

The stripping process can be slow: part I

Pu Gao* and Michael Molloy†

Department of Computer Science, University of Toronto

10 King's College Road, Toronto, ON

jane.gao@monash.edu

molloy@cs.toronto.edu

Abstract

Given an integer k , we consider the parallel k -stripping process applied to a hypergraph H : removing all vertices with degree less than k in each iteration until reaching the k -core of H . Take H as $\mathcal{H}_r(n, m)$: a random r -uniform hypergraph on n vertices and m hyperedges with the uniform distribution. Fixing $k, r \geq 2$ with $(k, r) \neq (2, 2)$, it has previously been proved that there is a constant $c_{r,k}$ such that for all $m = cn$ with constant $c \neq c_{r,k}$, with high probability, the parallel k -stripping process takes $O(\log n)$ iterations. In this paper we investigate the critical case when $c = c_{r,k} + o(1)$. We show that the number of iterations that the process takes can go up to some power of n , as long as c approaches $c_{r,k}$ sufficiently fast. A second result we show involves the depth of a non- k -core vertex v : the minimum number of steps required to delete v from $\mathcal{H}_r(n, m)$ where in each step one vertex with degree less than k is removed. We will prove lower and upper bounds on the maximum depth over all non- k -core vertices.

1 Introduction

Given a nonnegative integer k and a (hyper)graph H , the k -core of H , denoted by $\mathcal{C}_k(H)$, is the maximum subgraph of H with minimum degree at least k . The k -core was first studied by Bollobás[5] and has since become a major focus in random graph theory. Its many applications include erasure codes[25, 26], colouring[31], hashing[19], and graph orientability[18, 7, 13, 12]. We define $\mathcal{H}_r(n, m)$ to be a random r -uniform hypergraph on vertex set $[n] := \{1, 2, \dots, n\}$ and exactly m hyperedges, with the uniform distribution. We focus on sparse random hypergraphs with bounded average degree; thus the typical range of focus is $m = \Theta(n)$. The threshold for the appearance of a non-empty k -core in $\mathcal{H}_r(n, cn)$ was first determined by Pittel, Spencer and Wormald [34] for $r = 2$ and $k \geq 3$, whereas for general $(r, k) \neq (2, 2)$, the threshold is determined[30, 24] to be:

$$c_{r,k} = \inf_{\mu > 0} \frac{\mu}{r \left[e^{-\mu} \sum_{i=k-1}^{\infty} \mu^i / i! \right]^{r-1}} . \quad (1)$$

*Research supported by an NSERC Postdoctoral Fellowship and an NSERC Discovery Grant. This author is currently affiliated with Monash University.

†Research supported by an NSERC Discovery Grant and Accelerator Supplement.

The k -core of a (hyper)graph H can be obtained by repeatedly removing all vertices with degree less than k . In the parallel k -stripping process, all vertices with degree less than k are removed at once in each iteration until the k -core is reached. The number of rounds this process takes is called the k -stripping number, denoted by $s_k(H)$. This number is an important parameter associated with a (hyper)graph and its k -core. Jiang, Mitzenmacher and Thaler [23] discussed several applications of the parallel stripping process in computer science and the importance of analysing the k -stripping number. An upper bound $O(\log n)$ is given in [1] for $s_k(\mathcal{H}_r(n, cn))$ when $c > c_{r,k} + \epsilon$ and this bound is proved to be tight in [23]. Independently, [21] proved an upper bound of $\text{poly}(\log n)$. On the other hand, it was proved in [23, 15] that $s_k(\mathcal{H}_r(n, cn)) = O(\log \log n)$ if $c < c_{r,k} - \epsilon$ (in fact, results in [1, 23, 15] are presented for $\mathcal{H}_r(n, p)$, $p = c/n^{r-1}$, but they easily translate to $\mathcal{H}_r(n, m)$ as well). However, as we will prove in this paper, the stripping process can take a long time as $c \rightarrow c_{r,k}$: the number of rounds needed can go up to some power of n , depending on the rate at which c approaches $c_{r,k}$.

We give some intuitive explanation of this phenomenon. Note that $c_{r,k}$ is the emergence threshold of the k -core. When $c \rightarrow c_{r,k}$, the parallel k -stripping process undergoes a bottleneck. At a certain point, the number of vertices removed in each iteration (i.e. of degree less than k) becomes sublinear in n . If there is no k -core, then the process continues for a long enough time to eventually pass through the bottleneck: the number of vertices removed in each iteration becomes and stays linear in n again and eventually all vertices of the hypergraph will be removed, producing an empty k -core. If there is a k -core, then the process terminates after only $o(n)$ total vertices are removed during the bottleneck; what remains is the giant k -core. In both cases, the number of remaining vertices with degree less than k mimics a random walk whose expected change with each vertex deletion is very close to zero during the bottleneck. Hence a large number of vertices must be removed for it to either decrease to 0 or increase to linear in n . Since few vertices are removed in each iteration, it takes many iterations to remove this large number of vertices.

The analysis for the stripping number when $c \rightarrow c_{r,k}$ becomes subtle. For c inside the critical window $|c - c_{r,k}| = O(1/\sqrt{n})$, it is not certain that $\mathcal{H}_r(n, cn)$ has a non-empty k -core; i.e. the probability that $\mathcal{H}_r(n, cn)$ contains a non-empty k -core is bounded away from 0 and 1 as $n \rightarrow \infty$ [22]. This brings in certain complications in our analysis of the stripping number, especially for the upper bound. We will prove a lower bound for the stripping number when $c \leq c_{r,k} + n^{-1/2+\epsilon}$ (note this range contains the critical window), whereas the upper bound for c in this range will be studied in a subsequent paper. For c above $c_{r,k} + n^{-1/2+\epsilon}$, we will prove both upper and lower bounds that are tight in the asymptotic order. It is interesting to note that several graph parameters/properties related to the k -core have a radical change when c approaches to this threshold, including the robustness of the k -core [36].

Rather than removing all vertices with degree less than k at once in each iteration, we may remove one vertex at a time. This produces a stripping sequence: a sequence of vertices removed from the initial hypergraph H . Vertices in a stripping sequence can appear in an arbitrary order, as long as each vertex has degree less than k at the moment of its removal. Note that a stripping sequence does not necessarily need to terminate with the k -core: for a stripping sequence Ψ , a subsequence of Ψ composed of the first arbitrary number of vertices in Ψ is also a stripping sequence. Given a non- k -core vertex v , we are interested in the length of a shortest stripping sequence ending with v : roughly speaking, this is the minimum number

of vertices required to be deleted before deleting v . This number is called the *depth* of v . A formal definition is given in Section 2. The maximum depth of $\mathcal{H}_r(n, m)$ (over all non- k -core vertices) is bounded in [1] by $O(\log n)$ when c differs from $c_{r,k}$ by some absolute constant. In this paper, our goal is to bound (both from above and below) this parameter when $c \rightarrow c_{r,k}$.

The aforementioned bounds on the depth and stripping number from [1, 21] were motivated by applications to solution clustering in random XORSAT. We will provide analogous applications for our present results in a subsequent paper (a preliminary version containing partial results on their applications to clustering in random XORSAT is available in [17]).

2 Main results

We first give a formal definition of the parallel k -stripping process and the k -stripping number.

Definition 1. The *parallel k -stripping process*, applied to a hypergraph H , consists of iteratively removing *all* vertices of degree less than k at once along with any hyperedges containing any of those vertices, until no vertices of degree less than k remain; i.e. until we are left with the k -core of H . We use S_i to denote the vertices that are removed during iteration i . We use \widehat{H}_i to denote the hypergraph remaining after $i - 1$ iterations, i.e. after removing S_1, \dots, S_{i-1} .

We will analyze the number of rounds that this process takes:

Definition 2. The *k -stripping number* of H , denoted $s_k(H)$, is the number of iterations in the parallel k -stripping process applied to H . We often drop the “ k ” and speak of the stripping number and $s(H)$.

We use the following standard notation. Given a sequence of probability spaces Ω_n , we say a sequence of events A_n occurs *asymptotically almost surely* (a.a.s.) if the probability that A_n occurs in Ω_n tends to 1 as $n \rightarrow \infty$. With two sequences of real numbers a_n and b_n , we use $a_n = O(b_n)$ to denote that there is an absolute constant C such that $|a_n| \leq C|b_n|$. We write $a_n = o(b_n)$ if $\lim_{n \rightarrow \infty} a_n/b_n = 0$. We write $a_n = \Omega(b_n)$ if $b_n = O(a_n)$ and $a_n, b_n \geq 0$ eventually; $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $b_n = O(a_n)$ and $a_n, b_n \geq 0$ eventually. Thus, if $a_n = \Omega(b_n)$ or $a_n = \Theta(b_n)$ then a_n, b_n must both be positive (for large n). All asymptotics in this paper refer to $n \rightarrow \infty$.

We first present our main result on the stripping number of $\mathcal{H}_r(n, cn)$.

Theorem 3. Consider constants $r, k \geq 2$, $(r, k) \neq (2, 2)$ and $0 < \delta < 1/2$.

- (a) If $c = c_{r,k} + n^{-\delta}$ then a.a.s. $s(\mathcal{H}_r(n, cn)) = \Theta(n^{\delta/2} \log n)$.
- (b) If $|c - c_{r,k}| \leq n^{-\delta}$ then a.a.s. $s(\mathcal{H}_r(n, cn)) = \Omega(n^{\delta/2})$.

So if we roughly define the following three ranges of c , the behaviour of the stripping number is as follows:

- $c = c_{r,k} + n^{-\delta}$, $0 < \delta < 1/2$. Then the stripping number is $\Theta(n^{\delta/2} \log n)$.

- $c = c_{r,k} - n^{-\delta}$, $0 < \delta < 1/2$. Then the stripping number is at least $\Omega(n^{\delta/2})$.
- c is between $c_{r,k} - n^{-1/2+o(1)}$ and $c_{r,k} + n^{-1/2+o(1)}$. Then the stripping number is $\Omega(n^{1/4+o(1)})$.

In the first range, the stripping number is specified within a constant factor. In the other two ranges, we do not say anything about upper bounds on the stripping number; those upper bounds will be studied in a subsequent paper.

Remark. With the same proof, $n^{-\delta}$ in Theorem 3 (and in all relevant lemmas in Sections 3 and 4) can be replaced by ξ_n (and correspondingly $n^{\delta/2}$ is replaced by $\sqrt{1/\xi_n}$) for any $\xi_n = o(1)$ such that $\xi_n \geq n^{-1/2+\epsilon}$ for some constant $\epsilon > 0$ (e.g. $\xi_n = 1/\log n$ or $\xi_n = n^{-1/3} \log n$). We use the less general statement for a cleaner presentation.

We are also interested in the number of vertices that must be deleted in order to remove a particular vertex v . I.e. what is the smallest number of vertices that must be deleted in order to demonstrate that v is not in the k -core?

Definition 4. A k -stripping sequence is a sequence of vertices that can be deleted from a hypergraph, one-at-a-time, along with their incident hyperedges such that at the time of deletion each vertex has degree less than k . For any vertex v not in the k -core, the *depth* of v is the length of a shortest k -stripping sequence ending with v .

While every non- k -core vertex has depth $O(\log n)$ for any constant $c > c_{k,r}$, as proved in [1], one of our main contributions in this paper is to prove that when c approaches the k -core emergence threshold, the maximum depth can rise to $n^{\Theta(1)}$.

Theorem 5. Let $r, k \geq 2$, $(r, k) \neq (2, 2)$ be fixed. There are constants $Z = Z(k, r)$ and $\kappa = \kappa(k, r)$ such that: for any fixed $0 < \delta < 1/2$, if $c = c_{r,k} + n^{-\delta}$ then a.a.s. the maximum depth of all non- k -core vertices in $\mathcal{H}_r(n, cn)$ is between $Zn^{\delta/2}$ and $n^{\kappa\delta}$.

Remark.

(a) Again here and in Theorem 41 in Section 5 below, with the same proof, $n^{-\delta}$ above (and in all relevant lemmas in Sections 5 and 6) can be replaced by ξ_n for any $\xi_n = o(1/\log^7 n)$ and $\xi_n \geq n^{-1/2+\epsilon}$ for some constant $\epsilon > 0$. We did not try to optimize the power of $\log n$.

(b) For the upper bound in this theorem, we will actually prove a stronger statement, because the stronger statement will be used in another paper on solution clustering in random XOR-SAT. See Theorem 41 in Section 5.

(c) Both Theorems 3 and 5 translate to $\mathcal{H}_r(n, p)$ with $p = r!c/n^{r-1}$. Translations of a.a.s. properties from $\mathcal{H}_r(n, m)$ to $\mathcal{H}_r(n, p)$ are usually standard by conditioning the number of hyperedges X in $\mathcal{H}_r(n, p)$ on its typical values, as long the properties under consideration hold for all m such that $|m - \mathbb{E}X| = O(\sqrt{\text{Var}X})$. Our situation is a little subtle as our bounds on the stripping number depend on how close c is to $c_{r,k}$, and some of our results cover the case that $|(c - c_{r,k})n|$ is smaller than the standard deviation of X , e.g. if $|c - c_{r,k}| = o(n^{-1/2})$. The reason that the translation holds in this paper is that for c in the range in Theorem 3(b), the lower bound is uniform for all c and the size of the range is of order $n^{1-\delta}$, which is much greater than the deviation of X , for any $\delta < 1/2$.

The k -stripping number clearly bounds the maximum depth from below, and so the lower bound of $\Omega(n^{\delta/2})$ from Theorem 3 implies the lower bound in Theorem 5. The difficult part of the proof of Theorem 5 is to show the upper bound.

Our requirement that $c \geq c_{r,k} + n^{-1/2+\epsilon}$ in Theorem 5 comes from the similar restriction on the actual appearance of the k -core, as shown by Kim[24] (see also [9]; a more precise statement of Theorem 6(b) is given in Lemma 7 in Section 3):

Theorem 6. [24] For $r, k \geq 2, (r, k) \neq (2, 2)$ and for any constant $\epsilon > 0$.

- (a) For $c \leq c_{r,k} - n^{-1/2+\epsilon}$, a.a.s. the k -core of $\mathcal{H}_r(n, cn)$ is empty.
- (b) For $c \geq c_{r,k} + n^{-1/2+\epsilon}$, a.a.s. $\mathcal{H}_r(n, cn)$ has a k -core with $\Theta(n)$ vertices.

The most challenging difference between the setting of this paper and that of [1] is: when $c = c_{r,k} + \Theta(1)$, the number of vertices with degree less than k remaining after each round of the parallel stripping process drops geometrically (see the comment following Lemma 16). That easily implies that a.a.s. the stripping number is $O(\log n)$. Furthermore, the property which implies that this number drops geometrically is also key in the analysis of [1] proving that the depth is $O(\log n)$. For $c = c_{r,k} + n^{-\delta}$, the number of such vertices drops much more slowly, leading to an increase in the stripping number and hence in the maximum depth, which is easily seen to be bounded from below by the stripping number. This requires us to use a very different, and much more intricate, analysis.

One novelty in our analysis of the depth is as follows: The most straightforward approach to analyze the stripping process is to repeatedly expose the vertices and edges removed in each iteration. Instead, we only expose the vertex sets S_1, S_2, \dots . The advantage is that, when considering the depth of a vertex v in a particular level S_i , we can treat the edges removed in previous iterations as random. Thus even though we have exposed the fact that v is in S_i , there is still some randomness that we can make use of in the specific sequence of vertices whose deletions led to v being deleted in iteration i . Of course, exposing the vertex sets that are deleted in each step exposes something about the edges that are deleted, so we have to be careful about the conditional distribution of those edges. The details of how we do this are in Section 5.1.

We excluded the case $(r, k) = (2, 2)$ from our main results as this case has already been extensively studied. The 2-core of the random graph $G(n, cn)$ is well-studied[27, 28, 35, 10, 11, 14]. For $0 < \delta < \frac{1}{3}$ and $p = 1 + n^{-\delta}$ a.a.s. the largest component consists of a large 2-core C_2 , with a Poisson Galton-Watson tree of branching parameter $1 - n^{-\delta}$ rooted at each vertex. In addition, there are $\Theta(n)$ smaller components distributed essentially like Poisson Galton-Watson trees of branching parameter $1 - n^{-\delta}$, except that some of them contain a single cycle. We can define the stripping number to be the stripping number of the largest component, or to be the maximum of the stripping number of all components. Either way, it is of the order of the height of the tallest of those trees which is $n^\delta \text{poly}(\log n)$. Similarly, the maximum depth is of the order of the size of the largest tree, which is $n^{2\delta} \text{poly}(\log n)$.

We will prove Theorem 3(b), and the upper bound for Theorem 3(a) in Section 4, along with a weaker version of the lower bound for Theorem 3(a). The lower bound for Theorem 3(a) is proved in Section 6. We prove Theorem 5 in Section 5. In Section 3 we discuss the size of the k -core as a preparation for the analysis in Sections 4 – 6. A key lemma used in the proof of Theorems 3(a) and 5 will be presented in Section 6.

3 Size of the k -core

We give a more precise version of Theorem 6(b), where the size of the k -core is specified, when $c \geq c_{r,k} + n^{-1/2+\epsilon}$. We start by defining:

$$f_t(\mu) = e^{-\mu} \sum_{i \geq t} \frac{\mu^i}{i!} \quad (2)$$

$$h(\mu) = h_{r,k}(\mu) = \frac{\mu}{f_{k-1}(\mu)^{r-1}}. \quad (3)$$

Note that $f_t(\mu)$ is the probability that a Poisson variable with mean μ , denoted by $\mathbf{Po}(\mu)$, is at least t . Thus by (1),

$$c_{r,k} = \inf_{\mu > 0} \frac{h(\mu)}{r}.$$

Now for any $r, k \geq 2$, $(r, k) \neq (2, 2)$, we define $\mu_{r,k}$ to be the value of μ that minimizes $h(\mu)$; i.e.

$$\mu_{r,k} \text{ is the unique solution to } h(\mu_{r,k}) = r c_{r,k} \quad (4)$$

Define

$$\alpha = \alpha_{r,k} = f_k(\mu_{r,k}), \quad \beta = \beta_{r,k} = \frac{1}{r} \mu_{r,k} f_{k-1}(\mu_{r,k}). \quad (5)$$

For ease of notation, we drop most of the r, k subscripts. For any $c \geq c_{r,k}$, we define $\mu(c)$ to be the larger solution to

$$c = \frac{h(\mu)}{r}.$$

Then, $\mu_{r,k} = \mu(c_{r,k})$. Define

$$\alpha(c) = f_k(\mu(c)), \quad \beta(c) = \frac{1}{r} \mu(c) f_{k-1}(\mu(c)). \quad (6)$$

Theorem 1.7 of [24] yields the size of the k -core as follows (Theorem 1.7 of [24] is for $\mathcal{H}_r(n, p)$ but it easily translates to $\mathcal{H}_r(n, m)$ by a standard coupling argument for $\mathcal{H}_r(n, p)$ and $\mathcal{H}_r(n, m)$, and by the fact that having a non-empty k -core is an increasing graph property):

Lemma 7. *Fix $r, k \geq 2$, $(r, k) \neq (2, 2)$ and $\epsilon > 0$. If $c \geq c_{r,k} + n^{-1/2+\epsilon}$ then a.a.s. the k -core of $\mathcal{H}_r(n, cn)$ has*

- $\alpha(c)n + O(n^{3/4})$ vertices and
- $\beta(c)n + O(n^{3/4})$ edges.

While the proof of [24, Theorem 1.7] (i.e. Lemma 7) is very technical, we give a brief heuristic explanation of why $c_{r,k}$ is the emergence threshold for a non-empty k -core and why the k -core of $\mathcal{H}_r(n, cn)$ is expected to contain around $\alpha(c)n$ vertices, when $c > c_{r,k}$. For vertex $v \in \mathcal{H}_r(n, cn)$, let λ_t denote the probability that v survives after t iterations of the parallel stripping process. Intuitively, $\lambda := \lim_{t \rightarrow \infty} \lambda_t$ is the probability that v is in the k -core of

$\mathcal{H}_r(n, cn)$ and thus the k -core is expected to contain λn vertices. It is easy to prove that the neighbourhood of v locally converges in distribution to a hyper-tree generated by the Poisson branching process with parameter cr starting at v . Then, v survives after t iterations of the parallel stripping process if and only if v is incident with at least k hyperedges surviving from $t - 1$ iterations of the stripping process, i.e. for each of these surviving hyperedges x , all the other $r - 1$ vertices in x are incident with at least $k - 1$ other hyperedges that survive after $t - 2$ iterations of the stripping process, and so on. To compute λ_t , let ρ_t denote the probability that v is incident with at least $k - 1$ hyperedges surviving after t iterations of the stripping process. As mentioned before, the number of hyperedges incident with v is $\mathbf{Po}(cr)$. Each hyperedge survives after $t - 1$ iterations of the stripping process with probability ρ_{t-1}^{r-1} . It is then easy to prove that the number of hyperedges incident with v that survive after $t - 1$ iterations of the stripping process is $\mathbf{Po}(\rho_{t-1}^{r-1}cr)$. Hence, we derive the recursion for ρ_t and λ_t :

$$\begin{aligned}\rho_0 &= 1; \\ \rho_t &= \mathbf{Pr}(\mathbf{Po}(\rho_{t-1}^{r-1}cr) \geq k - 1); \\ \lambda_t &= \mathbf{Pr}(\mathbf{Po}(\rho_{t-1}^{r-1}cr) \geq k).\end{aligned}$$

Put $\lambda = \mathbf{Pr}(\mathbf{Po}(\rho^{r-1}cr) \geq k)$, where ρ satisfies $\rho = \mathbf{Pr}(\mathbf{Po}(\rho^{r-1}cr) \geq k - 1)$. It is then easy to show that $\lambda = \rho = 0$ if $c < c_{r,k}$ and thus, the heuristics above implies that the k -core is likely to be empty. For $c > c_{r,k}$, it is easy to show that ρ_t converges to the larger root of $\rho = \mathbf{Pr}(\mathbf{Po}(\rho^{r-1}cr) \geq k - 1)$ and then λ gives $\alpha(c)$ in (6). The expected sum of the degrees is $n \times \sum_{i \geq k} i \mathbf{Pr}(\mathbf{Po}(\rho^{r-1}cr) = i)$ which is 2β .

By a close analysis of $\alpha(c)$ and $\beta(c)$ we have:

Lemma 8. *Fix $r, k \geq 2$, $(r, k) \neq (2, 2)$ and $0 < \delta < 1/2$. There exist positive constants K_1, K_2 and K_3 such that if $c = c_{r,k} + n^{-\delta}$ then*

$$\begin{aligned}\mu(c) - \mu_{r,k} &= K_1 n^{-\delta/2} + O(n^{-\delta}), \\ \alpha(c) - \alpha &= K_2 n^{-\delta/2} + O(n^{-\delta}), \\ \beta(c) - \beta &= K_3 n^{-\delta/2} + O(n^{-\delta}).\end{aligned}$$

We present its proof in the Appendix.

Lemma 7, along with Lemma 8 above, together with the fact that $\mu f_{k-1}(\mu)/f_k(\mu)$ is an increasing function on $\mu > 0$ (see Lemma 27 below), yields the following bounds on the size of the k -core:

Corollary 9. *Fix $r, k \geq 2$, $(r, k) \neq (2, 2)$ and $0 < \delta < 1/2$. There exist positive constants $K_1 = K_1(r, k), K_2 = K_2(r, k)$ and $K_3 = K_3(r, k)$: if $c = c_{r,k} + n^{-\delta}$ then a.a.s. the k -core of $\mathcal{H}_r(n, cn)$ has*

- (a) $\alpha n + K_1 n^{1-\delta/2} + O(n^{1-\delta} + n^{3/4})$ vertices and
- (b) $\beta n + K_2 n^{1-\delta/2} + O(n^{1-\delta} + n^{3/4})$ hyperedges and
- (c) average degree $r\beta/\alpha + K_3 n^{-\delta/2} + O(n^{-\delta} + n^{-1/4})$.

We close this section by introducing two terms that are standard when analyzing the k -core stripping process:

Definition 10. The *light* vertices of a hypergraph are the vertices of degree less than k . The *heavy* vertices are those of degree at least k .

4 Bounding the stripping number

We restate the parallel stripping process in a slowed-down version, which will be more convenient to analyze. Rather than removing all vertices of S_i at once, we remove them one at a time. When removing a vertex, we slow down further by removing one edge at a time.

To facilitate this, we maintain a queue \mathcal{Q} containing all deletable vertices; i.e. all vertices of degree less than k :

SLOW-STRIP

Input: A hypergraph G .

Initialize: $t := 0, G_0 := G, \mathcal{Q}_1 = \mathcal{Q}$ is a list of all vertices of degree less than k in G , ordered uniformly at random.

While $\mathcal{Q} \neq \emptyset$:

Let v be the first vertex in \mathcal{Q} .

Remove a hyperedge e selected uniformly at random from all those containing v .

If any other vertex of e has its degree drop to below k then add that vertex to the end of \mathcal{Q} .

Repeatedly remove the vertex in the front of \mathcal{Q} if its degree is zero.

G_{t+1} is the resulting hypergraph; $\mathcal{Q}_{t+1} := \mathcal{Q}; t := t + 1$.

To be clear: Note that the removal of v might cause the degree of a vertex not at the front of \mathcal{Q} to drop to zero. That vertex remains in \mathcal{Q} until it reaches the front, at which point it will be removed. So it is possible that multiple vertices are removed from the front of \mathcal{Q} during one step of SLOW-STRIP.

At any point, \mathcal{Q} may contain some vertices from S_i and some from S_{i+1} . However, the vertices of S_i are removed before the vertices of S_{i+1} . Note also that when processing a vertex $v \in \mathcal{Q}$, all edges from v are removed (and hence v is removed) before moving to the next vertex of \mathcal{Q} . So this procedure removes vertices in the same order as the parallel stripping process. In particular, if t is the first step in SLOW-STRIP during which the vertex at the front of \mathcal{Q}_t is in S_i , then the set of vertices in \mathcal{Q}_t is exactly S_i minus possibly some vertices with degree zero in \widehat{H}_i . Therefore, the total degree of the light vertices in G_t (i.e. vertices in \mathcal{Q}_t) equals exactly the total degree of S_i in \widehat{H}_i .

Definition 11. We use $t(i)$ to denote the first iteration of SLOW-STRIP in which the vertex at the front of \mathcal{Q}_t is in S_i .

If every vertex of S_i has degree 0 in G_i then there is no iteration in which the vertex at the front of \mathcal{Q}_t is in S_i and so we define $t(i)$ to be the iteration during which all vertices of S_i are removed. If this is the case then i is the final iteration of the parallel stripping process and $t(i)$ is the final iteration of SLOW-STRIP.

Roughly speaking, we can view $t(i)$ as the iteration of SLOW-STRIP in which iteration i of the parallel stripping process begins. This is not quite accurate in that perhaps iteration i began during iteration $t(i) - 1$ of SLOW-STRIP if the first vertices of S_i to reach the front of \mathcal{Q} had degree zero.

We also define:

τ is the iteration in which SLOW-STRIP halts.

We will focus much of our analysis on the following parameters of G_t :

- L_t is the total degree of the light vertices in G_t ; i.e. of the vertices in \mathcal{Q}_t .
- N_t is the number of heavy vertices in G_t ; i.e. of the vertices outside of \mathcal{Q}_t .
- D_t is the total degree of the heavy vertices in G_t .

We denote the triple of these values as:

$$\mathcal{T}_t = (L_t, N_t, D_t),$$

and

$$\mathcal{F}_t = \{\mathcal{T}_s\}_{s \leq t} := \{\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_t\}.$$

We first give an overview of the proof of Theorem 3. As mentioned earlier, $L_{t(i)}$ equals the total degree of S_i in \widehat{H}_i . This will allow us to relate $|S_i|$ to $L_{t(i)}$. The key arguments are to rather precisely describe the evolution of $(L_t)_{t \geq 0}$, especially in a critical range of t . We will show that after a sufficiently large but bounded number B of iterations of the parallel stripping process, the number of vertices remaining in the hypergraph becomes very close to αn (see Lemma 15). Then, we will prove that in SLOW-STRIP, L_t decreases with at least a certain rate for all $t \geq t(B)$ (see Lemma 16). Using that we can bound from below the rate at which $L_{t(i)}$ (or $|S_i|$, roughly speaking) decreases for each $i \geq B$. This allows us to obtain the upper bound for $s_k(\mathcal{H}_r(n, cn))$ as in Theorem 3.

To obtain the lower bound of $s_k(\mathcal{H}_r(n, cn))$ in the supercritical case as in Theorem 3(a), we will tightly bound the rate (from both below and above) at which L_t decreases for $t \geq t(B)$ (See Section 6). This enables us to establish a rather precise description of $(|S_i|)_{i \geq B}$ (see Lemma 47), and hence deduce the desired lower bound on the stripping number.

However, we will first present a slightly weaker lower bound, i.e. without the logarithmic factor, with a much simpler proof in Section 4.3. We do so because this weaker bound is part of the proof of Theorem 3(b). The key idea is to focus on the steps of SLOW-STRIP during which the last $Kn^{1-\delta/2}$ vertices are removed before reaching the k -core, for some constant $K > 0$. We will show that with high probability $L_t = O(n^{1-\delta})$ in all these steps (see Lemma 19). We will then consider the iterations of the parallel stripping process during which the last $Kn^{1-\delta/2}$ vertices are removed. In each iteration, the total number of vertices being removed must be $O(n^{1-\delta})$ since their total degree is $O(n^{1-\delta})$. In order to remove $Kn^{1-\delta/2}$ vertices, at least $\Omega(n^{\delta/2})$ iterations of parallel stripping are required. This yields the slightly weaker lower bound. This same proof, combined with a coupling argument, will then yield the lower bound claimed in Theorem 3(b) (see Sections 4.4 and 4.5).

4.1 The allocation-partition model

We use the following *allocation-partition model* (*AP-model*), denoted by $AP_r(n, m)$. It was used in [18] and is a slight modification of what is called the pairing-allocation model in [6]. We are given a set of rm vertex-copies. We represent each vertex v as a bin. We choose two random objects: (i) a uniformly random partition of the vertex-copies into parts of size exactly r ; (ii) a uniform allocation of each vertex-copy into a bin. We call the output a *configuration*.

Having chosen a configuration from $AP_r(n, m)$, we can transform it into a hypergraph as follows: We contract the bins into vertices, and each part of the partition becomes a hyperedge. It is easy to show, with simple counting arguments, that all simple hypergraphs with n vertices and m edges are generated with equal probability. (*Simple* means that no two hyperedges are identical, and no vertex appears twice in the same hyperedge).

The AP-model differs from the configuration model of Bollobás [4] in that the degree sequence is not fixed in advance, and the vertex-copies are not initially assigned to actual vertices. Note that the random partition and the random allocation are orthogonal and they can be chosen independently of each other. This will be very helpful below when we condition on events (specifically values of L_t, N_t, D_t) which specify partial information about both the partition and the allocation.

$\mathcal{H}_r(n, m)$ only selects *simple* hypergraphs. Of course, the hypergraph formed by the AP-model might not be simple, but when $m = O(n)$, the probability of obtaining a simple hypergraph is at least ϵ for some absolute constant $\epsilon > 0$. This immediately yields:

Corollary 12. *If $m = O(n)$ and if property Q holds a.a.s. for $AP_r(n, m)$, then Q holds a.a.s. for $\mathcal{H}_r(n, m)$.*

4.2 SLOW-STRIP on the AP-model

We will analyse the running of SLOW-STRIP on a random configuration generated by the AP-model, and then use Corollary 12 to translate the results to $\mathcal{H}_r(n, m)$. We continue to use hypergraph terminology to refer to the configuration, and so strictly speaking, a “vertex” is a bin, a “hyperedge” is an r -tuple of the partition along with an allocation of the vertex-copies in that r -tuple, G_t is the configuration remaining after t iterations, etc. We say a vertex-copy is *light/heavy* if the bin it is allocated to is *light/heavy*. Where we say “graph” in this context, we mean the (not necessarily simple) hypergraph obtained from the configuration by contracting the bins.

SLOW-STRIP runs on a configuration in $AP_r(n, m)$ as follows: Initially, the queue \mathcal{Q}_0 contains all the light bins; i.e. the bins of size less than k . In each step, SLOW-STRIP removes a vertex-copy x of the bin u in the front of the queue, together with the $r - 1$ vertex-copies in the same part as x . Each time we delete one of the D_t vertex-copies not in \mathcal{Q}_t , we query whether the bin containing that copy now has size $k - 1$; if so then we move that bin to the queue. When all vertex-copies of u are removed then u is deleted from the queue. We define the k -core of a configuration to be what remains when SLOW-STRIP terminates.

We use G_t to denote the configuration remaining after t steps. In terms of a configuration, our three key parameters become:

- L_t is the number of light vertex-copies in G_t ; i.e. the total number of copies in \mathcal{Q}_t .
- N_t is the number of heavy bins in G_t .
- D_t is the number of heavy vertex-copies in G_t .

We will typically condition on the values of $\mathcal{F}_t = \{(L_s, N_s, D_s)\}_{s \leq t}$. The reader has likely noticed a lack of symmetry: we do not condition on the number of light bins; i.e. there is no light analogue to N_t . This is because the number and sizes of the light bins have no significant effect on the running of SLOW-STRIP - all that matters is the number of copies in \mathcal{Q}_t . Also note that $D_t + L_t = r(cn - t)$, as this is the number of remaining vertex-copies, and so it is not necessary for \mathcal{F}_t to record both D_t and L_t ; but we find it convenient to do so.

The following observation (first shown in [6]) enables our analysis. Recall that a configuration consists of (i) a partition of the vertex-copies into parts of size r , and (ii) an allocation of the vertex-copies to bins. Recall further that in our random model, this partition and allocation are chosen uniformly and independently of each other.

Observation 13. *Upon conditioning on \mathcal{F}_t ,*

- (a) *every partition of the remaining vertex-copies is equally likely; and*
- (b) *every allocation of the D_t heavy vertex-copies to the N_t heavy bins such that each bin has size at least k is equally likely.*

In other words, $\{(L_t, N_t, D_t)\}_{t \geq 0}$ is Markovian. This Markovian property will allow easy analysis of $\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t)$ for instance.

Proof *Part (a):* L_t, N_t, D_t say nothing about the partition; they are parameters of the allocation. So every partition of the remaining vertex-copies is equally likely. (The deleted vertex copies have already been assigned to parts.)

Part (b): We begin with some intuition:

N_t, D_t only change when one of the $r - 1$ copies that we choose to be in the same part as x , and hence delete, is heavy. Suppose that the chosen copy is in a bin of size k . Then that entire bin is moved to \mathcal{Q}_t , and (at least intuitively) we have not exposed anything new about the remaining bins. So any allocation in which each remaining heavy bin has size at least k is equally likely. The more subtle case is when the chosen copy is in a bin of size greater than k . It is important to note that we do not expose the size of that bin, only the fact that the size is at least $k + 1$. So when we delete the vertex-copy, our exposure only says that the bin now has size at least k . Again, any allocation in which each heavy bin has size at least k is equally likely.

And now a proof: Consider any configuration G with n vertices and rm vertex-copies. Let $G = G_0, G_1, \dots, G_t$ be the sequence of configurations obtained by running SLOW-STRIP on G for t iterations; define $\mathcal{F}_0, \dots, \mathcal{F}_t$ similarly.

Expose Θ , the D_t heavy vertex-copies in G_t , and \mathcal{B} , the N_t heavy bins. Let H_t be the allocation of Θ to \mathcal{B} in G . Let H'_t be any other allocation of Θ to \mathcal{B} such that each bin in \mathcal{B}

receives at least k members of Θ . We will prove that H'_t occurs with the same probability as H_t when conditioning on Θ, \mathcal{B} being the heavy vertex-copies and bins. This is what is asserted by part (b).

Let G' be the configuration obtained from G by replacing H_t with H'_t ; i.e. by allocating every vertex-copy in Θ according to H'_t and allocating every other vertex-copy according to G . Similarly, define G'_i to be the configuration obtained from G_i by replacing H_t with H'_t for each $0 \leq i \leq t$. We make two key observations:

Observation 1: If SLOW-STRIP on G' chooses the same initial ordering of the light bins as SLOW-STRIP on G , and at each step chooses the same vertex-copy from the bin at the front of the queue to be deleted, then it will produce the sequence G'_0, \dots, G'_t .

Observation 2: The probability of the sequence G_0, \dots, G_t is the same as the probability of the sequence G'_0, \dots, G'_t . To see this, note first that G, G' are chosen as our random configuration from $AP_r(n, m)$ with the same probability as they both have the same n bins and rm vertex-copies. Then note that G, G' have the same number of light bins and for $0 \leq i \leq t$ each bin has the same size in G_i as in G'_i . So Observation 1 implies that the two sequences occur with the same probability.

The probability that after t steps, the set of heavy vertex-copies is Θ , the set of heavy bins is \mathcal{B} and the allocation of Θ to \mathcal{B} is H_t , is the sum over all sequences G_0, \dots, G_t that end in a G_t with Θ, \mathcal{B}, H_t of the probability of that sequence. The analogous statement is true of H'_t . We have established a bijection from the sequences yielding Θ, \mathcal{B}, H_t to the sequences yielding $\Theta, \mathcal{B}, H'_t$ which preserves probabilities. So the probability of obtaining H_t is equal to the probability of obtaining H'_t . This is part (b). □

Since $\mathcal{H}_r(n, cn)$ is relatively hard to analyse directly; many of the existing proofs for the size of the k -core of $\mathcal{H}_r(n, cn)$ use simpler models and then the results are translated to $\mathcal{H}_r(n, cn)$; specifically [24] proved Lemma 7 for the Poisson cloning model. We will show that monotone results can be translated from the Poisson cloning model to the AP-model and so we have:

Lemma 14. *Both Lemma 7 and Corollary 9 hold for the AP-model.*

The proof is in the appendix.

The reader may notice that our result does not actually require Lemma 14. By Lemma 7 and Corollary 9, we may run SLOW-STRIP on a random hypergraph $H \in \mathcal{H}_r(n, cn)$ until the remaining subgraph G_t has the desired order as required in Section 4.3. It is easy to see that by conditioning on \mathcal{F}_t , G_t is uniformly distributed over all hypergraphs with graph parameters agreeing with \mathcal{T}_t . Therefore, we could have applied the AP-model to G_t and started our analysis from there. However, it makes the proof a little easier to understand if we start with $H \in AP_r(n, cn)$ and apply Lemma 14, which guarantees that with a high probability, we will obtain some configuration G_t of the desired order. This is how Section 4.3 will be presented.

4.3 The supercritical case: $c = c_{r,k} + n^{-\delta}$

We first consider the supercritical case in which we assume $c = c_{r,k} + n^{-\delta}$ where $0 < \delta < 1/2$. These conditions on δ are assumed in the rest of Section 4 as desired by the hypotheses in Theorem 3. Our goal is to prove that $s(\mathcal{H}_r(n, cn))$ is bounded above by $O(n^{\delta/2} \log n)$ and below by $\Omega(n^{\delta/2})$. Note that this lower bound is slightly weaker than that of Theorem 3(a), but the proof is much simpler. The tight bound of Theorem 3(a) will be proved in Section 6.

Recall that we are running SLOW-STRIP on a configuration generated from the AP-model.

The following lemma allows us to assume that the remaining subgraph is sufficiently close to the k -core. Roughly speaking, after a constant number of rounds of the parallel stripping process, we can get within any linear distance of the k -core. Recall that \widehat{H}_i is the subgraph remaining after $i - 1$ rounds of the parallel stripping process and $\mathcal{C}_k(H)$ is the k -core of H .

Lemma 15. *Let $\epsilon, \epsilon_0 > 0$ be fixed. Assume $c \geq c_{r,k} + n^{-1/2+\epsilon}$. Then, for $H \in AP_r(n, cn)$, there exists a constant $B = B(r, k, \epsilon_0) > 0$, such that a.a.s. $|\widehat{H}_B \setminus \mathcal{C}_k(H)| \leq \epsilon_0 n$.*

Proof. Let $\epsilon' > 0$ be a small constant to be determined later and let $c' = c + \epsilon'$. Choose H' according to the distribution of $AP_r(n, c'n)$ and generate H by removing $\epsilon'n$ edges chosen uniformly at random in H' . (I.e. we remove $\epsilon'n$ uniform parts and the vertex-copies contained in those parts.) Then $H \subseteq H'$ and H is distributed as $AP_r(n, cn)$. Run the parallel k -stripping process on H' , using \widehat{H}'_i to denote the subgraph remaining after $i - 1$ iterations. Several other papers (see eg. Proposition 31 of [1]), show that for any $\sigma > 0$ there exists a constant $B > 0$ such that $|\widehat{H}'_B \setminus \mathcal{C}_k(H')| \leq \sigma n$, if $H' \in \mathcal{H}_r(n, c'n)$; the same conclusion translates to the AP-model with a similar argument.

Note that every vertex removed during the first $B - 1$ iterations of the parallel stripping process applied to H' would also have been removed during the first $B - 1$ iterations of the parallel stripping process applied to H . Thus $\widehat{H}_B \subseteq \widehat{H}'_B$. Also, $\mathcal{C}_k(H) \subseteq \mathcal{C}_k(H')$ and, by Lemmas 7 and 14, $|\mathcal{C}_k(H') \setminus \mathcal{C}_k(H)| = (\alpha(c') - \alpha(c))n + o(n)$. Therefore:

$$\begin{aligned} |\widehat{H}_B \setminus \mathcal{C}_k(H)| &\leq |\widehat{H}'_B \setminus \mathcal{C}_k(H')| + |\mathcal{C}_k(H') \setminus \mathcal{C}_k(H)| \leq \sigma n + (\alpha(c') - \alpha(c))n + o(n) \\ &< 2(\sigma + \alpha(c') - \alpha(c))n < \epsilon_0 n, \end{aligned}$$

where the last inequality holds as long as ϵ', σ are sufficiently small (both depending on ϵ_0). \square

Lemma 15 says that for any $\epsilon_0 > 0$ there is a $B = B(\epsilon_0)$ such that after iteration $B - 1$ of the parallel process, or equivalently, at the beginning of step $t(B)$ of SLOW-STRIP, the size of the remaining hypergraph $G_{t(B)}$ is at most $\epsilon_0 n$ greater than the size of the k -core. This implies that various parameters are very close to those of the k -core, and this is where our analysis starts: we focus on $t \geq t(B)$.

A key part of our analysis is to control the change in L_t . We will express the expected change of L_t at step t as a function of L_t, N_t and D_t . Recall:

$$\mathcal{F}_t = \{\mathcal{T}_s\}_{s \leq t} = \{(L_s, N_s, D_s)\}_{s \leq t}$$

and we will estimate $\mathbb{E}(L_{t+1} \mid \mathcal{F}_t)$. The following lemma bounds this expectation.

Lemma 16. *There are constants B, K such that: If $c = c_{r,k} + n^{-\delta}$, then a.a.s. for every $t(B) \leq t < \tau$,*

$$\mathbb{E}(L_{t+1} \mid \mathcal{F}_t) \leq L_t - Kn^{-\delta/2}.$$

Remark. The key Lemma 32 of [1] implies that, when $c = c_{r,k} + \epsilon$, we have a.a.s. $\mathbb{E}(L_{t+1} \mid \mathcal{F}_t) \leq (1 - \zeta)L_t$ for a constant $\zeta > 0$. That difference is what causes the stripping number and depth to rise from $O(\log n)$ for $c > c_{r,k} + \epsilon$ to $n^{\Theta(1)}$ for $c = c_{r,k} + n^{-\delta}$. It is also the cause of most of the difficulties in this paper. However, the weaker fact that $\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t)$ remains bounded below $-Kn^{-\delta/2}$, no matter how small L_t gets, is still very useful to our analysis.

Let $\alpha = \alpha(r, k)$ be the constant given in (5) and $K_1 = K_1(r, k)$ be the constant specified in Corollary 9. Then, by Corollary 9, for $c = c_{r,k} + n^{-\delta}$, a.a.s. the size of the k -core is $\alpha n + K_1 n^{1-\delta/2} + o(n^{1-\delta/2})$.

Lemma 16 will be used to bound the stripping number from above. To obtain a lower bound, we will focus on the so-called Phase 2 of SLOW-STRIP, defined as follows:

$$\gamma = 3K_2 n^{1-\delta/2}.$$

Let i^* be the first iteration of the parallel stripping process at the beginning of which the number of hyperedges in the remaining hypergraph is at most $\lfloor \beta n + \gamma \rfloor$.

Definition 17. t_0 is the first step of SLOW-STRIP at the beginning of which the number of hyperedges in the remaining hypergraph is exactly $\lfloor \beta n + \gamma \rfloor$. We refer to steps $t = 0, \dots, t_0 - 1$ as *Phase 1* and the remaining steps as *Phase 2*.

Note that we can obtain a lower bound on the number of iterations in Phase 2:

Lemma 18. *If $c = c_{r,k} + n^{-\delta}$ then a.a.s. Phase 2 lasts at least $\gamma/3 = K_2 n^{1-\delta/2}$ iterations.*

Proof By Corollary 9(b), a.a.s. the k -core has fewer than $\beta n + 2K_2 n^{1-\delta/2}$ hyperedges and so SLOW-STRIP has to remove at least $K_2 n^{1-\delta/2}$ hyperedges to reach it. \square

We will show that, throughout Phase 2, L_t is small.

Lemma 19. *If $c = c_{r,k} + n^{-\delta}$, then a.a.s. for every $t_0 < t \leq \tau$, $L_t = O(n^{1-\delta})$.*

We defer the proofs of Lemmas 16 and 19 to Section 4.4.

We now show how these lemmas nearly yield a proof of Theorem 3(a). Note that Lemma 18 bounds from below the total number of hyperedges that are removed during Phase 2. Lemma 19 bounds the total degree of each S_i , which bounds from above the number of hyperedges that are deleted in each iteration. This implies that Phase 2 requires many iterations; specifically, it yields a lower bound of $\Omega(n^{-\delta/2})$ on the stripping number, which is within a factor of $\log n$ of the bound in Theorem 3(a) (see subsection 4.3.1 for details). For the upper bound: Lemma 16 bounds the rate at which L_t decreases and so implies that it drops to zero quickly; of course, when $L_t = 0$ then the stripping process has ended (see subsection 4.3.2 for details).

4.3.1 Theorem 3(a): proof of a weaker lower bound

We begin with a lower bound of $\Omega(n^{-\delta/2})$ on the stripping number when $c = c_{r,k} + n^{-\delta}$ for some $0 < \delta < \frac{1}{2}$. This lower bound is slightly weaker than that of Theorem 3(a), but the proof is much simpler. Furthermore, this weaker lower bound will be used in Section 4.5 to prove Theorem 3(b). The proof for the tight lower bound of Theorem 3(a) will be presented in Section 6.

We run SLOW-STRIP and recall that this can be viewed as also running the parallel stripping process slowly. Exactly one hyperedge is deleted in each iteration of SLOW-STRIP. By Lemma 18, at least $K_2 n^{1-\delta/2}$ hyperedges are removed from G_t during Phase 2. By Lemma 19, at most $|L_{t(j)}| = O(n^{1-\delta})$ of them belong to S_j for each j . Therefore, we require at least $K_2 n^{1-\delta/2} / O(n^{1-\delta}) = \Omega(n^{\delta/2})$ iterations of the parallel stripping process to remove them all.

This proves that a.a.s. the stripping number of $AP_r(n, cn)$ is at least $\Omega(n^{\delta/2})$; Corollary 12 implies that the same is true of $\mathcal{H}_r(n, cn)$. \square

4.3.2 Theorem 3(a): proof of the upper bound

Now we turn to the upper bound on the stripping number when $c = c_{r,k} + n^{-\delta}$ for some $0 < \delta < \frac{1}{2}$.

We will focus on the change in $L_{t(i)}$, the sum, over all $v \in S_i$ of the degree of v at the beginning of iteration i of the parallel stripping process.

From iterations $t(i)$ to $t(i+1) - 1$ of SLOW-STRIP, all hyperedges from all vertices in S_i must be deleted. One hyperedge is removed in each iteration, and it touches at least one and at most r members of S_i . Thus

$$\frac{1}{r} L_{t(i)} \leq t(i+1) - t(i) \leq L_{t(i)}. \quad (7)$$

Let B and K be constants specified in Lemma 16. Let τ^* be the first step t such that $t \geq t(B)$ and $\mathbb{E}(L_{t+1} | \mathcal{F}_t) > L_t - Kn^{-\delta/2}$; if there is no such step then we set $\tau^* = \tau$. Then, for all $t(B) \leq t < \tau^*$,

$$\mathbb{E}(L_{t+1} | \mathcal{F}_t) \leq L_t - Kn^{-\delta/2}.$$

Taking conditional expectation on $\mathcal{F}_{t(i)}$ of both sides, for $i \geq B$, yields

$$\mathbb{E}(L_{t+1} | \mathcal{F}_{t(i)}) \leq \mathbb{E}(L_t | \mathcal{F}_{t(i)}) - Kn^{-\delta/2}, \quad \text{for all } t(i) \leq t < \tau^*.$$

If $t(i+1) < \tau^*$ then inductively applying the above for all $t(i) \leq t \leq t(i+1)$ we get

$$\mathbb{E}(L_{t(i+1)} | \mathcal{F}_{t(i)}) \leq L_{t(i)} - \left(t(i+1) - t(i) \right) Kn^{-\delta/2} \leq L_{t(i)} - \frac{1}{r} L_{t(i)} Kn^{-\delta/2} = L_{t(i)} \left(1 - \frac{K}{r} n^{-\delta/2} \right).$$

Define the random process $(\mathcal{L}_i)_{i \geq 0}$ as follows. For all i such that $t(i) \leq \tau^*$, let $\mathcal{L}_i = L_{t(i)}$ and for all i such that $t(i) > \tau^*$, define $\mathcal{L}_i = \mathcal{L}_{i-1} (1 - (K/r)n^{-\delta/2})$. Let T denote the minimum integer such that $\mathcal{L}_T \leq 0$; thus a.a.s. $\tau = t(T)$. By Lemma 16 a.a.s. $\tau^* = \tau$, and

so a.a.s. $L_{t(i)} = \mathcal{L}_i$ for all $0 \leq i \leq T$ and $L_{t(T)} = \mathcal{L}_T = 0$. Hence, we only need to obtain an upper bound for T . Now we have that for every $i \geq B$,

$$\mathbb{E}(\mathcal{L}_{i+1} \mid \mathcal{L}_i) \leq \mathcal{L}_i \left(1 - \frac{K}{r} n^{-\delta/2}\right).$$

Taking expectation on both sides we obtain that

$$\mathbb{E}\mathcal{L}_{i+1} \leq \mathbb{E}\mathcal{L}_i \left(1 - \frac{K}{r} n^{-\delta/2}\right).$$

Hence, as $\mathcal{L}_B \leq n$, for each $i \geq B$,

$$\mathbb{E}\mathcal{L}_i \leq n \left(1 - \frac{K}{r} n^{-\delta/2}\right)^{i-B}.$$

Thus, for $i > B + \frac{2r}{K} n^{\delta/2} \log n$, $\mathbb{E}(\mathcal{L}_i) = o(1)$ and so a.a.s. $T \leq B + (2r/K)n^{\delta/2} \log n$. This implies that a.a.s. the process of $(\mathcal{L}_i)_{i \geq 0}$ reaches $\mathcal{L}_i \leq 0$ within $B + (2r/K)n^{\delta/2} \log n$ iterations and so a.a.s. the parallel stripping process halts within $B + (2r/K)n^{\delta/2} \log n$ iterations. Therefore a.a.s. the stripping number of $AP_r(n, cn)$ is at most $B + \frac{2r}{K} n^{\delta/2} \log n = O(n^{\delta/2} \log n)$; Corollary 12 implies that the same is true of $\mathcal{H}_r(n, cn)$. \square

4.4 Bounds on L_i : proof of Lemmas 16 and 19

In this section, we prove Lemmas 16 and 19. So throughout, we have $c = c_{r,k} + n^{-\delta}$ for some $0 < \delta < \frac{1}{2}$.

Let $H \in AP_r(n, cn)$ and run the SLOW-STRIP algorithm on H . Recall from Definition 17 that

$$\gamma = 3K_2 n^{1-\delta/2}.$$

and t_0 is the first iteration of SLOW-STRIP in which the number of hyperedges in the remaining hypergraph is exactly $\lfloor \beta n + \gamma \rfloor$. The Second Phase of SLOW-STRIP consists of iterations $t \geq t_0$. Lemma 18 enables us to focus on the case when the algorithm does enter this phase.

Definition 20. We say that G_{t_0} is *nice* if the number of heavy vertices in G_{t_0} is between αn and $\alpha n + 2K_1 n^{1-\delta/2} + (r-1)\gamma$, and the total degree of light vertices in G_{t_0} is at most $r\gamma$.

Each iteration of SLOW-STRIP deletes at most $r-1$ heavy vertices, and reduces L_t by at most r . Moreover, a.a.s. Phase 2 of SLOW-STRIP lasts less than γ steps by Lemma 18. So Corollary 9 immediately implies that a.a.s. G_{t_0} is nice.

Recall that for every $t \geq 0$, G_t is the hypergraph remaining at the beginning of iteration t of SLOW-STRIP. The light vertices in G_t are defined to be the vertices with degree less than k and L_t denotes the total degree of the light vertices in G_t . Recall that τ is the iteration in which SLOW-STRIP halts.

The following proposition follows immediately from Corollary 9 and Lemma 18.

Proposition 21. *A.a.s. $t_0 + \gamma/3 \leq \tau \leq t_0 + \gamma$.*

It will be convenient to define

$$\hat{\tau} = \min\{\tau, t_0 + \gamma\},$$

and so Proposition 21 implies that

$$a.a.s. \quad \tau = \hat{\tau}. \tag{8}$$

The following proposition follows from the definition of t_0 and $\hat{\tau}$.

Proposition 22. *Assume $\tau > t_0$ and G_{t_0} is nice. Then for all $t_0 \leq t \leq \hat{\tau}$,*

- (a) *the number of heavy vertices in G_t is $\alpha n + O(\gamma)$ and the number of hyperedges in G_t is $\beta n + O(\gamma)$;*
- (b) *$L_t = O(\gamma)$.*

Proof. By definition, the number of hyperedges in G_{t_0} is $\beta n + \gamma$. Since G_{t_0} is nice, the number of heavy vertices in G_{t_0} is $\alpha n + O(\gamma)$. Moreover, by the definition of $\hat{\tau}$, $\hat{\tau} - t_0 = O(\gamma)$. We remove one hyperedge in each step, so for all $t_0 \leq t \leq \hat{\tau}$, the number of hyperedges and heavy vertices in G_t change by $O(\gamma)$ from those in G_{t_0} . This immediately confirms part (a). Moreover, since G_{t_0} is nice, $L_{t_0} = O(\gamma)$. As L_t changes by at most k in each step, we have $L_t = O(\gamma)$ for every $t_0 \leq t \leq \hat{\tau}$ and this proves part (b). \square

4.4.1 Creating light vertices

The key to analyzing the evolution of L_t is determining the rate at which new vertices are added to \mathcal{Q}_t ; i.e. become light. We begin by examining the distribution from Observation 13.

Given positive integers N , D and $k \geq 0$ such that $D \geq kN$, define $Multi(N, D, k)$, the *truncated multinomial distribution*, to be the probability space consisting of integer vectors $\mathbf{X} = (X_1, \dots, X_N)$ with domain $\mathcal{I}_k := \{\mathbf{d} = (d_1, \dots, d_N) : \sum_{i=1}^N d_i = D, d_i \geq k, \forall i \in [N]\}$, such that for any $\mathbf{d} \in \mathcal{I}_k$,

$$\Pr(\mathbf{X} = \mathbf{d}) = \frac{D!}{N^D \Psi} \prod_{i \in [N]} \frac{1}{d_i!} = \frac{\prod_{i \in [N]} 1/d_i!}{\sum_{\mathbf{d} \in \mathcal{I}_k} \prod_{i \in [N]} 1/d_i!},$$

where

$$\Psi = \sum_{\mathbf{d} \in \mathcal{I}_k} \frac{D!}{N^D} \prod_{i \in [N]} \frac{1}{d_i!}.$$

The degree distribution of the heavy vertices of G_t , conditional on \mathcal{F}_t , is exactly $Multi(N_t, D_t, k)$, by Observation 13. It was proved in [6, Lemma 1] (also appeared in [2, eq. (7)]) that the truncated multinomial variables can be well approximated by truncated Poisson random variables with expectation D_t/N_t . The result is stated as follows. Recall the definition of $f_k(\lambda)$ from (2). We define:

$$g_k(\lambda) = \lambda f_{k-1}(\lambda) / f_k(\lambda). \tag{9}$$

Note that $g_k(\lambda)$ is the expectation of a Poisson random variable with parameter λ truncated at k .

Proposition 23. *Let $k \geq 0$ be fixed, and N and D satisfy $D - kN = \Omega(N)$. Assume $\mathbf{X} \sim \text{Multi}(N, D, k)$. For any $j \geq k$, let ρ_j denote the proportion of X that equals j . Then, with probability $1 - o(1/N)$,*

$$\rho_j = e^{-\lambda} \frac{\lambda^j}{f_k(\lambda)j!} + O(N^{-1/2} \log N), \quad (10)$$

where λ satisfies $g_k(\lambda) = D/N$.

By Lemma 27 (below), $g_k(x)$ is an increasing function on $x > 0$. It is easy to show that $\lim_{x \rightarrow 0} g_k(x) = k$. Hence, for any $D > kN$, there is a unique λ that satisfies $\lambda f_{k-1}(\lambda)/f_k(\lambda) = D/N$ in the above proposition.

It is easy to check from the definition of $\mu_{r,k}$ above (4) that

$$g_k(\mu_{r,k}) = r\beta/\alpha. \quad (11)$$

Define:

$$\zeta = \zeta_{r,k} = r\beta/\alpha. \quad (12)$$

The following lemma justifies that $\zeta > k$. The proof consists of some tedious calculus, so we defer it to the Appendix.

Lemma 24. *Suppose $r, k \geq 2$ and $(r, k) \neq (2, 2)$. Then, $k < \zeta < r(k - 1)$.*

Thus, Proposition 23 and Corollary 9 imply that a.a.s. the proportion of degree k vertices in the k -core is approximately

$$\bar{\rho}_{r,k} = e^{-\mu_{r,k}} \frac{\mu_{r,k}^k}{f_k(\mu_{r,k})k!}. \quad (13)$$

The following technical lemma has appeared in several other papers (e.g. in [1]), but as the proof is short we include it in the Appendix.

Lemma 25. *For every $r, k \geq 2$ ($r, k \neq (2, 2)$),*

$$\frac{k\bar{\rho}_{r,k} \cdot \alpha}{r\beta} = \frac{1}{(r-1)(k-1)}.$$

And now we can prove the key lemma governing the evolution of L_t . Recall from above (8) that $\hat{\tau} = \min\{\tau, t_0 + \gamma\}$.

Lemma 26. *Assume that $\tau > t_0$ and G_{t_0} is nice. For any $t_0 \leq t \leq \hat{\tau}$, \mathcal{F}_t must be such that: for each of the $r - 1$ random vertex-copies chosen during iteration t , the probability, conditional on \mathcal{F}_t , that the vertex-copy belongs to a heavy bin which becomes light in this iteration, is $1/(r - 1)(k - 1) + O(\gamma/n)$.*

Proof. By Observation 13, we can treat the allocation of the D_t heavy vertex-copies to the N_t heavy bins as a uniform allocation subject to each bin receiving at least k copies. We first choose the $r - 1$ vertex-copies, then we choose the allocation. Let x be any one vertex-copy and let b be the bin to which x is allocated; we wish to bound the probability that $|b| = k$.

Consider a different experiment: first conduct the allocation, and then choose x uniformly from amongst all vertex-copies, and let b be the bin to which x was allocated; clearly these two experiments are equivalent ways to choose b . Let λ^* be the unique root of $g_k(\lambda) = D_t/N_t$. By Proposition 23 the probability that we choose x from a bin of size k is:

$$q = \frac{k}{D_t} \times e^{-\lambda^*} \frac{\lambda^{*k}}{f_k(\lambda^*)k!} N_t + O(n^{-1/2} \log n).$$

(Note that the $o(1/n)$ failure probability in Proposition 23 is absorbed into the $O(n^{-1/2} \log n)$ term.)

Since G_{t_0} is nice, Proposition 22 yields:

$$N_t = \alpha n + O(\gamma), \quad D_t = r\beta n + O(\gamma). \quad (14)$$

So

$$g_k(\lambda^*) = \frac{r\beta n + O(\gamma)}{\alpha n + O(\gamma)} = \frac{r\beta}{\alpha} + O(\gamma/n) = g_k(\mu_{r,k}) + O(\gamma/n).$$

Let $\delta = g'_k(\mu_{r,k})$; recall that g'_k and $\mu_{r,k}$ are independent of γ, n and so δ is a constant independent of γ, n . Lemma 27 (below) implies that $\delta > 0$. Thus $g'_k(\lambda) > \frac{1}{2}\delta$ for all $\lambda = \mu_{r,k} + o(n)$; similarly, Lemma 27 implies that for any $\zeta > 0$, $g'_k(\lambda) > 0$ for all $\lambda > \zeta n$. These two bounds, along with (14) easily imply that

$$\lambda^* = \mu_{r,k} + O(\gamma/n).$$

Setting $h(x) = e^{-x} \frac{x^k}{f_k(x)k!}$, recalling (13), and noting that $h'(\mu_{r,k}) = O(1)$, and applying Lemma 25, we have

$$\begin{aligned} q &= \frac{kN_t}{D_t} h(\lambda^*) + O(n^{-1/2} \log n) = \frac{k\alpha}{r\beta} h(\mu_{r,k}) + O(\gamma/n) = \frac{k\alpha \bar{\rho}_{r,k}}{r\beta} + O(\gamma/n) \\ &= \frac{1}{(r-1)(k-1)} + O(\gamma/n). \end{aligned}$$

□

We close this subsection with a technical lemma proving the monotonicity of $g_k(x)$. The proof will be given in the Appendix.

Lemma 27. *For any $x > 0$, $g'_k(x) > 0$.*

4.4.2 Proof of Lemma 19

Recall that we are running SLOW-STRIP on the AP-model. Recall also from Section 4 that L_t denotes the total degree of the vertices with degree less than k in G_t , and recall from Definition 17 that t_0 is the beginning of Phase 2 of SLOW-STRIP. Corollary 9 ensures that a.a.s. G_{t_0} is nice and so Proposition 22 immediately yields a weaker version of Lemma 19: a.a.s. for all $t_0 \leq t \leq \hat{\tau}$, $L_t = O(n^{1-\delta/2}) = O(\gamma)$. This allows us to prove that a weaker form of Lemma 16 holds for $t \geq t_0$:

Lemma 28. Assume that $\hat{\tau} > t_0$ and G_{t_0} is nice. For every $t_0 \leq t < \hat{\tau}$,

$$\mathbb{E}(L_{t+1} | \mathcal{F}_t) = L_t \pm O(n^{-\delta/2}).$$

Proof. Let v be the vertex taken from \mathcal{Q}_t at iteration t . Let u_1, \dots, u_{r-1} denote the other vertex-copies (except the copy in v) that are selected to be deleted in this step. The removal of the vertex-copy in v contributes -1 to $\Delta L_t := L_{t+1} - L_t$ always. We consider the contribution to ΔL_t from the removal of u_i . Let $v(u_i)$ denote the bin that contains u_i . There are four cases.

Case 1: The removed edge contains two copies of the same vertex. The probability of this case is $O(1/n)$ and so the contribution of this case to $\mathbb{E}(L_{t+1} - L_t | \mathcal{F}_t)$ is $O(1/n)$.

For the remaining three cases, we can assume that no other copy in $v(u_i)$ has already been removed during iteration t .

Case 2: $v(u_i)$ was in \mathcal{Q}_t in G_t . In this case, the contribution of the removal of u_i to ΔL_t is -1 . Since u_i is chosen by the algorithm u.a.r. from all remaining vertex-copies, the probability of this event is $O(L_t/(D_t + L_t)) = O(\gamma/n)$ by (14) and Proposition 22(b).

Case 3: $v(u_i)$ enters \mathcal{Q}_t at iteration t . In this case, the size of $v(u_i)$ is k in G_t and the contribution to ΔL_t from the removal of u_i is $k - 1$. The probability of this event is $1/(r-1)(k-1) + O(\gamma/n)$ by Lemma 26.

Case 4: $v(u_i)$ was not in \mathcal{Q}_t and does not enter \mathcal{Q}_{t+1} at iteration $t+1$. In this case, the size of $v(u_i)$ is more than k in G_t and the contribution to ΔL_t from the removal of u_i is 0. The probability of this event is $1 - 1/(r-1)(k-1) + O(\gamma/n)$.

By the linearity of expectation, summing the contributions of u_1, \dots, u_{r-1} , we have

$$\begin{aligned} \mathbb{E}(L_{t+1} - L_t | \mathcal{F}_t) &= -1 + (r-1) \left((-1) \cdot O(\gamma/n) + (k-1) \left(\frac{1}{(r-1)(k-1)} + O(\gamma/n) \right) \right) \\ &\quad + O(1/n) = O(\gamma/n). \end{aligned}$$

The lemma follows by noting that $\gamma = \Theta(n^{1-\delta/2})$ by Definition 17. \square

The following lemma is a simple application of the Hoeffding-Azuma Inequality[3, 20].

Lemma 29. Let a_n and $c_n \geq 0$ be real numbers and $(X_{n,i})_{i \geq 0}$ be random variables with respect to a random process $(G_{n,i})_{i \geq 0}$ such that for all $t \geq 0$

$$\mathbb{E}(X_{n,i+1} | \{G_{n,s}\}_{s \leq i}) \leq X_{n,i} + a_n,$$

and $|X_{n,i+1} - X_{n,i}| \leq c_n$, for every $i \geq 0$ and all (sufficiently large) n . Then, for any real number $j \geq 0$,

$$\Pr(X_{n,t} - X_{n,0} \geq ta_n + j) \leq \exp\left(-\frac{j^2}{2t(c_n + |a_n|)^2}\right).$$

Proof. Define $Y_{n,i} = X_{n,i} - ia_n$ for all $i \geq 0$. Then,

$$\mathbb{E}(Y_{n,i+1} | \{G_{n,s}\}_{s \leq i}) = \mathbb{E}(X_{n,i+1} | \{G_{n,s}\}_{s \leq i}) - (i+1)a_n \leq X_{n,i} - ia_n = Y_{n,i}.$$

Thus, $(Y_{n,i})_{i \geq 0}$ is a supermartingale. Moreover, $|Y_{n,i+1} - Y_{n,i}| \leq c_n + |a_n|$. By the Hoeffding-Azuma Inequality,

$$\Pr(Y_{n,t} - Y_{n,0} \geq j) \leq \exp\left(-\frac{j^2}{2t(c_n + |a_n|)^2}\right).$$

This completes the proof of the lemma. \square

We now recall the statement of Lemma 19:

Lemma 19. If $c = c_{r,k} + n^{-\delta}$, then a.a.s. for every $t_0 \leq t \leq \tau$:

$$L_t = O(n^{1-\delta}).$$

Proof. Without loss of generality we may assume $t_0 < \tau$. By Corollary 9 we may also assume that G_{t_0} is nice. Recall that $\hat{\tau} = \min\{\tau, t_0 + \gamma\}$. By Lemma 28, there exists a nonnegative sequence $(a_n)_{n \geq 1}$ such that $a_n = O(n^{-\delta/2}) = O(\gamma/n)$ and for every $t_0 \leq t < \hat{\tau}$,

$$\mathbb{E}(L_{t+1} | \mathcal{F}_t) \leq L_t + a_n, \quad \mathbb{E}(-L_{t+1} | \mathcal{F}_t) \leq -L_t + a_n.$$

Define \mathcal{L}_t such that $\mathcal{L}_t = L_t$ for every $t \leq \hat{\tau}$ and $\mathcal{L}_{t+1} = \mathcal{L}_t$ for all $t \geq \hat{\tau}$. Now for every $t \geq t_0$,

$$\mathbb{E}(\mathcal{L}_{t+1} | \{\mathcal{L}_s\}_{s \leq t}) \leq \mathcal{L}_t + a_n, \quad \mathbb{E}(-\mathcal{L}_{t+1} | \{\mathcal{L}_s\}_{s \leq t}) \leq -\mathcal{L}_t + a_n.$$

It is clear that $|\mathcal{L}_{t+1} - \mathcal{L}_t| = O(1)$ always as $L_{t+1} = L_t + O(1)$ always. By Lemma 29 with $j = \gamma^{1/2} \log n$ and $c_n = O(1)$, for every $t_0 \leq t \leq t_0 + \gamma$,

$$\Pr(\mathcal{L}_t \geq \mathcal{L}_{t_0} + a_n(t-t_0) + \gamma^{1/2} \log n) = o(n^{-1}), \quad \Pr(-\mathcal{L}_t \geq -\mathcal{L}_{t_0} + a_n(t-t_0) + \gamma^{1/2} \log n) = o(n^{-1}).$$

We apply the union bound over all $t_0 \leq t < t_0 + \gamma$, along with the asymptotics $a_n = O(\gamma/n)$, and $\gamma^{1/2} \log n = o(\gamma^2/n)$ (since $\gamma = 3K_2 n^{1-\delta/2}$ and $\delta < 1/2$), to obtain that a.a.s. for all $t_0 \leq t \leq t_0 + \gamma$:

$$\mathcal{L}_t \geq \mathcal{L}_{t_0} - O(\gamma^2/n) \tag{15}$$

$$\mathcal{L}_t \leq \mathcal{L}_{t_0} + O(\gamma^2/n) \tag{16}$$

By (8), a.a.s. $\hat{\tau} = \tau$ and so a.a.s. $\tau = \hat{\tau} \leq t_0 + \gamma$ and $L_{\hat{\tau}} = L_{\tau} = \mathcal{L}_{\tau} = 0$. Therefore, (15) with $t = \tau$ yields

$$L_{t_0} = \mathcal{L}_{t_0} = O(\gamma^2/n). \tag{17}$$

Substituting that into (16) yields that for all $t_0 \leq t \leq \tau$, $L_t = \mathcal{L}_t = O(\gamma^2/n) = O(n^{1-\delta})$, thus establishing the lemma. \square

4.4.3 ζ_t and \bar{p}_t

In this section, we study two parameters ζ_t and \bar{p}_t to be defined below, which allow for a more careful analysis of the rate at which new light vertices are formed.

Recall that L_t is the total degree of the light vertices in G_t (i.e. those of degree less than k); eventually, our goal is to prove Lemma 16, i.e. to bound $\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t)$ from above. This quantity has been bounded from below in the previous section for $t \geq t_0$ (Lemma 28). The main challenge for Lemma 16 is that we need to show a uniform bound for all $t \geq t(B)$, as long as B is a sufficiently large constant (recall that t_0 grows with n). This also makes a significant difference from Lemma 28 which only requires $t \geq t_0$.

Now let $H \in AP_r(n, cn)$, where $c = c_{r,k} + n^{-\delta}$ and let B be a sufficiently large constant whose value is to be determined later. Recall that G_t is the hypergraph remaining after t iterations of SLOW-STRIP and so $G_{t(B)} = \widehat{H}_B$.

Recalling that N_t is the number of heavy vertices, and D_t is the total degree of the heavy vertices, we define:

$$\zeta_t = \zeta_t(\mathcal{F}_t) = D_t/N_t \text{ is the average degree of the heavy vertices in } G_t.$$

Recalling that, by Corollary 9, the k -core a.a.s. has $\alpha n + o(n)$ vertices and $\beta n + o(n)$ edges and $\zeta = r\beta/\alpha$. Therefore the k -core a.a.s. has average degree $\zeta + o(1)$ and so ζ_t approaches ζ . Note that $\zeta = g_k(\mu_{r,k})$ by (11).

We are interested in the probability that a particular heavy vertex-copy is allocated to a bin of size k since this tells the proportion of the heavy vertices in deleted hyperedges that become light. So, recalling Observation 13, we define:

Definition 30. $\bar{p}_t = \bar{p}_t(\mathcal{T}_t)$ is the probability that a given vertex-copy is assigned to a bin of size k in a uniformly random allocation of D_t points to N_t bins subject to each bin receiving at least k points.

We will deduce \bar{p}_t as a certain function of ζ_t , approximately. For all $x > k$, we define:

$$\lambda(x) \text{ is the root of } g_k(\lambda) = x; \quad \text{i.e. } \lambda f_{k-1}(\lambda) = x f_k(\lambda) \quad (18)$$

$$\psi(x) = \frac{e^{-\lambda(x)} \lambda(x)^{k-1}}{f_{k-1}(\lambda(x))(k-1)!} \quad (19)$$

With some basic calculations we can show that $\psi(x)$ is strictly decreasing for $x > k$.

Lemma 31. For all $x > k$, $\psi'_k(x) < 0$.

We defer the proof to the Appendix.

In the k -core, a.a.s. the proportion of the total degree that comes from vertices of degree k is approximately $\psi(\zeta)$ by Corollary 9 and Proposition 23.

Since \bar{p}_t approaches $\psi(\zeta)$, we should have $\bar{p}_t \approx \psi(\zeta_t)$. Our next lemma formalizes this approximation:

Lemma 32. Assume $N_t = \Omega(n)$ for every $t \leq \tau$. Then $\bar{p}_t = (1 + O(n^{-1/2} \log n))\psi(\zeta_t)$ for all $t \leq \tau$.

Proof. Let λ be chosen such that $\lambda f_{k-1}(\lambda) = \zeta_t f_k(\lambda)$. Then by Proposition 23, the following holds for all $t \leq \tau$:

$$\begin{aligned} \bar{p}_t &= \frac{k e^{-\lambda} \lambda^k N_t}{f_k(\lambda)(k-1)! D_t} (1 + O(n^{-1/2} \log n)) + o(n^{-1}) = \frac{e^{-\lambda} \lambda^k}{f_k(\lambda)(k-1)! \zeta_t} (1 + O(n^{-1/2} \log n)) \\ &= \frac{e^{-\lambda} \lambda^{k-1}}{f_{k-1}(\lambda)(k-1)!} (1 + O(n^{-1/2} \log n)) \quad \text{by (18)}. \quad \square \end{aligned}$$

We will analyze ζ_t, \bar{p}_t in order to prove Lemma 16.

Recall that $g_k(x) = x f_{k-1}(x)/f_k(x)$ as defined in (9) and that $\zeta = r\beta/\alpha$. Then, $\mu_{r,k}$ is the root of $g_k(x) = \zeta$ by (11). Recall also that

$$\psi(x) = \frac{e^{-\lambda(x)} \lambda(x)^{k-1}}{f_{k-1}(\lambda(x))(k-1)!},$$

where $\lambda(x)$ is the root of $g_k(\lambda) = x$. Now define $p^* = \psi(\zeta)$. Then, we have

$$p^* = \frac{e^{-\mu_{r,k}} \mu_{r,k}^{k-1}}{f_{k-1}(\mu_{r,k})(k-1)!}.$$

By the definition of α, β below (4) and $\bar{\rho}_{r,k}$ in (13), we have $p^* = k\bar{\rho}_{r,k}/\zeta$. Then, by Lemma 25, we have $p^* = 1/(r-1)(k-1)$. As a summary of the above discussion, we have the following equalities.

$$g_k(\mu_{r,k}) = \zeta = r\beta/\alpha; \tag{20}$$

$$p^* = \psi(\zeta) = \frac{1}{(r-1)(k-1)}. \tag{21}$$

The expected change in L_t , is closely tied to ζ_t . So in order to prove Lemma 16, we begin by bounding ζ_t over the next two lemmas. Recall that $\zeta_t = D_t/N_t$ denotes the average degree of heavy vertices in G_t . We will apply Lemma 32 to relate \bar{p}_t with $\psi(\zeta_t)$ for all t in the range we analyse and therefore we must restrict to a sequence of ‘‘typical’’ hypergraphs (G_t) such that $N_t = \Omega(n)$ holds in the whole process. To formalise the idea, we define:

$$G_t \text{ is normal if } |\bar{p}_t - \psi(\zeta_t)| \leq n^{-1/2} \log^2 n.$$

Note that whether G_t is normal is determined by \mathcal{F}_t so, equivalently, we could say \mathcal{F}_t is normal. We define the stopping times:

$$\begin{aligned} \tau_1 & \text{ is the minimum integer } t \text{ such that } G_t \text{ is not normal;} \\ \tau_1 & = \tau \text{ if } G_t \text{ is normal for all } 0 \leq t \leq \tau. \end{aligned} \tag{22}$$

We will focus on steps t such that ζ_t is close to ζ . Hence, we will restrict to sequences (G_t) such that ζ_t gets close to ζ eventually. Given a constant $\epsilon > 0$, define:

$$\begin{aligned} t_\epsilon & \text{ is the minimum integer that } ||V(G_t)| - \alpha n| \leq \epsilon n \text{ and } |\zeta_t - \zeta| \leq \epsilon; \\ t_\epsilon & = \tau \text{ if such an integer does not exist.} \end{aligned} \tag{23}$$

Observation 33. *For all $t_\epsilon \leq t \leq t_\epsilon + k\epsilon n$, we have $\zeta_t = \zeta + O(\epsilon)$.*

Proof. Each iteration of SLOW-STRIP removes exactly one hyperedge. So it reduces each of D_t and N_t by at most r . Since $D_{t_\epsilon} \geq r\beta n$ and $N_{t_\epsilon} \geq \alpha n$, each step can only change $\zeta_t = D_t/N_t$ by at most $O(\epsilon)$, and this will be true for at least $k\epsilon n$ steps. \square

Lemma 34.

- (a) There are a sufficiently small constant $\epsilon = \epsilon(r, k) > 0$ and two constants $\rho_1 = \rho_1(r, k) > 0$ and $\rho_2 = \rho_2(r, k) > 0$ such that for all $t_\epsilon \leq t < \min\{t_\epsilon + k\epsilon n, \tau_1\}$,

$$-\frac{\rho_1}{n} \leq \mathbb{E}(\zeta_{t+1} - \zeta_t \mid \mathcal{F}_t) \leq -\frac{\rho_2}{n}.$$

- (b) Given a constant $\epsilon > 0$, a.a.s. there is a large constant $B = B(r, k, \epsilon)$ for which: if $c = c_{r,k} + n^{-\delta}$ then $t_\epsilon \leq t(B)$. Moreover, a.a.s. $\tau_1 = \tau$ and $\tau < t_\epsilon + k\epsilon n$.

Proof. Let $\epsilon > 0$ be a small constant whose value is to be determined later. We first prove part (a). We may assume that $t_\epsilon < \tau_1$ since otherwise there is nothing to prove.

Consider any $t \geq t_\epsilon$, and let v be the vertex taken from \mathcal{Q}_t during iteration t . Recall that when SLOW-STRIP runs on the AP-model, it removes one vertex-copy from v and another $r - 1$ vertex-copies u_1, \dots, u_{r-1} chosen uniformly at random from the remaining ones. We will split the single step into $r - 1$ substeps $(T_{t,i})_{1 \leq i \leq r-1}$, such that u_i is removed in step $T_{t,i}$ for all $1 \leq i \leq r - 1$ and let $T_{t,0} = t$. We consider the contribution of deleting u_i , denoted by $C(u_i)$, to $\mathbb{E}(\zeta_{t+1} - \zeta_t \mid \mathcal{F}_t)$. If u_{i+1} is light then $C(u_i) = 0$. If it is heavy (which occurs with probability at least $1/2$ if ϵ is sufficiently small) then (extending the definition of D_t, N_t, \bar{p}_t to $D_{T_{t,i}}, N_{T_{t,i}}, \bar{p}_{T_{t,i}}$ in the obvious manner):

$$C(u_i) = \frac{D_{T_{t,i}} - 1}{N_{T_{t,i}}} (1 - \bar{p}_{T_{t,i}}) + \frac{D_{T_{t,i}} - k}{N_{T_{t,i}} - 1} \bar{p}_{T_{t,i}} - \frac{D_{T_{t,i}}}{N_{T_{t,i}}}. \quad (24)$$

In this case, dropping the subscript and substituting $1/(N - 1) = (1/N)(1 + 1/N + O(1/n^2))$ (as $N = \Omega(n)$), we have that uniformly for all $1 \leq i \leq r - 1$,

$$\begin{aligned} C(u_i) &= \frac{1}{N} \left((D - 1)(1 - \bar{p}) + \bar{p}(D - k)(1 + 1/N + O(n^{-2})) - D \right) \\ &= \frac{1}{N} \left(-1 - \bar{p}(D - 1) + \bar{p}(D + \frac{D}{N} - k + O(n^{-1})) \right) \\ &= \frac{1}{N} \left(-1 + \bar{p} \left(\frac{D}{N} - (k - 1) + O(n^{-1}) \right) \right). \end{aligned} \quad (25)$$

By the definition of t_ϵ and applying Observation 33, we have $\zeta_t = r\beta/\alpha + O(\epsilon) = \zeta + O(\epsilon)$ for every $t_\epsilon \leq t \leq t_\epsilon + k\epsilon n$. Recall from (21) that

$$p^* = \psi(\zeta) = \frac{1}{(r - 1)(k - 1)}.$$

By Lemma 24 $\zeta > k$. Since $\zeta = \zeta_{r,k}$ is a constant, Lemma 31 implies that $\psi'(\zeta) < 0$ and $\psi'(x)$ is uniformly bounded away from zero in a small neighbourhood of ζ . By the definition of τ_1 in (22), for all $t_\epsilon \leq t < \min\{t_\epsilon + k\epsilon n, \tau_1\}$ and $0 \leq i \leq r - 1$,

$$\bar{p}_{T_{t,i}} = \psi(\zeta_t) + o(1) = \psi(\zeta) + O(\epsilon) = p^* + O(\epsilon).$$

If we were to substitute $\bar{p} = p^*$, $D = r\beta n$, $N = \alpha n$ into the RHS of (25), and simplify it, we would obtain:

$$\begin{aligned}
C(u_i) &= \frac{1}{\alpha n} \left(-1 + \frac{1}{(r-1)(k-1)} \left(\frac{r\beta}{\alpha} - (k-1) + O(n^{-1}) \right) \right) \\
&= \frac{1}{\alpha n} \left(-1 + \frac{1}{(r-1)(k-1)} (\zeta - (k-1) + O(n^{-1})) \right). \tag{26}
\end{aligned}$$

By Lemma 24, there is a $\sigma > 0$ such that $\zeta < r(k-1) - \sigma$. Hence,

$$\begin{aligned}
-1 + \frac{1}{(r-1)(k-1)} (\zeta - (k-1) + O(n^{-1})) &< -1 + \frac{1}{(r-1)(k-1)} ((r-1)(k-1) - \sigma/2) \\
&< -\sigma/2(r-1)(k-1).
\end{aligned}$$

Since in every step, the quantity of variables in (25) (e.g. \bar{p} and D/N) differs from that in (26) by $O(\epsilon)$, by choosing $\epsilon > 0$ sufficiently small, there exists constants $\rho'_1, \rho'_2 > 0$ such that a.a.s.

$$-\frac{\rho'_1}{n} \leq C(u_i) \leq -\frac{\rho'_2}{n} \quad \text{for all } 1 \leq i \leq r-1,$$

for all $t_\epsilon \leq t < \min\{t_\epsilon + k\epsilon n, \tau_1\}$. Since there are $r-1$ u_i 's, part (a) follows with $\rho_1 = (r-1)\rho'_1, \rho_2 = (r-1)\rho'_2$.

Next, we prove part (b). By Lemma 15, there is a constant $B = B(r, k, \epsilon')$, such that a.a.s. $|V(G_{t(B)}) \setminus \mathcal{C}_k| \leq \epsilon' n$. By Corollary 9, a.a.s. the number of vertices and hyperedges in $G_{t(B)}$ is $\alpha n + O(\epsilon' n)$ and $r\beta n + O(\epsilon' n)$ respectively. Therefore, $|\zeta_{t(B)} - \zeta| < A\epsilon'$ for some constant $A > 0$. We choose ϵ' sufficiently small (correspondingly B sufficiently large), so that $\epsilon' < \epsilon$ and $\epsilon' < \epsilon/A$. Thus we have $t(B) \geq t_\epsilon$. Hence, a.a.s. there exists large constant $B = B(\epsilon)$ so that $t_\epsilon \leq t(B)$. The fact that a.a.s. $\tau_1 = \tau$ follows from the definition of τ_1 in (22) and Lemmas 32 and 7.

The fact that a.a.s. $\tau \leq t_\epsilon + k\epsilon n$ follows by the definition of t_ϵ in (23) and Corollary 9: a.a.s. there are at most $\epsilon n + o(n)$ vertices to be deleted from G_{t_ϵ} until SLOW-STRIP terminates, and it takes at most $k-1$ steps to remove each vertex in \mathcal{Q}_t . □

In the next lemma, we obtain a coarse bound on ζ_t , which will suffice to prove Lemma 16. We will refine this bound in the later part of this paper when we require a stronger form of Lemma 16 (see Lemma 50).

Lemma 35. *For any sufficiently small constant $\epsilon > 0$, there exists a sufficiently large constant B such that for every $\epsilon' > 0$, a.a.s. for all $t \geq t(B)$: $\zeta_t \geq \zeta_\tau - n^{-1/2+\epsilon'}$ and $|\zeta_t - \zeta| \leq \epsilon$.*

Proof. Take a sufficiently small constant ϵ_1 to satisfy Lemma 34(a); then a sufficiently large constant $B = B(r, k, \epsilon_1)$ as in Lemma 34(b). Then, a.a.s.

$$t_{\epsilon_1} \leq t(B) \quad \text{and} \quad \tau_1 = \tau \leq t_{\epsilon_1} + k\epsilon_1 n. \tag{27}$$

Moreover, by Lemma 34(a), for all $t_{\epsilon_1} \leq t \leq \tau' := \min\{t_{\epsilon_1} + k\epsilon_1 n, \tau_1\}$: ζ_t is a supermartingale and $\zeta_{t+1} - \zeta_t$ is at most $O(1/n)$. We couple ζ_t with another process Z_t as in the proof of

Lemma 19: for all $t_{\epsilon_1} \leq t \leq \tau'$, let $Z_t = \zeta_t$; for all $t \geq \tau'$, let $Z_{t+1} = Z_t$. Now, (Z_t) is a supermartingale for $t \geq t_{\epsilon_1}$ and $Z_{t+1} = Z_t + O(1/n)$. By Azuma's inequality, we have that for all $t_{\epsilon_1} \leq t_2 < t_1 \leq t_{\epsilon_1} + k\epsilon_1 n$ and for any $j > 0$,

$$\Pr(Z_{t_1} - Z_{t_2} \geq j) \leq \exp\left(-\Omega\left(\frac{j^2}{(t_1 - t_2)n^{-2}}\right)\right).$$

Taking $j = n^{-1/2+\epsilon'}$ and taking the union bound over all pairs $t_{\epsilon_1} \leq t_1 < t_2 \leq t_{\epsilon_1} + k\epsilon_1 n$, the probability that there is a pair $t_2 < t_1$ in the above range such that $Z_{t_1} - Z_{t_2} > n^{-1/2+\epsilon'}$ is at most $n^2 \exp(-\Omega(n^{2\epsilon'})) = o(1)$. Hence, a.a.s. $Z_{t_2} \geq Z_{t_1} - n^{-1/2+\epsilon'}$ for any pair $t_{\epsilon_1} \leq t_1 < t_2 \leq t_{\epsilon_1} + k\epsilon_1 n$.

By (27), a.a.s. $t(B) \geq t_{\epsilon_1}$ and $\tau' = \tau_1 = \tau$ and so a.a.s. $Z_t = \zeta_t$ for all $t_{\epsilon_1} \leq t \leq \tau$. So, a.a.s. $\zeta_t \geq \zeta_\tau - n^{-1/2+\epsilon'}$ for all $t \geq t(B)$.

To prove that a.a.s. $|\zeta_t - \zeta| \leq \epsilon$, we simply choose ϵ_1 to be sufficiently small in terms of ϵ (and the implicit constant in Observation 33). (27) yields that a.a.s. for all $t(B) \leq t \leq \tau$ we have $t_{\epsilon_1} \leq t \leq t_{\epsilon_1} + k\epsilon_1 n$, so Observation 33 yields

$$|\zeta_t - \zeta| \leq O(\epsilon_1) \leq \epsilon. \quad \square$$

This immediately yields the following corollary.

Corollary 36. *For any sufficiently small constant $\epsilon > 0$, there exist constants $B, K > 0$ such that a.a.s. for all $t \geq t(B)$:*

$$(a) \quad \zeta_t \geq \zeta + Kn^{-\delta/2};$$

$$(b) \quad |\zeta_t - \zeta| \leq \epsilon.$$

Proof. Choose B and ϵ to satisfy Lemma 35. Immediately, we have a.a.s. $|\zeta_t - \zeta| \leq \epsilon$ for all $t \geq t(B)$.

Recall the definition of $g_k(x)$ in (9). By Lemma 7 and since $\mu_{r,k} > 0$, a.a.s.

$$\begin{aligned} \zeta_\tau &= \frac{r\beta(c) + O(n^{3/4})}{\alpha(c) + O(n^{3/4})} = g_k(\mu(c)) + O(n^{-1/4}) \\ &= g_k(\mu_{r,k}) + g'_k(\mu_{r,k})(\mu(c) - \mu_{r,k}) + o(\mu(c) - \mu_{r,k}) + O(n^{-1/4}), \end{aligned}$$

By Lemma 27, $g'_k(\mu_{r,k}) > 0$. By Lemma 8, $\mu(c) - \mu_{r,k} = K_1 n^{-\delta/2} + o(n^{-\delta/2})$ for some constant $K_1 > 0$. Recall also that $\zeta = r\beta/\alpha = g_k(\mu_{r,k})$. So a.a.s.

$$\zeta_\tau = \zeta + \Theta(n^{-\delta/2}), \quad (28)$$

as $n^{-1/4}$ is absorbed by $o(n^{-\delta/2})$ since $\delta < 1/2$. Now applying Lemma 35 with some $\epsilon' < \frac{1}{4}$ so that $n^{-\frac{1}{2}+\epsilon'} = o(n^{-\delta/2})$, we have that a.a.s.

$$\text{for all } t \geq t(B), \quad \zeta_t \geq \zeta_\tau - n^{-\frac{1}{2}+\epsilon'} \geq \zeta + Kn^{-\delta/2},$$

for some appropriate constant $K > 0$. □

4.4.4 Proof of Lemma 16

Recall that we analyze the running of SLOW-STRIP on $AP_r(n, cn)$. In each step $t(B) \leq t \leq \tau$, the algorithm removes a vertex-copy in a light vertex and another $r - 1$ vertex-copies u_1, \dots, u_{r-1} chosen uniformly from all remaining ones. For each $1 \leq i \leq r - 1$, let $h_{t,i}$ denote the probability that u_i is light. Recall that \bar{p}_t is the probability that a heavy vertex-copy is allocated to a bin of size k . Then

$$\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) = -1 + \sum_{i=1}^{r-1} \left(-h_{t,i} + (1 - h_{t,i})(k - 1)\bar{p}_t + O(n^{-1}) \right), \quad (29)$$

where $O(n^{-1})$ accounts for the change of \bar{p}_t caused by the removal of the first $i - 1$ points and the possibility that we select two copies of the same vertex. (29) is maximized when $h_{t,i} = 0$ for all $1 \leq i \leq r - 1$. Thus,

$$\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) \leq -1 + (r - 1)(k - 1)\bar{p}_t + O(n^{-1}). \quad (30)$$

As in the proof of Lemma 34, we need a relation between \bar{p}_t and ζ_t and thus it is convenient to restrict to steps $t < \tau_1$ where τ_1 is defined in (22). Moreover, we want to restrict to sequences (G_t) such that Corollary 36(a,b) hold. To formalise the idea, we first choose constants B, ϵ and K to satisfy Corollary 36 (Note that ϵ can be chosen arbitrarily small which results in larger B). Then, we define:

$$\begin{aligned} \tau_2 \text{ is the minimum } t \geq t(B) \text{ such that } \zeta_t < \zeta + Kn^{-\delta/2} \text{ or } |\zeta_t - \zeta| > \epsilon; \\ \tau_2 = \tau \text{ if no such integer exists.} \end{aligned} \quad (31)$$

Define

$$\tau^* = \min\{\tau_1, \tau_2\}. \quad (32)$$

By Lemma 34(b) and Corollary 36, a.a.s. $\tau^* = \tau$.

By the definition of τ_1 in (22) and noting that $\tau^* \leq \tau_1$, we have for all $t(B) \leq t < \tau^*$:

$$\bar{p}_t = \psi(\zeta_t) + n^{-1/2} \log^2 n = \psi(\zeta) + \psi'(\zeta)(\zeta_t - \zeta) + O((\zeta_t - \zeta)^2) + n^{-1/2} \log^2 n, \quad (33)$$

by expanding $\psi(\zeta_t)$ at ζ .

By Lemmas 31 and 24, and since $\zeta = \zeta(r, k)$ is a constant, we have $\psi'(\zeta) < -C$ for some constant $C > 0$. By the definition of τ_2 and τ^* , for all $t(B) \leq t < \tau^*$, $\zeta_t - \zeta$ can be assumed sufficiently small (by choosing sufficiently small ϵ) so that

$$\psi'(\zeta)(\zeta_t - \zeta) + O((\zeta_t - \zeta)^2) \leq -\frac{C}{2}(\zeta_t - \zeta).$$

By (21), $\psi(\zeta) = \frac{1}{(r-1)(k-1)}$. By Corollary 36, there is a constant $K_1 > 0$ such that a.a.s. for all $t(B) \leq t < \tau^*$, $\zeta_t \geq \zeta + K_1 n^{-\delta/2}$. Putting all this together yields:

$$\begin{aligned} \bar{p}_t &\leq \frac{1}{(r-1)(k-1)} - \frac{C}{2} K_1 n^{-\delta/2} + n^{-1/2} \log^2 n \\ &\leq \frac{1}{(r-1)(k-1)} - K_2 n^{-\delta/2}, \end{aligned} \quad (34)$$

for any constant $K_2 < K_1 C/2$, as $\delta/2 < \frac{1}{2}$.

It follows then from (30) that there is a $K > 0$ such that for all $t(B) \leq t < \tau^*$,

$$\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) \leq -Kn^{-\delta/2}. \quad (35)$$

By (33), this holds a.a.s. for all $t(B) \leq t < \tau^*$, and by (32) a.a.s. $\tau = \tau^*$; this proves the lemma. \square

4.5 Proof of Theorem 3(b)

In this section, we prove Theorem 3(b). So $|c - c_{r,k}| \leq n^{-\delta}$ for some $0 < \delta < \frac{1}{2}$. Our goal is to show that a.a.s. the stripping number of $H \in \mathcal{H}_r(n, cn)$ is $\Omega(n^{\delta/2})$. Similarly to the supercritical case, we will analyse the process on $H \in AP_r(n, cn)$ instead.

We will define $\delta' \approx \delta$ so that c is not very far from $c' = c_{r,k} + n^{-\delta'}$ and then argue that the stripping number of H is not much smaller than the stripping number of $H' = AP_r(n, c'n)$. We have already proven that the latter stripping number is $\Omega(n^{\delta'/2})$ in Section 4.3.1.

We specify a small $\epsilon > 0$ and define δ' such that:

$$n^{-\delta'} = \begin{cases} n^{-\delta} & \text{if } c = c_{r,k} - n^{-\delta} \text{ where } \delta \leq 1/2 - \epsilon \\ 2n^{-1/2+\epsilon} & \text{if } |c - c_{r,k}| < n^{-1/2+\epsilon}. \end{cases}$$

Remark So if $|c - c_{r,k}| < n^{-1/2+\epsilon}$, then δ' is a function of n , rather than a constant. We can still apply the results of Section 4.3.1 to H' as the proofs work even for non-constant δ , see the remark following the statement of Theorem 3.

It follows then that in both cases,

$$\frac{1}{2}n^{-\delta'} \leq n^{-\delta'} - (c - c_{r,k}) \leq 2n^{-\delta'}. \quad (36)$$

To prove Theorem 3 for these ranges of c , it suffices to show that a.a.s. $s(\mathcal{H}_r(n, cn)) = \Omega(n^{\delta'/2})$ with δ' defined above, since $n^{\delta'/2} = \Omega(n^{\delta/2})$.

Let $H' = AP_r(n, c'n)$ where $c' = c_{r,k} + n^{-\delta'}$, and we generate H by removing $(c' - c)n$ edges (i.e. r -tuples), chosen uniformly at random in H' . This way we couple $H \subseteq H'$ and H is distributed as $AP_r(n, cn)$. Consider the following stripping procedure to find the k -core of H :

1. Run Phase 1 of SLOW-STRIP on H' , thus obtaining $G'_{t_0} \subseteq H'$.
2. For each vertex v removed from H' in Step 1, we also remove v from H . We call the remaining hypergraph G_{t_0} .
3. Run SLOW-STRIP on G_{t_0} .

To be clear: γ and t_0 are defined for the process of running SLOW-STRIP on H' , not for H . So:

$$\gamma = 3K_2 n^{1-\delta'/2},$$

and t_0 is the first step at which exactly $\beta n + \gamma$ hyperedges remain in H' .

Note that since $H \subseteq H'$, the k -core of H is contained in the k -core of H' , and so the k -core of H is contained in G'_{t_0} . G_{t_0} contains every vertex of G'_{t_0} and contains every edge of G'_{t_0} that is an edge of H . So the k -core of H is contained in G_{t_0} . Thus, this is a valid way to obtain the k -core of H .

Note also that this is not equivalent to running SLOW-STRIP on H , since doing so could remove the vertices in a different order. Nevertheless, we still have $G_{t_0} \subseteq H$ and so the stripping number of H is at least the stripping number of G_{t_0} .

The number of hyperedges in H' but not in H is $(c' - c)n$, which is $\Theta(n^{1-\delta'})$ by (36). We use G_i to denote the subgraph of H remaining after $i - t_0$ iterations of SLOW-STRIP on G_{t_0} . We define L_i, D_i, N_i as we did in Sections 4.4.1 and 4.4.2. Similarly we define $L'_{t_0}, D'_{t_0}, N'_{t_0}$ to be the values of the same parameters for G'_{t_0} .

G'_{t_0} is the result of carrying out Phase 1 of SLOW-STRIP on $H' = AP_r(n, c'n)$. So a.a.s. $L'_{t_0}, D'_{t_0}, N'_{t_0}$ satisfy the bounds in (14) and (17). Clearly, $|L_{t_0} - L'_{t_0}|, |D_{t_0} - D'_{t_0}|, |N_{t_0} - N'_{t_0}|$ are bounded by r times the number of edges in $E(G'_{t_0}) \setminus E(G_{t_0})$. Note that $|E(G'_{t_0}) \setminus E(G_{t_0})|$ is at most the number of edges in $E(H') \setminus E(H)$, which is $\Theta(n^{1-\delta'}) = o(\gamma)$. Combining this with the bounds (14) and (17) on $L'_{t_0}, D'_{t_0}, N'_{t_0}$, we have:

$$\begin{aligned} N_{t_0} &= N'_{t_0} + o(\gamma) = \alpha n + O(\gamma) \\ D_{t_0} &= D'_{t_0} + o(\gamma) = r\beta n + O(\gamma) \\ L_{t_0} &= L'_{t_0} + O(\gamma^2/n) = O(\gamma^2/n) \end{aligned}$$

Next we prove that a.a.s. SLOW-STRIP applied to G_{t_0} lasts for at least $\gamma/4$ steps. Applying Proposition 21 to H' tells us that applying SLOW-STRIP to G'_{t_0} would take at least $\gamma/3$ iterations. Each iteration removes one hyperedge, and the removed hyperedge is not in the k -core of G'_{t_0} and hence not in the k -core of G_{t_0} . We have shown that at most $\Theta(n^{1-\delta'}) = o(\gamma)$ of those hyperedges are not in G_{t_0} . Therefore, SLOW-STRIP takes at least $\gamma/3 - o(\gamma) > \gamma/4$ iterations on G_{t_0} (c.f. Lemma 18). It is then convenient to define $\tau' = \min\{t_0 + \gamma/4, \tau\}$ and we have just shown that a.a.s. $\tau \geq \tau'$.

Corollary 9 implies that a.a.s. G'_{t_0} is nice. Since G_{t_0} contains all but $o(\gamma)$ hyperedges of G'_{t_0} , a.a.s. G_{t_0} is also nice. Thus, the analysis of Section 4.4.2 applies to the running of SLOW-STRIP on G_{t_0} . The analysis is similar except that δ is replaced by δ' , and yields the conclusion of Lemma 28; i.e. for every $t_0 \leq t \leq \tau'$,

$$\mathbb{E}(L_{t+1} | \mathcal{F}_t) = L_t \pm O(n^{-\delta'/2}).$$

The same analysis as in the proof of Lemma 19 yields that (16) holds; i.e. a.a.s. for every $t_0 \leq t \leq \tau'$,

$$L_t \leq L_{t_0} + O(\gamma^2/n) = O(n^{1-\delta'}). \quad (37)$$

The rest of the proof follows as in Section 4.3.1. We argued above that SLOW-STRIP takes at least $\gamma/3 - o(\gamma) > \gamma/4$ iterations on G_{t_0} (thus, a.a.s. $\tau' = t_0 + \gamma/4$). It takes at most $k - 1$ iterations to remove a vertex, and so the parallel stripping process, applied to G_{t_0} removes at least $\gamma/(4(k - 1))$ vertices. By (37) each iteration i removes $|L_{t(i)}| = O(n^{1-\delta'})$ vertices. So there must be at least $\gamma/O(n^{1-\delta'}) = \Omega(n^{\delta'/2})$ (recalling that $\gamma = \Theta(n^{1-\delta'/2})$) iterations of the parallel stripping process. I.e. the stripping number of G_{t_0} is at least $\Omega(n^{\delta'/2})$ and hence the stripping number of H is also $\Omega(n^{\delta'/2}) = \Omega(n^{\delta/2})$.

This proves that the lemma holds for $AP_r(n, cn)$. Corollary 12 implies that it also holds for $\mathcal{H}_r(n, cn)$. \square

5 Bounding the maximum depth: proof of Theorem 5

We first note that the stripping number provides a lower bound on the maximum depth over all non- k -core vertices.

Lemma 37. *For any vertex $v \in S_i$, the depth of v is at least i .*

Proof We prove by induction that every vertex $v \in \cup_{j \geq i} S_j$ has depth at least i . This is trivial for $i = 1$. Suppose it is true for i , and consider $v \in \cup_{j \geq i+1} S_j$. Since $v \notin S_i$, v has at least k neighbours which are either in the k -core or in $\cup_{j \geq i} S_j$. At least one of those neighbours must be removed before v can be removed, and by our induction hypothesis, each such neighbour has depth at least i . So any stripping sequence which removes v must first include a sequence of length at least i which removes a neighbour of v . Thus it must have length at least $i + 1$; i.e. the depth of v is at least $i + 1$. \square

Therefore, Theorem 3(a) provides the lower bound of Theorem 5. We will focus on the upper bound.

Recall that S_i is the set of vertices removed during iteration i of the parallel stripping process, and \widehat{H}_i is the subhypergraph remaining after S_1, \dots, S_{i-1} are removed.

Define:

I_{\max} is the number of iterations carried out by the parallel stripping process.

We define the following hypergraph formed by the vertices of S_i :

Definition 38. The vertices of \mathcal{S}_i are S_i . For any hyperedge f in \widehat{H}_i that includes at least one vertex of S_i , $f' = f \cap S_i$ is a hyperedge of \mathcal{S}_i . If $|f'| = a$ then f' is said to be an a -edge.

Note that \mathcal{S}_i may contain hyperedges of size one (in fact, for large i , most of the edges will have size one).

We wish to bound the depth of a non- k -core vertex v . We begin by defining a set $R(v)$ that contains such a stripping sequence.

Definition 39. For each $1 \leq i \leq I_{\max}$ and any $v \in S_i$, we set $R'_i = R'_i(v) := \{v\}$ and for each $j = i$ to 1:

- (a) we set $R_j = R_j(v)$ to be the union of the vertex sets of all components of \mathcal{S}_j that contain vertices of R'_j .
- (b) we set R'_{j-1} to be the set of all vertices $v \in S_{j-1}$ that are adjacent to $\cup_{\ell=i}^j R_\ell$.

We define $R(v) = \cup_{\ell=i}^1 R_\ell$.

Remark. For the purposes of this paper, we could have omitted step (a) and replaced R' by R elsewhere. The set $R(v)$ would still have contained a stripping sequence leading to the removal of v . We define, and bound, this larger set for our application to clusters in random XOR-SAT in another paper[16]. (A preliminary version is in [17]).

Observation 40. $R(v)$ contains a stripping sequence ending with v .

Proof We prove this for each $v \in S_i$ using a simple induction on i . For $i = 1$, v is a stripping sequence of length 1. For $i \geq 2$, note that $R(v)$ contains every neighbour of v lying in levels $1, \dots, i - 1$; call these neighbours u_1, \dots, u_q . The recursive construction ensures that $R(v)$ also contains $R(u_1), \dots, R(u_q)$ and so, by induction, contains stripping sequences ending with u_1, \dots, u_q . After the deletion of u_1, \dots, u_q , the degree of v drops below r (since $v \in S_i$), and so adding v to the concatenation of those stripping sequences produces a stripping sequence ending in v . \square

Theorem 41. Let $r, k \geq 2$, $(r, k) \neq (2, 2)$ be fixed. There is a constant $\kappa = \kappa(r, k)$ such that: for any $0 < \delta < 1/2$, if $c = c_{r,k} + n^{-\delta}$, then a.a.s. for every $v \in \mathcal{H}_r(n, cn)$: $|R(v)| \leq n^{\kappa\delta}$.

We will prove the upper bound in Theorem 41 by showing that a.a.s. $|R(v)| = n^{O(\delta)}$ for every non- k -core vertex v . Here and in the following sections, the implicit constant involved in $O(\cdot)$ is always independent of δ . Theorem 5 follows immediately from Theorem 41 and Observation 40.

We close this section with an overview of the proof for Theorem 41. One natural approach to bound $R(v)$ is to: (a) carry out the stripping process until v is removed in some iteration i , then (b) explore $R(v)$ starting with $R_i(v)$ and working through $R_{i-1}(v), R_{i-2}(v), \dots, R_1(v)$ according to Definition 39. However, after exposing all the edges deleted in iterations $1, \dots, i$ in part (a), we have no randomness left to facilitate the analysis of part (b).

To overcome this difficulty, we only expose a minimal amount of information in part (a). We expose the vertices of each S_i ; some information about the deleted edges, such as the number of vertices an edge has in S_i and in S_{i+1} ; and some degree information. But crucially we do not expose the actual vertices in each deleted edge. When we carry out part (b), we expose the vertices of the edges relevant to $R(v)$. This is enough randomness for us to bound $|R(v)|$.

To carry out part (a), we complete the entire stripping process using a procedure called *EXPOSURE*. The parameters we expose in this phase are listed at the beginning of the next subsection. Lemma 47 bounds these parameters. We then expose the vertices in all deleted edges using a procedure called *EDGE-SELECTION*. We use this procedure to expose the vertices in $R_i(v), R_{i-1}(v), \dots, R_1(v)$ and obtain a recursive bound on $|R_j(v)|$ in terms of $R_{j+1}(v), \dots, R_i(v)$; see Section 5.2 and its proof in Section 5.4.

5.1 Vertex-exposure and edge-selection

As in Section 4, we will analyze the running of the parallel stripping process on the AP-model. We will work with the AP-model for the remainder of Section 5. Again, when we use a graph theoretic term, we mean the obvious analogue for a configuration. Recall:

I_{\max} is the number of iterations carried out by the parallel stripping process.

We parameterize the hyperedges removed during iteration i as follows:

Definition 42. A hyperedge removed during iteration i is called an (a, b) -edge, where a is the number of vertex-copies it contains from S_i and b is the number of vertex-copies it contains from S_{i+1} . Thus $a \geq 1$ and $0 \leq b \leq r - a$ and if $i = I_{\max}$ then we must have $b = 0$.

For every $i, 1 \leq a \leq r, 0 \leq b \leq r - a$ we define:

$$M_i^{a,b} \text{ is the number of } (a, b)\text{-edges in } \widehat{H}_i.$$

Definition 43. For each $v \in S_i$:

- $d^+(v)$ is the degree of v in \widehat{H}_i ; i.e. the number of vertex-copies of v that are removed during iteration i .
- (if $i < I_{\max}$), $d^-(v)$ is the total degree of v amongst the hyperedges in \widehat{H}_{i-1} that contain at least one vertex of S_{i-1} ; i.e. the number of vertex-copies of v in hyperedges that are removed during iteration $i - 1$.

As described above, we expose the hyperedges removed during the parallel stripping process in two phases. First, we expose the vertices that are removed in each iteration, along with some degree and edge-count information:

EXPOSURE:

1. Expose I_{\max} .
2. Expose the vertices in $S_1, \dots, S_{I_{\max}}$.
3. For each $0 \leq i \leq I_{\max}$ and each vertex $v \in S_i, u \in S_{i+1}$, expose $d^+(v), d^-(u)$.
4. For each $0 \leq i \leq I_{\max}, 1 \leq a \leq r, 0 \leq b \leq r - a$, expose $M_i^{a,b}$.

Of course, this also exposes the vertices and the number of edges in the k -core $\mathcal{C}_k = \mathcal{C}_k(H)$.

To clarify what is exposed in terms of the AP-model: We have exposed the bins (vertices) in $S_1, \dots, S_{I_{\max}}$. For each bin $v \in S_i$, we have exposed the number of copies of v that are removed in iteration $i - 1$ and in iteration i respectively. In fact, for convenience we expose those actual copies. Other vertex-copies may still be allocated to bin v , but any such vertex-copy must be deleted during iterations $1, \dots, i - 2$. For each relevant i, a, b , we have exposed the number of r -tuples removed in iteration i that contain a vertex-copies from S_i and b vertex-copies from S_{i+1} ; but for each such r -tuple, we do not expose the actual bins (vertices) those vertex-copies lie in, nor do we expose the remaining $r - a - b$ vertex-copies in the r -tuple.

Next we expose the actual hyperedges (i.e. r -tuples of vertex-copies which are allocated to bins) that are removed during each iteration. We define:

\mathcal{E}_i is the set of hyperedges removed during iteration i .

\mathcal{E}_i must satisfy the following conditions for each $1 \leq i \leq I_{\max}$:

(P1) Each $v \in S_i$ has exactly $d^+(v)$ vertex-copies in \mathcal{E}_i .

- (P2) Each $u \in S_{i+1}$ has exactly $d^-(u)$ vertex-copies in \mathcal{E}_i .
(P3) \mathcal{E}_i contains exactly $M_i^{a,b}$ (a, b) -edges for each a, b .
(P4) In each of those $M_i^{a,b}$ (a, b) -edges, the $r - a - b$ vertex-copies not in $S_i \cup S_{i+1}$ must be allocated to a bin in $\mathcal{C}_k \bigcup_{j=i+2}^{I_{\max}} S_j$.

Lemma 44. *Conditional on the outcome of EXPOSURE, any set of hyperedges (r -tuples) satisfying properties (P1–P3) is equally likely to be \mathcal{E}_i .*

Proof Fix all parameters that are exposed in EXPOSURE (i.e. $I_{\max}, S_1, \dots, S_{I_{\max}}, d^+(v)$, etc). Consider two sets of hyperedges (i.e. r -tuples of vertex-copies, and an allocation of those vertex-copies into bins) E_1, E_2 satisfying properties (P1–P4). We need to show that E_1, E_2 are equally likely to be \mathcal{E}_i .

Recall that a configuration is a partition of the rm vertex-copies into r -tuples, and an allocation of the rm vertex-copies into the n bins. Let H_1 be a configuration that is consistent with the outcome of EXPOSURE (i.e. applying the parallel stripping process to H_1 will result in the various parameters being equal to what was exposed in EXPOSURE), and such that applying the parallel stripping process to H_1 will result in $\mathcal{E}_i = E_1$. Let H_2 be the hypergraph obtained from H_1 by replacing E_1 with E_2 .

We claim that: H_2 is also consistent with the outcome of EXPOSURE, and that applying the parallel stripping process to H_2 will result in $\mathcal{E}_i = E_2$. To verify this claim, we only need to show that, when we apply the stripping process to H_2 , we take I_{\max} iterations, and for each $1 \leq j \leq I_{\max}$, the set of vertices removed in iteration j is S_j (the fact that E_2 satisfies (P1–P4) confirms the remainder of the claim.) This follows easily from the fact that E_1, E_2 do not include any hyperedges that contain vertices from S_1, \dots, S_{i-1} and that E_1, E_2 each consist of all the hyperedges that contain vertices from S_i in what remains after removing S_1, \dots, S_{i-1} from H_1, H_2 , resp.

So we have a bijection between the configurations which yield $\mathcal{E}_i = E_1$ and $\mathcal{E}_i = E_2$. Furthermore, every configuration is equally likely to be chosen as $AP_r(n, cn)$. Thus E_1 and E_2 are equally likely to be \mathcal{E}_i , which establishes the lemma. \square

So our goal is to choose a uniform set of r -tuples satisfying (P1–P4), for each $1 \leq i \leq I_{\max}$.

EDGE-SELECTION: For each $1 \leq i \leq I_{\max}$, we expose \mathcal{E}_i , the hyperedges deleted in iteration i : We have already exposed $M_i^{a,b}$, the number of (a, b) -edges. Each such edge will be assigned a vertex-copies from S_i , b vertex-copies from S_{i+1} , and $r - a - b$ vertex-copies from $\mathcal{C}_k \bigcup_{j=i+2}^{I_{\max}} S_j$. We assign those copies as follows:

1. Each vertex $v \in S_i$ has $d^+(v)$ copies. Noting that $\sum_{v \in S_i} d^+(v) = \sum_{a,b} aM_i^{a,b}$, we take a uniformly random partition of all vertex-copies in S_i so that each (a, b) -edge receives a part of size a .
2. Each vertex $u \in S_{i+1}$ has $d^-(u)$ copies. Noting that $\sum_{u \in S_{i+1}} d^-(u) = \sum_{a,b} bM_i^{a,b}$, we take a uniformly random partition of all vertex-copies in S_{i+1} so that each (a, b) -edge receives a part of size b .
3. For each (a, b) -edge, we choose the remaining $r - a - b$ vertex-copies uniformly from those that have not yet been allocated to bins. Then we allocate each of these vertex-copies to a bin selected uniformly from $\mathcal{C}_k \bigcup_{j=i+2}^{I_{\max}} S_j$.

Any two sets of hyperedges (i.e. r -tuples of vertex-copies which are allocated to bins) satisfying (P1–P4) are equally likely to be chosen by EDGE-SELECTION. Therefore, if we carry out EDGE-SELECTION, then by Lemma 44, we have chosen the hyperedges from the correct distribution.

5.2 A recursive bound

Consider some vertex $v \in S_i$ and recall Definition 39 where we define $R(v) = R_i(v) \cup R_{i-1}(v) \cup \dots \cup R_1(v)$. That definition naturally lends itself to a recursive bound of $|R_j(v)|$ in terms of $R_{j-1}(v), \dots, R_i(v)$. We present that definition in this subsection, prove it in Section 5.4, and then use it to bound $|R(v)|$ in Section 5.5. It will be convenient to restrict our analysis to deletion rounds which are late enough that certain bounds on various parameters hold. So we let $B = B(r, k)$ be a sufficiently large constant to be named later (it will come from Lemma 47 below) and set

$$R^{(B)}(v) = \cup_{j=i}^B R_j = R(v) \setminus (\cup_{1 \leq i < B} S_i) \quad (38)$$

Lemma 49 bounds $|R(v)| = O(|R^{(B)}(v)| + \log n)$ and so bounding $R^{(B)}(v)$ will suffice to prove Theorem 41.

The expected size of R_{j-1} depends not just on the size of R_j , but also on the d^- values of the vertices in R_j . So we will recursively bound the sum of those values, rather than bound the (nearly equal) $|R_j|$:

Definition 45. For $X \subseteq S_i$, we define

$$D^-(X) = \sum_{u \in X} d^-(u).$$

Note that if $i > 1$ then $d^-(u) \geq 1$ for all $u \in S_i$ as otherwise, u would have been deleted in iteration $i - 1$, and so $D^-(R_j(u)) \geq |R_j(u)|$ for all $j \leq i$. Therefore, bounding $D^-(R_j(u))$ will suffice to bound $|R_j(u)|$.

Given our non- k -core vertex v , we define:

I_v is the integer i such that $v \in S_i$; i.e. the iteration during which v is removed.

As described above, we are restricting our analysis to vertices deleted after iteration B , so we will assume that $I_v \geq B$. We will bound $D^-(R^{(B)}(v))$. The recursion starts with a base case bound on $D^-(R_{I_v}(v))$, and then bounds $D^-(R_{I_v-1}(v)), D^-(R_{I_v-2}(v)), \dots, D^-(R_B(v))$. Thus, we express $D^-(R_j(v))$ in terms of $D^-(R_\ell(v))$ for $\ell > j$, rather than the usual $\ell < j$.

Lemma 46. *There are constants $B = B(r, k), Z = Z(r, k) > 0$ such that with probability at least $1 - n^{-3}$: for all $B \leq j \leq I_v$ with $|S_j| \geq n^\delta \log^2 n$, we have*

$$D^-(R_j(v)) \leq D^-(R_{j+1}(v)) + Z \frac{|S_j|}{n} \sum_{\ell=I_v}^{j+1} D^-(R_\ell(v)) + \log^{14} n. \quad (39)$$

We will analyze the recursive equation in Lemma 46 to obtain a bound on $D^-(R_j)$ for each j , in Section 5.5.

We prove (39) by analyzing the exposure process from Section 5.1. We begin by analyzing EXPOSURE: we bound I_{\max} and prove that the parameters $d^+(v), d^-(v), M_i^{a,b}$ satisfy certain properties (Sections 5.3 and 6). Next, we analyze EDGE-SELECTION one iteration at a time, with i decreasing from I_{\max} . When analyzing R_j to prove (39), iterations $i = I_{\max}, \dots, j+1$ have already been completed. So for $v \in S_{I_v}$, we have already exposed $R_{I_v}, R_{I_v-1}, \dots, R_{j+1}$.

5.3 Properties of S_i

We first run EXPOSURE. We prove that a.a.s. the sets S_i satisfy certain properties.

Lemma 47. *There exist constants B, Y_1, Y_2, Z_1 , dependent only on r, k , such that a.a.s. for every $B \leq i < I_{\max}$ with $|S_i| \geq n^\delta \log^2 n$:*

- (a) if $|S_i| < n^{1-\delta}$ then $(1 - Y_1 n^{-\delta/2})|S_i| \leq |S_{i+1}| \leq (1 - Y_2 n^{-\delta/2})|S_i|$;
- (b) if $|S_i| \geq n^{1-\delta}$ then $(1 - Y_1 \sqrt{\frac{|S_i|}{n}})|S_i| \leq |S_{i+1}| \leq (1 - Y_2 \sqrt{\frac{|S_i|}{n}})|S_i|$;
- (c) $\sum_{j \geq i} |S_j| \leq Z_1 |S_i| n^{\delta/2}$.
- (d) $|S_i| \leq \sum_{u \in S_i} d^-(u) < |S_i| + Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (e) $|S_{i+1}| \leq \sum_{a,b} ab M_i^{a,b} \leq |S_{i+1}| + Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (f) $\sum_{a \geq 2, b \leq r-a} ab M_i^{a,b} \leq Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (g) $\sum_{u \in S_i} d^+(u) d^-(u) \leq \sum_{u \in S_i} d^+(u) + Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (h) $\sum_{u \in S_i} (d^-(u))^2 \leq Z_1 |S_i|$;
- (i) $d^-(u) < \log n$ for all $u \in \cup_{i=2}^{I_{\max}} S_i$.

The proof is deferred until Section 6.

5.4 Proving the recursive bound

Here, we will prove the recursive bound from Section 5.2. Recall that we specified a vertex v , and $I_v \leq I_{\max}$ is the iteration during which v was deleted.

Our goal is to prove Lemma 46, and so we can assume $j \geq B$ and $|S_j| > n^\delta \log^2 n$. For each j , we will prove the probability that (39) fails is less than n^{-4} , and so a union-bound implies that the probability it holds for every j is at least $1 - I_v n^{-4} > 1 - n^{-3}$.

The setting is that we have carried out EXPOSURE, and we assume that all properties in Lemma 47 hold. We have also carried out EDGE-SELECTION for iterations $I_{\max}, \dots, j+1$, and in particular, we have exposed R_{I_v}, \dots, R_{j+1} .

The random experiment in this section is iteration j of EDGE-SELECTION. At this point, we know that there are $M_j^{a,b}$ (a,b) -edges for each $a \geq 1, 0 \leq b \leq r - a - b$, but we don't know which vertices are in these edges. Each (a,b) -edge consists of r blanks which will be filled with vertex-copies: the a level- j blanks will receive vertex-copies from S_j ; the b level- $(j+1)$ blanks will receive vertex-copies from S_{j+1} ; the remaining $r - a - b$ blanks will receive vertex-copies which will then be allocated to bins in $\mathcal{C}_k \bigcup_{i=j+2}^{I_{\max}} S_i$.

We fill in these blanks with three independent steps, in reverse order of how they are listed above.

Step 1: For each (a,b) -edge, we choose the $r - a - b$ vertex-copies and then allocate them to $\mathcal{C}_k \bigcup_{i=j+2}^{I_{\max}} S_i$.

In order to bound $D^-(R_j)$, we will first bound

$$\Lambda : \quad \begin{array}{l} \text{the total number of level-}j \text{ blanks, amongst all hyperedges} \\ \text{deleted in iteration } j \text{ that contain a vertex-copy from } \bigcup_{i=I_v}^{j+1} R_i. \end{array} \quad (40)$$

Note that Λ is an upper bound on $|R'_j|$.

We let Λ_1 denote the number of these level- j blanks whose hyperedge contains a vertex from $\bigcup_{i=I_v}^{j+2} R_i$. Note that Λ_1 is determined by Step 1: After allocating these vertex-copies to $\mathcal{C}_k \bigcup_{i=j+2}^{I_{\max}} S_i$, we know exactly which (a,b) -edges contain vertex-copies that are allocated to $\bigcup_{i=I_v}^{j+2} R_i$.

Each time we allocate one of these copies, the probability that we allocate it to a bin from $\bigcup_{i=I_v}^{j+2} R_i$ is at most

$$\frac{|\bigcup_{i=I_v}^{j+2} R_i|}{|\mathcal{C}_k \bigcup_{i=j+2}^{I_{\max}} S_i|} \leq \frac{1}{\alpha(c)n + o(n)} \sum_{i=I_v}^{j+2} |R_i|, \quad \text{where } \alpha(c) \text{ is from Lemma 7.}$$

There are at most $(k-1)|S_j|$ such hyperedges, and for each one we choose at most $r-1$ vertices. So the expected number of hyperedges which have at least one vertex-copy allocated to $\bigcup_{i=I_v}^{j+2} R_i$ is at most $\frac{(r-1)(k-1)}{\alpha(c)n + o(n)} |S_j| \sum_{i=I_v}^{j+2} |R_i|$. Standard concentration bounds on binomial variables imply that the probability this number exceeds $\frac{rk}{\alpha(c)n} |S_j| \sum_{i=I_v}^{j+2} |R_i| + \log^2 n$ is less than n^{-5} . Each such hyperedge contains fewer than r vertex-copies from S_j . So with probability at least $1 - n^{-5}$, we have

$$\Lambda_1 \leq \frac{r^2 k}{\alpha(c)} \frac{|S_j|}{n} \sum_{i=I_v}^{j+2} |R_i| + r \log^2 n. \quad (41)$$

Step 2: For each (a,b) -hyperedge, we choose the b vertex-copies from S_{j+1} .

We let $\Lambda_2 = \Lambda - \Lambda_1$. So Λ_2 is the number of level- j blanks whose hyperedge contains a vertex-copy from R_{j+1} and was not counted in Λ_1 . Note that Λ_2 is determined by Steps 1 and 2.

It will be easier to focus on $\Lambda_2^* \geq \Lambda_2$ which is defined to be the number of pairs of vertex-copies $v \in S_j, u \in R_{j+1}$ that both lie in an (a, b) -edge for some a, b . We only require an upper bound on Λ_2 , so it will suffice to bound Λ_2^* .

Each (a, b) -edge contributes a to Λ_2^* for each of its b copies from S_{j+1} that lie in R_{j+1} . The total number of copies in R_{j+1} is $D^-(R_{j+1})$ so

$$\begin{aligned}
\mathbb{E}(\Lambda_2^*) &= \sum_{a,b} abM_j^{a,b} \frac{D^-(R_{j+1})}{\sum_{u \in S_{j+1}} d^-(u)} \\
&\leq \frac{D^-(R_{j+1})}{|S_{j+1}|} \sum_{a,b} abM_j^{a,b} \quad \text{since } d^-(u) \geq 1 \text{ for all } u \in S_{j+1} \\
&\leq D^-(R_{j+1}) \left(1 + 2Z_1 \frac{|S_j|}{n} + \frac{\log^2 n}{|S_{j+1}|} \right) \quad \text{by Lemma 47(e,a,b)} \\
&\leq D^-(R_{j+1}) + 2Z_1 \frac{|S_j|}{n} D^-(R_{j+1}) + Z_1 \log^2 n, \tag{42}
\end{aligned}$$

since, by Lemma 47(g), $D^-(R_{j+1}) \leq D^-(S_{j+1}) \leq \sum_{v \in S_{j+1}} (d^-(v))^2 \leq Z_1 |S_{j+1}|$.

To bound Λ_2^* , we will prove that it is concentrated. Note that $\Lambda_2^* \geq D^-(R_{j+1})$ since every vertex-copy of R_{j+1} lies in an (a, b) -edge for some $a \geq 1$, and hence contributes at least one of the pairs counted by Λ_2^* . This allows us to focus instead on proving the concentration of

$$X = \Lambda_2^* - D^-(R_{j+1}).$$

X is typically much smaller than Λ_2^* . This will be an advantage when we use $\mathbb{E}(X)$ rather than $\mathbb{E}(\Lambda_2^*)$ in our calculations in (43) below. Note that $D^-(R_{j+1})$ is fixed.

To prove concentration of X , we apply McDiarmid's variation on Talagrand's Inequality [37]. We use the version stated in [33]:

McDiarmid's Inequality [29] *Let X be a non-negative random variable determined by independent trials T_1, \dots, T_m and independent permutations Π_1, \dots, Π_m . We call the outcome of one trial T_i , or the mapping of a single element in a permutation Π_i , a choice. Suppose that for every set of possible outcomes of the trials and permutations, we have:*

- (i) *changing the outcome of any one trial can affect X by at most ϱ ;*
- (ii) *interchanging two elements in any one permutation can affect X by at most ϱ ; and*
- (iii) *for each $s > 0$, if $X \geq s$ then there is a set of at most qs choices whose outcomes certify that $X \geq s$.*

Then for any $t \geq 0$, we have

$$\Pr(|X - \mathbb{E}(X)| > t + 25\varrho\sqrt{q\mathbb{E}(X)} + 128\varrho^2q) \leq 4 \exp\left(-\frac{t^2}{32\varrho^2q(\mathbb{E}(X) + t)}\right).$$

Our random choice in Step 2 is an assignment of the vertex-copies of S_{j+1} to the (a, b) -edges. This can be done by taking a uniform permutation of those vertex-copies: There is an implicit listing of all (a, b) -edges over all a, b : For each edge, we know the values of

a, b corresponding to that edge, but we don't yet know exactly which a copies from S_j and b copies from S_{j+1} are assigned to each edge; the permutation of Step 2 determines the assignment of the copies of S_{j+1} . Thus the choice corresponding to a particular vertex-copy specifies the hyperedge to which it is assigned. We will apply McDiarmid's Inequality using this permutation.

If we exchange the position of two vertex-copies in the permutation, we are swapping the hyperedges to which they were assigned. This can change Λ_2^* , and hence X , by at most $r - 1$ (the extreme case is when the swap involves an $(r - 1, 1)$ -edge). So we take $\varrho = r - 1$.

To certify $X \geq s$, we can always present $\ell^* \leq s$ vertex-copies u_1, \dots, u_{ℓ^*} such that each u_ℓ is assigned to an (a_ℓ, b_ℓ) -edge with $a_\ell \geq 2$ and $\sum_{\ell=1}^{\ell^*} (a_\ell - 1)b_\ell \geq s$. So this can be certified by the outcomes of $\ell^* \leq s$ choices, and we can take $q = 1$.

Setting $t = \max\{\mathbb{E}(X), \log^2 n\}$, it is easy to see that $25\varrho\sqrt{\mathbb{E}(X)} + 128\varrho^2 < t$ and $\mathbb{E}(X) + t \leq 2t$. So McDiarmid's Inequality yields:

$$\Pr(X > \mathbb{E}(X) + 2t) \leq 4 \exp\left(-\frac{t^2}{32\varrho^2(\mathbb{E}(X) + t)}\right) < 4 \exp\left(-\frac{t}{64\varrho^2}\right) = o(n^{-5}). \quad (43)$$

Therefore, applying (42), with probability at least $1 - n^{-5}$, we have

$$\Lambda_2 \leq \Lambda_2^* \leq D^-(R_{j+1}) + 6Z_1 \frac{|S_j|}{n} D^-(R_{j+1}) + 3Z_1 \log^2 n. \quad (44)$$

Step 3: For each (a, b) -hyperedge, we choose the a vertex-copies from S_j .

In most of the analysis for this step, the probabilities are in terms only of the random choices in Step 3; it is assumed that the choices from Steps 1 and 2 are fixed. We will use the notation \Pr_3 and \mathbb{E}_3 to indicate this.

Recall from Definition 38 that \mathcal{S}_j is the hypergraph formed by treating each (a, b) -edge as a hyperedge on a vertices of S_j ; we call this an a -edge. At this point, each a -edge consists of a blanks, and we have a pool containing $d_j(u)$ vertex-copies of each $u \in S_j$. The random choice in Step 3 is a uniformly random bijection from these vertex-copies to these blanks.

We define

$$\Phi : \text{the hyperedges of } \mathcal{S}_j \text{ that belong to } (a, b)\text{-edges containing vertex-copies from } \cup_{i=I_v}^{j+1} R_i; \quad (45)$$

thus R'_j is the set of vertices of Φ . Note that Φ was determined by Steps 1,2, and that $\Lambda = \Lambda_1 + \Lambda_2$ is the total number of blanks contained in the hyperedges of Φ .

For each hyperedge $f \in \mathcal{S}_j$, C_f denotes the component of \mathcal{S}_j that contains f . Thus R_j is the union over all $f \in \Phi$ of the vertices in C_f , and $D^-(R_j) \leq \sum_{f \in \Phi} |D^-(C_f)|$ (there is an inequality here as some pairs $f, f' \in \Phi$ may lie in the same component). In order to bound $\mathbb{E}(D^-(R_j))$, we will bound $\mathbb{E}_3(D^-(C_f))$ for each f .

Consider any particular hyperedge $f \in \mathcal{S}_j$. We will expose C_f using a branching process as in [32]:

Initially, every blank of f is labelled *open*.

While there are open blanks:

Take any open blank, and choose a uniform vertex-copy for that blank;
the blank is labelled *closed*.

Assign every other copy of the same vertex to a blank chosen uniformly
from those not yet assigned vertex-copies.

For each blank g that is chosen:

if it is open (i.e. already known to be in C_f), we label it *closed*;
else, every other blank in the same hyperedge as g is labelled *open*.

Intuition: By Lemma 47(e,f), the vast majority of blanks lie in 1-edges. So when we assign copies to blanks, we usually do not create any new open blanks. Thus, the process tends to die out very quickly and C_f typically has few (in fact, no) vertices outside of f .

To analyze this branching process, we consider the following experiment. Select a sequence of vertices $u_1, \dots, u_{\log^2 n}$, each chosen from S_j without replacement, where the probability that $w \in S_j \setminus \{u_1, \dots, u_{\ell-1}\}$ is chosen to be u_ℓ is proportional to $d_j(w)$.

To run the branching process: each time that we need to select a new vertex-copy x for an open blank, we simply take the next vertex on the list u_1, \dots and choose one of its vertex-copies and let x be that vertex-copy. Note that these vertices are chosen with the correct distribution.

We will prove that there is a constant $W_1 > 0$ such that for any $y \geq 0$ and every $1 \leq t < \log^2 n$:

$$\mathbb{E}_3 \left(\sum_{\ell=1}^t d^-(u_\ell) \right) < t \left(1 + 2Z_1 \frac{|S_j|}{n} + 2 \frac{\log^2 n}{|S_j|} \right); \quad (46)$$

$$\text{for each } a\text{-edge } f: \quad \Pr_3 \left(|C_f| \geq t \mid \sum_{i=1}^t d^-(u_i) = y \right) < \left(\frac{W_1 |S_j|}{n} \right)^{\lceil \frac{t-a}{(r-1)(k-1)} \rceil}. \quad (47)$$

Proof of (46): From Lemma 47(g) and the fact that $\sum_{u \in S_j} d_j(u) \geq |S_j|$, initially the expected value of $d^-(u_\ell)$ is:

$$\frac{\sum_{u \in S_j} d_j(u) d^-(u)}{\sum_{u \in S_j} d_j(u)} \leq 1 + Z_1 \frac{|S_j|}{n} + \frac{\log^2 n}{|S_j|}.$$

After removing at most $\log^2 n$ of the vertices of S_j , the expected value of $d^-(u_\ell)$ rises to at most

$$\left(1 + Z_1 \frac{|S_j|}{n} + \frac{\log^2 n}{|S_j|} \right) \frac{|S_j|}{|S_j| - \log^2 n} < 1 + 2Z_1 \frac{|S_j|}{n} + 2 \frac{\log^2 n}{|S_j|},$$

since $|S_j| \geq n^\delta \log^2 n$.

Proof of (47): Conditioning on the event $\sum_{i=1}^t d^-(u_i) = y$, exposes information about $d^-(u_1), \dots, d^-(u_t)$. This affects our conditional probability only through its effect on the conditional distribution of $d_j(u_1), \dots, d_j(u_t)$, as EXPOSURE determined $(d_j(u), d^-(u))$ for each $u \in S_j$. So we can deal with this conditioning by proving that for *any* choice of $\delta_1, \dots, \delta_t$:

$$\Pr_3(|C_u| \geq t \mid d_j(u_1) = \delta_1, \dots, d_j(u_t) = \delta_{t-1}) < \left(\frac{W_1 |S_j|}{n} \right)^{\lceil \frac{t-a}{(r-1)(k-1)} \rceil}. \quad (48)$$

Suppose that we selected vertex u_ℓ . We place one of its vertex-copies into the open blank that we are filling. In order to create at least one new open blank, we must assign at least one of the other $d_j(u_\ell) - 1$ copies to a blank in an a -edge for some $a \geq 2$; we call such an edge a *plural edge*.

The total number of blanks in S_j is at least $\frac{1}{r-1}$ times the number of vertex-copies in S_{j+1} (the extreme case is when every hyperedge is a $(1, r-1)$ -edge) and thus is at least $\frac{|S_{j+1}|}{r-1} \geq \frac{|S_j|}{2(r-1)}$ by Lemma 47(a,b). If we have assigned the copies of fewer than $\log^2 n$ vertices, then the number of unfilled blanks is at least $\frac{|S_j|}{2(r-1)} - (k-1)\log^2 n \geq \frac{|S_j|}{2r}$. Lemma 47(f) says that the total number of blanks in all plural edges is at most $Z_1|S_j|^2/n + \log^2 n$. So the probability that at least one of those $d_j(u_\ell) - 1 \leq k-2$ copies of u_ℓ is assigned to a plural edge at most

$$(k-2) \left(\frac{Z_1|S_j|^2}{n} + \log^2 n \right) \bigg/ \frac{|S_j|}{2r} < 2krZ_1 \frac{|S_j|}{n}.$$

If at least one copy of u_ℓ is assigned to a plural edge, then we create at most $(k-2)(r-1)$ new open blanks. For an a -edge f , if $|C_f| \geq t$, then we must have created at least $t-a$ open blanks (in addition to the a initial open blanks in f) during the exposure of the first t vertices u_1, \dots, u_t . So at least $\lceil \frac{t-a}{(r-1)(k-1)} \rceil$ of the vertices u_1, \dots, u_t must have a copy assigned to a plural edge. The probability of this occurring is at most:

$$\left(\binom{t}{\lceil \frac{t-a}{(r-1)(k-1)} \rceil} \right) \left(2krZ_1 \frac{|S_j|}{n} \right)^{\lceil \frac{t-a}{(r-1)(k-1)} \rceil} < \left(\frac{et \cdot 2krZ_1|S_j|}{\lceil \frac{t-a}{(r-1)(k-1)} \rceil n} \right)^{\lceil \frac{t-a}{(r-1)(k-1)} \rceil} < \left(W_1 \frac{|S_j|}{n} \right)^{\lceil \frac{t-a}{(r-1)(k-1)} \rceil},$$

for an appropriate constant W_1 , thus establishing (47).

Having proven (46) and (47), these imply that there is a constant W such that

$$\text{for each } a\text{-edge } f: \quad \mathbb{E}_3(D^-(C_f)) \leq a \left(1 + W \frac{|S_j|}{n} + 5 \frac{\log^2 n}{|S_j|} \right). \quad (49)$$

Proof First note that (47) implies:

$$\Pr(|C_f| > \log^2 n) \leq \left(\frac{W_1|S_j|}{n} \right)^{\log^2 n / ((r-1)(k-1))} = o(n^{-10}). \quad (50)$$

Next, note that $D^-(C_f) \leq rcn$, the number of vertex-copies in the entire configuration. So

$$\begin{aligned}
\mathbb{E}_3(D^-(C_f)) &= \sum_{y=1}^{rcn} y \Pr_3(D^-(C_f) = y) \\
&\leq \sum_{y=1}^{rcn} y \left(\Pr_3(|C_f| \geq \log^2 n) + \sum_{t=a}^{\log^2 n} \Pr_3 \left((|C_f| \geq t) \wedge \left(\sum_{i=1}^t d^-(u_i) = y \right) \right) \right) \\
&\leq O(n^2) \Pr_3(|C_f| \geq \log^2 n) \\
&\quad + \sum_{y=1}^{rcn} y \sum_{t=a}^{\log^2 n} \Pr_3 \left(\sum_{\ell=1}^t d^-(u_\ell) = y \right) \left(\frac{W_1 |S_j|}{n} \right)^{\lceil \frac{t-a}{(k-1)(r-1)} \rceil} \text{ by (47)} \\
&\leq o(n^{-1}) + \sum_{t=a}^{\log^2 n} \left(\frac{W_1 |S_j|}{n} \right)^{\lceil \frac{t-a}{(k-1)(r-1)} \rceil} \mathbb{E}_3 \left(\sum_{\ell=1}^t d^-(u_\ell) \right) \text{ by (50)} \\
&< o(n^{-1}) + \sum_{t=a}^{\log^2 n} \left(\frac{W_1 |S_j|}{n} \right)^{\lceil \frac{t-a}{(k-1)(r-1)} \rceil} t \left(1 + 2Z_1 \frac{|S_j|}{n} + 2 \frac{\log^2 n}{|S_i|} \right) \text{ by (46)} \\
&< o(n^{-1}) + \left(a + 2k^2 r^2 \frac{W_1 |S_j|}{n} \right) \left(1 + 2Z_1 \frac{|S_j|}{n} + 2 \frac{\log^2 n}{|S_i|} \right) \\
&< a + Wa \frac{|S_j|}{n} + 5a \frac{\log^2 n}{|S_j|}, \tag{51}
\end{aligned}$$

for an appropriate constant W ; this is (49).

Our goal is to bound $D^-(R_j) \leq \sum_{f \in \Phi} D^-(C_f)$ in order to obtain (39). As in Step 2, we will instead focus on a related quantity: We use $|f|^*$ to denote the number of *vertices* in f ; i.e. if f contains multiple copies of a vertex u then u is counted only once in $|f|^*$. We define

$$X = \sum_{f \in \Phi} \min\{D^-(C_f), \log^3 n\} - |f|^*.$$

Note that $D^-(C_f) \geq |f|^*$ as every vertex $u \in f$ satisfies $d^-(u) \geq 1$. Also, $\log^3 n > r \geq |f|^*$ and so X is non-negative.

We will show that, with very high probability, $D^-(C_f) < \log^3 n$ for every f , which allows us to work with X . The advantages of doing so are twofold: (i) subtracting $|f|^*$ from each term in the summand has the same advantage as switching to X in the analysis of Step 2; (ii) by introducing an upper bound of $\log^3 n$ on the contribution of each term, we bound the effect of each random choice on X . Since $\sum_{f \in \Phi} |f| \leq \Lambda$, (49) implies

$$\mathbb{E}_3(X) \leq \sum_{f \in \Phi} |f| \left(W \frac{|S_j|}{n} + \frac{5 \log^2 n}{|S_j|} \right) \leq \Lambda \left(W \frac{|S_j|}{n} + \frac{\log^2 n}{|S_j|} \right). \tag{52}$$

To prove concentration of X , we again use McDiarmid's Inequality.

Our random choice in Step 3 is an assignment of the vertex-copies of S_j to the blanks in the a -edges. As in the analysis of Step 2, we do this by taking a uniform permutation of those vertex-copies.

We first bound the amount by which exchanging two vertex-copies can affect X . Suppose that we replace a copy of u in an edge f with a copy of w in an edge f' . We will show that this cannot increase the contribution of the edges in C_f to X by more than $2(k-1)\log^6 n$.

Let C be the component containing f after the copy of u is replaced by a copy ω of w . Consider removing ω from f ; so the size of f has been reduced by one. Let C' be the hypergraph obtained from C by the removal of ω . Let C_1 be the component of C' containing f (if f had size 1 and hence is now empty, then $C_1 = \emptyset$) and let C_2 be the union of the other components of C' . Note that *before the copy of u was replaced with ω* : C_1 was a subgraph of C_f and C_2 was a subgraph of the component containing w .

Case 1: $|C_1|, |C_2| \geq \log^3 n$. Then for every hyperedge $f' \in C_1$, $\min(|C_{f'}|, \log^3 n) = \log^3 n$ both before and after the switch. The same is true of every hyperedge in C_2 . So the contribution of those edges to X was unchanged by the switch.

Case 2: $|C_1| \geq \log^3 n, |C_2| < \log^3 n$. The contribution of the edges of C_1 was unchanged by the switch, but the contribution of each edge in C_2 may have increased by up to $\log^3 n$. There are at most $(k-1)|C_2|$ vertex-copies amongst those edges and so there are at most $(k-1)|C_2| < (k-1)\log^3 n$ such edges. So the total contribution from those edges increases by less than $(k-1)\log^6 n$.

In each of the remaining two cases, similar reasoning shows that the total contribution of the edges in C_f increases by at most $2(k-1)\log^6 n$. The same argument shows that the total contribution of the edges in $C_{f'}$, the component containing f' (the other hyperedge involved in the switch) is at most $2(k-1)\log^6 n$. Thus no switch can increase X by more than $4(k-1)\log^6 n$. By considering the inversion of a switch, no switch can decrease X by more than $4(k-1)\log^6 n$. So we can take $\varrho = 4k\log^6 n$.

We next show that the event $X \geq s$ can be certified by revealing the outcomes of at most $2rs$ choices. If $X \geq s$ then there must be a subset of the edges of Φ , say f_1, \dots, f_{ℓ^*} , with $D^-(C_{f_1}) \geq |f_1|^* + s_1, \dots, D^-(C_{f_{\ell^*}}) \geq |f_{\ell^*}|^* + s_{\ell^*}$, and $s_1 + \dots + s_{\ell^*} = s$ with each $s_\ell \geq 1$. For each $1 \leq \ell \leq \ell^*$, we will certify that $D^-(C_{f_\ell}) \geq |f_\ell|^* + s_\ell$ by revealing at most $2rs_\ell$ choices as follows. We discuss two cases.

Case 1: $|C_{f_\ell}| \leq |f_\ell|^* + s_\ell$. In this case, we reveal a spanning tree of C_{f_ℓ} . That is, we reveal the choice of vertex-copy assigned to each hyperedge of a spanning tree of C_{f_ℓ} . This spanning tree only needs to consist of f_ℓ , and at most s_ℓ additional edges, and so the total number of vertex-copies whose assignments are revealed is at most $r(1+s_\ell) \leq 2rs_\ell$, as $s_\ell \geq 1$.

Case 2: $|C_{f_\ell}| > |f_\ell|^* + s_\ell$. In this case, we will reveal a connected subgraph of C_{f_ℓ} containing f_ℓ and at least s_ℓ additional vertices. As described in Case 1, we can do so using at most $s_\ell + 1$ edges and hence by revealing the assignments of at most $r(1+s_\ell) \leq 2rs_\ell$ vertex-copies. Since each of these vertices has $d^- \geq 1$, this will certify that $D^-(C_{f_\ell}) \geq |f_\ell|^* + s_\ell$.

This shows that we may certify $X \geq s$ by exposing at most $2rs$ choices so we can take $q = 2r$. Setting $t = \max\{\mathbb{E}(X), 6 \times 2048k^2r \log^{13} n\}$, it is easy to see that $25\varrho\sqrt{q}\mathbb{E}(X) + 128 \times \varrho^2 q < t$ and $\mathbb{E}(X) + t \leq 2t$. So applying McDiarmid's Inequality with $\varrho = 4k\log^6 n$ and $q = 2r$ yields:

$$\begin{aligned} \Pr(X > \mathbb{E}(X) + 2t) &\leq 4 \exp\left(-\frac{t^2}{1024k^2r \log^{12} n (\mathbb{E}(X) + t)}\right) < 4 \exp\left(-\frac{t}{2048k^2r \log^{12} n}\right) \\ &< 4 \exp(-6 \log n) = o(n^{-5}). \end{aligned} \tag{53}$$

To use our bound on X to obtain a bound on $D^-(R_j)$, we first note that by (50), with probability at least $1 - n^{-9}$: every C_f has size at most $\log^2 n$ and hence by Lemma 47(i), $D^-(C_f) < \log^3 n$ for all f . This, (52) and (53) yield that with probability at least $1 - n^{-5}$:

$$\begin{aligned}
D^-(R_j) &\leq X + \sum_{f \in \Phi} |f^*| \leq X + \sum_{f \in \Phi} |f| \\
&\leq X + \Lambda \leq \mathbb{E}_3(X) + 2t + \Lambda \\
&\leq 3\mathbb{E}_3(X) + 12 \times 2048k^2r \log^{13} n + \Lambda \\
&\leq \Lambda \left(1 + 3W \frac{|S_j|}{n} + 3 \frac{\log^2 n}{|S_j|} \right) + 12 \times 2048k^2r \log^{13} n \\
&\leq \Lambda \left(1 + 3W \frac{|S_j|}{n} \right) + 13 \times 2048k^2r \log^{13} n,
\end{aligned}$$

since $\Lambda \leq (k-1)|S_j|$, fix and so $\Lambda \log^2 n / |S_j| = o(\log^{13} n)$. This, with (41) and (44), yields that with probability at least $1 - 3n^{-5}$, we have:

$$\begin{aligned}
D^-(R_j) &\leq \left(D^-(R_{j+1}) + Z'_1 \frac{|S_j|}{n} D^-(R_{j+1}) + \frac{r^2 k}{\alpha(c)} \frac{|S_j|}{n} \sum_{i=I_v}^{j+2} |R_i| + Z'_1 \log^2 n \right) \\
&\quad \times \left(1 + 3W \frac{|S_j|}{n} \right) + 13 \times 2048k^2r \log^{13} n \\
&\leq D^-(R_{j+1}(v)) + Z \frac{|S_j|}{n} \sum_{\ell=I_v}^{j+1} D^-(R_\ell(v)) + \log^{14} n,
\end{aligned}$$

for suitable constants Z'_1 and Z . This is (39).

So the probability that (39) holds for a given j is at least $1 - \frac{3}{\epsilon} n^{-5} > 1 - n^{-4}$. Taking the union bound for $B \leq j \leq I_v$ yields Lemma 46. \square

5.5 Solving the recurrence

Because Lemma 46 holds only for $|S_j| > n^\delta \log^2 n$ and for $j \geq B$, we define:

$$i_0 = \min\{i : |S_{i+1}| < n^\delta \log^2 n\};$$

i.e. i_0 is the largest index so that Lemma 46 applies to S_B, \dots, S_{i_0} . Note that Lemma 47(a,b) implies

$$|S_{i_0}| < 2n^\delta \log^2 n. \quad (54)$$

Recall that

$$R^{(B)}(v) = R(v) \setminus \left(\cup_{1 \leq i < B} S_i \right).$$

We will prove:

Lemma 48. *There is a constant $X = X(r, k) > 0$ such that for any $1 \leq i \leq I_{\max}$ and any $v \in S_i$:*

$$\Pr(|R^{(B)}(v)| > n^{X\delta}) \leq \frac{1}{n^2}.$$

Proof. Taking B and Z as specified in Lemma 46, and taking the union bound over all vertices v , we have that with probability at least $1 - n^{-2}$: (39) holds for all v and all $B \leq j \leq \min\{I_v, i_0\}$.

We fix a particular $i \geq B$ and $v \in S_i$. As in the previous section, we say R_ℓ instead of $R_\ell(v)$. Restating (39) and replacing j with j' (because of an index change below) yields:

$$D^-(R_{j'}) \leq D^-(R_{j'+1}) + Z \frac{|S_{j'}|}{n} \sum_{\ell=i}^{j'+1} D^-(R_\ell) + \log^{14} n, \quad \text{for all } B \leq j' \leq \min\{i, i_0\}. \quad (55)$$

This recursive equation bounds $D^-(R_{j'})$ in terms of $D^-(R_\ell)$ for values of ℓ that are *larger* than j' . In order to analyze this using a recursive equation where values are bounded by values with *smaller* indices, we will perform the change of indices:

$$j := i - j'.$$

We will recursively define r_j such that:

$$r_j \geq D^-(R_{i-j}) \quad (56)$$

First, we set up our base cases. Because (55) only holds for $j' \leq \min\{i, i_0\}$, i.e. for $j \geq \max\{0, i - i_0\}$, we need a base case for all smaller values of j . We set

$$j_0 = \max\{0, i - i_0\}$$

and define:

$$\text{For } 0 \leq j \leq j_0 : \quad r_j = n^{2\delta}.$$

We verify that something even stronger than (56) holds for this base case. Applying Lemma 47(i), we have $D^-(R_{i-j}) \leq D^-(S_{i-j}) < \log n \cdot |S_{i-j}|$. Lemma 47(c) and (54) yield

$$\sum_{j=0}^{i-i_0} D^-(R_{i-j}) \leq \log n \sum_{j=0}^{i-i_0} |S_{i-j}| \leq \log n \sum_{\ell \geq i_0} |S_\ell| \leq \log n \cdot Z_1 |S_{i_0}| n^{\delta/2} = O(n^{3\delta/2} \log^3 n). \quad (57)$$

This immediately yields (56) for all $j \leq j_0$.

We define the following recursion.

$$\forall j_0 + 1 \leq j \leq i - B : r_j = r_{j-1} + Z \frac{|S_{i-j}|}{n} \sum_{\ell=j_0}^{j-1} r_\ell + n^{2\delta}. \quad (58)$$

We argue inductively that (56) holds for all $j_0 \leq j \leq i - B$. We have already seen that it holds for the base case $j = j_0$. For higher values of j , (55) yields:

$$\begin{aligned}
D^-(R_{i-j}) &\leq D^-(R_{i-(j-1)}) + Z \frac{|S_{i-j}|}{n} \sum_{\ell=i}^{i-j+1} D^-(R_\ell) + \log^{14} n \\
&\leq r_{j-1} + Z \frac{|S_{i-j}|}{n} \sum_{\ell=j_0}^{j-1} r_\ell + \sum_{\ell=0}^{j_0-1} D^-(R_{i-\ell}) + \log^{14} n \quad \text{by Lemma 47(d) and induction on (56)} \\
&\leq r_{j-1} + Z \frac{|S_{i-j}|}{n} \sum_{\ell=j_0}^{j-1} r_\ell + n^{2\delta} \quad \text{by (57) and noting that } \log^{14} n \text{ is absorbed by } n^{2\delta} \\
&= r_j.
\end{aligned}$$

Thus r_j bounds $|R_{i-j}|$:

$$|R_{i-j}| \leq D^-(R_{i-j}) \leq r_j, \quad \forall 0 \leq j \leq i - B,$$

It will be convenient to define:

$$t_j = \sum_{\ell=j_0}^j r_\ell.$$

Therefore, by (57),

$$|R^{(B)}(v)| = \sum_{j=0}^{i-B} |R_{i-j}| \leq \sum_{j=0}^{i-B} D^-(R_{i-j}) = \sum_{j=j_0}^{i-B} D^-(R_{i-j}) + \sum_{j=0}^{j_0-1} D^-(R_{i-j}) \leq t_{i-B} + n^{2\delta}. \quad (59)$$

To prove Theorem 41, it is sufficient to show that $t_{i-B} = n^{O(\delta)}$.

Since $r_j = t_j - t_{j-1}$, rearranging (58), we have

$$t_j - t_{j-1} = t_{j-1} - t_{j-2} + Z \frac{|S_{i-j}|}{n} t_{j-1} + n^{2\delta}, \quad \forall j \geq j_0 + 1,$$

where $t_{j_0} = n^{2\delta}$ and $t_{j_0-1} = 0$. We solve this recurrence. It will be helpful to find sequences $(a_j)_{j \geq j_0}$ and $(b_j)_{j \geq j_0+1}$ that satisfy

$$t_j - a_j t_{j-1} = b_j (t_{j-1} - a_{j-1} t_{j-2}) + n^{2\delta}, \quad \forall j \geq j_0 + 1. \quad (60)$$

Rearranging gives

$$t_j - t_{j-1} = (a_j - 1 + b_j) t_{j-1} + (-a_{j-1} b_j) t_{j-2} + n^{2\delta},$$

so we require that for all $j \geq j_0 + 1$,

$$\begin{aligned}
a_j - 1 + b_j &= 1 + Z \frac{|S_{i-j}|}{n} \\
-a_{j-1} b_j &= -1,
\end{aligned}$$

and we may set initial condition $a_{j_0} = b_{j_0+1} = 1$.

So we define a_j, b_j recursively as:

$$b_j = 1/a_{j-1}, \quad a_j = 2 - \frac{1}{a_{j-1}} + Z \frac{|S_{i-j}|}{n},$$

and (60) holds. Since $a_j + \frac{1}{a_{j-1}} \geq 2$ for each j and $a_{j_0} = 1$, it follows that $a_j \geq 1$ for every $j \geq j_0$. Next, we show that there is a constant $D = D(r, k) > 0$ such that for every $j \geq j_0 + 1$,

$$1 + Z \frac{|S_{i-j}|}{n} \leq a_j \leq 1 + D \sqrt{\frac{|S_{i-j}|}{n}}. \quad (61)$$

The lower bound follows immediately from $a_{j-1} \geq 1$ and the recursion $a_j = 2 - 1/a_{j-1} + Z|S_{i-j}|/n$. We prove the upper bound by induction. By taking $D \geq Z$ we ensure that (61) holds for $j = j_0 + 1$. Now assume that $j \geq j_0 + 2$ and that (61) holds for $j - 1$, and so:

$$\frac{1}{a_{j-1}} \geq 1 - D \sqrt{\frac{|S_{i-j+1}|}{n}}.$$

Since $a_j = 2 + Z|S_{i-j}|/n - 1/a_{j-1}$, we have

$$a_j \leq 1 + \frac{Z|S_{i-j}|}{n} + D \sqrt{\frac{|S_{i-j+1}|}{n}}.$$

By the definition of j_0 and i_0 , we have $|S_{i-j}| \geq n^\delta \log^2 n$ for all $j \geq j_0$. So by Lemma 47(b), we have that for all $j_0 \leq j \leq i - B$, $|S_{i-j+1}| \leq (1 - Y_2 \sqrt{|S_{i-j}|/n})|S_{i-j}|$ for constant $Y_2 = Y_2(r, k) > 0$. Thus,

$$\begin{aligned} 1 + Z \frac{|S_{i-j}|}{n} + D \sqrt{\frac{|S_{i-j+1}|}{n}} &\leq 1 + Z \frac{|S_{i-j}|}{n} + \left(1 - \frac{Y_2}{2} \sqrt{|S_{i-j}|/n}\right) D \sqrt{\frac{|S_{i-j}|}{n}} \\ &= 1 + D \sqrt{\frac{|S_{i-j}|}{n}} - \left(\frac{DY_2}{2} - Z\right) \frac{|S_{i-j}|}{n} \leq 1 + D \sqrt{\frac{|S_{i-j}|}{n}} \end{aligned}$$

where the last inequality holds by choosing $D > 2Z/Y_2$. Thus, (61) holds also for j and thus it holds for every $j_0 \leq j \leq i - B$.

Now we continue to solve the recurrence (58). Let $c_j = t_j - a_j t_{j-1}$. Then, since $b_j = 1/a_{j-1} \leq 1$ for every $j \geq j_0 + 1$, we have

$$c_j = b_j c_{j-1} + n^{2\delta} \leq c_{j-1} + n^{2\delta} \leq c_{j_0} + (j - j_0)n^{2\delta} \leq n^{2\delta} + (j - j_0)n^{2\delta}.$$

Since $j < i = O(n^{\delta/2} \log n)$ by Theorem 3(b), this yields

$$t_j - a_j t_{j-1} \leq n^{2\delta} + O(n^{5\delta/2} \log n) \leq U := n^{3\delta}, \quad \text{for all } j \geq j_0 + 1. \quad (62)$$

Let ℓ_0 be the maximum integer for which $|S_{\ell_0}| \geq n^{1-\delta}$. Again applying Theorem 3(b), we have $\ell_0 = O(n^{\delta/2} \log n)$. Note that, due to the monotonicity of $|S_\ell|$ by Lemma 47(a,b), $\ell_0 < i_0$ trivially by the definition of ℓ_0 and i_0 and by the fact that $0 < \delta < 1/2$.

Now is a good time to recall that our goal is to show $t_{i-B} = n^{O(\delta)}$.

(62) says $t_j \leq a_j t_{j-1} + U$. Applying this recursively yields that for every $1 \leq \ell \leq i - B$,

$$t_{i-B} \leq t_{i-B-\ell} \prod_{h=0}^{\ell-1} a_{i-B-h} + U \left(1 + \sum_{h_2=0}^{\ell-2} \prod_{h=0}^{h_2} a_{i-B-h} \right). \quad (63)$$

Since $a_h \geq 1$ for each h , we have $1 + \sum_{h_2=0}^{\ell-2} \prod_{h=0}^{h_2} a_{i-B-h} \leq \ell \prod_{h=0}^{\ell-2} a_{i-B-h}$. First assume that $i \geq \ell_0 + 2$. Taking $\ell = \ell_0 - B + 2$ in (63) and noting that $a_{i-\ell_0-1} < U\ell_0$ by (61) yields

$$t_{i-B} \leq t_{i-\ell_0-2} \left(\prod_{j=0}^{\ell_0-B+1} a_{i-B-j} \right) + U\ell_0 \prod_{j=0}^{\ell_0-B} a_{i-B-j} \leq (1 + t_{i-\ell_0-2})U\ell_0 \prod_{j=0}^{\ell_0-B} a_{i-B-j}. \quad (64)$$

Again applying (61), we have:

$$\prod_{j=0}^{\ell_0-B} a_{i-B-j} \leq \exp \left(D \sum_{j=0}^{\ell_0-B} \sqrt{\frac{|S_{j+B}|}{n}} \right) = \exp \left(D \sum_{j=B}^{\ell_0} \sqrt{\frac{|S_j|}{n}} \right). \quad (65)$$

Next, we bound $\exp \left(D \sum_{j=B}^{\ell_0} \sqrt{\frac{|S_j|}{n}} \right)$. By Lemma 47(b), we have for all $j \leq \ell_0$,

$$|S_{j+1}| \leq \exp \left(-Y_2 \sqrt{\frac{|S_j|}{n}} \right) |S_j|,$$

and so

$$|S_{\ell_0+1}| \leq \exp \left(-Y_2 \sum_{j=B}^{\ell_0} \sqrt{\frac{|S_j|}{n}} \right) |S_B|.$$

By the definition of ℓ_0 and Lemma 47(a), we have $|S_{\ell_0+1}| > \frac{1}{2}n^{1-\delta}$, so

$$\exp \left(Y_2 \sum_{j=B}^{\ell_0} \sqrt{\frac{|S_j|}{n}} \right) \leq \frac{|S_B|}{|S_{\ell_0+1}|} < \frac{n}{|S_{\ell_0+1}|} = O(n^\delta),$$

and so

$$\exp \left(D \sum_{j=B}^{\ell_0} \sqrt{\frac{|S_j|}{n}} \right) = n^{O(\delta)},$$

since both D and Y_2 are positive constants. This, (64), (65), $\ell_0 = O(n^{\delta/2} \log n)$ and $U = n^{3\delta}$ yield

$$t_{i-B} = n^{O(\delta)} t_{i-\ell_0-2}. \quad (66)$$

It is easy to see that if $i < \ell_0 + 2$, then the above argument already proves that $t_{i-B} = n^{O(\delta)}$, by taking $\ell = i - B$ in (58). So we assume that $i \geq \ell_0 + 2$. It only remains to show that $t_{i-\ell_0-2} = n^{O(\delta)}$. The same recursion that produced (63) yields that for every $1 \leq \ell \leq i - \ell_0 - 2 - j_0$,

$$t_{i-\ell_0-2} \leq t_{i-\ell_0-2-\ell} \prod_{h=0}^{\ell-1} a_{i-\ell_0-2-h} + U \left(1 + \sum_{h_2=0}^{\ell-2} \prod_{h=0}^{h_2} a_{i-\ell_0-2-h} \right).$$

Arguing as for (64), this time taking $\ell = i - \ell_0 - 2 - j_0$ yields (noting that $t_{j_0} \leq n^{2\delta}$ always)

$$t_{i-\ell_0-2} \leq (t_{j_0} + Ui) \prod_{j=0}^{i-\ell_0-3-j_0} a_{i-\ell_0-2-j} = n^{O(\delta)} \prod_{j=0}^{i-\ell_0-3-j_0} a_{i-\ell_0-2-j}, \quad (67)$$

and arguing as for (65) yields

$$\prod_{j=0}^{i-\ell_0-3-j_0} a_{i-\ell_0-2-j} \leq \exp \left(D \sum_{j=\ell_0+2}^{i-1-j_0} \sqrt{\frac{|S_j|}{n}} \right). \quad (68)$$

By the definition of ℓ_0 we have $|S_{\ell_0+2}| < n^{1-\delta}$. By our definition of j_0 , for every $\ell_0 + 2 \leq j \leq i - 1 - j_0$, we have $|S_j| \geq n^\delta \log^2 n$, and so we can apply Lemma 47(a) to show

$$|S_j| \leq (1 - Y_2 n^{-\delta/2})^{j-(\ell_0+2)} |S_{\ell_0+2}| \leq (1 - Y_2 n^{-\delta/2})^{j-(\ell_0+2)} n^{1-\delta}, \quad \forall \ell_0 + 2 \leq j \leq i - 1 - j_0,$$

which implies that

$$\sum_{j=\ell_0+2}^{i-1-j_0} \sqrt{\frac{|S_j|}{n}} \leq n^{-\delta/2} \sum_{j \geq 0} (1 - Y_2 n^{-\delta/2})^j = O(1).$$

This proves that $t_{i-\ell_0-2} = n^{O(\delta)}$ by (67) and (68). So by (66) it completes our proof that $t_{i-B} = n^{O(\delta)}$. Since $|R^{(B)}(v)| \leq t_{i-B} + n^{2\delta}$ by (59), this proves the lemma. \square

5.6 Bounding the maximum depth

Lemma 48 bounds $|R^{(B)}(v)|$ for any non- k -core vertex v in $AP_r(n, m)$. Corollary 12 implies that the same bound holds for $\mathcal{H}_r(n, m)$.

The following lemma allows us to bound $|R(v)|$ using the bound on $|R^{(B)}(v)|$. To present the lemma, we define $N^s(A)$ to be the set of vertices with distance at most s from A . The following easily proved lemma is from [1, Lemma 34].

Lemma 49. *Assume $s, c > 0$ are $O(1)$. A.a.s. for every subset of vertices A in $\mathcal{H}_r(n, cn)$ such that A induces a connected subgraph, $|N^s(A)| = O(|A| + \log n)$.*

Applying Lemma 49 with $A = R^{(B)}(v)$ yields our bound on the maximum depth as follows:

Proof of Theorem 41: By Lemma 48, there is a constant $X > 0$ such that in $AP_r(n, m)$ a.a.s. there are no vertices v with $R^{(B)}(v) > n^{X\delta}$. Any vertices in $R(v)$ that are not contained in $R^{(B)}(v)$, must have been removed during the first B rounds of the parallel stripping process, and must be within distance B of some member of $R(v)$. This bounds each $R_B(v)$ in $AP_r(n, m)$; Corollary 12 shows that the same bound holds for $\mathcal{H}_r(n, m)$. Now we can apply Lemma 49, to show that a.a.s. for each $v \in \mathcal{H}_r(n, m)$, $|R(v)| \leq Y(|R^{(B)}(v)| + \log n) < n^{2X\delta}$, for some constant $Y = Y(B, r, k)$. \square

6 Proofs of Lemma 47 and Theorem 3(a): the tight lower bound

In this section, we will prove the key Lemma 47. Throughout this section, we have $c = c_{r,k} + n^{-\delta}$ for some $0 < \delta < \frac{1}{2}$. We have already seen that Lemma 47 is needed to complete the proof of Theorem 5. Lemma 47(a,b) will also immediately imply the lower bound in Theorem 3(a):

Proof of Theorem 3(a): The upper bound was proved in Section 4.3.2, so we only need to establish the lower bound: that a.a.s. the stripping number is $\Omega(n^{\delta/2} \log n)$.

By Lemma 47(a,b), there must be an iteration i_0 in the parallel process such that $\frac{1}{2}n^{1-\delta} < |S_{i_0}| < n^{1-\delta}$. Taking the constant Y_1 from Lemma 47 and noting that $2\delta < 1$, we let $a > 0$ be a constant satisfying $1 - 2Y_1a > 2\delta$. Applying Lemma 47(a) recursively for all $i_0 \leq i \leq \ell = an^{\delta/2} \log n$, we have

$$|S_{i_0+\ell}| \geq (1 - Y_1n^{-\delta/2})^\ell |S_{i_0}| \geq \exp(-2Y_1\ell n^{-\delta/2}) \cdot \frac{1}{2}n^{1-\delta} = \frac{1}{2}n^{-2aY_1+1-\delta}.$$

This is valid since $|S_{i_0+j}| \geq |S_{i_0+\ell}| \geq n^\delta \log^2 n$ for all $0 \leq j \leq \ell$ (which is desired in order to apply Lemma 47) by our choice of a . This shows that the number of iterations in the parallel stripping process applied to $AP_r(n, m)$ is at least $an^{\delta/2} \log n$; Corollary 12 implies that the same bound holds for $\mathcal{H}_r(n, m)$. This confirms the lower bound in Theorem 3(a). \square

Most of the work in our proof of Lemma 47 goes towards proving parts (a,b). This requires a very tight analysis of the evolution of L_t , the total degree of the light vertices (i.e. vertices of degree less than k) in G_t , the hypergraph remaining after t iterations of SLOW-STRIP. Much of this work can be viewed as a strengthening of the analysis from Section 4. A key result from that section is Lemma 16 which bounds $\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) < -K_1n^{-\delta/2}$. This sufficed to prove the upper bound for Theorem 3(b), but to prove the lower bound, we need to strengthen Lemma 16 in two ways: (i) when $L_t > n^{1-\delta}$, the bound on the expected change is decreased to $-\Theta(\sqrt{L_t/n})$; and (ii) we obtain matching lower bounds on the expected change. This is Corollary 54.

Of course, the expected change in L_i depends on the rate at which new light vertices are created. In Section 4.4.3 we studied this rate by examining two key closely related parameters: roughly speaking, ζ_t is the average degree of the heavy vertices in G_t (i.e. vertices of degree at least k) and \bar{p}_t is the probability that a uniform vertex-copy from the heavy vertices is a copy of a degree k vertex. We obtained a coarse bound on ζ_t (Lemma 35, Corollary 36) which was sufficient to obtain Lemma 16. For our tighter analysis in this section, we will require a much tighter bound on ζ_t (Lemma 50). In addition, we will introduce two more parameters.

Recall that G_t is the hypergraph (i.e. configuration) remaining after t iterations of SLOW-STRIP on $AP_r(n, m)$, and \widehat{H}_i is the hypergraph (i.e. configuration) remaining after $i - 1$ iterations of the parallel stripping process. Recall also that $t(i)$ is the iteration of SLOW-STRIP corresponding to the beginning of iteration i of the parallel stripping process. Recall that τ denotes the step when SLOW-STRIP terminates.

Recall the definitions of L_t, D_t, N_t, \bar{p}_t from Sections 4 and 4.4.3, and that $\zeta_t = D_t/N_t$. Recall that $\mathcal{F}_t = \{(L_s, N_s, D_s)\}_{s \leq t}$. Our first new parameter is a very close approximation of $L_{t+1} - L_t$:

$$br_t = -1 + (r-1)(k-1)\bar{p}_t. \quad (69)$$

This is a key parameter in analyzing the evolution of $(L_t)_{t \geq 0}$. If we view (L_t) as a branching process then $1+br_t$ approximates the branching parameter of (L_t) ; i.e. the expected number of new light vertices formed during step t of SLOW-STRIP. Note that for $r=2$, this is exactly the expected number of new light vertices formed when the other endpoint of a deleted edge is heavy. But we need to account for (i) the possibility of both endpoints being light, and (ii) the fact that when $r > 2$ we typically remove multiple heavy vertex-copies. By (29) and (30), and noting that $h_{t,i} \leq L_t/(L_t + D_t - (r-2)) = O(L_t/n)$ (by Corollary 9), we have that a.a.s. for every $t(B) \leq t \leq \tau$,

$$\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) = br_t + O(L_t/n), \quad \mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) \leq br_t + O(n^{-1}). \quad (70)$$

The second part of (70) above is applied when we only require an upper bound on $\mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t)$. However, in some cases we need a lower bound as well, and we will use the first part.

By (32), (33) and (34), there is a constant $K > 0$ such that

$$\text{a.a.s. for every } t(B) \leq t \leq \tau, \quad br_t \leq -Kn^{-\delta/2}. \quad (71)$$

By (28), a.a.s. $\zeta_\tau = \zeta + \Theta(n^{-\delta/2})$. By Lemmas 32 and 31 and considering the Taylor expansion of ψ around ζ , a.a.s.

$$\bar{p}_\tau = \psi(\zeta_\tau) + O(n^{-1/2} \log n) = \psi(\zeta) + \psi'(\zeta)(\zeta_\tau - \zeta) + O((\zeta_\tau - \zeta)^2 + n^{-1/2} \log n) = \psi(\zeta) - \Theta(n^{-\delta/2}).$$

By (21), $\psi(\zeta) = 1/(r-1)(k-1)$, which implies that a.a.s.

$$br_\tau = -\Theta(n^{-\delta/2}) \quad (72)$$

by (69).

Our second new parameter, $\pi(G_t)$, roughly counts the number of vertices in G_t which are not in the k -core and so will be removed before the process ends. Recall from Corollary 9 that a.a.s. the k -core has size $\alpha n + (K_1 + o(1))n^{1-\delta/2}$, for a particular constant $K_1 > 0$. So the number of non- k -core vertices is simply $|V(G_t)|$ minus that number.

We will prove (Lemma 52) that br_t is the same multiplicative order as the negation of the proportion of vertices in G_t that are not in the k -core, until that proportion drops below $n^{-\delta/2}$; from that point on, br_t remains at $-\Theta(n^{-\delta/2})$ (see (72)). To reflect this, we define $\pi(G_t)$ to stay at $\Theta(n^{1-\delta/2})$ once the remaining number of vertices drop below that. So we define:

$$\pi(G_t) := |V(G_t)| - \alpha n - \frac{K_1}{2} n^{1-\delta/2}.$$

Over the next few subsections, we obtain tight bounds on br_t in terms of L_t (Lemma 53); coupled with (70), these will yield our bounds on the expected change in L_t . To do so, we will link br_t and π_t , using ζ_t . Our first step will be to obtain a tighter bound on ζ_t .

6.1 Controlling ζ_t

We list below a few facts that we will use. Since $c = c_{r,k} + n^{-\delta}$, $\delta < \frac{1}{2}$, we can assume by Lemma 7 that there is a k -core on a linear number of vertices. At each step of SLOW-STRIP, we remove at most one hyperedge. Thus there is a constant $Q = Q(r, k)$ such that in every step, the average degree of the heavy vertices is changed by at most $\pm Q/n$; i.e.

$$\zeta_{t+1} - \zeta_t = O(1/n) \text{ uniformly for all } 0 \leq t < \tau. \quad (73)$$

Therefore, for any $\epsilon > 0$, Lemma 15 allows us to choose sufficiently large B such that $|\zeta_t - \zeta_\tau| \leq \epsilon/2$ for all $t \geq t(B)$. Recalling the definition of ζ from (12), we also know that a.a.s. $|\zeta_\tau - \zeta| = o(1)$ by Corollary 9. Hence, for all $t \geq t(B)$, $|\zeta_t - \zeta| \leq |\zeta_t - \zeta_\tau| + |\zeta_\tau - \zeta| \leq \epsilon$. This immediately gives (Fa) below.

(Fa) For every $\epsilon > 0$, we can choose B sufficiently large such that a.a.s. for all $t \geq t(B)$, $|\zeta_t - \zeta| < \epsilon$.

(Fb) A.a.s. for every $t \geq t(B)$, $\bar{p}_t = (1 + O(n^{-1/2} \log n))\psi(\zeta_t)$, by Lemma 32 and Theorem 6.

(Fc) We can choose $\epsilon > 0$ sufficiently small so that there are $c_1, c_2 > 0$ such that $-c_1 < \psi'(x) < -c_2$ for all x such that $|x - \zeta| < \epsilon$ by Lemma 31 and since $\zeta = \zeta_{r,k} > k$.

By (Fa) and (Fc), we may assume B is chosen so that

(Fc') $-c_1 < \psi'(\zeta_t) < -c_2$ uniformly for all $t \geq t(B)$.

By (73) we have

(Fd) $\zeta_i = \zeta_t + O((i - t)/n)$ uniformly for every $0 \leq t \leq i \leq \tau$.

We will apply Lemma 29 several times. To formalise its application, we need to define normal configurations as in Section 4.4.3, and define several stopping times when various a.a.s. events such as (Fa) and (Fb) fail. We skip such tedious settings of stopping times, and note that by the same treatment as in Section 4.4.3, we will only create an $o(1)$ error in all probability bounds if we apply Lemma 29 with events in (Fa)–(Fd) assumed.

In the next lemma, we prove a more precise form of (Fd). This is our strengthening of Lemma 35.

Lemma 50. *A.a.s. for every $t(B) \leq t < i \leq \tau$,*

(a) $\zeta_i \leq \zeta_t + O(\log n/n)$ uniformly for every $0 \leq t \leq i \leq \tau$;

(b) if $i - t \geq \log n$, then $\zeta_i = \zeta_t - \Theta((i - t)/n)$ uniformly.

Proof. By Lemma 34, there exist two constants $\rho_1 > \rho_2 > 0$ such that a.a.s. for every $t(B) \leq t < \tau$

$$-\frac{\rho_1}{n} \leq \mathbb{E}(\zeta_{t+1} - \zeta_t \mid \mathcal{F}_t) \leq -\frac{\rho_2}{n}.$$

Moreover, $|\zeta_{t+1} - \zeta_t| = O(1/n)$ by (73). We apply Lemma 29 with $c_n = O(1/n)$ and $a_n = \rho_2/n$ and with a similar argument as in the proof of Lemma 19 to show, for every $i > t$ and $j \geq 0$,

$$\Pr(\zeta_i \geq \zeta_t - (i-t)\rho_2/n + j) \leq \exp(-\Omega(j^2 n^2 / (i-t))), \quad (74)$$

$$\Pr(\zeta_i \leq \zeta_t - (i-t)\rho_1/n - j) \leq \exp(-\Omega(j^2 n^2 / (i-t))). \quad (75)$$

Then by the union bound (by taking $j = (i-t)\rho_2/2n$ in (74) and taking $j = (i-t)\rho_1/2n$ in (75) for each t and each $i \geq t + \log n$), we obtain (b). Part (a) follows by (b) and the fact that for each $i \leq t + \log n$, we always have $\zeta_i = \zeta_t + O(\log n/n)$ by (Fd). \square

6.2 Relations between L_t , $\pi(G_t)$ and br_t

The next Lemma essentially says that if we can bound the expected change in L_i then we can show L_i is concentrated. Recall that τ is the stopping time of SLOW-STRIP; i.e. the first iteration t for which $L_t = 0$.

Lemma 16 says that there are constants $K, B > 0$ such that a.a.s. for every $t(B) \leq t < \tau$,

$$\mathbb{E}(L_{t+1} \mid \mathcal{F}_t) \leq L_t - Kn^{-\delta/2}. \quad (76)$$

It is convenient to consider a random process $(\mathcal{L}_t)_{t \geq t(B)}$ such that $\mathbb{E}(\mathcal{L}_{t+1} \mid \mathcal{L}_t) \leq \mathcal{L}_t - Kn^{-\delta/2}$ for all $t \geq t(B)$. The process (\mathcal{L}_t) can be defined in various ways depending on each application and in many applications we will let $\mathcal{L}_t = L_t$ (or sometimes with a shift of the subscript) for all steps in which (76) holds (c.f. the proof of Lemma 19 and the arguments in Section 4.3.2). We first prove some a.a.s. properties of such processes.

Lemma 51. *Let $(\mathcal{L}_i)_{i \geq 0}$ be a random process and define T to be the minimum integer such that $\mathcal{L}_T \leq 0$. Suppose there is a constant $C > 0$ and functions $C > a = a(n) > b = b(n) \geq \Theta(n^{-\delta/2})$ such that the following bounds always hold for every $i \geq 0$:*

- (i) $n^\delta \log^2 n \leq \mathcal{L}_0 \leq n$;
- (ii) $|\mathcal{L}_{i+1} - \mathcal{L}_i| \leq C$;
- (ii) $-a \leq \mathbb{E}(\mathcal{L}_{i+1} - \mathcal{L}_i \mid \{\mathcal{L}_s\}_{s \leq i}) \leq -b$.

Then, with probability at least $1 - o(n^{-1})$,

- (a) $\mathcal{L}_0 - 2ai < \mathcal{L}_i < \mathcal{L}_0 - \frac{1}{2}bi$ for all $n^\delta \log^{1.5} n \leq i \leq n^2$;
- (b) $\mathcal{L}_i < 2\mathcal{L}_0$ for all $0 \leq i \leq n^2$;
- (c) $\frac{1}{2a}\mathcal{L}_0 < T < \frac{2}{b}\mathcal{L}_0$.

Proof. We start with the upper bound in part (a). We will apply Lemma 29 with $X_{n,\ell} = \mathcal{L}_\ell - \mathcal{L}_0$. So we can take $a_n = -b$. By (ii) we can choose $c_n = C$.

Consider $i \geq n^\delta \log^{1.5} n$. If $\mathcal{L}_i \geq \mathcal{L}_0 - \frac{1}{2}bi$ then $X_{n,i} - X_{n,0} = \mathcal{L}_i - \mathcal{L}_0 \geq ia_n + \frac{1}{2}ib$. By Lemma 29 with $j = \frac{1}{2}ib$, the probability of this is at most

$$\exp\left(-\frac{(\frac{1}{2}ib)^2}{2i(C + |b|)^2}\right) \leq \exp(-\Omega(b^2i)) \leq \exp(-\Omega(\log^{1.5} n)) = o(n^{-3}).$$

The lower bound in part (a) is nearly identical, but this time we apply Lemma 29 with $X_{n,i} = \mathcal{L}_0 - \mathcal{L}_i$, $a_n = a$ and $j = ia$. Applying the union bound for the at most n^2 choices for i shows that (a) holds with probability at least $1 - o(n^{-1})$.

For part (b): If $i < \mathcal{L}_0/C$ then the fact that $\mathcal{L}_{j+1} \leq \mathcal{L}_j + C$ implies that $\mathcal{L}_i < 2\mathcal{L}_0$. If $i \geq \mathcal{L}_0/C$, then $i > n^\delta \log^{1.5} n$, then part (a) implies $\mathcal{L}_i < \mathcal{L}_0$.

For part (c): Note that $\frac{2}{b}\mathcal{L}_0 \geq n^\delta \log^{1.5} n$, and so part (a) implies that \mathcal{L}_i reaches 0 for some $i \leq \frac{2}{b}\mathcal{L}_0$ (noting here that $\frac{2}{b}\mathcal{L}_0 = o(n^2)$ by assumption (i)); this yields the upper bound on T . For the lower bound on T , we apply the same argument used for the lower bound in part (a) with $i = \frac{1}{2a}\mathcal{L}_0$ but with $j = \frac{1}{2}\mathcal{L}_0$. This time we get

$$\Pr(\mathcal{L}_i < \frac{1}{2}\mathcal{L}_0) < \exp(-\Omega(\mathcal{L}_0^2/i)) \leq \exp(-\Omega(a\mathcal{L}_0)) = o(n^{-2}),$$

thus providing the desired lower bound on T . □

Our goal is to bound br_t in terms of L_t . First we bound it in terms of $\pi(G_t)$.

Lemma 52. *There are two constants $C_1, C_2 > 0$ such that a.a.s. $-C_1\pi(G_t)/n \leq br_t \leq -C_2\pi(G_t)/n$ for every $t \geq t(B)$.*

Proof. Recall that ζ_t denotes the average degree of heavy vertices in G_t . Let t' be the maximum integer such that $\pi(G_{t'}) = K_1 n^{1-\delta/2}$, where K_1 comes from Corollary 9. Note that $\pi(G_t)$ is a non-increasing function of t , which decreases by at most 1 in each step. Hence, for all $t \leq t'$, $\pi(G_t) \geq K_1 n^{1-\delta/2}$ and for all $t > t'$, $\pi(G_t) < K_1 n^{1-\delta/2}$.

By Corollary 9, a.a.s. $\pi(G_\tau) \sim (K_1/2)n^{1-\delta/2}$. Since for every t , $|\pi(G_t) - \pi(G_{t+1})| \leq 1$ as at most one vertex is removed in each step, we have that a.a.s. for all $t \leq t'$, $\tau - t \geq \pi(G_t) - \pi(G_\tau) = \Omega(\pi(G_t))$. This is because $\pi(G_\tau) \sim (K_1/2)n^{1-\delta/2}$ and so $\pi(G_\tau) \sim \frac{1}{2}\pi(G_{t'}) < \frac{2}{3}\pi(G_t)$ for every $t \leq t'$. The constant bounds involved in the asymptotic notation above and below will be uniform for all t . In particular, $\tau - t' \geq \pi(G_{t'}) - \pi(G_\tau) \geq (K_1/3)n^{1-\delta/2}$. On the other hand, for every t , $\tau - t \leq k(\pi(G_t) - \pi(G_\tau)) \leq k\pi(G_t)$, since every light vertex in the queue \mathcal{Q}_t takes less than k steps to be removed. So a.a.s. uniformly for every $t \leq t'$, we have

$$(K_1/3)n^{1-\delta/2} \leq \tau - t' \leq \tau - t \leq k\pi(G_t); \quad (77)$$

$$\tau - t = \Theta(\pi(G_t)). \quad (78)$$

If $t \leq t'$ then $\tau - t = \Omega(n^{1-\delta/2}) > \log n$ by (77). Thus by Lemma 50(b) we have $\zeta_t = \zeta_\tau + \Theta((\tau - t)/n)$. By (Fb) and (Fc') and by (69), a.a.s. for all $t(B) \leq t \leq \tau$,

$$br_t - br_\tau = \Theta(1)(\bar{p}_t - \bar{p}_\tau) = \Theta(1)(\psi(\zeta_t) - \psi(\zeta_\tau)) = -\Theta(1)(\zeta_t - \zeta_\tau), \quad (79)$$

and then by the fact that $\zeta_t = \zeta_\tau + \Theta((\tau - t)/n)$ for all $t \leq t'$ and (78), a.a.s. for each $t \leq t'$,

$$br_t - br_\tau = -\Theta((\tau - t)/n) = -\Theta(\pi(G_t)/n). \quad (80)$$

The constants in the asymptotic notations above are uniform for all $t \leq \tau$. By (72), a.a.s. $br_\tau = -\Theta(n^{-\delta/2})$. Since $\pi(G_\tau) \sim (K_1/2)n^{1-\delta/2}$, it follows then that a.a.s. for all $t(B) \leq t \leq t'$,

$br_t = -\Theta(\pi(G_t)/n)$; note that $-\Theta(n^{-\delta/2})$ is absorbed as $\pi(G_t)/n \geq \pi(G_\tau)/n = \Theta(n^{-\delta/2})$ for every t . This proves that our lemma holds for all $t \leq t'$.

Now we consider $t > t'$. Applying (77) with $t := t'$, we obtain that a.a.s. $\tau - t' \leq k\pi(G_{t'}) = kK_1n^{1-\delta/2}$. This implies that for all $t' \leq t \leq \tau$, $\tau - t \leq \tau - t' = O(n^{1-\delta/2})$. Then by (73) we have

$$\zeta_\tau - \zeta_t = O(n^{-\delta/2}) \quad (81)$$

So by (79), $br_t - br_\tau = -\Theta(\zeta_t - \zeta_\tau) = O(n^{-\delta/2})$ for all $t' \leq t \leq \tau$. Then, we must have $br_t = -\Theta(n^{-\delta/2})$ for all $t' < t \leq \tau$ by (71) and (72). The definition of t' implies that $\pi(G_t)/n = \Theta(n^{-\delta/2})$ for all $t > t'$. This implies that our lemma holds also for $t > t'$. \square

And now we are ready to bound br_t in terms of L_t :

Lemma 53. *There are constants $D_1, D_2, B > 0$ such that a.a.s. for all $t \geq t(B)$,*

$$(a) \text{ for all } t \text{ such that } L_t \geq n^{1-\delta}, -D_1\sqrt{L_t/n} \leq br_t \leq -D_2\sqrt{L_t/n}.$$

$$(b) \text{ for all } t \text{ such that } L_t < n^{1-\delta}, -D_1n^{-\delta/2} \leq br_t \leq -D_2n^{-\delta/2}.$$

Combining Lemma 53 with (70), and applying a slight adjustment to D_1, D_2 yields our strengthening of Lemma 16:

Corollary 54. *There are constants $D_1, D_2, B > 0$ such that a.a.s. for all $t \geq t(B)$,*

$$(a) \text{ for all } t \text{ such that } L_t \geq n^{1-\delta}, -D_1\sqrt{L_t/n} \leq \mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) \leq -D_2\sqrt{L_t/n}.$$

$$(b) \text{ for all } t \text{ such that } L_t < n^{1-\delta}, -D_1n^{-\delta/2} \leq \mathbb{E}(L_{t+1} - L_t \mid \mathcal{F}_t) \leq -D_2n^{-\delta/2}.$$

This strengthening is what we need to prove Lemma 47(a,b).

Proof of Lemma 53 We take B large enough so that the relevant preceding results hold.

We have $br_t \leq -Kn^{-\delta/2}$ for some constant $K > 0$, for all $t(B) \leq t \leq \tau$ by (71). For any $t(B) \leq t \leq \tau$, we will consider i such that $t \leq i \leq \tau$. By Lemma 50(a), a.a.s. for any such pair of t, i , we have $\zeta_i \leq \zeta_t + O(\log n/n)$. Thus by (Fb) and (Fc') and the definition of br_i , a.a.s. for all pairs of t, i such that $t(B) \leq t < i \leq \tau$,

$$\begin{aligned} br_i &= br_t - \Theta(1)(\bar{p}_t - \bar{p}_i) = br_t - \Theta(1)(\psi(\zeta_t) - \psi(\zeta_i)) = br_t - \Theta(1)(\zeta_i - \zeta_t) \\ &\geq br_t - \Theta\left(\frac{\log n}{n}\right) \geq 2br_t, \end{aligned}$$

since $\log n/n = o(|br_t|)$. By (70), a.a.s. for every $t(B) \leq t < i \leq \tau$,

$$2br_t - O(L_i/n) \leq \mathbb{E}(L_{i+1} - L_i \mid \mathcal{F}_i) \leq br_i + O(n^{-1}).$$

We know that a.a.s. for every $t(B) \leq t < i \leq \tau$, we have $br_t, br_i \leq -Kn^{-\delta/2}$ by (71). So for some constant $A > 0$, we have a.a.s. for all $t(B) \leq t < i \leq \tau$:

$$2br_t - AL_i/n \leq \mathbb{E}(L_{i+1} - L_i \mid \mathcal{F}_i) \leq -\frac{K}{2}n^{-\delta/2}. \quad (82)$$

Let t_1 be the smallest t such that $L_{t+1} < n^\delta \log^2 n$. We first prove the lemma for all $t \leq t_1$.

Note that since $br_t \geq -1$ (by 69), we can take B large enough so that for $i \geq B$, L_i/n is small enough that the LHS of (82) is at least -3 . So we let τ^* denote the first step that the condition $-3 < \mathbb{E}(L_{i+1} - L_i \mid \mathcal{F}_i) \leq -\frac{K}{2}n^{-\delta/2}$ fails; define $\tau^* = \tau$ if it never fails. Assume $t \leq t_1$. Define $(\mathcal{L}_j)_{j \geq 0}$ by letting $\mathcal{L}_0 = L_t$ and $\mathcal{L}_{i-t} = L_i$ for each $t < i \leq \tau^*$ and $\mathcal{L}_{i+1-t} = \mathcal{L}_{i-t} - Kn^{-\delta/2}$ for all $i \geq \tau^*$. Moreover, (82) implies that a.a.s. $\mathcal{L}_{i-t} = L_i$ for all $t \leq i \leq \tau$.

Now the process (\mathcal{L}_i) satisfies the hypotheses in Lemma 51(i) (by the definition of t_1), (ii) with $C = kr$ and (iii) with $a = 3$ and $b = Kn^{-\delta/2}$. Now by Lemma 51(b) (and by taking the union bound over t), a.a.s. $L_i < 2L_t$ (corresponding to $\mathcal{L}_{i-t} < 2\mathcal{L}_0$ for each given t) for all pairs of i, t such that $t(B) \leq t \leq i < \tau$. Then (82) becomes

$$2br_t - 2AL_t/n \leq \mathbb{E}(L_{i+1} - L_i \mid \mathcal{F}_i) \leq -\frac{K}{2}n^{-\delta/2}. \quad (83)$$

So we can apply Lemma 51 again to $(\mathcal{L}_i)_{i \geq 0}$ defined as above (except that we modify τ^* to reflect (83) rather than (82)) with $a = -2br_t + 2AL_t/n$. Letting T denote the first step that \mathcal{L}_i becomes at most zero, Lemma 51(c) shows that with probability $1 - o(n^{-1})$, $T \geq 4AL_t/(|br_t| + L_t/n)$; i.e. $\tau \geq t + 4AL_t/(|br_t| + L_t/n)$. Taking the union bound over $t(B) \leq t \leq t_1$ and using that by (83) a.a.s. $\mathcal{L}_{i-t} = L_t$ for all $t(B) \leq t < i \leq \tau$ shows that a.a.s. for all $t(B) \leq t \leq t_1$:

$$\tau \geq t + 4AL_t/(|br_t| + L_t/n).$$

We delete at least one vertex for every $k - 1$ steps of SLOW-STRIP, and so $|V(G_t) \setminus V(G_\tau)| \geq (\tau - t)/(k - 1)$. Therefore, applying Corollary 9, we have that a.a.s. for all $t(B) \leq t \leq t_1$:

$$\begin{aligned} \pi(G_t) &= |V(G_t)| - \alpha n - \frac{K_1}{2}n^{1-\delta/2} \geq |V(G_t)| - |V(G_\tau)| + \frac{K_1}{3}n^{1-\delta/2} \\ &\geq \frac{4AL_t}{(k-1)(|br_t| + L_t/n)} + \frac{K_1}{3}n^{1-\delta/2}. \end{aligned} \quad (84)$$

By Lemma 52, there exists a constant $C_2 > 0$ such that a.a.s. for all $t(B) \leq t \leq t_1$,

$$br_t \leq -C_2 \frac{\pi(G_t)}{n} \leq -C_2 \left(\frac{4AL_t/n}{(k-1)(|br_t| + L_t/n)} + \frac{K_1}{3}n^{-\delta/2} \right),$$

and so

$$|br_t| \geq C_2 \max \left(\frac{4AL_t/n}{(k-1)(|br_t| + L_t/n)}, \frac{K_1}{3}n^{-\delta/2} \right). \quad (85)$$

Taking B large enough that L_t/n is sufficiently small for $t \geq t(B)$ (by Lemma 15), and rearranging (85), there is a constant $D_2 > 0$, such that a.a.s. for all $t(B) \leq t \leq t_1$:

$$|br_t| \geq D_2 \max \left(\sqrt{\frac{L_t}{n}}, n^{-\delta/2} \right). \quad (86)$$

This yields the upper bounds in our lemma for $t \leq t_1$. Next we prove the lower bounds; i.e., we wish to prove that for some constant $D_1 > 0$ for all $t(B) \leq t < i \leq t_1$:

$$|br_t| \leq D_1 \max \left(\sqrt{\frac{L_t}{n}}, n^{-\delta/2} \right). \quad (87)$$

The proof is similar to that for the upper bound so we briefly describe the arguments. Let $A_2, A_3 > 0$ be the implicit constants in (Fd), (79), and set $A_1 = 1/(2A_2A_3)$. Applying (79), (Fd) we get that a.a.s. for any $t(B) \leq t \leq i \leq \min\{t + A_1|br_t|n, \tau\}$:

$$br_i - br_t \leq A_3(\zeta_t - \zeta_i) \leq A_2A_3((i - t)/n) \leq A_1A_2A_3|br_t| = -\frac{1}{2}br_t,$$

where the first inequality above comes from (Fb), (Fc') and (69) (in the same way that (79) is deduced), the second inequality follows by (Fd) and the third inequality follows by the upper bound of i ; the equality holds by the definition of A_1 . Thus, $br_i \leq \frac{1}{2}br_t$. So, by (70):

$$\mathbb{E}(L_{i+1} | \mathcal{F}_i) - L_i \leq \frac{1}{2}br_t + O(1/n), \quad \forall t(B) \leq t < i \leq \min\{t + A_1|br_t|n, \tau\}. \quad (88)$$

We can assume $|br_t| > \sqrt{(8/A_1)L_t/n}$, as otherwise (87) holds with $D_1 = \sqrt{8/A_1}$. Therefore we have (i) $8L_t/|br_t| < A_1|br_t|n$ and (ii) the RHS of (88) is at most $\frac{1}{4}br_t$ (noting that $|br_t| \gg 1/n$ by (71)).

Define $(\mathcal{L}_i)_{i \geq 0}$ analogously to how it was defined above, and apply Lemma 51(c) with $b = \frac{1}{4}br_t$ (and with $a = -2br_t + 2AL_t/n$, as argued above). By the a.a.s. correspondence between \mathcal{L}_{i-t} and L_t , and noting that (i) above allows us to apply (88) for all $i \leq \min\{t + 8L_t/|br_t|, \tau\}$, we can show that with probability $1 - o(n^{-1})$, the stopping time

$$\tau < t + \frac{2}{b}L_t = t + 8L_t/|br_t|. \quad (89)$$

Taking the union bound over $t(B) \leq t \leq t_1$ shows that a.a.s. (89) holds for all $t(B) \leq t \leq t_1$ for which $|br_t| > \sqrt{(8/A_1)L_t/n}$.

Since we remove at most one vertex during each iteration of SLOW-STRIP, it follows that G_t contains at most $8L_t/|br_t|$ non- k -core vertices by (89). Recalling that $\pi(G_t)$ is approximately the number of non- k -core vertices in G_t plus $\frac{1}{2}K_1n^{-\delta/2}$, this implies that $\pi(G_t) \leq A_4(L_t/|br_t| + n^{1-\delta/2})$ for some constant $A_4 > \max\{8, \frac{1}{2}K_1\}$. Now, by Lemma 52, we have that there is a constant $C_1 > 0$ such that a.a.s. for all $t(B) \leq t \leq t_1$ with $|br_t| > \sqrt{(8/A_1)L_t/n}$ we have:

$$br_t \geq -C_1\pi(G_t)/n \geq -C_1A_4 \left(\frac{L_t/n}{|br_t|} + n^{-\delta/2} \right).$$

This implies (87) for $t(B) \leq t \leq t_1$ with $D_1 = \max\{\sqrt{8/A_1}, C_1A_4\}$.

Now we consider $t > t_1$. Since $L_{t_1} \geq n^\delta \log^2 n$, by (89), a.a.s. $\tau - t_1 < 8L_{t_1}/|br_{t_1}|$. By the definition of t_1 we also have that $L_{t_1} \leq L_{t_1+1} + r < 2n^\delta \log^2 n$. (86) says that a.a.s. $|br_{t_1}| \geq D_2n^{-\delta/2}$. So we obtain: a.a.s. for all $t > t_1$,

$$\tau - t < \tau - t_1 < (16/D_2)n^{3\delta/2} \log^2 n.$$

This implies that a.a.s. for all $t_1 < t \leq \tau$, $t - t_1 < (16/D_2)n^{3\delta/2} \log^2 n$. Then, recalling that A_2, A_3 are the implicit constants from (Fd) and (79), and since $|br_{t_1}| \geq D_2n^{-\delta/2}$ (from (86)), a.a.s. for all $t_1 < t \leq \tau$,

$$|br_t - br_{t_1}| \leq A_2A_3(t - t_1)/n \leq A_2A_3(16/D_1)n^{3\delta/2-1} \log^2 n = o(n^{-\delta/2}) = o(br_{t_1}),$$

as $\delta < 1/2$. Since (86), (87) hold for $t = t_1$ then by increasing D_1 and decreasing D_2 , these bounds hold for $t > t_1$, and hence for all $t(B) \leq t \leq \tau$. \square

6.3 Proof of the key lemma

And now we can prove Lemma 47, which we restate.

Recall that we are carrying out SLOW-STRIP on $AP_r(n, m)$, and so implicitly, this yields the steps of the parallel stripping process applied to $AP_r(n, m)$. I_{\max} is the number of iterations carried out by the parallel stripping process, and S_i is the set of vertices removed in iteration i . The number of hyperedges removed in iteration i that contain u is $d^+(u)$ for each $u \in S_i$, and is $d^-(u)$ for each $u \in S_{i+1}$ (if $i < I_{\max}$).

Lemma 47. There exist constants B, Y_1, Y_2, Z_1 dependent only on r, k , such that a.a.s. for every $B \leq i < I_{\max}$ with $|S_i| \geq n^\delta \log^2 n$:

- (a) if $|S_i| < n^{1-\delta}$ then $(1 - Y_1 n^{-\delta/2})|S_i| \leq |S_{i+1}| \leq (1 - Y_2 n^{-\delta/2})|S_i|$;
- (b) if $|S_i| \geq n^{1-\delta}$ then $(1 - Y_1 \sqrt{\frac{|S_i|}{n}})|S_i| \leq |S_{i+1}| \leq (1 - Y_2 \sqrt{\frac{|S_i|}{n}})|S_i|$;
- (c) $\sum_{j \geq i} |S_j| \leq Z_1 |S_i| n^{\delta/2}$.
- (d) $|S_i| \leq \sum_{u \in S_i} d^-(u) < |S_i| + Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (e) $|S_{i+1}| \leq \sum_{a,b} abM_i^{a,b} \leq |S_{i+1}| + Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (f) $\sum_{a \geq 2, b \leq r-a} abM_i^{a,b} \leq Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (g) $\sum_{u \in S_i} d^+(u)d^-(u) \leq \sum_{u \in S_i} d^+(u) + Z_1 \frac{|S_i|^2}{n} + \log^2 n$;
- (h) $\sum_{u \in S_i} (d^-(u))^2 \leq Z_1 |S_i|$;
- (i) $d^-(u) < \log n$ for all $u \in \cup_{i=2}^{I_{\max}} S_i$.

Proof. We take B large enough so that the relevant preceding results hold.

Part (i): We allocate rcn vertex-copies to n bins. So the probability that at least one bin has size at least $\log n$ is at most $n \Pr[\mathbf{Bin}(rcn, \frac{1}{n}) \geq \log n]$ which is easily computed to be $o(1)$. Part (i) follows since $d^-(u)$ is less than the size of bin u .

Part (d): We first prove (d) assuming (e). The first inequality follows since $d^-(u) \geq 1$ for all $u \in S_i$ (with $i \geq 2$). For the second inequality, note that $\sum_{u \in S_i} d^-(u) = \sum_{a \geq 1, b \leq r-a} bM_{i-1}^{a,b} \leq \sum_{a,b} abM_{i-1}^{a,b}$, and so the bound follows from Lemma 47(e) and the fact that a.a.s. $|S_{i+1}| = \Theta(|S_i|)$ by Lemma 47(a,b).

Part (f): $\sum_{a \geq 2, b \leq r-a} abM_i^{a,b} \leq r^2 \sum_{a \geq 2, 1 \leq b \leq r-a} M_i^{a,b} = r^2 Y$, where Y is the number of hyperedges (i.e. r -tuples) that contains at least two vertex-copies in S_i and at least one vertex-copy in S_{i+1} .

Consider SLOW-STRIP from step $t(i)$ to $t(i+1) - 1$, i.e. the steps when the vertices in S_i are removed. When SLOW-STRIP removes a hyperedge incident with $u \in S_i$, it removes a vertex-copy from u and another $r - 1$ vertex-copies chosen uniformly at random. This edge counts towards Y only if at least one of these $r - 1$ vertex-copies are from S_i . Regardless of what happened during the removal of previous vertices from S_i , there are at most $(k - 1)|S_i|$ remaining vertex-copies from S_i , and there are a total of $\Theta(n)$ available vertex-copies (by

Corollary 9). So the probability that at least one vertex-copy is selected from S_i is at most $O(|S_i|/n)$. We remove at most $(k-1)|S_i|$ hyperedges during this phase. So Y is dominated in distribution by a binomial variable $\mathbf{Bin}((k-1)|S_i|, O(|S_i|/n))$ and so the Chernoff bound yields that with probability at least $1 - n^{-2}$, $Y = O(|S_i| \cdot |S_i|/n + \log n)$. Multiplying by the at most n choices for i completes the proof for part (f).

Part (e): The lower bound is trivial as every vertex in S_{i+1} is incident with at least one hyperedge counted by $\sum_{a,b} M_i^{a,b}$, and each (a,b) -hyperedge contains b vertices of S_{i+1} . We now prove the upper bound.

Let Λ denote the sum of $d^-(u) - 1$ over all $u \in S_{i+1}$ with $d^-(u) \geq 2$. We note that

$$\sum_{a,b} abM_i^{a,b} \leq \sum_b bM_i^{1,b} + \sum_{a \geq 2, b \leq r-a} abM_i^{a,b} \quad (90)$$

$$= \sum_{u \in S_{i+1}} d^-(u) + O(|S_i|^2/n) \quad \text{by part (f)} \quad (91)$$

$$= |S_{i+1}| + \Lambda + O(|S_i|^2/n) \quad \text{since } d^-(u) \geq 1 \text{ for all } u \in S_{i+1}. \quad (92)$$

To bound Λ , we bound X_j which is defined to be the number of vertices $u \in S_{i+1}$ with $d^-(u) \geq j$. To do so, we expose the degree sequence of the configuration remaining after i iterations of the parallel stripping process, and then consider choosing the configuration using Bollobás' configuration model. We emphasize that this exposure of the degree sequence, and choice of the configuration are only carried out for the purposes of the proof of (93) below; the exposure does not carry on outside of this lemma - in particular to the analysis in Section 5.4.

If a vertex u is counted by X_j , then we must have $j \leq \deg(u) \leq k-1+j$, where $\deg(u)$ is its degree in the remaining configuration, and j copies of u must be selected for removal. At most $(r-1)(k-1)|S_i|$ vertex-copies are randomly selected for removal, and there are at least αn vertex-copies to choose from (by Corollary 9). So the probability that u is counted by X_j is at most

$$\binom{\deg(u)}{j} \left(\frac{(r-1)(k-1)|S_i|}{\alpha n} \right)^j \leq \binom{k-1+j}{j} \left(\frac{(r-1)(k-1)|S_i|}{\alpha n} \right)^j < \left(Z \frac{|S_i|}{4n} \right)^j,$$

for a suitable constant $Z = Z(r, k)$.

The probability that u, u' both count towards X_j are negatively correlated - if u counts towards X_j then at least j copies are chosen from u and hence are not chosen from u' . (Note: the reason that X_j is defined to be the number of $u \in S_{i+1}$ with $d^-(u) \geq j$ rather than $d^-(u) = j$ is that it makes this negative correlation easier to see.) Noting that there are at most n vertices with $j \leq \deg(u) \leq k-1+j$, this implies that X_j is dominated by the binomial $BIN(n, 2^{-j} \left(Z \frac{|S_i|}{n} \right)^j)$. Applying the Chernoff Bound and summing over all choices of j implies that with probability at least $1 - n^{-2}$

$$\text{for all } j \geq 2: \quad X_j \leq n \left(Z \frac{|S_i|}{2n} \right)^j + \log n < \frac{Z}{2^j} \frac{|S_i|^2}{n}. \quad (93)$$

(93) implies that $\Lambda \leq \sum_{j \geq 2} (j-1)X_j = O(|S_i|^2/n + \log n)$. Multiplying by the at most n choices for i completes the proof for part (e) and hence of part (d).

Part (g): Here, we focus again on Λ from part (e), this time using the value from iteration $i - 1$. So Λ is the sum of $d^-(u) - 1$ over all $u \in S_i$ with $d^-(u) \geq 2$. As we showed in part (e), a.a.s. $\Lambda < Z|S_i|^2/n + \log^2 n$ (for every i). Part (g) now follows since:

$$\sum_{u \in S_i} d^+(u)d^-(u) = \sum_{u \in S_i} d^+(u) + \sum_{u \in S_i} d^+(u)(d^-(u) - 1) \leq \sum_{u \in S_i} d^+(u) + (k - 1)\Lambda.$$

For the remaining parts, we focus on:

$$\widehat{L}_i = \sum_{v \in S_i} d^+(v); \quad \text{i.e. } \widehat{L}_i = L_{t(i)}.$$

We prove the following relation between \widehat{L}_i and $|S_i|$: a.a.s. for all i under the assumptions of this lemma,

$$\widehat{L}_i = (k - 1)|S_i|(1 + O(|S_i|/n + \log^2 n/|S_i|)). \quad (94)$$

Since $d^+(v) \leq k - 1$ for every $v \in S_i$, immediately we have $\widehat{L}_i \leq (k - 1)|S_i|$. If a vertex v in S_i has $d^+(v) < k - 1$ then $d^-(v)$ must be at least two since otherwise v should have been removed before the i -th iteration of the process. Let I_2 denote the set of vertices $v \in S_i$ with $d^-(v) \geq 2$ then

$$\widehat{L}_i \geq (k - 1)|S_i \setminus I_2|.$$

By (d),

$$|S_i \setminus I_2| + 2|I_2| \leq \sum_{u \in S_i} d^-(u) \leq |S_i| + O(|S_i|^2/n + \log^2 n).$$

It yields $|I_2| = O(|S_i|^2/n + \log^2 n)$ and so $|S_i \setminus I_2| = |S_i| + O(|S_i|^2/n + \log^2 n) = |S_i|(1 + O(|S_i|/n + \log^2 n/|S_i|))$. It follows then that $\widehat{L}_i \geq (k - 1)|S_i|(1 + O(|S_i|/n + \log^2 n/|S_i|))$. Combining with the upper bound that $\widehat{L}_i \leq (k - 1)|S_i|$, (94) follows.

Parts (a,b): For (a), we have $\widehat{L}_i = \Omega(n^\delta \log^2 n)$ and $\widehat{L}_i = O(n^{1-\delta})$ by the assumptions of the lemma and by (94). We will bound $\mathbb{E}(L_{j+1} - L_j \mid \mathcal{F}_j)$ in order to apply Lemma 51 as in the proof for Lemma 53.

Since SLOW-STRIP removes at least one and at most r vertex-copies from S_i in every iteration $t(i) \leq j < t(i + 1)$, we have $t(i + 1) - t(i) = \Theta(\widehat{L}_i)$, uniformly over i satisfying the conditions of the lemma. So $t(i + 1) > t(i) + n^\delta \log^{1.5} n$. Furthermore, by (Fd), for all $t(i) \leq j \leq t(i + 1) - 1$, $\zeta_j - \zeta_{t(i)} = O((j - t(i))/n) = O(\widehat{L}_i/n)$, and so by (79), we have $br_j = br_{t(i)} + O(\widehat{L}_i/n) = br_{t(i)} + O(n^{-\delta})$. Corollary 53(b) says that a.a.s.:

$$-D_1 n^{-\delta/2} \leq \mathbb{E}(L_{j+1} - L_j \mid \mathcal{F}_j) \leq -D_2 n^{-\delta/2}, \quad (95)$$

for all $t(i) \leq j \leq t(i + 1) - 1$. This allows us to apply Lemma 51 in a similar manner as in the proof of Lemma 53. I.e. we define a process \mathcal{L}_j that is equal to $L_{j+t(i)}$ so long as (95) holds, and such that we can apply Lemma 51 to \mathcal{L}_j ; then we translate what this says about \mathcal{L}_j to L_j since (95) a.a.s. holds for all j . This yields that for each i satisfying the conditions of the lemma, with probability $1 - o(n^{-1})$,

$$L_{t(i+1)} = L_{t(i)} - \Theta(n^{-\delta/2})(t(i + 1) - t(i)) = L_{t(i)}(1 - \Theta(n^{-\delta/2})).$$

Taking the union bound over all i yields part (a) with $|S_i|$ replaced by \widehat{L}_i . Then part (a) follows by (94) by noting that $n^{-\delta/2}$ dominates the other errors $|S_i|/n$ and $\log^2 n/|S_i|$ for $|S_i|$ in this range.

The proof of part (b) is nearly identical, applying Lemma 53(a) rather than Lemma 53(b).

Part (c): Recall that τ is the stopping time of SLOW-STRIP. As in the proofs of parts (a,b), we will apply Lemma 51. There are two possible ranges for L_j and in both, we can take $b = \Theta(n^{-\delta/2})$: if $L_j < n^{1-\delta}$ we use $b = \Theta(n^{-\delta/2})$; if $L_j \geq n^{1-\delta}$ we use $b = \Theta(\sqrt{L_j/n}) \geq \Theta(n^{-\delta/2})$. Lemma 51(c) yields that with probability $1 - o(n^{-1})$, $\mathcal{L}_t \leq \mathcal{L}_0 + t\Theta(n^{-\delta/2})$ for each $t \geq n^\delta \log^{1.5} n$, which implies (since w.h.p. $L_{t(i)+t} = \mathcal{L}_t$ until step τ when \mathcal{L}_t drops to zero) $\tau - t(i) \leq \Theta(n^{\delta/2})\widehat{L}_i$.

Since the total degree in \mathcal{Q}_t decreases by at most r in every iteration of SLOW-STRIP, we have

$$\sum_{j \geq i} \widehat{L}_j \leq r(\tau - t(i)) \leq \Theta(n^{\delta/2})\widehat{L}_i.$$

Taking the union bound over all i and then applying (94) yields part (c).

Part (h): By (93) applied to iteration $i - 1$ and using parts (a,b), we have that with probability at least $1 - n^{-2}$ there is a constant Z_1 such that:

$$\sum_{u \in S_i} (d^-(u))^2 \leq |S_i| + \sum_{j \geq 2} (j^2 - 1)X_j < |S_i| + O(|S_{i-1}|) < Z_1|S_i|. \quad (96)$$

Part (h) then follows by multiplying the failing probability of at most n^{-2} by the at most n choices for i .

□

Acknowledgement We thank two anonymous referees for their careful reading and several corrections.

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Appendix

Proof of Lemma 8. First we bound $y := \mu(c) - \mu_{r,k}$. Recall that $\mu(c)$ is the larger root of $h(\mu) = c$ and $\mu_{r,k}$ is the unique root of $h(\mu) = c_{r,k}$. As $h(x)$ is convex over $x \in (0, +\infty)$ and has derivative 0 at $x = \mu_{r,k}$. The Taylor expansion at $\mu = \mu_{r,k}$ gives

$$c = h(\mu_{r,k}) + \frac{h''(\mu_{r,k})}{2}y^2 + O(y^3) = c_{r,k} + \frac{h''(\mu_{r,k})}{2}y^2(1 + O(y)).$$

Thus, $y = \sqrt{2/h''(\mu_{r,k})}n^{-\delta/2} + O(n^{-\delta}) = K_1n^{-\delta/2} + O(n^{-\delta})$, by letting $K_1 = \sqrt{2/h''(\mu_{r,k})}$. Next, we bound

$$f_k(\mu(c)) - \alpha_{r,k} = f_k(\mu(c)) - f_k(\mu_{r,k}) = f'_k(\mu_{r,k})y + O(y^2).$$

Recall that

$$f_k(x) = e^{-x} \sum_{i \geq k} x^i / i!.$$

Thus, $f'_k(x) = e^{-x}x^{k-1}/(k-1)!$. Hence, $f'_k(\mu_{r,k}) > 0$. It follows then that there is a constant $K_2 > 0$, such that

$$f_k(\mu(c)) - \alpha = K_2n^{-\delta/2} + O(n^{-\delta}).$$

Similarly, there is a $K_3 > 0$ such that

$$\frac{1}{r}\mu(c)f_{k-1}(\mu(c)) - \beta = K_3n^{-\delta/2} + O(n^{-\delta}). \quad \square$$

Proof of Lemma 25. By the definitions of $\mu_{r,k}$ and $h(\mu)$ from (4), $h'(\mu_{r,k}) = 0$. Since

$$h'(x) = \frac{f_{k-1}(x)^{r-1} - x(r-1)f_{k-1}(x)^{r-2}f'_{k-1}(x)}{f_{k-1}(x)^{2(r-1)}},$$

and $f'_k(x) = f_{k-1}(x) - f_k(x)$ for all $k \geq 1$, we have

$$f_{k-1}(\mu_{r,k}) = \mu_{r,k}(r-1)(f_{k-2}(\mu_{r,k}) - f_{k-1}(\mu_{r,k})),$$

i.e.,

$$\frac{f_{k-2}(\mu_{r,k})}{f_{k-1}(\mu_{r,k})} = \frac{1 + \mu_{r,k}(r-1)}{\mu_{r,k}(r-1)} = 1 + \frac{1}{\mu_{r,k}(r-1)}.$$

On the other hand,

$$f_{k-2}(\mu_{r,k}) = f_{k-1}(\mu_{r,k}) + e^{-\mu_{r,k}} \frac{\mu_{r,k}^{k-2}}{(k-2)!},$$

it follows immediately that

$$e^{-\mu_{r,k}} \frac{\mu_{r,k}^{k-2}}{(k-2)!f_{k-1}(\mu_{r,k})} = \frac{1}{\mu_{r,k}(r-1)}. \quad (97)$$

By the definition of $\bar{\rho}_{r,k}(k)$, α and β , the assertion follows thereby. \square

Proof of Lemma 14. The Poisson Cloning Model for a random r -uniform configuration on n vertices, $H_{PC}(n, p; r)$ is defined in [24] as follows: For each vertex v , select $d(v)$ to be a Poisson variable with mean $p\binom{n-1}{r-1}$ where these n Poisson variables are independent. Create $d(v)$ copies (i.e. clones) of each vertex v . If $D = \sum d(v)$ is a multiple of r then take a uniformly random partition of the vertex-copies into parts of size r , as in the AP-model. If D is not a multiple of r then choose one of the parts to have size $D \bmod r$, and so the resulting hypergraph will have one edge of size less than r . Note that the probability that r divides D is bounded from below by a positive constant. This implies that if a property Q holds w.h.p. for $H_{PC}(n, p; r)$ then it holds w.h.p. when conditioning on the event that r divides D .

[24] notes that this model is equivalent to: Choose D to be a Poisson variable with mean $np\binom{n-1}{r-1}$, create D vertex-copies, and assign each vertex-copy to a uniformly chosen vertex. Then take a uniformly random partition as described above. Thus, the AP-model $AP_r(n, m)$ is exactly $H_{PC}(n, p; r)$ conditioned on $D = rm$.

We say that a property Q of configurations is monotone decreasing (resp. increasing) if: Consider any configuration for which Q holds. Create a new configuration by removing any part and its vertex-copies (resp. adding a part containing r new vertex-copies and assigning those vertex-copies to any bin). Then Q holds for the new configuration. We will argue that if a monotone property Q holds w.h.p. in $H_{PC}(n, p; r)$ then Q holds w.h.p. in $AP_r(n, m)$ if $m = \frac{np}{r}\binom{n-1}{r-1}$.

Note that $AP_r(n, m - 1)$ is equivalent to: choose a configuration from $AP_r(n, m)$, select a uniformly random part, and remove it along with its vertex-copies. It follows that if Q is monotone increasing (resp. decreasing) then the probability that Q holds in $AP_r(n, m)$ is at least (resp. at most) the probability that Q holds in $AP_r(n, m - 1)$. Suppose Q is monotone increasing and that Q holds w.h.p. in $H_{PC}(n, p; r)$. Note that the probability that $D = Po(p\binom{n-1}{r-1})$ is a multiple of r that is at least $p\binom{n-1}{r-1}$ is bounded from below by a positive constant. This implies that Q holds w.h.p. for $H_{PC}(n, p; r)$ when conditioning that D is a multiple of r that is at least $p\binom{n-1}{r-1}$. Since $AP_r(n, m)$ is $H_{PC}(n, p; r)$ conditioned on $D = rm = p\binom{n-1}{r-1}$ and since the probability that Q holds in $AP_r(n, m)$ is monotone increasing with m it follows that Q holds w.h.p. for $H_{PC}(n, p; r)$ when conditioning that $D = p\binom{n-1}{r-1}$, i.e. for $AP_r(n, m)$. The argument for Q monotone decreasing is almost identical.

[24] proves that Lemma 7 holds for $H_{PC}(n, p; r)$ where $\frac{np}{r}\binom{n-1}{r-1} = m$. Note that each part of Lemma 7 is the intersection of two monotone properties, one increasing and the other decreasing. For example, the first part says that there is a positive function $g(n) = O(n^{3/4})$ such that (i) the k -core has at least $\alpha(c)n - g(n)$ vertices and (ii) the k -core has at most $\alpha(c)n + g(n)$ vertices. Since all of these properties hold w.h.p. for $H_{PC}(n, p; r)$ they hold w.h.p. for $AP_r(n, m)$. Corollary 9 follows from Lemma 7 and Lemma 8, so it also holds w.h.p. for $AP_r(n, m)$. □

Proof of Lemma 27. Set $f(x) = e^x f_k(x)$, $h(x) = xe^x f_{k-1}(x)$. So we wish to show that $f'(x)h(x) < f(x)h'(x)$.

$$f(x) = \sum_{i \geq k} \frac{x^i}{i!}, \quad f'(x) = \sum_{i \geq k} \frac{x^{i-1}}{(i-1)!}.$$

$$h(x) = \sum_{i \geq k} \frac{x^i}{(i-1)!}, \quad h'(x) = \sum_{i \geq k} \frac{ix^{i-1}}{(i-1)!}.$$

$$f(x)h'(x) = \sum_{i,j \geq k} \frac{ix^{i+j-1}}{(i-1)!j!}, \quad f'(x)h(x) = \sum_{i,j \geq k} \frac{x^{i+j-1}}{(i-1)!(j-1)!}.$$

When $i = j$, the contribution to each sum is the same: $\frac{x^{2i-1}}{(i-1)!(i-1)!}$. When $i \neq j$, consider the contribution of (i, j) plus the contribution of (j, i) . The sum of these contributions to $f(x)h'(x)$ and $f'(x)h(x)$ is

$$x^{i+j-1} \left(\frac{i}{(i-1)!j!} + \frac{j}{i!(j-1)!} \right), \quad x^{i+j-1} \frac{2}{(i-1)!(j-1)!}.$$

The contribution to $f(x)h'(x)$ is larger since $\frac{i}{j} + \frac{j}{i} > 2$. □

Proof of Lemma 31. Let $h(\lambda) = e^{-\lambda} \lambda^{k-1} / f_{k-1}(\lambda)(k-1)!$. Recall $g_k(x)$ above Lemma 27 and recall the definition of $\psi(x)$ in (19). Then,

$$\psi(x) = h(g_k^{-1}(x)),$$

where g_k^{-1} is the inverse of g_k . It is easy to see and we have mentioned before that $\lim_{x \rightarrow 0} g_k(x) = k$. So for every $x > k$, $g_k^{-1}(x)$ exists, and by Lemma 27 and the chain rule, the derivative of $g_k^{-1}(x)$ is positive. Thus, in order to show that $\psi'(x) < 0$, it is sufficient to show that for every $\lambda > 0$, $h'(\lambda) < 0$.

Showing that $h'(\lambda) < 0$ is equivalent to showing that for every $\lambda > 0$,

$$(k-1) \sum_{j \geq k-1} \frac{\lambda^j}{j!} - \lambda \sum_{j \geq k-2} \frac{\lambda^j}{j!} = \frac{\lambda^{k-2}}{(k-2)!} \left(-\lambda + (k-1-\lambda) \sum_{j \geq k-1} \frac{\lambda^{j-k+2}}{[j]_{j-k+2}} \right)$$

is negative. It is trivially true if $\lambda \geq k-1$. Now assume that $\lambda < k-1$. Since for every $j \geq k-1$, we have

$$\frac{\lambda^{j-k+2}}{[j]_{j-k+2}} \leq \left(\frac{\lambda}{k-1} \right)^{j-k+2},$$

where the inequality is strict except for $j = k-1$. Thus,

$$\begin{aligned} -\lambda + (k-1-\lambda) \sum_{j \geq k-1} \frac{\lambda^{j-k+2}}{[j]_{j-k+2}} &< -\lambda + (k-1-\lambda) \sum_{j \geq 0} \left(\frac{\lambda}{k-1} \right)^{j+1} \\ &= -\lambda + (k-1-\lambda) \cdot \frac{\lambda}{k-1} \cdot \frac{1}{1 - \frac{\lambda}{k-1}} = 0. \end{aligned}$$

This completes the proof that $h'(\lambda) < 0$. □

We need the following technical lemma before proving Lemma 24.

Lemma 55. For all $k, r \geq 2$ with $(r, k) \neq (2, 2)$, $h(x) < 1/(r-1)$ for all $x \geq x^*$ where

$$h(x) = \frac{e^{-x}x^{k-1}}{f_{k-1}(x)(k-2)!}, \quad \text{and } x^* = r(k-1) - \frac{r}{r-1}.$$

Proof. It is easy to see that $h(x)$ is a decreasing function on $x > 0$ (dividing $e^{-x}x^{k-1}$ from both the numerator and the denominator and then the numerator is constant whereas the denominator is an increasing function of x). Hence if we can prove $h(x) < 1/(r-1)$ for some $x \leq x^*$ then we are done.

We first prove a well-known inequality with respect to $f_k(\mu)$.

Claim 56. For any integer $k < \lfloor \mu \rfloor$, $f_k(\mu) > 1/2$.

Proof. Let M be the maximum integer such that $M \leq \mu - 1$. Then we must have $k - 1 \leq M - 1$. By the definition of $f_k(\mu)$, we only need to prove that

$$\sum_{i=0}^{k-1} e^{-\mu} \frac{\mu^i}{i!} \leq \sum_{i=0}^{M-1} e^{-\mu} \frac{\mu^i}{i!} < 1/2.$$

Let $p(i) = e^{-\mu} \mu^i / i!$; it suffices then to show that for every $1 \leq i \leq M$, $p(M-i) \leq p(M+i)$. This follows easily from

$$\frac{p(M-i)}{p(M+i)} = \prod_{j=-i+1}^i \frac{M+j}{\mu} \leq \prod_{j=-i+1}^i \left(1 + \frac{-1+j}{\mu}\right) \leq \exp\left(\sum_{j=-i}^{i-1} \frac{j}{\mu}\right) = \exp\left(-\frac{i}{\mu}\right) < 1. \quad \square$$

Now we continue to prove Lemma 55. We split our discussion into two cases. In both, we will reduce the lemma to checking a finite number of pairs (k, r) , which is straightforward.

Case 1: $k = 2$, $r \geq 3$. Now $h(x) = e^{-x}x/(1 - e^{-x})$ and $x^* = r - r/(r-1)$. By computing the derivative of $(r-1) \cdot h(x^*(r))$ with respect to r , it is easy to see that $h(x)$ is a decreasing function on $r \geq 3$. Hence it suffices to verify that $h(x^*(3)) < 1/2$ which can be easily done.

Case 2: $k \geq 3$. We may easily verify the lemma for $(k, r) = (3, 2)$. So we assume that $(k, r) \neq (3, 2)$. Let $x_1 = r(k-2)$. Clearly $x_1 \leq x^*$ and $\lfloor x_1 \rfloor > k-1$. By Claim 56, we have $f_{k-1}(x_1) > 1/2$. Since $h(x)$ is decreasing on $x > 0$, it suffices to prove that $h(x_1) < 1/(r-1)$ and thus it suffices to prove that $e^{-x_1}x_1^{k-1}/(k-2)! < \frac{1}{2(r-1)}$. Define

$$\phi(x, r) = \frac{e^{-rx}(rx)^{x+1}}{x!}.$$

Then, $e^{-x_1}x_1^{k-1}/(k-2)! = \phi(k-2, r)$ and so it suffices to prove

$$(r-1)\phi(k-2, r) < \frac{1}{2}. \quad (98)$$

By computing the derivative of $(r-1)\phi(x, r)$ with respect to r , we see that

$$\text{for } (r-1)^2 \geq r^2/(x+1): \quad (r-1) \cdot \phi(x, r) \text{ is non-increasing on } r. \quad (99)$$

It is easy to see that

$$\frac{\phi(x+1, r)}{\phi(x, r)} = \frac{e^{-r}(rx+r)^{x+2}}{(x+1)(rx)^{x+1}} = re^{-r} \left(1 + \frac{1}{x}\right)^{x+1} \leq re^{-r+1+1/x} \leq re^{-r+2},$$

which is strictly less than one for any $x \geq 4$ and $r \geq 2$. So

$$\text{for } x \geq 4, r \geq 2: \quad (r-1) \cdot \phi(x, r) \text{ is decreasing on } x. \quad (100)$$

$k \geq 6$: We can easily verify that $\phi(4, 2) < 1/2$, and so by (99), this establishes (98) for $k = 6, r \geq 2$. Then (100) establishes (98) for all $k \geq 6, r \geq 2$.

$k = 5$: We can easily verify that $2\phi(3, 3) < 1/2$, and so by (99), this establishes (98) for $k = 5, r \geq 2$.

$k = 4$: We can easily verify that $3\phi(2, 4) < 1/2$, and so by (99), this establishes (98) for $k = 4, r \geq 3$. For $r = 2$, we verify the lemma directly from the definitions of $h(x)$ and x^* .

$k = 3$: We can easily verify that $5\phi(1, 6) < 1/2$, and so by (99), this establishes (98) for $k = 3, r \geq 4$. For $r \in \{2, 3\}$, we verify the lemma directly from the definitions of $h(x)$ and x^* . □

Proof of Lemma 24. We first verify that $\zeta > k$. Recall that $g_k(x) = x f_{k-1}(x)/f_k(x)$. By definition of ζ, α and β in (12) and (5),

$$\zeta = g_k(\mu_{r,k}).$$

It is easy to see that $\lim_{x \rightarrow 0} g_k(x) = k$. Lemma 27 says that $g_k(x)$ is strictly increasing on $x > 0$. It is easy to see that for any $(r, k) \neq (2, 2)$, $h_{r,k}(\mu)$ in (3) tends to infinity both when $\mu \rightarrow 0$ and when $\mu \rightarrow \infty$. This implies that $\mu_{r,k}$ is a positive real number. Thus, $g_k(\mu_{r,k}) > k$.

Next we prove that $\zeta < r(k-1)$. By (97),

$$e^{-\mu_{r,k}} \frac{\mu_{r,k}^{k-1}}{(k-1)! f_{k-1}(\mu_{r,k})} = \frac{1}{(k-1)(r-1)}. \quad (101)$$

As we said above, $\zeta = g_k(\mu_{r,k})$ and so by (11), $\zeta = \mu_{r,k} f_{k-1}(\mu_{r,k})/f_k(\mu_{r,k})$. Hence, $\zeta < r(k-1)$ is equivalent to

$$\frac{f_k(\mu_{r,k})}{\mu_{r,k} f_{k-1}(\mu_{r,k})} > \frac{1}{r(k-1)}.$$

Note that $f_k(\mu_{r,k}) = f_{k-1}(\mu_{r,k}) - e^{-\mu_{r,k}} \mu_{r,k}^{k-1}/(k-1)!$; combined with (101) the left hand side above is equal to

$$\frac{1}{\mu_{r,k}} \left(1 - e^{-\mu_{r,k}} \frac{\mu_{r,k}^{k-1}}{(k-1)! f_{k-1}(\mu_{r,k})} \right) = \frac{1}{\mu_{r,k}} \left(1 - \frac{1}{(k-1)(r-1)} \right).$$

Hence $\zeta < r(k-1)$ is equivalent to

$$\frac{1}{\mu_{r,k}} \left(1 - \frac{1}{(k-1)(r-1)} \right) > \frac{1}{r(k-1)} \quad \text{i.e. } \mu_{r,k} < \mu^* := r(k-1) - \frac{r}{r-1}. \quad (102)$$

We consider the function $h(x)$ from Lemma 55; i.e.

$$h(x) = \frac{e^{-x}x^{k-1}/(k-2)!}{f_{k-1}(x)}.$$

By (101) (and multiplying $k-1$ on both sides), $h(\mu_{r,k}) = 1/(r-1)$. So by Lemma 55, $\mu_{r,k} < \mu^*$, thus establishing (102) and hence the lemma. \square