Mantel's Theorem for Random Hypergraphs

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Abstract

A classical result in extremal graph theory is Mantel's Theorem, which states that every maximum triangle-free subgraph of K_n is bipartite. A sparse version of Mantel's Theorem is that, for sufficiently large p, every maximum triangle-free subgraph of $G(n,p)$ is w.h.p. bipartite. Recently, DeMarco and Kahn proved this for $p > K\sqrt{\log n/n}$ for some constant K, and apart from the value of the constant this bound is best possible.

We study an extremal problem of this type in random hypergraphs. Denote by F_5, which sometimes called as the generalized triangle, the 3-uniform hypergraph with vertex set $\{a,b,c,d,e\}$ and edge set $\{abc, ade, bde\}$. One of the first extremal results in extremal hypergraph theory is by Frankl and Füredi, who proved that the maximum 3-uniform hypergraph on n vertices containing no copy of F_5 is tripartite for $n > 3000$. A natural question is for what p is every maximum F_5-free subhypergraph of $G^3(n,p)$ w.h.p. tripartite. We show this holds for $p > K\log n/n$ for some constant K and does not hold for $p = 0.1\sqrt{\log n}/n$.

Keywords: Turán number, random hypergraphs, extremal problems.

1 Introduction

A classical result in extremal graph theory is Mantel's Theorem [13], which states that every K_3-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges. Furthermore, the complete bipartite graph whose partite sets differ in size by at most one is the unique K_3-free graph that achieves this bound. In other words, every maximum (with respect to the number of edges) triangle-free subgraph of K_n is bipartite.

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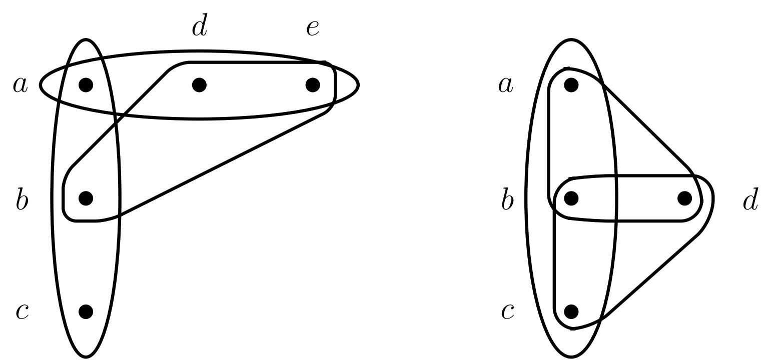
A sparse version of Mantel's Theorem has recently been proved by DeMarco and Kahn [8]: Let $G(n,p)$ be the usual Erdős-Rényi random graph. An event occurs *with high probability* (w.h.p.) if the probability of that event approaches 1 as n tends to infinity. We are interested to determine for what p every maximum triangle-free subgraph of $G(n,p)$ is w.h.p. bipartite. DeMarco and Kahn proved that every maximum triangle-free subgraph of $G(n,p)$ is w.h.p. bipartite if $p > K\sqrt{\log n/n}$ for some large constant K. If $p = 0.1\sqrt{\log n/n}$, then w.h.p. there is a C_5 in $G(n,p)$ whose edges are not in any triangle, therefore any maximum triangle-free subgraph of $G(n,p)$ contains this C_5 and is not bipartite. So apart from the value of the constant the result of DeMarco and Kahn is best possible.

Problems of this type were first considered by Babai, Simonovits and Spencer [1]. Brightwell, Panagiotou and Steger [5] proved the existence of a constant c, depending only on ℓ, such that whenever $p \geq n^{-c}$, w.h.p. every maximum K_ℓ-free subgraph of $G(n,p)$ is $(\ell-1)$-partite, and recently, DeMarco and Kahn [9] found the appropriate range of p for this problem. Here, we study an extremal problem of this type in random hypergraphs.

Definition. For $n \in \mathbb{N}$ and $p \in [0,1]$, let $G^r(n,p)$ be a random r-uniform hypergraph with n vertices and each element of $\binom{[n]}{r}$ occurring as an edge with probability p independently of each other. In particular, $G^2(n,p) = G(n,p)$. Denote by F_5 the 3-uniform hypergraph with vertex set $\{a,b,c,d,e\}$ and edge set $\{abc, ade, bde\}$. Denote by K_4^- the 3-uniform hypergraph with 4 vertices and 3 edges.



(a) F_5 (b) K_4^-

Figure 1: The 3-uniform hypergraph F_5 and K_4^-.

The *Turán hypergraph* $T_r(n)$ is the complete n-vertex r-uniform r-partite hypergraph whose partite sets are as equally-sized as possible. In particular, Mantel's Theorem states that the maximum triangle-free graph on n vertices is $T_2(n)$. Finding extremal graphs for 3-uniform hypergraphs is much more difficult; even the extremal hypergraph of K_4^- is not known. The best known construction is due to Frankl and Füredi [11], with asymptotically $n^3/21$ edges, so a maximum K_4^--free 3-uniform hypergraph is not tripartite. One of the first extremal results in extremal hypergraph theory is determining the extremal hypergraph of F_5. This problem F_5 was first considered by Bollobás [4], who proved results for cancellative hypergraphs, i.e., that the maximum $\{K_4^-, F_5\}$-free hypergraph is tripartite. Frankl and

Füredi [10] proved that the maximum 3-uniform hypergraph on n vertices containing no copy of F_5 is $T_3(n)$ for $n > 3000$. The hypergraph F_5 is the smallest 3-uniform hypergraph whose extremal hypergraph is $T_3(n)$.

Our main result is a random variant of the theorem of Frankl and Füredi [10], i.e., that for sufficiently large p a largest F_5-free subgraph of $G^3(n, p)$ is w.h.p. tripartite, and our p is close to best possible.

Theorem 1. *There exists a positive constant K such that w.h.p. the following is true. If $\mathcal{G} = G^3(n, p)$ is a 3-uniform random hypergraph with $p > K \log n/n$, then every maximum (with respect to the number of edges) F_5-free subhypergraph of $\mathcal{G}$ is tripartite.*

If p is very small, then an F_5-free subhypergraph of $\mathcal{G}$ is also w.h.p. tripartite since $\mathcal{G}$ itself is likely to be tripartite, so this case is not so interesting for us.

If $p = 0.1\sqrt{\log n}/n$, then w.h.p. there is a maximum F_5-free subhypergraph of $G^3(n, p)$ that is not tripartite. To see this, first using the second moment method one can prove that w.h.p. there are $n/5$ vertex disjoint copies of K_4^- in $G^3(n, p)$. Then using Janson's inequality (which computation is quite delicate, we omit the details), one can prove that w.h.p. one of them has the property that none of its edges are in an F_5. Then a maximum F_5-free subhypergraph of $G^3(n, p)$ would contain this K_4^-, and note that K_4^- is not tripartite.

We consider that the above reasoning might be optimal, therefore we conjecture that $\sqrt{\log n}/n$ is the correct order of p.

Conjecture 2. *There exists a positive constant K such that w.h.p. the following is true. If $\mathcal{G} = G^3(n, p)$ is a 3-uniform random hypergraph with $p > K\sqrt{\log n}/n$, then every maximum F_5-free subhypergraph of $\mathcal{G}$ is tripartite.*

Note that a weaker result than Theorem 1 appeared in the thesis of the second author [6]. To improve the results of [6], some ideas of [8], see Lemma 14, are used in this paper, but there are several differences as well. Our result, similar to [8], characterizes the precise structure of the extremal subgraph of the random hypergraph.

Recently, powerful general asymptotic statements were proved about extremal substructures of random discrete structures, see Balogh–Morris–Samotij [3], Conlon–Gowers [7], Samotij [14], Saxton–Thomason [15] and Schacht [16].

We shall use Theorem 1.8 of Samotij [14], which transferred a stability theorem of Keevash and Mubayi [12]:

Theorem 3. *For every $\delta > 0$ there exist positive constants K and ε such that if $p_n \geq K/n$, then w.h.p. the following holds. Every F_5-free subgraph of $G^3(n, p_n)$ with at least $(2/9 - \varepsilon)\binom{n}{3}p_n$ edges admits a partition (A_1, A_2, A_3) of $[n]$ such that all but at most $\delta n^3 p_n$ edges have one vertex in each A_i.*

The hypergraph F_5 is an example of what Balogh, Butterfield, Hu, Lenz, and Mubayi [2] call a "critical hypergraph"; they proved that if H is a critical hypergraph, then for sufficiently large n the unique largest H-free hypergraph with n vertices is the Turán hypergraph. We could prove results analogous to Theorem 1 for the family of critical hypergraphs, as some ideas of our proofs are from [2], but this extension to critical hypergraphs is likely to be very technical, and probably we would not be able to determine the whole range of p where the sparse extremal theorem is valid.

The rest of the paper is organized as follows. In Section 2 we introduce some more notation and state some standard properties of $G^3(n,p)$. In Section 3 we provide our main lemmas and prove them. We prove our main result, Theorem 1, in Section 4. To simplify the formulas, we shall often omit floor and ceiling signs when they are not crucial.

2 Notations and Preliminaries

For the remainder of the paper, $\mathcal{G}$ will always denote the 3-uniform random hypergraph $G^3(n,p)$. The *size* of a hypergraph (graph) $\mathcal{H}$, denoted $|\mathcal{H}|$, is the number of hyperedges (edges) it contains. We denote by $t(\mathcal{G})$ the size of a largest tripartite subhypergraph of $\mathcal{G}$.

We write $x = (1 \pm \varepsilon)y$ when $(1-\varepsilon)y \leq x \leq (1+\varepsilon)y$. We say $\Pi = (A_1, A_2, A_3)$ is a *balanced partition* if $|A_i| = (1 \pm 10^{-10})n/3$ for all i. Given a partition $\Pi = (A_1, A_2, A_3)$ and a 3-uniform hypergraph $\mathcal{H}$, we say that an edge e of $\mathcal{H}$ is *crossing* if $e \cap A_i$ is non-empty for every i. We use $\mathcal{H}[\Pi]$ to denote the set of crossing edges of $\mathcal{H}$.

The *link graph* $L(v)$ of a vertex v in $\mathcal{G}$ is the graph induced by the edge set $\{xy : xyv \in \mathcal{G}\}$. The *crossing link graph* $L_\Pi(v)$ of a vertex v is the subgraph of $L(v)$ whose edge set is $\{xy : xyv$ is a crossing edge of $\mathcal{G}\}$. The *degree* $d(v)$ of v is the size of $L(v)$ (i.e. the number of edges in $L(v)$), and the *crossing degree* $d_\Pi(v)$ of v is the size of $L_\Pi(v)$. The *common link graph* $L(u,v)$ of two vertices u and v is $L(u) \cap L(v)$ and the *common degree* $d(u,v)$ is the size of $L(u,v)$. The *common crossing link graph* $L_\Pi(u,v)$ of two vertices u and v is $L_\Pi(u) \cap L_\Pi(v)$ and the *common crossing degree* $d_\Pi(u,v)$ is the size of $L_\Pi(u,v)$. Given two vertices u and v, their *co-neighborhood* $N(u,v)$ is $\{x : xuv \in \mathcal{G}\}$; the *co-degree* of u and v is the number of vertices in their co-neighborhood.

Given two disjoint sets A and B, we use $[A,B]$ to denote the set $\{a \cup b : a \in A, b \in B\}$. We will use this notation in two contexts. First, if both A and B are vertex sets, then $[A,B]$ is a complete bipartite graph. Second, if A is a subset of a vertex set V and B is a set of *pairs* of vertices of $V \setminus A$, then $[A,B]$ is a 3-uniform hypergraph. In these two contexts, given a graph or hypergraph $\mathcal{H}$, let $\mathcal{H}[A,B]$ denote the set $\mathcal{H} \cap [A,B]$. Note that in the first case $\mathcal{H}[A,B]$ is the bipartite subgraph of $\mathcal{H}$ induced by A and B. In the second case, $\mathcal{H}[A,B]$ is the 3-uniform subhypergraph of $\mathcal{H}$ whose edges have exactly one vertex in A and contain a pair of vertices from B.

We say a vertex partition Π with three classes, which we will call a 3-*partition*, is *maximum* if $|\mathcal{G}[\Pi]| = t(\mathcal{G})$. Let $\mathcal{F}$ be a maximum F_5-free subhypergraph of $\mathcal{G}$. Clearly $t(\mathcal{G}) \leq |\mathcal{F}|$. To prove Theorem 1, we will show that w.h.p. $|\mathcal{F}| \leq t(\mathcal{G})$ is also true for an appropriate choice of p. Moreover, we will prove that if $\mathcal{F}$ is not tripartite, then w.h.p. $|\mathcal{F}| < t(\mathcal{G})$.

We will make use of the following Chernoff-type bound to prove Propositions 5-11, which state useful properties of $G^3(n,p)$. The proofs of those propositions are standard applications of the Chernoff bound, therefore we omit some of them.

Lemma 4. *Let Y be the sum of mutually independent indicator random variables, and let $\mu = E[Y]$. For every $\varepsilon > 0$,*

$$P[|Y - \mu| > \varepsilon\mu] < 2e^{-c_\varepsilon \mu},$$

where $c_\varepsilon = \min\{-\ln\left(e^\varepsilon(1+\varepsilon)^{-(1+\varepsilon)}\right), \varepsilon^2/2\}$.

For the rest of this paper, we always use c_ε (which depends on ε) to denote the constant in Lemma 4.

Proposition 5. *For any $\varepsilon > 0$, there exists a constant K such that if $p > K\log n/n$, then w.h.p. the co-degree of any pair of vertices in $\mathcal{G}$ is $(1 \pm \varepsilon)pn$.*

Proposition 6. *For any $\varepsilon > 0$, there exists a constant K such that if $p > K\sqrt{\log n}/n$, then w.h.p. the common degree $d(x, y)$ of any pair of vertices (x, y) in $\mathcal{G}$ is $(1 \pm \varepsilon)p^2n^2/2$.*

Proposition 7. *For any $\varepsilon > 0$, there exists a constant K such that if $p > K\log n/n^2$, then w.h.p. for any vertex v of $\mathcal{G}$, we have $d(v) = (1 \pm \varepsilon)pn^2/2$.*

Proposition 8. *For any $\varepsilon > 0$, there exists a constant K such that if $p > K/n$, then w.h.p. for any 3-partition $\Pi = (A_1, A_2, A_3)$ with $|A_2|, |A_3| \geq n/20$ and any vertex $v \in A_1$, we have $d_\Pi(v) = (1 \pm \varepsilon)p|A_2||A_3|$.*

For a vertex v and a vertex set S, let $\mathcal{E}$ be a subset of $\{vxw \in \mathcal{G} : x \in S\}$ satisfying that for every $x \in S$, there exists an $e \in \mathcal{E}$ such that $x \in e$, and let T be a subset of $L(v)$. Define

$$\mathcal{K}_{v,\mathcal{E}}[S, T] = \{xyz : x \in S, yz \in T, \exists e \in \mathcal{E} \text{ s.t. } x \in e, \quad y, z \notin e\},$$

and $\mathcal{G}_{v,\mathcal{E}}[S, T] := \mathcal{K}_{v,\mathcal{E}}[S, T] \cap \mathcal{G}$. Observe, since $\mathcal{G}$ is the random hypergraph, we have

$$\mathbb{E}[|\mathcal{G}_{v,\mathcal{E}}[S, T]|] = p\,|\mathcal{K}_{v,\mathcal{E}}[S, T]|\,.$$

Then for any $xyz \in \mathcal{G}_{v,\mathcal{E}}[S, T]$ with $x \in S, yz \in T$, we can find an $F_5 = \{vxw, vyz, xyz\}$ where $vxw \in \mathcal{E}$. The condition $y, z \notin e$ in the definition of $\mathcal{G}_{v,\mathcal{E}}[S, T]$ guarantees that we can find such an F_5 instead of only a K_4^-. The somewhat artificial definition of $\mathcal{G}_{v,\mathcal{E}}[S, T]$ is needed for given v, $\mathcal{E}$, S and T to forbid many hyperedges, which could create an F_5.

Proposition 9. *For any constants $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$, there exists a constant K such that if $p > K\log n/n$, then w.h.p. for every choices of $\{v, S, \mathcal{E}, T\}$ as above with $|S| \geq \varepsilon_1 n$ and $|T| \geq \varepsilon_2 pn^2$, we have $|\mathcal{G}_{v,\mathcal{E}}[S, T]| = (1 \pm \varepsilon)p|S||T|$.*

Proof. Note that here we first reveal the edges containing v; given this choice we fix S, T and $\mathcal{E}$. We also assume that the conclusions of the previous propositions hold. For $x \in S$, let $d_\mathcal{E}(x) = |\{e \in \mathcal{E} : x \in e\}|$ and $T_x = \{yz \in T : vxy \in \mathcal{E}\}$. If $d_\mathcal{E}(x) > 2$, then clearly $[x, T] \subseteq \mathcal{K}_{v,\mathcal{E}}[S, T]$. If $d_\mathcal{E}(x) \leq 2$, then by Proposition 5, we have $|T_x| \leq 2 \cdot 2pn = 4pn$, and clearly $[x, T \setminus T_x] \subseteq \mathcal{K}_{v,\mathcal{E}}[S, T]$. Therefore,

$$|[S, T]| - |\mathcal{K}_{v,\mathcal{E}}[S, T]| \leq \sum_{x \in S, d_\mathcal{E}(x) \leq 2} |T_x| \leq |S| \cdot 4pn.$$

We have $|[S,T]| = |S||T| \geq |S_1|\varepsilon_2 pn^2$, so $|\mathcal{K}_{v,\mathcal{E}}[S,T]| = (1-o(1))|S||T|$. Let $\mu = E[|\mathcal{G}_{v,\mathcal{E}}[S,T]|] = p\,|\mathcal{K}_{v,\mathcal{E}}[S,T]| = (1-o(1))p|S||T|$. By Lemma 4 we have

$$\mathbb{P}\left[||\mathcal{G}_{v,\mathcal{E}}[S,T]| - \mu| > \varepsilon\mu\right] < 2e^{-c_\varepsilon \mu}.$$

For sets S, T with $|S| = s$ and $|T| = t$, we have at most n choices for v, $\binom{n}{s}$ choices for S, 2^{2spn} choices for $\mathcal{E}$ and $\binom{pn^2}{t}$ choices for T. Then by the union bound, the probability that the statement of Proposition 9 does not hold is bounded by

$$\sum_{s \geq \varepsilon_1 n} \sum_{t \geq \varepsilon_2 pn^2} n\binom{n}{s} 2^{2spn}\binom{pn^2}{t} 2e^{-c_\varepsilon \mu} \leq \sum_{s \geq \varepsilon_1 n} \sum_{t \geq \varepsilon_2 pn^2} n\binom{n}{s} 2^{2spn}\binom{pn^2}{t} 2e^{-c_\varepsilon st/2} = o(1). \quad \square$$

The proof of the following proposition is based on Theorem 3.

Proposition 10. *Let δ be a small positive constant. Then there is an $\varepsilon > 0$ and a large constant $K = K(\delta, \varepsilon)$ such that if $p > K/n$ and $\mathcal{F}$ is a maximum F_5-free subhypergraph of $\mathcal{G}$, then $|\mathcal{F}| \geq (2/9 - \varepsilon)\binom{n}{3}p$. Furthermore, for every F_5-free subhypergraph $\mathcal{F}$ of $\mathcal{G}$ and a 3-partition Π maximizing $|\mathcal{F}[\Pi]|$, where $|\mathcal{F}| \geq (2/9-\varepsilon)\binom{n}{3}p$, the partition $\Pi = (A_1, A_2, A_3)$ is balanced and all but at most $\delta n^3 p$ edges have one vertex in each A_i.*

Proof. We may assume that $\delta, \varepsilon < 10^{-100}$, where ε is determined by Theorem 3 given δ. For a partition $\Pi = (A_1, A_2, A_3)$, Proposition 8 implies that w.h.p. $|\mathcal{G}[\Pi]| = (1 \pm \varepsilon)p|A_1||A_2||A_3|$ if $|A_2|, |A_3| \geq n/20$. Clearly $|\mathcal{F}| \geq t(\mathcal{G}) \geq |\mathcal{G}[\Pi]|$. If $|A_1| = |A_2| = |A_3| = n/3$, then we have $|\mathcal{F}| \geq (2/9-\varepsilon)\binom{n}{3}p$. Theorem 3 implies that if Π maximizes $|\mathcal{F}[\Pi]|$, then $\mathcal{G}[\Pi] \geq \mathcal{F}[\Pi] \geq (2/9 - \varepsilon - \delta)\binom{n}{3}p$, and the number of non-crossing edges is at most δpn^3.

If Π is not balanced and $|A_2|, |A_3| \geq n/20$, then $|\mathcal{G}[\Pi]| \leq (1+\varepsilon)p|A_1||A_2||A_3| < (2/9 - 2\varepsilon)\binom{n}{3}p$. If Π is not balanced and one of $|A_1|, |A_2|, |A_3|$ is less than $n/20$, then Proposition 7 implies that $|\mathcal{G}[\Pi]| < n/20 \cdot (1+\varepsilon)pn^2/2 < (2/9 - \varepsilon - \delta)\binom{n}{3}p$. Therefore, if Π maximizes $|\mathcal{F}[\Pi]|$, then Π is balanced. $\square$

Given a balanced partition $\Pi = (A_1, A_2, A_3)$, let $Q(\Pi) = \{(u,v) \in \binom{A_1}{2} : d_\Pi(u,v) < 0.8n^2p^2/9\}$. In words, $Q(\Pi)$ is the set of pairs of vertices in A_1 that have low common crossing degree (in $\mathcal{G}$).

Proposition 11. *There exists a constant K such that if $p > K/n$, then w.h.p. for every balanced partition Π and every vertex v, we have $d_{Q(\Pi)}(v) < 0.001/p$.*

Proof. Let $\varepsilon = 0.1$. By Proposition 8, we may assume that $d_\Pi(v) \geq (1-\varepsilon)pn^2/9$, and therefore, $d_\Pi(u,v) \leq \frac{0.8}{1-\varepsilon} d_\Pi(v)p$ for $(u,v) \in Q(\Pi)$.

If a vertex v and a balanced partition Π violate the statement of Proposition 11, then there are $S \subseteq V$ and $T = L_\Pi(v)$ with $|S| := s = \lceil 0.001/p \rceil$ and $|\mathcal{G}[S,T]| \leq \frac{0.8}{1-\varepsilon}|S||T|p$. We have at most 3^n choices of $|\Pi|$, n choices of v, $\binom{n}{s}$ choices of S, so the probability of such a violation is at most

$$3^n n\binom{n}{s} \exp(-c \cdot 0.001/p \cdot pn^2 \cdot p) = o(1),$$

where c is some small constant. $\square$

The following statement is heavily used in the proof of Lemma 13, which is one of the two main lemmas we use to prove our main theorem.

Lemma 12. *Let a and r be positive integers. For any $\varepsilon > 0$, there exists a constant K such that if $p > K \log n/n, a \leq \varepsilon n$ and*

$$\binom{n}{a} \cdot \binom{n^2/2}{r} \cdot \exp(-c_1 \varepsilon pnr) = o(1), \tag{1}$$

then w.h.p. the following holds. For any set of vertices A with $|A| = a$, there are at most r pairs $\{u, v\} \in \binom{V(G)\setminus A}{2}$ such that $|N(u, v) \cap A| > 2\varepsilon pn$.

Proof. Fix a set A of size a. We shall show that there are at most r pairs u, v in $\binom{V(G)\setminus A}{2}$ for which $|N(u, v) \cap A|$ is large. For each pair of vertices u and v, let $B(u, v)$ be the event that $|N(u, v) \cap A| > 2\varepsilon pn \geq 2pa$. By (a slight variant of) Chernoff's inequality,

$$\mathbb{P}[B(u, v)] < e^{-c\varepsilon pn}$$

for $c = c_1$ in Lemma 4. If $\{u, v\} \neq \{u', v'\}$ then $B(u, v)$ and $B(u', v')$ are independent events. Consequently, the probability that $B(u, v)$ holds for at least r pairs is at most

$$\binom{n^2/2}{r} e^{-c\varepsilon pnr}.$$

There are $\binom{n}{a}$ choices of A. Therefore, if (1) holds, then w.h.p. there are at most r pairs $\{u, v\} \in \binom{V(G)\setminus A}{2}$ such that $|N(u, v) \cap A| > 2\varepsilon pn$. $\square$

3 Key Lemmas for Theorem 1

Let $\mathcal{F}$ be an F_5-free subhypergraph of $\mathcal{G}$; we want to show that $|\mathcal{F}| \leq t(\mathcal{G})$. The following lemma proves this with some additional conditions on $\mathcal{F}$. The *shadow graph* of a 3-uniform hypergraph $\mathcal{H}$ is the graph with xy an edge if and only if there exists some hyperedge of $\mathcal{H}$ that contains both x and y.

Lemma 13. *Let $\mathcal{F}$ be an F_5-free subhypergraph of $\mathcal{G}$ and $\Pi = (A_1, A_2, A_3)$ be a balanced partition maximizing $|\mathcal{F}[\Pi]|$. Let $\mathcal{B}_i = \{e \in \mathcal{F} : |e \cap A_i| \geq 2\}$ for $1 \leq i \leq 3$. There exist positive constants K and δ such that if $p > K \log n/n$ and if the following conditions hold:*
(i) $\sum_i |\mathcal{B}_i| \leq \delta pn^3$,
(ii) the shadow graph of $\mathcal{B}_1$ is disjoint from $Q(\Pi)$,
then w.h.p. $|\mathcal{F}[\Pi]| + 3|\mathcal{B}_1| \leq |\mathcal{G}[\Pi]|$, where equality is possible only if $\mathcal{F}$ is tripartite.

If $\mathcal{F}$ is a maximum F_5-free subhypergraph of $\mathcal{G}$ and not tripartite, then by Proposition 10, for every $\delta > 0$, w.h.p. Condition (i) of Lemma 13 holds. Without loss of generality, we may assume that $|\mathcal{B}_1| \geq |\mathcal{B}_2|, |\mathcal{B}_3|$, in particular $\mathcal{B}_1 \neq \emptyset$. If $Q(\Pi) = \emptyset$, then Condition (ii) of

Lemma 13 holds and we get $|\mathcal{F}| < t(\mathcal{G})$, a contradiction. If $Q(\Pi) = \emptyset$ for every balanced partition $\Pi = (A_1, A_2, A_3)$, then the proof would be completed. Unfortunately, we are only able to prove this property for $p > K/\sqrt{n}$ with some large K, so Lemma 13 implies that Theorem 1 is true for $p > K/\sqrt{n}$. To improve the bound on p from the order of $1/\sqrt{n}$ to $\log n/n$, we prove that $Q(\Pi) = \emptyset$ for every maximum 3-partition Π. (Recall that a 3-partition Π is maximum if $t(\mathcal{G}) = |\mathcal{G}[\Pi]|$.) This is stated in the following lemma, which says that if $Q(\Pi) \neq \emptyset$, then Π is far from being a maximum 3-partition. The proof of Lemma 14 is along the lines of the proof of Lemma 5.1 in DeMarco–Kahn [8].

Lemma 14. *There exist positive constants K and δ such that if $p > K \log n/n$ and the 3-partition Π is balanced, then w.h.p.*

$$t(\mathcal{G}) \geq |\mathcal{G}[\Pi]| + |Q(\Pi)|\delta n^2 p^2,$$

where equality is possible only if $Q(\Pi) = \emptyset$.

Note that if $Q(\Pi) = \emptyset$, then by definition $t(\mathcal{G}) \geq |\mathcal{G}[\Pi]|$. We will use Lemmas 13 and 14 to prove Theorem 1. In the next two subsections we prove these two lemmas.

3.1 Proof of Lemma 13

We will begin with a sketch of the proof of Lemma 13, which will motivate the following lemmas.

Let $\mathcal{M}$ be the set of crossing edges of $\mathcal{G} \setminus \mathcal{F}$, and assume that $|\mathcal{B}_1| \geq |\mathcal{B}_2|, |\mathcal{B}_3|$. If $|\mathcal{B}_1| = 0$, then we have $|\mathcal{F}[\Pi]| + 3|\mathcal{B}_1| \leq |\mathcal{G}[\Pi]|$, so to prove Lemma 13, it suffices to prove that if $\mathcal{B}_1 \neq \emptyset$, then $3|\mathcal{B}_1| < |\mathcal{M}|$. So we assume for contradiction that $|\mathcal{M}| \leq 3|\mathcal{B}_1| \leq 3\delta pn^3$, where the second inequality follows from Condition (i) of Lemma 13.

For each edge $e = w_1w_2w_3 \in \mathcal{B}_1$ with $w_1, w_2 \in e \cap A_1$, because $w_1w_2 \notin Q(\Pi)$, there exist at least $0.8p^2n^2/9$ choices of $y \in A_2$ and $z \in A_3$ such that w_1yz and w_2yz are both crossing edges of $\mathcal{G}$. By Proposition 5, the co-degree of w_1 and w_3 is w.h.p. at most $2pn$. Therefore, there are at least $0.8p^2n^2/9 - 2pn \geq p^2n^2/12$ choices of such pairs (y, z) such that $w_3 \notin \{y, z\}$, and then each of these pairs (y, z) together with e form a copy of $F_5 = \{w_1w_2w_3, w_1yz, w_2yz\}$ in $\mathcal{G}$. Since $\mathcal{F}$ contains no copy of F_5, at least one of w_1yz, w_2yz must be in $\mathcal{M}$.

We will count elements of $\mathcal{M}$ by counting the embeddings of F_5 in $\mathcal{G}$ that contain some $e \in \mathcal{B}_1$. Each such F_5 contains at least one edge from $\mathcal{M}$, and this will provide a lower bound on the size of $\mathcal{M}$ in terms of $|\mathcal{B}_1|$. Instead of counting copies of F_5 itself, we will count copies of $\hat{F}_5$ which is a 4-set $\{w_1, w_2, y, z\}$ such that there exists $e \in \mathcal{B}_1$ with $w_1, w_2 \in e \cap A_1$, $y, z \notin e$ and w_1yz, w_2yz being crossing edges. It is easy to see that each $\hat{F}_5$ yields at least one F_5 containing some $e \in \mathcal{B}_1$. So for each pair $w_1, w_2 \in e \cap A_1$ for any $e \in \mathcal{B}_1$, there are at least $p^2n^2/12$ copies of $\hat{F}_5$ containing w_1, w_2. We will count copies of $\hat{F}_5$ in $\mathcal{G}$ by considering several cases, based on the relative sizes of the sets C_1 and C_2, defined below.

Let

$$\varepsilon_1 = \frac{1}{960}, \qquad \varepsilon_2 = \frac{1}{400}, \qquad \delta = \frac{\varepsilon_1^2 \varepsilon_2}{108 \cdot 160} \qquad \text{and} \qquad \varepsilon_3 = \frac{108\delta}{\varepsilon_1}. \tag{2}$$

Denote by J the shadow graph of $\mathcal{B}_1$ on the vertex set A_1. Let $C = \{x \in A_1 : d_J(x) \geq \varepsilon_1 n\}$, C_1 be the set of every vertex in C that is in at least $\varepsilon_2 pn^2$ crossing edges of $\mathcal{F}$ and let $C_2 = C \setminus C_1$.

With these definitions in hand, we are prepared to prove the following lemmas, which will lead to a proof of Lemma 13 at the end of this subsection.

Lemma 15. *$|C| \leq \varepsilon_3 n$.*

Proof. For each edge $wx \in E(J)$, since $wx \notin Q(\Pi)$, there are at least $p^2n^2/12$ choices of $y \in A_2$, $z \in A_3$ such that $\{w, x, y, z\}$ spans an $\hat{F}_5$ in $\mathcal{G}$. Then $xyz, wyz \in \mathcal{G}$ and one of these two edges must be in $\mathcal{M}$, otherwise $\mathcal{F}$ contains a copy of F_5. On the other hand, by Proposition 5 with $\varepsilon = 0.5$, for each edge $xyz \in \mathcal{M}$ with $x \in A_1, y \in A_2, z \in A_3$, there are at most $3pn/2$ choices of w such that $wyz \in \mathcal{G}$. Therefore, we have $\frac{3}{2}pn|\mathcal{M}| \geq |J|p^2n^2/12$. We assume $3\delta pn^3 \geq |\mathcal{M}|$, so $\frac{3}{2}pn \cdot 3\delta pn^3 \geq \frac{3}{2}pn|\mathcal{M}|$. It follows that $54\delta n^2 \geq |J|$.

Now, every vertex in C has degree at least $\varepsilon_1 n$ in J, so $\varepsilon_1 n|C| \leq 2|J| \leq 108\delta n^2$ implies that $|C| \leq 108\delta\varepsilon_1^{-1}n = \varepsilon_3 n$. $\square$

Lemma 16. *$|\mathcal{M}| \geq 20pn^2|C_1|$.*

Proof. Assume $|C_1| \geq 1$, otherwise this inequality is trivial. For each $x \in C_1$, let $T_x := \{(y,z) \in A_2 \times A_3 \,|\, xyz \in \mathcal{F}\} \subseteq L_\Pi(x)$. By the definition of C_1, we have $|N_J(x)| \geq \varepsilon_1 n$ and $|T_x| \geq \varepsilon_2 pn^2$ for each $x \in C_1$. We will count the number of copies of $\hat{F}_5 : \{x, w, y, z\}$ in $\mathcal{G}$ with $x \in C_1, w \in N_J(x), xyz \in \mathcal{F}$ and $wyz \in \mathcal{G}$. By Proposition 9 with $v = x$, $S = N_J(x)$, $\mathcal{E} = \{e \in \mathcal{B}_1 : x \in e\}$ and $T = T_x$, there are at least $\frac{1}{2}d_J(x)|T_x|p$ such copies of $\hat{F}_5$ for each $x \in C_1$. Therefore, the total number of such copies of $\hat{F}_5$ is at least

$$\sum_{x \in C_1} \frac{1}{2}d_J(x)|T_x|p \geq \frac{1}{2}|C_1| \cdot \varepsilon_1 n \cdot \varepsilon_2 pn^2 \cdot p = \frac{1}{2}\varepsilon_1\varepsilon_2 p^2n^3|C_1|. \tag{3}$$

Say that an edge $wyz \in \mathcal{M}$ is *bad* if $w \in A_1$, $y \in A_2$, $z \in A_3$, and there are at least $2\varepsilon_3 pn$ vertices $x \in C_1$ for which $xyz \in \mathcal{G}$. Because $|C_1| \leq |C|$, which by Lemma 15 has size at most $\varepsilon_3 n$, we can apply Lemma 12 with $\varepsilon = \varepsilon_3, a = \varepsilon n, r = (\log\log n)/p$ and $A = C_1$ to show that there are at most $(\log\log n)/p$ pairs $(y,z) \in A_2 \times A_3$ that are in some bad edge. By Proposition 5, the co-degree of each such pair (y,z) is at most $2pn$. Therefore, each (y,z) is in at most $\binom{2pn}{2}$ $\hat{F}_5$'s, and so the number of copies of $\hat{F}_5$ estimated in (3) that contain a non-bad edge from $\mathcal{M}$ is at least

$$\frac{1}{2}\varepsilon_1\varepsilon_2 p^2n^3|C_1| - \binom{2pn}{2} \cdot \frac{\log\log n}{p}.$$

Now,

$$\binom{2pn}{2} \cdot \frac{\log\log n}{p} \leq 2pn^2\log\log n \leq \frac{1}{4}\varepsilon_1\varepsilon_2 p^2n^3 \leq \frac{1}{4}\varepsilon_1\varepsilon_2 p^2n^3|C_1|,$$

where the second inequality follows from $p \geq \log n/n$. Therefore, at least

$$\frac{1}{2}\varepsilon_1\varepsilon_2 p^2n^3|C_1| - \frac{1}{4}\varepsilon_1\varepsilon_2 p^2n^3|C_1| = \frac{1}{4}\varepsilon_1\varepsilon_2 p^2n^3|C_1|$$

of the copies of $\hat{F}_5$ estimated in (3) contain a non-bad edge from $\mathcal{M}$. Each such non-bad edge from $\mathcal{M}$ is contained in at most $2\varepsilon_3 pn = 216\delta\varepsilon_1^{-1}pn$ such copies of $\hat{F}_5$, and so

$$|\mathcal{M}| \geq \frac{\varepsilon_1^2\varepsilon_2 p^2 n^3 |C_1|}{4 \cdot 216\delta pn} = \frac{\varepsilon_1^2\varepsilon_2}{8 \cdot 108\delta} pn^2|C_1| = 20pn^2|C_1|.$$

□

Lemma 17. *If J' is a subgraph of J such that $\Delta(J') \leq \varepsilon_1 n$, then*

$$|\mathcal{M}| \geq 20pn|J'|.$$

Proof. For each $wx \in E(J')$, since $wx \notin Q(\Pi)$, there are at least $p^2n^2/12$ choices of $(y, z) \in A_2 \times A_3$ such that $\{w, x, y, z\}$ spans an $\hat{F}_5$ in $\mathcal{G}$. There are therefore at least $\frac{1}{12}|J'|p^2n^2$ copies of $\hat{F}_5$, and at least one of wyz, xyz must be in $\mathcal{M}$ for each of these copies of $\hat{F}_5$.

Consider an edge $xyz \in \mathcal{M}$ with $x \in V(J')$. We will count how many of these copies of $\hat{F}_5$ in $\mathcal{G}$ contain xyz. Say that xyz is *bad* if there exist at least $2\varepsilon_1 pn$ vertices $w \in N_{J'}(x)$ with $wyz \in \mathcal{G}$. For each $x \in V(J')$, let $d_x = d_{J'}(x)$ and denote by r_x the number of pairs (y, z) such that xyz is bad. By Proposition 5, the co-degree of each such pair (y, z) is at most $2pn$, so there exist at most $\min\{2pn, d_x\}$ vertices $w \in N_{J'}(x)$ with $wyz \in \mathcal{G}$. Then the number of copies of $\hat{F}_5$ that contain a non-bad edge from $\mathcal{M}$ is at least

$$\frac{1}{2}\sum_{x \in V(J')} d_x \frac{p^2n^2}{12} - \sum_{x \in V(J')} r_x \cdot \min\{2pn, d_x\}. \tag{4}$$

We will prove $\frac{1}{2}d_x \cdot p^2n^2/12 \geq 2r_x \cdot \min\{2pn, d_x\}$ for every vertex $x \in V(J')$ by applying Lemma 12 with $\varepsilon = \varepsilon_1, A = N_{J'}(x)$ and various choices of a and r depending on d_x. Note that $d_x \leq \Delta(J') \leq \varepsilon_1 n$. So d_x will fall into one of the following three cases.

1. $d_x > 2pn$ and $\frac{\log n}{p^2 n} \leq d_x \leq \varepsilon_1 n$. We apply Lemma 12 with $a = \varepsilon_1 n$ and $r = (\log\log n)/p$ to obtain that $r_x \leq (\log\log n)/p$.
2. $d_x > 2pn$ and $\frac{\log n}{p^{k+2}n^{k+1}} \leq d_x \leq \frac{\log n}{p^{k+1}n^k}$ for some integer $k \in [1, \frac{\log n}{\log\log n}]$. We apply Lemma 12 with $a = \frac{\log n}{p^{k+1}n^k}$ and $r = a/100$ to obtain that $r_x \leq \frac{\log n}{100p^{k+1}n^k} \leq pnd_x/100$.
3. $d_x \leq 2pn$. We apply Lemma 12 with $a = 2pn$ and $r = p^2n^2/50$ to obtain that $r_x \leq p^2n^2/50$.

As long as a and r are positive integers and satisfy (1), we can apply Lemma 12. For each of these three cases, we can easily check that

$$\frac{1}{2}d_x\frac{p^2n^2}{12} \geq 2r_x \cdot \min\{2pn, d_x\}.$$

Therefore, the number of copies of $\hat{F}_5$ estimated in (4) is at least

$$\frac{1}{2}\sum_{x \in V(J')} d_x \frac{p^2n^2}{12} - \sum_{x \in V(J')} r_x \cdot \min\{2pn, d_x\} \geq \frac{1}{4}\sum_{x \in V(J')} d_x \frac{p^2n^2}{12} = \frac{1}{24}|J'|p^2n^2.$$

By definition, an edge that is not bad is in at most $2\varepsilon_1 pn$ of the copies of $\hat{F}_5$ estimated in (4). Therefore,

$$|\mathcal{M}| \geq \frac{1}{24} \cdot \frac{|J'|p^2n^2}{2\varepsilon_1 pn} = \frac{1}{48\varepsilon_1} \cdot pn|J'| = 20pn|J'|.$$

$\square$

Lemma 18. $|\mathcal{M}| \geq \frac{1}{20}pn^2\,|C_2|$.

Proof. For every vertex $x \in C_2$, the number of edges in $\mathcal{F}[\Pi]$ that contain x is at most $\varepsilon_2 pn^2$, but by Proposition 8, w.h.p. the crossing degree of x in $\mathcal{G}$, $d_\Pi(x)$, is at least $pn^2/10$. Thus, there are at least $pn^2/20$ edges of $\mathcal{M}$ incident to x, so $|\mathcal{M}| \geq |C_2|\,pn^2/20$. $\square$

Proof of Lemma 13. Let δ be as defined in (2) and K sufficiently large that all the previous lemmas and propositions are applicable. We now have three different lower bounds on the size of $\mathcal{M}$. We will show that $|\mathcal{M}| > 3\,|\mathcal{B}_1|$ by proving that no matter how the edges of $\mathcal{B}_1$ are arranged, one of the above lower bounds on $\mathcal{M}$ is larger than $3\,|\mathcal{B}_1|$. To do this, we divide the edges of $\mathcal{B}_1$ into three classes. Let $D = A_1 \setminus C$.

I. $\mathcal{B}_1(1) = \{e \in \mathcal{B}_1 : |e \cap C| \geq 2 \text{ or } |e \cap D| \geq 2\}$.
II. $\mathcal{B}_1(2) = \{e \in \mathcal{B}_1 \setminus \mathcal{B}_1(1) : |e \cap C_1| = 1\}$. Note that every edge in $\mathcal{B}_1(2)$ contains a vertex in C_1, one in D and one outside of A_1.
III. $\mathcal{B}_1(3) = \mathcal{B}_1 \setminus \mathcal{B}_1(1) \setminus \mathcal{B}_1(2)$. Note that every edge in $\mathcal{B}_1(3)$ contains a vertex in C_2, one in D and one outside of A_1.

We now consider the following three cases on $|\mathcal{B}_1(i)|$.

Case 1. $3|\mathcal{B}_1(1)| \geq |\mathcal{B}_1|$.

Let $J' = J[C] \cup J[D]$. By definition, vertices $x \in D$ have degree at most $\varepsilon_1 n$. For $x \in C$, Lemma 15 shows that x has degree in J' at most $|C| \leq \varepsilon_3 n < \varepsilon_1 n$. Proposition 5 shows that $|\mathcal{B}_1(1)| \leq 2pn|J'|$. Combined with Lemma 17, this shows that $|\mathcal{M}| \geq 20pn|J'| \geq 10\,|\mathcal{B}_1(1)| > 3|\mathcal{B}_1|$.

Case 2. $3|\mathcal{B}_1(2)| \geq |\mathcal{B}_1|$.

For each vertex $x \in C_1$ and each $y \in D$, by Proposition 5, the co-degree of x and y is at most $2pn$. Since $|D| \leq n$, there are at most $2pn^2$ edges of $\mathcal{B}_1 \setminus \mathcal{B}_1(1)$ containing x. Thus $|\mathcal{B}_1(2)| \leq 2pn^2\,|C_1|$, so Lemma 16 implies that $|\mathcal{M}| \geq 20pn^2\,|C_1| \geq 10\,|\mathcal{B}_1(2)| > 3\,|\mathcal{B}_1|$.

Case 3. $3|\mathcal{B}_1(3)| \geq |\mathcal{B}_1|$.

Every $x \in C_2$ is in less than $\varepsilon_2 pn^2$ crossing edges of $\mathcal{F}$. Note that every edge in $\mathcal{B}_1(3)$ has at least one vertex in C_2 and is not completely contained in A_1 (edges completely contained in A_1 are in $\mathcal{B}_1(1)$.) If there exist at least $\varepsilon_2 pn^2$ edges of $\mathcal{B}_1$ which contain x and have a vertex in A_2, we could move x to A_3 and increase the number of edges across the partition. Similarly, there are at most $\varepsilon_2 pn^2$ edges of $\mathcal{B}_1$ which contain x and have a vertex in A_3, since otherwise we could move x to A_2. Thus $|\mathcal{B}_1(3)| \leq 2\varepsilon_2 pn^2\,|C_2| = \frac{1}{200}pn^2\,|C_2|$. Then Lemma 18 implies that $|\mathcal{M}| \geq \frac{1}{20}pn^2\,|C_2| \geq 10\,|\mathcal{B}_1(3)| > 3\,|\mathcal{B}_1|$.

In each case we verified $|\mathcal{M}| > 3|\mathcal{B}_1|$, then since one of these three cases must hold, we have $|\mathcal{M}| > 3|\mathcal{B}_1|$. $\square$

3.2 Proof of Lemma 14

Proof. If $Q(\Pi) = \emptyset$, then clearly Lemma 14 is true, so we may assume $Q(\Pi) \neq \emptyset$.

Let

$$\varepsilon = 0.1,\ \zeta = 0.001,\ \gamma = 0.1,\ \alpha = \frac{8}{9} \text{ and } \varphi = 0.001.$$

Recall that a partition $\Pi = (A_1, A_2, A_3)$ is balanced if $|A_i| = (1 \pm 10^{-10})n/3$ for every i, and for a balanced partition Π, $Q(\Pi) = \{(u, v) \in \binom{A_1}{2} : d_\Pi(u, v) < 0.8n^2p^2/9\}$.

By Propositions 6 and 8, for any balanced partition Π we have w.h.p. $d_\Pi(v) \geq (1 - \varepsilon)pn^2/9 = \gamma pn^2$ for every vertex v and $d(u, v) \leq (1+\varepsilon)n^2p^2/2$ for every pair (u, v) of distinct vertices, and therefore $d_\Pi(u, v) \leq \alpha p d_\Pi(v)$ for every $(u, v) \in Q(\Pi)$.

Fix any positive $\delta < \varphi\gamma/2$, and let K be sufficiently large that all previous lemmas and propositions are applicable. Let $A = A(\delta)$ be the event that for $\delta > 0$, there exists a balanced partition Π such that $t(\mathcal{G}) \leq |\mathcal{G}(\Pi)| + |Q(\Pi)|\delta n^2p^2$. To prove Lemma 14, we will show that $\mathbb{P}[A] = o(1)$. Since $Q(\Pi)$ contains a bipartite subgraph R with at least half of the edges of $Q(\Pi)$, the event A implies that $t(\mathcal{G}) \leq |\mathcal{G}(\Pi)| + 2|R|\delta n^2p^2$ for some bipartite $R \subseteq Q(\Pi)$. By Proposition 11, we have $d_{Q(\Pi)}(v) \leq \zeta/p$ for every vertex v, and therefore, we have

$$d_R(v) \leq \zeta/p. \tag{5}$$

Let X, Y be disjoint subsets of V, R be a spanning subgraph of $[X, Y]$ satisfying (5), and f be a function from X to $\{k \in \mathbb{N} : k \geq \gamma pn^2\}$. Denote by $E(R, X, Y, f)$ the event that there is a balanced partition Σ of $\mathcal{G}$ such that for every vertex x in X, we have

$$d_\Sigma(x) = f(x), \qquad R \subseteq Q(\Sigma) \qquad \text{and} \qquad t(\mathcal{G}) \leq |\mathcal{G}[\Sigma]| + \varphi|R|\gamma n^2p^2, \tag{6}$$

where we should emphasize that $Q(\Sigma)$ should be in the first partition class of Σ. Since $\delta < \varphi\gamma/2$, the event A implies event $E(R, X, Y, f)$ for some choice of (R, X, Y, f).

We will show that there exists a constant c such that

$$\mathbb{P}[E(R, X, Y, f)] \leq e^{-c|R|n^2p^2}. \tag{7}$$

There are at most $\binom{\binom{n}{2}}{t}2^t n^{2t}$ ways to choose (R, X, Y, f) with $|R| = t$. Then by the union bound, we have

$$\mathbb{P}[A] \leq \sum_{t \geq 1} \binom{\binom{n}{2}}{t} 2^t n^{2t} e^{-ctn^2p^2} \leq \sum_{t \geq 1} \left(\frac{en^4}{t \cdot e^{cn^2p^2}}\right)^t = o(1).$$

Now we prove (7), which completes the proof of Lemma 14. We consider revealing the edges of $\mathcal{G}$ in stages:

(i) Reveal the triplets of vertices of $\mathcal{G}$ that contain $x \in X$.

(ii) Reveal the rest of the triplets of vertices of $\mathcal{G}$ except those belonging to $\bigcup_{y \in Y}[y, \cup_{xy \in R} L(x)]$.

(iii) Reveal the rest of the triplets of vertices of $\mathcal{G}$.

Let $\mathcal{G}'$ be the subhypergraph of $\mathcal{G}$ consisting of the edges chosen in (i) and (ii), and let Γ be a balanced partition of $\mathcal{G}'$ maximizing $|\mathcal{G}'[\Sigma]|$ among balanced partitions Σ satisfying (6). Recall that for any balanced partition Σ, we have $d_\Sigma(x,y) < \alpha p d_\Sigma(x)$ for all $(x,y) \in Q(\Sigma)$. So for any balanced partition Σ satisfying (6), we have

$$\begin{aligned} |\mathcal{G}[\Sigma]| &\leq |\mathcal{G}'[\Sigma]| + \sum_{y\in Y}\sum_{xy\in R} d_\Sigma(x,y) \leq |\mathcal{G}'[\Gamma]| + \sum_{y\in Y}\sum_{xy\in R} d_\Sigma(x,y) \\ &\leq |\mathcal{G}'[\Gamma]| + \sum_{y\in Y}\sum_{xy\in R} \alpha p d_\Sigma(x) \leq |\mathcal{G}'[\Gamma]| + \alpha p \sum_{y\in Y}\sum_{xy\in R} f(x). \end{aligned} \tag{8}$$

Note that the right hand side of (8) does not depend on the partition Σ, so it gives an upper bound on $|\mathcal{G}[\Sigma]|$ for all Σ satisfying (6). On the other hand, we look at Γ. For each $y \in Y$, set $M(y) = \cup_{xy\in R} L_\Gamma(x)$. We have

$$t(\mathcal{G}) \geq |\mathcal{G}[\Gamma]| = |\mathcal{G}'[\Gamma]| + \sum_{y\in Y} |\mathcal{G}[y, M(y)]|. \tag{9}$$

Recall that $d_\Gamma(x) = f(x) \geq \gamma p n^2$, so for any two vertices x and x', we have $d_\Gamma(x,x') \leq d(x,x') \leq (1+\varepsilon)n^2p^2/2 \leq p d_\Gamma(x)/\gamma$. Also recall that R satisfies (5), so for each $y \in Y$ we have $d_R(y) \leq \zeta/p$. It follows that for each $y \in Y$, we have

$$\begin{aligned} |M(y)| &\geq \sum_{xy\in R}\left[d_\Gamma(x) - \sum_{x\neq x'\in N_R(y)} d_\Gamma(x,x')\right] &&\geq \sum_{xy\in R}\left[d_\Gamma(x) - d_R(y)\cdot \max_{x\neq x'\in N_R(y)} d_\Gamma(x,x')\right] \\ &\geq \sum_{xy\in R}\left[d_\Gamma(x) - \zeta/p\cdot p d_\Gamma(x)/\gamma\right] &&\geq (1-\zeta/\gamma)\sum_{xy\in R} f(x). \end{aligned}$$

Let μ be the expectation of the sum in (9). Then we have

$$\mu = p\sum_{y\in Y}|M(y)| \geq (1-\zeta/\gamma)p\sum_{y\in Y}\sum_{xy\in R} f(x).$$

Then using Lemma 4, we know that with probability at least $1 - e^{-c_\varepsilon \mu} \geq 1 - e^{-c|R|n^2p^2}$ for constant $c = c_\varepsilon(\gamma - \zeta)$, the sum in (9) is at least $(1-\varepsilon)\mu$, and when this happens, (8) and (9) imply that

$$t(\mathcal{G}) - |\mathcal{G}[\Sigma]| \geq ((1-\varepsilon)(1-\zeta/\gamma) - \alpha)\, p\sum_{y\in Y}\sum_{xy\in R} f(x) > \varphi|R|\gamma n^2 p^2,$$

which proves (7). □

4 Proof of Theorem 1

Proof of Theorem 1. Let $\tilde{\mathcal{F}}$ be a maximum F_5-free subhypergraph of $\mathcal{G}$, so $|\tilde{\mathcal{F}}| \geq t(\mathcal{G})$. Suppose to the contrary that $\tilde{\mathcal{F}}$ is not tripartite; then to prove Theorem 1, it suffices to show that $|\tilde{\mathcal{F}}| < t(\mathcal{G})$. Let $\Pi = (A_1, A_2, A_3)$ be a 3-partition maximizing $\tilde{\mathcal{F}}[\Pi]$. By Proposition 10 we know that Π is balanced and $\tilde{\mathcal{F}}$ and Π satisfy Condition (i) of Lemma 13. For $1 \leq i \leq 3$, let $\tilde{\mathcal{B}_i} = \{e \in \tilde{\mathcal{F}}, |e \cap A_i| \geq 2\}$. Without loss of generality, we may assume $|\tilde{\mathcal{B}_1}| \geq |\tilde{\mathcal{B}_2}|, |\tilde{\mathcal{B}_3}|$. Let $\mathcal{B}(\Pi) = \{e \in \mathcal{G} : \exists (u,v) \in Q(\Pi) \text{ s.t. } \{u,v\} \subset e\}$ and $\mathcal{F} = \tilde{\mathcal{F}} - \mathcal{B}(\Pi)$. Observe that Π is a maximal partition of $\tilde{\mathcal{F}}$, and $\mathcal{F}$ was obtained by removing some non-crossing edges of $\tilde{\mathcal{F}}$, therefore Π is a maximal partition of $\mathcal{F}$ as well. Now $\mathcal{F}$ and Π satisfy all conditions of Lemma 13. For $1 \leq i \leq 3$, let $\mathcal{B}_i = \{e \in \mathcal{F} : \ |e \cap A_i| \geq 2\}$. Then we have:

$$\begin{aligned}
|\tilde{\mathcal{F}}| &\leq |\tilde{\mathcal{F}}[\Pi]| + 3|\tilde{\mathcal{B}}_1| \\
&= |\mathcal{F}[\Pi]| + 3|\mathcal{B}_1| + 3|\tilde{\mathcal{F}} \cap \mathcal{B}(\Pi)| \\
&\leq |\mathcal{G}[\Pi]| + 3|\mathcal{B}(\Pi)| \qquad (10) \\
&\leq |\mathcal{G}[\Pi]| + 3 \cdot 2|Q(\Pi)|np \qquad (11) \\
&\leq |\mathcal{G}[\Pi]| + |Q(\Pi)|\delta n^2 p^2 \\
&\leq t(\mathcal{G}). \qquad (12)
\end{aligned}$$

Here we apply Lemma 13 to $\mathcal{F}$ and Π to get (10); note that equality is only possible when $\mathcal{F}$ is tripartite. We apply Proposition 5 to get (11) and Lemma 14 to get (12). If $\mathcal{F}$ is tripartite, but $\tilde{\mathcal{F}}$ is not, then $Q(\Pi) \neq \emptyset$, and so equality in (12) would fail. We therefore know that $|\tilde{\mathcal{F}}| < t(\mathcal{G})$, a contradiction, which means $\tilde{\mathcal{F}}$ is tripartite. $\square$

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