat{G}$, it follows that this copy has to contain at least one edge which belongs to Breaker. Secondly, consider an H-copy $\hat{H}$ which is contained in G_i but not in G_{i+1}, for some $1 \leq i \leq k$. It follows from the construction of S_i that $S_i \subset \hat{H}$, and since Breaker claims at least one edge from S_i, he also claims at least one edge from $\hat{H}$. □

It remains to show that there exists a winning strategy for $\hat{G}$. In order to state the argument concisely, we introduce some notation.

Definition 21. An *H-core* of G is a maximal subgraph $G' \subseteq G$ (with respect to inclusion) that has the following two properties: every edge of G' is contained in at least one copy of H and every copy of H in G' is problematic.

Recall that, by construction, $\hat{G}$ is an H-core. The following claim shows that it is the unique H-core.

Claim 22. *There exists a unique H-core.*

Proof. Let us assume that there exist two different H-cores, say G' and G''. Then $G' \not\subseteq G''$ and $G'' \not\subseteq G'$, so $G_s = G' \cup G''$ is a proper superset of G' and G''. Therefore, to reach a contradiction to the maximality of G' and G'' it suffices to show that G_s is an H-core.

First, it is easy to see that every edge of G_s is contained in at least one copy of H. Further, observe that every H-copy which is problematic in G' or G'' remains problematic in G_s as well. Thus, if an H-copy in G_s is unproblematic then it cannot be contained in G' nor in G''. Consider such an H-copy $\hat{H}$ and consider an arbitrary edge $e \in \hat{H}$. Then e is contained in at least one of G' and G'' and thus, by the definition of G' and G'', e is also contained in a copy of H different from $\hat{H}$. Therefore e is closed in G_s, and thus $\hat{H}$ is problematic implying that G_s is an H-core. □

We say that a subgraph G' of the H-core of G is *H-closed* if every copy of H from the H-core is either contained in G' or edge-disjoint with G'. It is easy to see that the edges of the H-core can be partitioned into minimal H-closed subgraphs where minimal is with respect to subgraph inclusion. Furthermore, as all minimal H-closed subgraphs are edge disjoint, Breaker can consider each such subgraph independently.

The core of our argument is the following lemma which states that with high probability every minimal H-closed subgraph in the H-core of $G_{n,p}$ has constant size.

Lemma 23. *Let H be a strictly 2-balanced graph which is not a tree or a triangle. Then there exist constants $c > 0$ and $L > 0$ such that w.h.p. every minimal H-closed subgraph of the H-core of $G_{n,p}$ has size at most L, provided that $p \leq cn^{-1/m_2(H)}$.*

Before we prove Lemma 23, we first show how it implies the 0-statement of Theorem 2.

Proof of the 0*-statement of Theorem 2.* Let $G := G_{n,p}$, and let Breaker play as described. Recall that, by Claim 20, it suffices to show that there exists a winning strategy for the H-core $\hat{G}$ of G. Furthermore, by the definition of H-closed subgraphs, we only have to find a winning strategy for all minimal H-closed subgraphs of the H-core.

From Lemma 23 we know that w.h.p. the graph G is such that all minimal H-closed subgraphs have size at most $L = L(H)$. From Lemma 13 we know that w.h.p. the graph G is such that this implies that all minimal H-closed subgraphs have density at most $m_2(H)$. Theorem 18 thus implies that there exists a winning strategy for Breaker for all minimal H-closed subgraphs – and thus also for the H-core $\hat{G}$, which together with Claim 20 finishes the proof. □

It remains to prove Lemma 23. We do this in the remainder of this section.

Actually, our proof of Lemma 23 follows the proof of Lemma 6 from [9]. The main difference is that in [9] a problematic copy of H was defined as a copy of H in which *all* edges are contained in two copies of H, while the definition in this paper allows the existence of one (but only one) edge that may be open. As we shall see, this difference in definition is responsible for the fact that the proof goes through for triangles in [9], but does not here. Of course, this is no coincidence: for the Random Ramsey result that was considered in [9] the threshold for triangles is $p = n^{-1/m_2(K_3)} = n^{-1/2}$ [6], while for the Maker-Breaker game considered in this paper the threshold for triangles is $n^{-5/9}$ [14]. In the following we repeat the main arguments from [9], for the convenience of the reader.

We define a process that generates H-closed structures iteratively starting from a single copy of H. Assume that we have fixed an (arbitrary) total ordering ω of the edges of $G_{n,p}$, and let G' be a minimal H-closed subgraph of the H-core of $G_{n,p}$. Then G' can be generated by starting with an arbitrary H-copy in G' and repeatedly attaching H-copies to the graph constructed so far, as described in the following procedure.

Let H_0 be an H-copy in G',
$k \leftarrow 0$; $\hat{G} \leftarrow H_0$;
while $\hat{G} \neq G'$ **do**
    $k \leftarrow k+1$;
    if $\hat{G}$ contains a copy of H that is unproblematic in $\hat{G}$ **then**
        let $\ell < k$ be the smallest index such that H_ℓ is
            a copy of H that is unproblematic in $\hat{G}$;
        let e be the ω-minimum edge in H_ℓ which
            is open in $\hat{G}$ and closed in G';
        let H_k be an H-copy in G' that contains e but is
            not contained in $\hat{G}$;
    else
        let H_k be an H-copy in G' that is not contained
            in $\hat{G}$ and intersects $\hat{G}$ in at least one edge;
    $\hat{G} \leftarrow \hat{G} \cup H_k$;

In order to show that w.h.p. the highest value the parameter k reaches is bounded by a constant, we first collect some properties of this process. Consider the H-copy H_i. We distinguish two cases: a) if H_i intersects $\bigcup_{j<i} H_j$ in *exactly* one edge, we call this a *regular* copy, and b) if H_i intersects $\bigcup_{j<i} H_j$ in some subgraph D with $v_D \geq 3$, we call this a *degenerate* copy. Let us denote with $\mathrm{reg}(\ell)$ and $\deg(\ell)$ the number of H-copies H_i, $1 \leq i \leq \ell$, which are regular, resp. degenerate. Furthermore, for $0 \leq i \leq \ell$ we say that the copy H_i is *fully-open* at time ℓ if H_i is a regular copy and no new vertex of H_i, i.e., no vertex of $V(H_i) \setminus (\bigcup_{j<i} V(H_j))$, is touched by

any of the copies $H_{i+1}, \ldots, H_\ell$. Let us denote with $f_o(\ell)$ the number of fully-open copies at time ℓ. The following lemma implies that every fully-open copy at time ℓ contains exactly $e_H - 1$ open edges.

Lemma 24 (Lemma 8 in [9])**.** *Let H be strictly 2-balanced, let G be an arbitrary graph and let h_e be an edge of G. Construct a graph G_H by attaching H to an edge h_e. Then G_H has the property that if $\hat{H}$ is an H-copy in G_H that contains at least one vertex from H that is not incident with h_e, then $\hat{H} = H$.*

For $\ell \geq 1$, let

$$\Delta(\ell) := |\{i < \ell \ : \ H_i \text{ fully-open at time } \ell - 1, \ \text{ but not at time } \ \ell\}|.$$

Clearly, $\Delta(\ell) \leq 1$ if H_ℓ is a regular copy, and $\Delta(\ell) \leq v_H - 1$ if H_ℓ is a degenerate copy. The following claim is from [9] (Claim 10); the only difference is that we here have $e_H - 3$ while in [9] we had $e_H - 2$. (This difference comes from the fact the we now allow one open edge.)

Claim 25. *For any sequence $H_i, \ldots, H_{i+e_H-3}$ of consecutive regular copies such that $\Delta(i) = 1$ we have $\Delta(i+1) = \ldots = \Delta(i + e_H - 3) = 0$.* □

Similarly, the next claim is proven exactly as Claim 11 in [9], with $e_H - 1$ (there) replaced by $e_H - 2$ (here).

Claim 26. *For every $\ell \geq 1$, assuming the process does not stop before adding the ℓ-th copy, we have*

$$f_o(\ell) \geq \text{reg}(\ell) \left(1 - \frac{1}{e_H - 2}\right) - \deg(\ell) \cdot v_H.$$

□

Observe that this bound on $f_o(\ell)$ is only meaningful if $e_H \geq 4$. This is the reason why the proof does not go through for the case of triangles.

If $f_o(\ell) > 0$ for some $\ell \geq 1$, then H_ℓ cannot be the last copy in the process, as there exists at least one H-copy with at least $e_H - 1 \geq 2$ open edges, which cannot be by the definition of the H-core. Furthermore, from Claim 26 we have that after adding L copies, out of which at most ξ are degenerate, there are still at least

$$(L - \xi)(1 - 1/(e_H - 2)) - \xi \cdot v_F \tag{5}$$

fully-open copies at time L.

In a first moment calculation we have to multiply the number of choices for H_ℓ with the probability that the chosen H-copy is in $G_{n,p}$. For a regular copy where H_ℓ is attached to an open edge, the open edge to which it is attached is given deterministically by the design of our algorithm, provided that $f_o(\ell) > 0$. We just have to choose the edge (and orientation) in the new copy that we attach to it. Thus, this term is bounded by

$$2e_H \cdot n^{v_H - 2} \cdot p^{e_H - 1} \leq 2e_H \cdot c < \tfrac{1}{2}, \tag{6}$$

for $0 < c < 1/(4e_H)$. For a regular copy H_ℓ that is either attached to a closed edge or to an open edge and $f_o(\ell) = 0$, the edge to which we attach the regular copy is not given deterministically so we have to choose two vertices to which we attach H_ℓ, which we can do in at most $(\ell \cdot v_H)^2$ ways.

To bound the term for degenerate copies one first easily checks (see [9]) that there exists an $\alpha > 0$ such that

$$(v_H - v_J) - \tfrac{e_H - e_J}{m_2(H)} < -\alpha, \qquad \text{for all } J \subsetneq H \text{ with } v_J \geq 3.$$

Thus, we can bound the case that the copy H_ℓ is a degenerate copy by

$$\sum_{J \subsetneq H, v_J \geq 3} (\ell \cdot v_H)^{v_J} \cdot n^{v_H - v_J} \cdot p^{e_H - e_J} < (\ell \cdot v_H \cdot 2^{e_H})^{v_H} \cdot n^{-\alpha}, \tag{7}$$

with room to spare.

With these preparations at hand we can now finish the proof exactly as in [9] by a union bound argument, choosing ξ such that $\xi \cdot \alpha > v_H + 1$ and L such that the term in (5) is positive. Informally, in [9] it is shown that there are w.h.p. at most ξ degenerate steps within the first $\Theta(\log n)$ steps. Furthermore, if the process doesn't stop before the L-th step then the term in (5) stays positive until at least $(\xi + 1)$ degenerate steps occur, and by the previous observation this doesn't happen before the $\Theta(\log n)$-th step. Finally, we show that w.h.p. the process cannot run for $\Theta(\log n)$ steps. We skip the details.

6. Proof of Theorem 3

In the following proof we use M to denote Maker's graph.

Proof of Theorem 3. If $m_2(H) \geq 2$, then H_P satisfies the condition of Theorem 2, and the conclusion of the theorem trivially follows. Therefore, we can assume that $m_2(H) < 2$.

Assume $p \ll n^{-t}$. If $t = 5/9$, then by Theorem 6 Breaker can prevent Maker from creating a copy of K_3, and if $t = 1/m_2(H) < \frac{5}{9}$, then by Theorem 2 Breaker can prevent Maker from creating a copy of H. In any case, there exists a subgraph of H_P which Maker cannot create, thus Breaker wins in the H_P-game.

So, let now $p \gg n^{-t}$ and let $G := G_{n,p}$ be such that it satisfies the property given in Lemma 14 with $\varepsilon = 1/2$. As $t > \frac{1}{2}$, without loss of generality we can add a technical assumption that $p \ll n^{-1/2}$. We split the strategy of Maker into several phases.

Phase 1. Since $m(K_5^-) = 5/9$ (where K_5^- is a complete graph on 5 vertices with one arbitrary edge removed), G contains w.h.p. a copy of K_5^-. Denote with $\hat{K}$ one such copy. It is not hard to check that playing only on the edges of $\hat{K}$, Maker can create a copy of K_3 in at most 4 moves [14]. Let $K = \{v_1, v_2, v_3\}$ be the vertices of the obtained K_3-copy.
Phase 2. It follows from Lemma 14 that w.h.p. every vertex has at least $np/2 \gg 1/(np^2)$ incident edges in G. Thus, in the next $8/np^2$ rounds Maker can claim edges such that the set $N_1 = N_M(v_1) \setminus K$ has size $8/np^2$.
Phase 3. Again, from Lemma 14 and $1/p \gg |N_1|$ we have that w.h.p. $|N_G(N_1)| \geq \frac{1}{2}|N_1|np \geq 4/p$ and thus, with room to spare, $|N_G(N_1) \setminus K| \geq 3/p$. Therefore, regardless of Breaker's moves so far, in the next $1/p$ rounds Maker can claim edges such that the set $N_2 = N_M(N_1) \setminus K$ is of size $1/p$.
Phase 4. It again follows from Lemma 14 that $|N_G(N_2)| \geq \frac{1}{2}|N_2|np \geq n/2$ w.h.p. Again, regardless of Breaker's moves, in the next $n/6$ rounds Maker can easily claim edges such that the set $N_3 = N_M(N_2) \setminus (N_1 \cup K)$ is of size $n/6$.
Phase 5. Maker creates a copy of H in the induced subgraph $G[N_3]$.

It remains to show that the last step (Phase 5) is indeed w.h.p. possible. First, observe that until this phase, only $o(n^2p)$ rounds have been played. In other words, assuming that n is sufficiently large, we know that less than $\frac{\gamma}{6^2} \cdot n^2p$ rounds have been played to this point, where γ is the constant given by Theorem 15. On the other hand, it follows by a union bound that statement of Theorem 15 holds w.h.p.

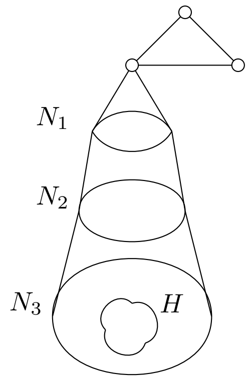

FIGURE 2. Evolution of Maker's graph in H_p-game.

for every induced subgraph of G on $n/6$ vertices,

$$\Pr[\exists S \subseteq V(G) \;:\; |S| = n/6,\ G[S] \text{ does not satisfy Theorem 15 }] \leq$$
$$\leq \binom{n}{n/6} \cdot e^{-\Theta(n^2p)} \leq e^{n-\Theta(n^2p)} = o(1).$$

Therefore, we can assume that $G[N_3]$ satisfies the statement of Theorem 15. Let $R \subset E(G)$ the set of Breaker's edges, and by previous observation we have $|R| \leq \frac{\gamma}{6^2} \cdot n^2p$. Therefore, Maker can create a copy of H in $G[N_3] \setminus R$, and by construction of set N_3 any such copy of H closes a copy of H_P in Maker's graph, see Figure 2. This completes the proof of Theorem 3. $\square$

We close this section by mentioning that the phenomena of Theorem 3 do hold for 2-connected graphs as well. For example, if we connect two vertices of the triangle by a path, then the threshold of the resulting graph will also depend on the length of this path. Let C_3^+ and C_6^+ be as defined in Figures 3 and 4.

Adapting the proof of the 0-statement of Theorem 2 one can show that Breaker wins the C_3^+-game on $G_{n,p}$ whenever $p \leq n^{-1/2-\varepsilon}$ for some $\varepsilon > 0$. In addition, it follows from Theorem 15 that there exists a positive constant C such that w.h.p. Maker has a winning strategy in the C_3^+-game, provided that $p \geq Cn^{-1/m_2(C_3^+)} = Cn^{-1/2}$.

For C_6^+ it follows from Theorem 6 that Breaker can prevent Maker from obtaining a copy of K_3 (and thus of C_6^+ as well), whenever $p \ll n^{-5/9}$. On the other hand, adapting the ideas of the proof of Theorem 3 one can show that for $p \gg n^{-5/9}$ Maker has a winning strategy.

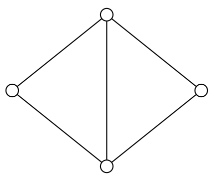

FIGURE 3. C_3^+ graph

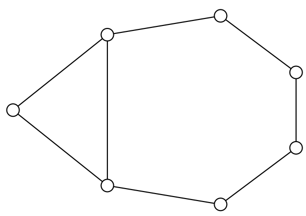

FIGURE 4. C_6^+ graph

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