

UPPER TRIANGULAR MATRIX WALK: CUTOFF FOR FINITELY MANY COLUMNS

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ABSTRACT. We consider random walk on the group of uni-upper triangular matrices with entries in \mathbb{F}_2 which forms an important example of a nilpotent group. Peres and Sly [12] proved tight bounds on the mixing time of this walk up to constants. It is well known that the single column projection of this chain is the one dimensional East process. In this article, we complement the Peres-Sly result by proving a cutoff result for the mixing of finitely many columns in the upper triangular matrix walk at the same location as the East process of the same dimension. Moreover, we also show that the spectral gaps of the matrix walk and the East process are equal. The proof of the cutoff result is based on a recursive argument which uses a local version of a dual process appearing in [12], various combinatorial consequences of mixing and concentration results for the movement of the front in the one dimensional East process.

1. INTRODUCTION

Random walks on the group $\mathcal{M}_n(q)$ of $n \times n$ -upper triangular matrices with ones along the diagonal and entries from the finite field \mathbb{F}_q , q a prime, have received quite a lot of attention. The random walk, sometimes called the *upper triangular matrix walk*, in the case $q = 2$ is the Markov chain whose generic step consists in choosing uniformly a row among the first $(n - 1)$ rows and adding to it the next row mod 2. It is easy to check that this chain is reversible w.r.t. the uniform measure on $\mathcal{M}_n(2)$. A natural variant, called the *lazy upper triangular matrix walk*, entails to perform the above addition with probability $1/2$. This is equivalent to adding the to-be-added row only after multiplying by a uniform Bernoulli variable. We refer the interested reader to [12] for a nice review of the literature related to this walk and other related variants.

In this article, we consider the continuous time version of the upper triangular matrix walk where each row at rate one is updated by adding the row below it with probability $1/2$. Sharp bounds on the spectral gap were proven by Stong [13] implying, in particular, that the spectral gap $\lambda_2(n)$ is positive uniformly in n . Using an elegant argument, Peres and Sly [12] proved that the total variation mixing time $t_{\text{mix}}(n) = \Theta(n)$. From the above results it follows that $\lim_{n \rightarrow \infty} \lambda_2(n) \times t_{\text{mix}}(n) = +\infty$, a known necessary condition for the occurrence of the so called *mixing time cutoff* [1], i.e. a sharp transition in the total variation distance from equilibrium which drops from being close to one to being close to zero in a very small time window compared to the mixing time scale.

In [11], Y. Peres conjectured that, for many natural classes of reversible Markov chains, the above condition (sometimes referred to as the *product condition*) is also sufficient for the occurrence of cutoff, despite of the fact that, in full generality, this is known to be false (cf. [9, Chapter 18]). Thus it is a natural and interesting problem to decide whether the upper triangular matrix walk exhibits cutoff or not.

It has been observed before and was crucially used in [12], in the proof of their mixing time result, that the marginal process on a given column coincides with the East process [2, 4, 7] at density $1/2$, a well known constrained interacting particle system.

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The East chain is known to exhibit cutoff (cf. [8]), a result which, combined with the previous observation, suggests that the upper triangular matrix walk in continuous time might do the same.

In this paper we extend and complement the Peres-Sly result by proving that (i) the spectral gap of the upper triangular matrix walk is equal to the spectral gap of the East process on $n - 1$ vertices; (ii) the marginal chain on *finitely* many columns exhibits mixing time cutoff at the mixing time of the column with the largest index, among the chosen ones.

Note that, even though at stationarity the elements of the matrix are independent random bits, the dynamics of the walk makes the column evolutions highly correlated so that the joint mixing of a group of columns is by no means a straightforward consequence of the mixing of only one column (the East process). We also remark that, perhaps surprisingly, certain numerical experiments suggest that the mixing time of the full chain is strictly larger (at the linear scale, n) than the mixing time of one column (see Section 2.4). We conclude by mentioning that recent progress on the temporal mixing of other finite dimensional statistics in a similar setting was also made in [6].

2. SETUP, MOTIVATIONS AND MAIN RESULTS

2.1. Setting and notation. Throughout this article for any $n \in \mathbb{N}$, we will write \mathcal{X}_n for the vector space \mathbb{F}_2^n , with the usual addition operation mod 2 denoted by “+” in the sequel, and standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$. Elements of \mathcal{X}_n will usually be denoted by \mathbf{Y}, \mathbf{Z} etc with coordinates $\{\mathbf{Y}(x)\}_{x=1}^n$ and they will always be thought as column vectors. Sometimes a vector \mathbf{X} will depend on a time parameter t and randomness variable ω . In that case we will write $\mathbf{X}(t; \omega; x)$ for its x -coordinate. Row vectors will always be denoted by $\mathbf{Y}^\top, \mathbf{Z}^\top$ etc and their vector space will be denoted by \mathcal{X}_n^\top . Multiplication between \mathbf{Z}^\top and \mathbf{Y} will be denoted by $\mathbf{Z}^\top \cdot \mathbf{Y}$.

We will write $[n]$ for the set $\{1, 2, \dots, n\}$ and we will use the letter c to denote a universal numerical constant whose value might change from line to line inside the same proof. Finally for any random variable ξ with distribution \mathcal{P} and any other law ν we will sometimes write $d_{TV}(\xi, \nu)$ for the total variation distance between \mathcal{P} and ν .

2.1.1. The Markov chain. Our state space is the set of all upper triangular matrices with entries in \mathbb{F}_2 :

$$\mathcal{M}_n = \{M \in \mathbb{F}_2^{n \times n} : M(i, j) = 0, \text{ for } i > j, M(i, i) = 1\},$$

i.e.

$$M = \begin{bmatrix} 1 & M(1, 1) & M(1, 2) & \dots \\ 0 & 1 & \dots & \dots \\ \vdots & \vdots & \dots & M(n-1, n) \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

When $n = 3$, this is well known as the Heisenberg group. Let $E^{i,j}$ be the matrix which is one at the position (i, j) and zero everywhere else. On \mathcal{M}_n we consider the continuous time Markov chain (random walk) $\{M(t)\}_{t \geq 0} = \{M^{(n)}(t)\}_{t \geq 0}$ (we will drop the dependence on n in the notation whenever there is no scope of confusion) whose generator is given by

$$(\mathcal{L}f)(M) = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{a=0}^1 [f((\mathbb{I} + aE^{i,i+1})M) - f(M)]. \quad (2.1)$$

Notice that for $\mathbf{Y} \in \mathcal{X}_n$ and $x \in [n]$:

$$\begin{aligned} ((\mathbb{I} + E^{i,i+1})\mathbf{Y})(x) &= \mathbf{Y}(x) + \delta_{i,x}\mathbf{Y}(i+1), \\ (\mathbf{Y}^\top(\mathbb{I} + E^{i,i+1}))(x) &= \mathbf{Y}(x) + \delta_{x,i+1}\mathbf{Y}(i). \end{aligned}$$

Thus the Markov chain can be informally described as follows: every row but the last one updates itself at rate one by first sampling a fair Bernoulli variable ξ and then by adding (mod 2) to itself the next row multiplied by ξ . Clearly if $\xi = 0$ nothing happens, while if $\xi = 1$ all the entries of the updating row, with the entry below them equal to 1, change values ($1 \rightarrow 0$ or $0 \rightarrow 1$). For some purposes it will be convenient to write $M(t) \equiv [\mathbf{M}_1(t), \mathbf{M}_2(t), \dots, \mathbf{M}_n(t)]$ where $\mathbf{M}_i(t) \in \mathcal{X}_n$ denotes the i^{th} column at time t . In the sequel we will sometimes refer to the Markov chain as the *matrix walk* (MW).

Remark 2.1.

- (1) It is easy to check that the Markov chain is ergodic and reversible w.r.t. to the uniform measure $\pi = \pi^{(n)}$, on \mathcal{M}_n . In particular at equilibrium all the entries of M above the diagonal are i.i.d Bernoulli(1/2) random variables.
- (2) The marginal process of every column is also a Markov chain and on the i^{th} column it coincides with the East process ¹ on $[i-1]$ at density 1/2, a well known kinetically constrained interacting particle system (cf. [4] and references therein).

2.2. Mixing time and open problems. It is natural to consider the ε -mixing time of the process $\{M(t)\}_{t \geq 0}$

$$T_{\text{mix}}(n, \varepsilon) = \inf \left\{ t : \max_{M \in \mathcal{M}_n} d_{TV}(M(t), \pi) \leq \varepsilon \right\}, \quad \varepsilon \in (0, 1),$$

where $M(0) = M$. If $\varepsilon = 1/4$ we will simply refer to $T_{\text{mix}}(n) \equiv T_{\text{mix}}(n, 1/4)$ as the mixing time. Using Remark 2.1 we immediately get that

$$T_{\text{mix}}(n, \varepsilon) \geq \max_{j=2, \dots, n} T_{\text{mix}}^{\text{East}}(j, \varepsilon), \quad (2.2)$$

where $T_{\text{mix}}^{\text{East}}(j, \varepsilon)$ is the ε -mixing time of the East process at density 1/2 on $[j]$. For the latter the following sharp result was recently established by E. Lubetzky and the authors:

Theorem 2.2 ([8]). *For any $p \in (0, 1)$ there exists a positive constant $v = v(p)$ such that the East process on $[n]$ at density p exhibits cutoff at $v^{-1}n$ with an $O(\sqrt{n})$ -window. More precisely, for any fixed $0 < \varepsilon < 1$ and large enough n ,*

$$T_{\text{mix}}^{\text{East}}(n, \varepsilon) = v^{-1}n + O(\Phi^{-1}(1 - \varepsilon)\sqrt{n}),$$

where Φ is the c.d.f. of $\mathcal{N}(0, 1)$.

By (2.2) it follows that $T_{\text{mix}}(n, \varepsilon) \geq v_*^{-1}n + O(\Phi^{-1}(1 - \varepsilon)\sqrt{n})$ where $v_* \equiv v(1/2)$. A comparable *upper bound*, a much harder task, was proved using a very elegant argument by Y. Peres and A. Sly:

Theorem 2.3 ([12]). $T_{\text{mix}}(n) = \Theta(n)$.

Two natural questions arise at this point:

- (1) Is $\limsup_n T_{\text{mix}}(n)/n = v_*^{-1}$?

¹In the standard definition of the East process on $[n]$ at density $p \in (0, 1)$, for every $x \in [n]$ with rate one the spin $\sigma(t; x) \in \{0, 1\}$, is replaced by a fresh Bernoulli(p) variable iff $x = 1$ or $x \in [2, n]$ and the constraint $\sigma(t; x-1) = 0$ is satisfied. It is easy to check that for $p = 1/2$ and after the change of variables $\eta(x) = 1 - \sigma(x)$ the generator of the East process coincides with that of the $\mathbf{M}^{(n)}(t)$ process.

- (2) Does $T_{\text{mix}}(n, \varepsilon)$ exhibit the cutoff phenomenon similar to the single column process (i.e. the East process) ?

The main purpose of this article is to prove a positive answer to the above questions for certain finite dimensional projections (e.g. for the marginal process of the last k columns, where k is either bounded in n or it grows very slowly). We also provide certain numerical findings which suggest that perhaps $\limsup_n T_{\text{mix}}(n)/n > v_*^{-1}$.

Before stating our results more formally we recall that in order to have the cutoff phenomenon the *relaxation time* $T_{\text{rel}}(n)$ of the chain, defined as the inverse spectral gap of the Markov generator, must satisfy $T_{\text{rel}}(n) = o(T_{\text{mix}}(n))$ as $n \rightarrow \infty$ (cf. [9]). That MW satisfies this condition follows from work of Stong [13] as mentioned previously. However we prove a stronger comparison result showing that in fact $T_{\text{rel}}(n) = T_{\text{rel}}^{\text{East}}(n-1)$, where $T_{\text{rel}}^{\text{East}}(n-1)$ is the relaxation time of the East process on $[n-1]$ at density $1/2$. The latter was shown to be bounded in n (cf. [2] or [7]).

2.3. Main results. For our arguments it will be convenient to generalize our initial setting. For $1 \leq i_1 < i_2 < \dots < i_k \leq n$ let

$$\mathcal{M}_{[i_1, \dots, i_k]}^{(n)} = \{A \in \mathbb{F}_2^{n \times k} : A(i, j) = 0, \text{ for } i > i_j, A(i_j, i_j) = 1\},$$

so that $\mathcal{M}_n = \mathcal{M}_{[1, 2, \dots, n]}^{(n)}$. As before we will simply write $\mathcal{M}_{[i_1, \dots, i_k]}$ for $\mathcal{M}_{[i_1, \dots, i_k]}^{(n)}$ if no confusion arises. Observe that for any $M \in \mathcal{M}_n$ the sub-matrix obtained from M by retaining only the columns i_1, i_2, \dots, i_k belongs to $\mathcal{M}_{[i_1, \dots, i_k]}$.

Using the above observation, it is not hard to see that the definition of the random walk $M(t)$ on \mathcal{M}_n naturally extends to a continuous time ergodic reversible Markov chain on $\mathcal{M}_{[i_1, i_2, \dots, i_k]}$, with reversible measure the uniform measure $\pi_{[i_1, i_2, \dots, i_k]}$ on $\mathcal{M}_{[i_1, i_2, \dots, i_k]}$. We will denote by $T_{\text{mix}}(n; [i_1, i_2, \dots, i_k], \varepsilon)$ and $T_{\text{rel}}(n; [i_1, i_2, \dots, i_k])$ its mixing time and relaxation time respectively. Our two main results read as follows.

Theorem 1. For any $1 \leq i_1 < i_2 < \dots < i_k \leq n$

$$T_{\text{rel}}(n; [i_1, i_2, \dots, i_k]) = T_{\text{rel}}^{\text{East}}(i_k - 1).$$

In particular the relaxation time of the process $\{M(t)\}_{t \geq 0}$ on \mathcal{M}_n coincides with that of the East process on $[n]$ at density $1/2$.

Theorem 2. There exists $C > 0$ such that, given k and $\varepsilon \in (0, 1)$, for all large enough n and all $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that $i_k = n$,

$$T_{\text{mix}}(n, \varepsilon; [i_1, i_2, \dots, i_k]) \leq n/v_* + Ck2^k \sqrt{n} \log^2(n).$$

The ε dependence in the statement of the theorem is hidden in the choice of large enough n . A comparable lower bound $T_{\text{mix}}(n, \varepsilon) \geq n/v_* + O(\Phi^{-1}(1-\varepsilon)\sqrt{n})$ was stated right after Theorem 2.2.

Remark 2.4. Note that the dependence on k is exponential and hence one can only hope to push this method of proof to prove sharp mixing, up to $k = \log(n)/2$ (ignoring smaller order terms) dimensional projections. The theorem holds also for $i_k = (1 - o(1))n$ but then the error term beyond n/v_* will depend on the detailed form of the error term $o(1)$.

Remark 2.5. Our method can be easily extended to cover the case when the entries of the matrices belong to the field \mathbb{F}_q , with q a prime, and the matrix walk is as follows (cf. [12]): at rate one, the $(i+1)^{\text{th}}$ -row is multiplied by a uniformly chosen element of \mathbb{F}_q and added to the i^{th} -row where $i \in [1, 2, \dots, n-1]$. In this setting, for any column, the projection obtained by denoting each non zero entry by 1, performs an East process with density $(q-1)/q$ [12, Section 2.1]. In this case we have the analog of the above theorems

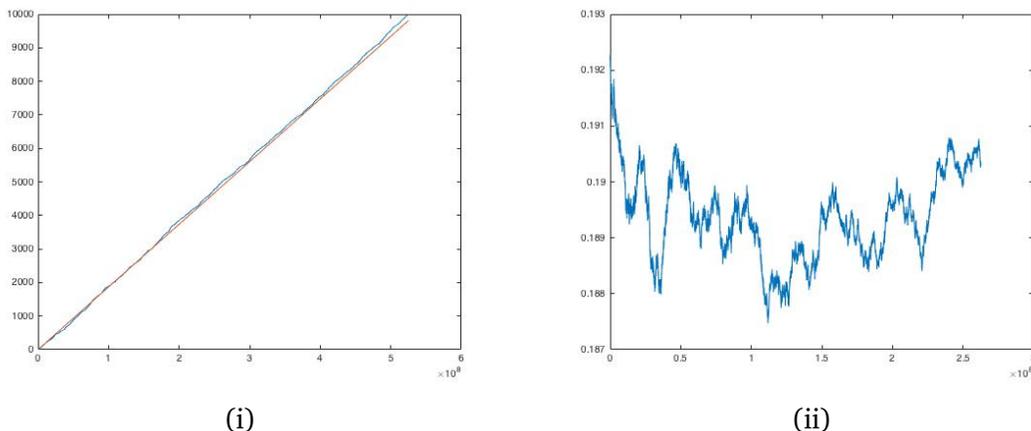


FIGURE 1. (i) The location of the front of an East process (the rightmost one) for $p = \frac{1}{2}$ en-route from 1 to 10^4 and the straight line approximating its trajectory under discrete time dynamics where at every discrete time the east process dynamics is performed at an uniformly chosen site in $[10^4]$. (ii) Average velocity of the front (location/time) along its journey (on a discrete time interval of order 10^8 : note that the discrete time dynamics on an interval on length 10^4 is slower than the continuous time dynamics by a factor 10^4). The above figures are scaled taking this into account.

with the relaxation time $T_{\text{rel}}^{\text{East}}(i_k - 1)$ and the velocity v^* replaced by the corresponding quantities for the new East process.

2.4. Numerical Results: In [8] it was proved that the front (the rightmost one in the East process, (see (4.9)) for formal definition) behaves like a random walk with velocity v_* and has gaussian concentration. The figure above simulates the front process.

We ran an experiment on Matlab where the MW is run starting from the 1000×1000 identity matrix. The statistic we keep track of is the rank of the rectangular block comprised of rows $[1, \dots, 333]$ and columns $[747, \dots, 1000]$. It is well known that a rectangular block of size $m \times n$ with $m < n$ and independent $\text{Ber}(1/2)$ entries has rank m with probability approximately $1 - 2^{n-m}$. Thus our block only has a chance of around 2^{-20} of having rank less than 333 under equilibrium. The above figure shows the velocity of the front to be in the window $[\cdot 185, \cdot 195]$ (which implies that the mixing time in discrete time steps for the last column of the matrix is at most $(1/\cdot 18)10^6$ taking into account fluctuations). We ran the MW for $(1/\cdot 15)10^6$ discrete time steps and found the rank to be less than 200 for five independent rounds. The numbers are chosen to ensure that $(1/\cdot 15)10^6$ is indeed larger than $(1/\cdot 18)10^6$ taking into account fluctuations.

3. MAIN IDEAS AND ORGANIZATION OF THE ARTICLE

In this section we briefly describe the key ideas behind the proofs of the main results and how the article is organized. The proof of Theorem 2 is based on a recursive argument. The case $k = 1$ (the East process) is the content of [8]. Assuming the theorem for k we sketch how to prove it for $k + 1$. It suffices to consider the matrix $A = \mathcal{M}_{[n-k, n-k+1, \dots, n]}$, (see 2.3). Note that the column evolutions are highly correlated and at a high level the approach is to find combinatorial situations which help us decouple

the columns: for eg: if at any particular time t when the i^{th} row of A is $\underbrace{[0, \dots, 0, 1]}_{k \text{ times}}$, i.e.

$$(\mathbf{M}_{n-k}(t; i), \dots, \mathbf{M}_n(t; i)) = (0, \dots, 0, 1),$$

the $(i-1)^{\text{th}}$ row is updated by adding the i^{th} row times a Bernoulli(1/2) variable $\xi_i(t)$, then forever in the future the random bit $\xi_i(t)$ only affects the last column and is independent of the first k columns. We then crucially use the underlying linearity in the dynamics (see discussions after Remark 4.2) to show that under the assumption that the following marginals on collection of k columns,

$$[\mathbf{M}_{n-k}, \dots, \mathbf{M}_{n-1}], [\mathbf{M}_{n-k}, \dots, \mathbf{M}_{n-2}\mathbf{M}_n], [\mathbf{M}_{n-k}, \dots, \mathbf{M}_{n-2}, \mathbf{M}_{n-1} + \mathbf{M}_n]$$

where the addition is mod 2, are well mixed, such patterns occur frequently at various rows which are not too far apart. Thus, at all the clock rings at such rows at such times, the last column receives random bits which are independent from the first k column evolutions (see discussion after (4.5)). In order to use these bits to prove that indeed the last column mixes independently of the first k columns, we rely on a local version of the argument appearing in [12]. The heart of this argument is based on the crucial observation that a natural adjoint process to the upper triangular matrix walk also has the law of an East process. This fact, coupled with concentration results on the movement of the front (the rightmost one of the East process in infinite volume) obtained by refining results appearing in [8], is enough to conclude the proof.

Theorem 1 is proved by determining the asymptotic exponential decay rate of the total variation distance from equilibrium measure and relating it to the so called ‘persistence function’ of the one dimensional East process. We then use a crucial input from [5], which relates the persistence function to spectral gap. This is done in Section 5.

In Section 4 we define rigorously the underlying noise space, the primal and the adjoint processes, and collect various results which are crucially used, in particular results about combinatorial patterns helping mixing are discussed in Section 4.4 and concentration results are discussed in Section 4.3. The observation about the adjoint process relating it to the East process is discussed in Section 4.1.1. The local mixing results and the complete proof of Theorem 2 appears in Section 6.

4. PRELIMINARY RESULTS

In this section we collect several results that will be used for the proof of the main results.

4.1. Graphical construction, primal and adjoint maps. To each $x \in \{1, \dots, n-1\}$ we associate a rate one Poisson process and, independently, a family of independent $\frac{1}{2}$ -Bernoulli random variables $\{\xi_{x,k} : k \in \mathbb{N}\}$. The occurrences of the Poisson process associated to x will be denoted by $\{t_{x,k} : k \in \mathbb{N}\}$ and we assume independence as x varies. That fixes the probability space Ω_n . Notice that almost surely all the occurrences $\{t_{x,k}\}$ as x varies are different.

On Ω_n we construct a Markov chain with state space \mathcal{M}_n according to the following rules. Starting from $M \in \mathcal{M}_n$ and given $\omega \in \Omega_n$, at each time $t = t_{x,k}$, we update the state of the chain $M(t^-)$ to $M(t) = (\mathbb{I} + \xi_{x,k} E^{x,x+1}) M(t^-)$, i.e. we add to the x^{th} -row of $M(t^-)$ the $(x+1)^{\text{th}}$ -row multiplied by the Bernoulli variable $\xi_{x,k}$. It is easy to check that the above rule defines a Markov chain $\{M(t; \omega)\}_{t \geq 0}$ whose generator coincides with that defined in (2.1).

Given $\omega \in \Omega_n$, a row interval $I = [i, i+1, \dots, j] \subset [n]$ and a time interval $\Delta = [s, t]$, $0 < s < t$, we denote by $\omega_I(\Delta)$, the part of ω (i.e. the clock rings and the Bernoulli

variables) restricted to indices in the interval I and clock rings in the time interval Δ . More explicitly

$$\omega_I(\Delta) := \{(t_{x,k}, \xi_{x,k}) : x \in I, t_{x,k} \in \Delta\}. \quad (4.1)$$

If $I = [n]$ we will simply write $\omega(\Delta)$. We will denote by $\mathcal{F}_I(\Delta)$ the σ -algebra generated by the above variables and by $\tau_1 < \tau_2 < \dots < \tau_m$ the successive rings of $\omega_I(\Delta)$. The position and the Bernoulli variable corresponding to the k^{th} -ring τ_k will be denoted x_k and ξ_k respectively. Later on, we will need an extension of this space where the finite set $[n]$ is replaced by the whole lattice \mathbb{Z} . In that case the interval I can be any finite interval in \mathbb{Z} .

Definition 4.1 (Primal and adjoint map). *Given a row and time intervals I, Δ and $\omega \in \Omega_n$, we define the primal map $\Phi_{I,\Delta}(\omega, \cdot) : \mathcal{X}_n \mapsto \mathcal{X}_n$ by*

$$\Phi_{I,\Delta}(\omega, \mathbf{Y}) = (\mathbb{I} + \xi_m E^{x_m, x_m+1}) \dots (\mathbb{I} + \xi_1 E^{x_1, x_1+1}) \mathbf{Y}. \quad (4.2)$$

The adjoint map $\Phi_{I,\Delta}^*(\omega, \cdot) : \mathcal{X}_n^\top \rightarrow \mathcal{X}_n^\top$ is given instead by

$$\Phi_{I,\Delta}^*(\omega, \mathbf{Y}^\top) = \mathbf{Y}^\top (\mathbb{I} + \xi_m E^{x_m, x_m+1}) \dots (\mathbb{I} + \xi_1 E^{x_1, x_1+1}). \quad (4.3)$$

The maps $\Phi_{I,\Delta}(\omega, \cdot), \Phi_{I,\Delta}^*(\omega, \cdot)$ can be naturally extended to act on any matrix $M \in \mathcal{M}_n$ by setting the i^{th} -column of $\Phi_{I,\Delta}(\omega, M)$ equal to $\Phi_{I,\Delta}(\omega, \mathbf{M}_i)$ and the i^{th} -row of $\Phi_{I,\Delta}^*(\omega, M)$ equal to $\Phi_{I,\Delta}^*(\omega, \mathbf{R}_i^\top)$, where $\mathbf{R}_1^\top, \dots, \mathbf{R}_n^\top$ are the row vectors of M .

Remark 4.2. Note that the adjoint map is appropriately named since for any $\mathbf{Z}^\top, \mathbf{Y}$,

$$\mathbf{Z}^\top \Phi_{I,\Delta}(\omega, \mathbf{Y}) = \Phi_{I,\Delta}^*(\omega, \mathbf{Z}^\top) \cdot \mathbf{Y}. \quad (4.4)$$

If $I = [n]$ and $\Delta = [0, t]$ then, for any $M \in \mathcal{M}_n$, $\Phi_{I,\Delta}(\omega, M) = M(t; \omega)$ where $M(t; \omega)$ is the matrix determined by the graphical construction such that $M(0; \omega) = M$. In particular the MW has the remarkable linearity feature, namely the evolution starting from $M + M' \in \mathcal{X}_n$ is the sum (in \mathcal{X}_n) of the evolutions starting from M and M' separately.

The matrix $\phi_{I,\Delta}^*(\omega, M)$ coincides instead with the matrix $\hat{M}(t; \omega)$ constructed as follows. Let $m = m(\omega, t)$ be the number of rings in ω before t and define the adjoint randomness $\omega^* = \omega_t^*$ as the collection of times $t_{x,k}^* := t - t_{x, m+1-k}$, $k \leq m$, and Bernoulli(1/2) variables $\xi_{x,k}^* := \xi_{x, m+1-k}$. Then, starting from $\hat{M}(0; \omega) = M$, at each time $t_{x,k}^*$ the matrix $\hat{M}(t_{x,k}^*; \omega^*)$ is updated by adding $\xi_{x,k}^* \times (x_{m+1-k})^{\text{th}}$ -column to the $(x_{m+1-k} + 1)^{\text{th}}$ -column. For consistency we think that the ring at time $t_{x,k}^*$ is associated to the $(x_{m+1-k} + 1)^{\text{th}}$ -column. Clearly, for any $i \in [n]$ and any $s \leq t$, the i^{th} -row of $\hat{M}(s; \omega^*)$ has the first $i - 1$ entries fixed equal to 0, the i^{th} -one equal to 1 and the other ones evolving as the East process on $[n - i - 1]$ in the time interval $[0, t]$.

The graphical construction together with the linearity of the MW allows the following representation of the marginal process on e.g. the last column, a cornerstone for [12] and also for us (cf. the proof of Proposition 6.1 in Section 6.3).

Fix $i \in [n - 1]$ and a set of columns $\mathcal{C} \subset [n]$. Without loss of generality we assume that the last column of \mathcal{C} is the n^{th} -one. Given a time interval $[t_1, t_2]$ let

$$t_1 < \tau_1 < \tau_2 < \dots < \tau_\nu < t_2 < \tau_{\nu+1} \quad (4.5)$$

be those rings $t_{i,k}$ in the graphical construction such that $t_{i,k} \in [t_1, t_2]$ and the $(i + 1)^{\text{th}}$ -entries of the columns in \mathcal{C} satisfy some a priori condition (e.g. they are all equal to zero except the last one). Let $\xi_j \equiv \xi_{i, \tau_j}$ be the Bernoulli variable associated to τ_j and let $\hat{\mathcal{F}}_t$, $t \geq t_2$, be the σ -algebra generated by all the variables generating $\mathcal{F}_{[1,n]}([0, t])$

(see (4.1)) except the variables $\{\xi_j\}_{j=1}^\nu$. Let also α_j be the indicator function that $M_{i+1,n}(\tau_j) = 1$. Then

$$\mathbf{M}_n(t; \omega) = \mathbf{A}_0(t; \omega) + \sum_{j=1}^{\nu} \alpha_j(\omega) \xi_j(\omega) \mathbf{A}_j(t, \omega), \quad (4.6)$$

where $\mathbf{A}_0(t; \omega) = \mathbf{M}_n(t; \hat{\omega})$ with $\hat{\omega}$ obtained from ω by removing all the rings in (4.5) at the i^{th} -row between time t_1 and t_2 and, for any $j \in [\nu]$ such that $\alpha_j = 1$, $\mathbf{A}_j(t; \omega) = \Phi_{I, \Delta_j}(\omega, \mathbf{e}_i)$ with $I = [1, i-1]$ and $\Delta_j = (\tau_j, t]$. The important feature of $\{(\mathbf{A}_0, \mathbf{A}_j, \alpha_j)\}_{j=1}^\nu$ is that they *do not depend* on the Bernoulli variables $\{\xi_j\}_{j=1}^\nu$.

The last, purely deterministic, result exploiting the interplay between the primal and adjoint processes is as follows (cf. the proof of Proposition 3.2 in [12]).

4.1.1. The adjoint process argument. Given $\omega \in \Omega_n$ and $t > 0$, choose an interval $I = [a, b] \subset [n]$ together with a subset $t_1 < t_2 < \dots < t_k$ of the Poisson times $\{\tau_i\}_{i=1}^{m(t)}$ such that each t_i corresponds to an update of some row in I . Define \mathbf{X}_j to be the restriction to I of the vector $\Phi_{I, [t_j, t]}(\omega, \mathbf{e}_b)$ (i.e. the column formed by the co-ordinates in I). The vectors $\{\mathbf{X}_j\}_{j=1}^k$ span the whole space \mathcal{X}_{b-a+1} iff for all non-zero row vector $\mathbf{Y}^\top \in \mathcal{X}_{b-a+1}^\top$ there exists $j \in [k]$ such that $\mathbf{Y}^\top \cdot \mathbf{X}_j = 1$. Using the adjoint map we can rewrite $\mathbf{Y}^\top \cdot \mathbf{X}_j$ as follows.

Let $\mathbf{Z}_x^\top = \mathbf{Y}_x$ for $x \in I$ and $\mathbf{Z}_x^\top = 0$ otherwise. Then

$$\mathbf{Y}^\top \cdot \mathbf{X}_j = \mathbf{Z}^\top \cdot \Phi_{I, [t_j, t]}(\omega, \mathbf{e}_b) = \Phi_{I, [t_j, t]}^*(\omega, \mathbf{Z}^\top)(b).$$

Using the connection between $\Phi_{I, [t_j, t]}^*(\omega, \mathbf{Z}^\top)$ and the East process on the rows (cf. the discussion after (4.4)), we conclude that the vectors $\{\mathbf{X}_j\}_{j=1}^k$ span the whole space iff for all row vector \mathbf{Z}^\top which is identically equal to zero outside I there exists $j \in [k]$ such that the (graphical representation of) East process on $[n]$, evolving with the ‘‘adjoint’’ randomness ω^* and initial condition \mathbf{Z} , at time $t - t_j$ is equal to 1 at the vertex b .

4.2. Chernoff bounds for continuous time Markov chains. We give here a slight generalization of [10, Theorem 3.4] proving Chernoff bounds for the time average of continuous time Markov chains.

Lemma 4.3. *Let $\{w(t)\}_{t \geq 0}$ be a continuous time finite Markov chain with reversible measure μ . Assume that μ is positive and that the chain has a positive spectral gap γ . Let $A \subset [0, t]$ be of the form $A = \cup_{i=1}^N [s_i, t_i]$, where $t_{i-1} \leq s_i < t_i$, and set $|A| = \sum_i (t_i - s_i)$. Then for any $\delta \in (0, 1)$ and any f with $\|f\|_\infty \leq 1$, $\mu(f) = 0$, $\mu(f^2) \leq b^2$,*

$$\max_{w_0} \mathbb{P}_{w_0} \left[\left| \int_0^t f(w(s)) \mathbf{1}(s \in A) ds \right| \geq \delta |A| \right] \leq \frac{2}{\mu_{\min}} \exp(-\gamma \delta^2 |A| / (1 + 2b)^2), \quad (4.7)$$

where $\mu_{\min} = \min_{w \in \Omega} \mu(w)$ and $\mathbb{P}_{w_0}(\cdot)$ is the law of the chain starting from w_0 .

Proof. For any event E in path space $\max_{w_0} \mathbb{P}_{w_0}(w(\cdot) \in E) \leq \frac{1}{\mu_{\min}} \mathbb{P}_\mu(w(\cdot) \in E)$, so that it suffices to prove that

$$\mathbb{P}_\mu \left(\int_0^t ds f(w(s)) \mathbf{1}(s \in A) \geq \delta |A| \right) \leq \exp(-\gamma \delta^2 |A| / (1 + 2b)^2). \quad (4.8)$$

The exponential Chebyshev inequality gives

$$\mathbb{P}_\mu \left(\int_0^t ds f(\eta(s)) \mathbf{1}(s \in A) \geq \delta |A| \right) \leq e^{-\lambda \delta |A|} \mathbb{E}_\mu \left(e^{\lambda \int_0^t ds f(\eta(s)) \mathbf{1}(s \in A)} \right), \quad \lambda > 0.$$

Let H_λ be the self-adjoint operator on $\mathcal{H} = L^2(\mu)$ given by $H_\lambda = \mathcal{L} + \lambda f$, where \mathcal{L} is the generator of the chain. If $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathcal{H} then

$$\mathbb{E}_\mu \left(e^{\lambda \int_0^t ds f(\eta(s)) \mathbf{1}(s \in A)} \right) = \langle \mathbf{1}, e^{s_1 \mathcal{L}} e^{(t_1 - s_1) H_\lambda} e^{(s_2 - t_1) \mathcal{L}} \dots e^{(t_N - s_N) H_\lambda} e^{(t - t_N) \mathcal{L}} \mathbf{1} \rangle.$$

Since $\|e^{s\mathcal{L}}\| = 1$ we get that

$$\mathbb{E}_\mu \left(e^{\lambda \int_0^t ds f(\eta(s)) \mathbf{1}(s \in A)} \right) \leq \exp(|A| \beta(\lambda)),$$

where $\beta(\lambda)$ denotes the supremum of the spectrum of H_λ .

We now bound from above $\beta(\lambda)$. For any function $h \in \mathcal{H}$ with $\langle h, h \rangle = 1$ we write $h = \alpha \mathbf{1} + g$, so that $\pi(g) = 0$ and $\langle g, g \rangle = 1 - \alpha^2$. Then

$$\begin{aligned} \langle h, H_\lambda h \rangle &= \langle g, \mathcal{L}g \rangle + \lambda \langle g, fg \rangle + 2\alpha \lambda \langle \mathbf{1}, fg \rangle \\ &\leq -\gamma(1 - \alpha^2) + \lambda \sqrt{1 - \alpha^2} + 2b\lambda \alpha \sqrt{1 - \alpha^2} \leq \frac{\lambda^2(1 + 2b)^2}{4\gamma}, \end{aligned}$$

which implies $\beta(\lambda) \leq \frac{\lambda^2(1+2b)^2}{4\gamma}$. In conclusion

$$\mathbb{P}_\mu \left(\int_0^t ds f(w(s)) \mathbf{1}(s \in A) \geq \delta |A| \right) \leq e^{-\lambda \delta |A| + \frac{\lambda^2(1+2b)^2}{4\gamma} |A|} \leq \exp(-\gamma \delta^2 |A| / (1 + 2b)^2),$$

once we choose $\lambda = 2\delta\gamma/(1 + 2b)^2$. \square

4.3. Concentration and mixing time for the East process. As we explained before the East process coincides with the marginal process of each column $\mathbf{M}^{(j)}(\cdot)$. It is therefore important to develop a detailed analysis of its mixing time $T_{\text{mix}}^{\text{East}}$. In this section we collect some refinements of the results of [8, Theorem 1].

We begin by considering the natural extension of the East process on \mathbb{Z} which is the unique Markov process on $\Gamma = \{0, 1\}^{\mathbb{Z}}$ with (formal) generator²

$$(\mathcal{L}^{\text{East}} f)(\eta) = \sum_{x \in \mathbb{Z}} \eta_{x-1} [\mathbb{E}_x(f)(\eta) - f(\eta)],$$

where $\mathbb{E}_x(\cdot)$ denotes expectation over the variable $\eta_x \in \{0, 1\}$ with the fair Bernoulli distribution (see [8] for the formal definitions). In Γ we consider the set Γ^* of those configurations such that the variable

$$X(\eta) := \sup\{x : \eta_x = 1\} \tag{4.9}$$

is finite. In the sequel, for any $\eta \in \Gamma^*$ we will refer to $X(\eta)$ as the *front* of η .

In order to understand the behavior of the process behind the front it is convenient to move to an evolving reference frame in which the front is always at the origin. More precisely let Γ_F denote the set of configurations such that $X(\eta) = 0$ and let $\eta^F(\cdot)$ be the Markov process on Γ_F constructed as follows. The process behind the front evolves as the usual East process. If the East dynamics tries to move (by ± 1) the front way from the origin then the whole configuration is shifted by ∓ 1 in order to keep to keep its front at the origin. Blondel [3] showed that the law of $\eta^F(t)$ as $t \rightarrow +\infty$ converges to an invariant measure ν on $(-\infty, -1] \cap \mathbb{Z}$ whose marginal on $(-\infty, -N] \cap \mathbb{Z}$ approaches exponentially fast (in N) the same marginal of the Bernoulli(1/2) product measure. In the same limit she also proved that $\frac{1}{t} X(\eta(t))$ converges in probability to

²Notice that, for the reader's convenience, we adopt the convention that the 1's are the facilitating vertices, as it is the case for the upper triangular matrix walk, and not the 0's as it is customary in the East model literature.

v_* , where v_* was defined right after Theorem 2.2. In [8] Blondel's result was improved by establishing a CLT for the variables $(X(\eta(t)) - v_*t)/\sqrt{t}$ as $t \rightarrow +\infty$.

Using concentration properties for sums of weakly mixing sequence of random variables in Appendix 1.1 we sharpen the CLT result of [8] as follows.

Lemma 4.4. *There exists universal constants c, C such that for any $a > C\sqrt{t} \log^{3/2}(t)$, and $\eta(0) \in \Gamma_F$,*

$$\mathbb{P}(|X(\eta(t)) - v_*t| \geq a) \leq e^{-c(\frac{a^2}{t})^{1/3}}. \quad (4.10)$$

Corollary 4.5. *For any $\gamma > 0$, all $\varepsilon \geq 1/n^\gamma$ and n large enough,*

$$T_{\text{mix}}^{\text{East}}(n, \varepsilon) \leq n/v_* + \sqrt{n} \log^2(n). \quad (4.11)$$

Proof of the Corollary. A natural coupling argument of [8] (cf. the beginning of Section 4.2. there) proves that, for any $\eta \in \Gamma^F$ and any $\varepsilon \in (0, 1)$,

$$d_{TV}(\eta(t), \pi) \leq \mathbb{P}_\omega(X(\eta(t)) < n).$$

The bound (4.11) is now an immediate consequence of (4.10). \square

4.4. Special row patterns and their large deviations. For reason that will appear clearly in the proof of Theorem 2.2, given a set of k columns of the matrix $M(t)$ it will be important to have a good control on the statistics at time t of the number of rows where the entries of the chosen columns exhibit certain special patterns (e.g. they are all equal to 0 or only the last one is equal to 1, etc). This is the main motivation for what follows.

Consider a k tuple of random vectors $(\mathbf{X}_1, \dots, \mathbf{X}_k)$ with joint distribution \mathcal{P} and let

$$\mathcal{N} = \sum_{x=1}^n \mathbf{1}((\mathbf{X}_1(x), \dots, \mathbf{X}_k(x)) = (0, \dots, 0, 1)). \quad (4.12)$$

Thus \mathcal{N} counts the number of coordinates where all but the last vector are zero.

Lemma 4.6. *Let π_j be the uniform measure on \mathcal{X}_n^j . Then for any positive integer k ,*

$$\begin{aligned} \mathcal{P}(|\mathcal{N} - n/2^k| \geq n/2^{k+1}) &< 6e^{-n/(9 \cdot 2^{2k-1})} \\ &+ d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-1}), \pi_{k-1}) \\ &+ d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-2}, \mathbf{X}_k), \pi_{k-1}) \\ &+ d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-2}, \mathbf{X}_{k-1} + \mathbf{X}_k), \pi_{k-1}) \\ &+ d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-2}), \pi_{k-2}). \end{aligned}$$

Proof. For the ease of reading, we first discuss the case $k = 2$. Let $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ be defined as the variable \mathcal{N} in (4.12) but with other possible patterns of $(\mathbf{X}_1(x), \mathbf{X}_2(x))$ as specified below:

$$\mathcal{N}_1 \leftrightarrow (1, 0), \quad \mathcal{N}_2 \leftrightarrow (0, 0), \quad \mathcal{N}_3 \leftrightarrow (1, 1). \quad (4.13)$$

Clearly $\mathcal{N}_1 + \mathcal{N}_2 + \mathcal{N}_3 + \mathcal{N} = n$ so that $|\mathcal{N} - n/4| \leq \sum_{i=1}^3 |\mathcal{N} + \mathcal{N}_i - n/2|/2$. In particular

$$\mathcal{P}(|\mathcal{N} - n/4| \geq n/8) \leq \sum_{i=1}^3 \mathcal{P}(|\mathcal{N} + \mathcal{N}_i - n/2| \geq n/12). \quad (4.14)$$

The following is a standard consequence of the Azuma- Hoeffding's inequality and the definition of total variation distance. For any random vector $\mathbf{X} \in \mathcal{X}_n$ distributed

according to \mathcal{P} and $a > 0$

$$\mathcal{P}\left(\left|\sum_{x=1}^n \mathbf{X}(x) - n/2\right| \geq a/2\right) \leq 2e^{-\frac{a^2}{2n}} + d_{TV}(\mathbf{X}, \pi). \quad (4.15)$$

This implies:

$$\begin{aligned} \mathcal{P}(|\mathcal{N} + \mathcal{N}_2 - n/2| \geq a/2) &\leq 2e^{-\frac{a^2}{2n}} + d_{TV}(\mathbf{X}_1, \pi) \\ \mathcal{P}(|\mathcal{N} + \mathcal{N}_3 - n/2| \geq a/2) &\leq 2e^{-\frac{a^2}{2n}} + d_{TV}(\mathbf{X}_2, \pi) \\ \mathcal{P}(|\mathcal{N} + \mathcal{N}_1 - n/2| \geq a/2) &\leq 2e^{-\frac{a^2}{2n}} + d_{TV}(\mathbf{X}_1 + \mathbf{X}_2, \pi). \end{aligned}$$

Choosing $a = n/6$ and summing the above three inequalities we finally get by (4.14)

$$\mathcal{P}(|\mathcal{N} - n/4| \geq n/8) \leq 6e^{-n/72} + \sum_{i=1}^3 d_i, \quad (4.16)$$

where d_1, d_2, d_3 are the total variation distance terms above.

For general k , we define $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ as before according to the following patterns of the array $(\mathbf{X}_1(x), \dots, \mathbf{X}_k(x))$. The first $k-2$ elements are equal to zero while the last two elements $(\mathbf{X}_{k-1}(x), \mathbf{X}_k(x))$ are as in (4.13). Thus

$$\mathcal{N} \leftrightarrow \underbrace{(0, \dots, 0)}_{k-2}, (0, 1), \quad \mathcal{N}_1 \leftrightarrow \underbrace{(0, \dots, 0)}_{k-2}, (1, 0), \quad \mathcal{N}_2 \leftrightarrow \underbrace{(0, \dots, 0)}_{k-2}, (0, 0), \quad \mathcal{N}_3 \leftrightarrow \underbrace{(0, \dots, 0)}_{k-2}, (1, 1).$$

In this case $\mathcal{N} + \sum_{i=1}^3 \mathcal{N}_i$ is not deterministic; however by triangle inequality we still get

$$|\mathcal{N} - n/2^k| \leq \frac{1}{2} \left(\sum_{i=1}^3 |\mathcal{N} + \mathcal{N}_i - n/2^{k-1}| + |\mathcal{N} + \sum_{i=1}^3 \mathcal{N}_i - n/2^{k-2}| \right).$$

Using again (4.15) we have:

$$\begin{aligned} \mathcal{P}(|\mathcal{N} + \mathcal{N}_2 - n/2^{k-1}| \geq a/2) &\leq 2e^{-\frac{a^2}{2n}} + d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-1}), \pi_{k-1}) \\ \mathcal{P}(|\mathcal{N} + \mathcal{N}_3 - n/2^{k-1}| \geq a/2) &\leq 2e^{-\frac{a^2}{2n}} + d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-2}, \mathbf{X}_k), \pi_{k-1}) \\ \mathcal{P}(|\mathcal{N} + \mathcal{N}_1 - n/2^{k-1}| \geq a/2) &\leq 2e^{-\frac{a^2}{2n}} + d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-2}, \mathbf{X}_{k-1} + \mathbf{X}_k), \pi_{k-1}) \\ \mathcal{P}(|\mathcal{N} + \sum_{i=1}^3 \mathcal{N}_i - n/2^{k-2}| \geq a/2) &\leq 2e^{-\frac{a^2}{2n}} + d_{TV}((\mathbf{X}_1, \dots, \mathbf{X}_{k-2}), \pi_{k-2}). \end{aligned}$$

We finally get the thesis as in the $k = 2$ case by summing the above inequalities computed for $a = \frac{1}{3} \frac{n}{2^{k-1}}$. \square

4.4.1. Application to the matrix walk. Let μ be a probability measure on \mathcal{M}_n and consider the MW $M(t)$ with initial measure μ and law $\mathbb{P}_\mu(\cdot)$. Given a set of k columns with indices $i_1 < i_2 < \dots < i_k$ we denote by $\mu_1, \mu_2, \mu_3, \mu_4$ the marginals of μ on

$$(\mathbf{M}_{i_1}, \dots, \mathbf{M}_{i_{k-1}}), (\mathbf{M}_{i_1}, \dots, \mathbf{M}_{i_{k-2}}, \mathbf{M}_{i_k}), (\mathbf{M}_{i_1}, \dots, \mathbf{M}_{i_{k-1}} + \mathbf{M}_{i_k}), (\mathbf{M}_{i_1}, \dots, \mathbf{M}_{i_{k-2}}),$$

respectively.

Definition 4.7. Given a time interval $\Delta = [t_1, t_2]$ and $k, n \in \mathbb{N}$, we say that the x^{th} -row is good for Δ if

$$\int_{\Delta} \mathbf{1}\left((\mathbf{M}_{i_1}(t; x), \dots, \mathbf{M}_{i_k}(t; x)) = (0, \dots, 0, 1)\right) dt \geq \frac{t_2 - t_1}{2^{k+1}},$$

otherwise we term it bad for Δ . We will sometimes suppress the reference to Δ if clear from the context.

Recall that π_j is the uniform measure on \mathcal{X}_n^j . The result that will play a crucial role in getting a sharp bound on the mixing time of the marginal of the MW on k columns is as follows.

Lemma 4.8. *There exists a universal constant $c > 0$ such that for any time interval $[t_1, t_2]$, any $k, n \in \mathbb{N}$ and any probability measure μ on \mathcal{X}_n^k*

$$\begin{aligned} & \mathbb{P}_\mu(\text{all rows are bad for } [t_1, t_2]) \\ & \leq \exp(-c(t_2 - t_1)) + d_{TV}(\mu_1, \pi_{k-1}) + d_{TV}(\mu_2, \pi_{k-1}) + d_{TV}(\mu_3, \pi_{k-1}) + d_{TV}(\mu_4, \pi_{k-2}). \end{aligned}$$

Proof. Fix $I = [t_1, t_2]$, k, n, μ and recall the notation of Lemma 4.6 and of its proof. For $s \in I$ let $\mathcal{N}(s)$, $\mathcal{N}_1(s)$, $\mathcal{N}_2(s)$, $\mathcal{N}_3(s)$ be the random variables defined after (4.16) computed for the vectors $\mathbf{M}_{i_1}(s), \dots, \mathbf{M}_{i_k}(s)$ and let for $a > 0$

$$\begin{aligned} Y_1(s) &= \mathbf{1}(|\mathcal{N}(s) + \mathcal{N}_1(s) - n/2^{k-1}| \geq a/2) \\ Y_2(s) &= \mathbf{1}(|\mathcal{N}(s) + \mathcal{N}_2(s) - n/2^{k-1}| \geq a/2) \\ Y_3(s) &= \mathbf{1}(|\mathcal{N}(s) + \mathcal{N}_3(s) - n/2^{k-1}| \geq a/2) \\ Y_4(s) &= \mathbf{1}(|\mathcal{N}_1(s) + \mathcal{N}_2(s) + \mathcal{N}_3(s) + \mathcal{N}(s) - n/2^{k-2}| \geq a/2). \end{aligned}$$

When μ is the uniform measure π we denote the expectation $\mathbb{E}_\pi(Y_i(s))$ by p_i . Notice that p_i does not depend on s because of reversibility w.r.t. π and that $p_i \leq 2e^{-a^2/2n}$ because of (4.15). Let also

$$J_i = \int_I Y_i(s) ds, \quad i = 1, \dots, 4,$$

Using (4.8) together with the fact that the spectral gap of the chain $A(\cdot)$ is positive uniformly in k, n (cf. Theorem 1 and [12, 13]), it follows that there exists $c > 0$ such that, for $i = 1, 2, 3$ and any $b_i \in (0, 1)$,

$$\mathbb{P}_\mu(|J_i - p_i t| \geq b_i t) \leq \exp(-ctb_i^2) + d_{TV}(\mu_i, \pi_{k-1}),$$

where $t = t_2 - t_1$. On the event that $J_i < (p_i + b_i)t$ for all $i = 1, \dots, 4$, the fraction of times s in Δ such that $Y_i(s) = 0$ for all $i = 1, \dots, 4$ is at least $1 - \sum_{i=1}^4 (p_i + b_i)$. For all such times s , a simple analysis shows that $|\mathcal{N}(s) - \frac{n}{2^k}| \leq a$. Therefore

$$\int_\Delta \mathcal{N}(s) \geq t(n/2^k - a)(1 - \sum_{i=1}^4 (p_i + b_i)).$$

Hence by definition of \mathcal{N} (cf. (4.12)), there exists $x \in [n]$ such that

$$\begin{aligned} & \int_\Delta \mathbf{1}\left(\left(\mathbf{M}_{i_1}(s; x), \dots, \mathbf{M}_{i_k}(s; x)\right) = (0, 0, \dots, 0, 1)\right) ds \\ & \geq \frac{t}{n}(n/2^k - a)(1 - \sum_{i=1}^4 (p_i + b_i)). \end{aligned}$$

Choosing a to be $n/(3 \cdot 2^k)$ and all the $b_i = \frac{1}{17}$, we see that $\sum_{i=1}^4 (p_i + b_i) \leq 1/4$ and hence x is good for I . \square

5. PROOF OF THEOREM 1

Proof. For notational convenience and without loss of generality we consider only the full matrix walk (i.e. $(i_1, \dots, i_k) = [n]$). Let us denote the spectral gaps of the MW $M(t) = (\mathbf{M}_1(t), \dots, \mathbf{M}_n(t))$ and $\mathbf{M}_n(t)$ (the last column which as mentioned before has the law of the East process with $p = 1/2$) by $\lambda_1(n)$ and $\lambda_2(n)$ respectively. Notice that $\lambda_2(n)$ is a non-increasing sequence (cf. [4]). That $\lambda_2(n) \geq \lambda_1(n)$ follows from the fact that $\mathbf{M}_n(t)$ is a projection chain of $M(t)$. We now prove the other direction by combining the approach introduced in [12] together with a crucial input from [5].

Choose $i = n - 1$ and $[t_1, t_2] = [0, t]$ in (4.6) to write

$$\mathbf{M}_n(t; \omega) = \mathbf{A}_0(t; \omega) + \sum_{j=1}^{\nu} \alpha_j(\omega) \xi_j(\omega) \mathbf{A}_j(t, \omega) = \mathbf{A}_0(t; \omega) + \sum_{j=1}^{\nu} \xi_j(\omega) \mathbf{A}_j(t, \omega),$$

where for the last equality we use the fact that, as $M_{n,n}(t) = 1$ deterministically, $\alpha_j = 1 \forall j \in [n]$. Define the event $\mathcal{C}_n = \mathcal{C}_n(t)$ that the vectors $\{\mathbf{A}_j(t, \omega)\}$ restricted to the first $n - 1$ rows span the vector space \mathcal{X}_{n-1} . Using Lemma A.1, on the event \mathcal{C}_n , we can couple $\mathbf{M}_n(t)$ to a vector distributed according the equilibrium measure independent of $[\mathbf{M}_1(t), \mathbf{M}_2(t), \dots, \mathbf{M}_{n-1}(t)]$. Thus

$$d_{TV}(M(t), \pi) \leq d_{TV}((\mathbf{M}_1(t), \dots, \mathbf{M}_{n-1}(t)), \pi) + \mathbb{P}(\mathcal{C}_n^c),$$

i.e.

$$d_{TV}(M^{(n)}(t), \pi) \leq \sum_{i=1}^n \mathbb{P}(\mathcal{C}_i^c).$$

The key new input is now the following result.

Lemma 5.1. *For each $n \geq 2$, $\lim_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{P}(\mathcal{C}_n^c)) \geq -\lambda_2(n)$.*

Assuming the lemma and recalling that $\lambda_2(i) \geq \lambda_2(n)$ for $i \leq n$, we get immediately that $\lim_{t \rightarrow \infty} \frac{1}{t} \log(d_{TV}(M(t), \pi)) \leq -\lambda_2(n)$. A standard functional analysis argument now implies $\lambda_1(n) \geq \lambda_2(n)$. \square

Proof of Lemma 5.1. For the stationary East process $\sigma(t) = [\sigma_1(t), \sigma_2(t), \dots, \sigma_n(t)]$ on $[n]$, let τ be the first time there is a legal ring at n , i.e. the clock rings at n and $\sigma_{n-1}(\tau) = 1$. In [5, Theorem A.1] it was proved that the exponential decay rate of $\mathbb{P}_\pi(\tau \geq t)$ is governed by the spectral gap λ_2 :

$$\lim_{t \rightarrow \infty} \frac{\log \mathbb{P}_\pi(\tau \geq t)}{t} = -\lambda_2.$$

Recall the notation from (4.6) that $0 < \tau_1 < \tau_2 < \dots < \tau_\nu < t$ are the ring times for the $n - 1^{th}$ row. Thus in the ‘‘adjoint’’ randomness ω^* ,

$$0 < t - \tau_\nu < t - \tau_{\nu-1} < \dots < t - \tau_1 < t$$

are the ring times for the n^{th} bit. Now by the discussion of Section 4.1.1, choosing $I = [1, n - 1]$ it follows that the event \mathcal{C}_n occurs iff for any row vector \mathbf{Z}^\top there exists $j \in [\nu]$ such that the (graphical representation of) East process on $[2, \dots, n]$, evolving with the adjoint randomness ω^* and initial condition \mathbf{Z}^\top (note that the first coordinate is frozen), at time $t - \tau_j$ is equal to 1 at the vertex $n - 1$ for some $1 \leq j \leq \nu$, and hence in the terminology of the East process, the ring at $t - \tau_j$ is legal for the n^{th} bit. Thus putting the above together along with a shift of coordinate to transfer $[2, n]$ to $[n - 1]$ we get

$$\mathbb{P}(\mathcal{C}_n^c) \leq \sum_{\mathbf{Z}^\top \in \mathcal{X}_n} \mathbb{P}_{\mathbf{Z}^\top}(\tau > t) \leq 2^{2n} \mathbb{P}_\pi(\tau > t)$$

and the lemma follows. \square

6. LOCAL MIXING AND THE PROOF OF THEOREM 2.

In this section we first prove that certain patterns of bits as in Section 4.4 facilitate some form of local mixing of the matrix walk (cf. Proposition 6.1 below). We then prove that these patterns are likely to occur (cf. Lemma 6.2) and, as a consequence, we are able to establish a precise recursion relation between the mixing times of the marginal of the MW on k and $k - 1$ columns (cf. Lemma 6.3). Putting all together we will finally deduce Theorem 2.

6.1. Local mixing. We first need some initial setup. Recall the definition given in Section 2.3 of the space $\mathcal{M}_{[i_1, \dots, i_k]}$ as the set of the columns $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and let, for any $g : \mathcal{M}_{[i_1, \dots, i_k]} \mapsto [-1, 1]$ and any subset \mathcal{I} of entries of the chosen columns, $g_{\mathcal{I}}$ be the π -average of g over the entries in \mathcal{I} .

Fix a burn-in time $t_1 > 0$ together with a positive time lag $\Delta = o(n)$ and let $t_2 = t_1 + \kappa\Delta$, where κ is a large constant (possibly depending on k ³) to be fixed later on. Recall Definition 4.7 and let, for any $a \leq n - 2\Delta$, $\mathcal{B}_a \equiv \mathcal{B}_a(t_1, \Delta)$ be the event that there is a row $x \in [a + \Delta, a + 2\Delta]$ which is good for the time interval $[t_1, t_2]$. Finally let $\mathcal{I} \equiv \mathcal{I}_{i_k, a, \Delta} = \{i_k\} \times [1, a + \Delta]$.

In the above setting the local mixing result reads as follows:

Proposition 6.1. *There exists a universal constant $c > 0$ and a choice of $\kappa = \kappa(k) > 0$ such that the following holds. For any $g : \mathcal{M}_{[i_1, \dots, i_k]} \rightarrow [-1, 1]$ which does not depend on the entries $\{(i_k, j) : 1 \leq j < a\}$, any $t > t_2$ and any initial condition M of the matrix walk:*

$$|\mathbb{E}[g(M(t)) - g_{\mathcal{I}}(M(t))]| \leq 2[e^{-c\Delta} + \mathbb{P}(\mathcal{B}_a^c)].$$

The error term $\mathbb{P}(\mathcal{B}_a^c)$ can be bounded from above using Lemma 4.8 provided that the burn-in time t_1 is large enough. Recall the definition of $T_{\text{mix}}(n; [i_1, \dots, i_k], \varepsilon)$ as the ε -mixing time of the marginal matrix walk on the columns i_1, i_2, \dots, i_k and let

$$T_{\text{mix}}(n, k, \varepsilon) = \max_{i_1, \dots, i_k} T_{\text{mix}}(n; [i_1, \dots, i_k], \varepsilon).$$

Lemma 6.2. *There exists a universal constant c such that, for any $\varepsilon \in (0, 1)$ and any $a \leq n - 2\Delta$, the following holds. If $t_1 \geq T_{\text{mix}}(n, k - 1, \varepsilon)$ then*

$$\mathbb{P}(\mathcal{B}_a^c) \leq 4\varepsilon + e^{-c\Delta}.$$

Before proving the lemma and the proposition we prove Theorem 2.

6.2. Proof of Theorem 2. At the basis of the argument there is the following recursive relation between the mixing times of k and $k - 1$ columns.

Lemma 6.3. *There exists $c > 0$ such that the following holds. Fix $\varepsilon \in (0, 1)$ and let $\varepsilon' = \varepsilon/(9\sqrt{n})$. Then, for any given $k \in \mathbb{N}$ and n large enough,*

$$T_{\text{mix}}(n, k, \varepsilon) \leq T_{\text{mix}}(n, k - 1, \varepsilon') + c.2^k \sqrt{n}.$$

Assuming the lemma we proceed as follows. Fix k, n together with $\varepsilon \in (0, 1)$ and let $\varepsilon' = 9^{-k} n^{-k/2} \varepsilon$. Using Corollary 4.5 we get

$$T_{\text{mix}}(n, 1, \varepsilon') \leq n/v_* + \sqrt{n} \log^2(n)$$

³The constant κ will be taken to be a large universal constant times 2^k for large k .

for all n large enough depending on ε, k . Now applying Lemma 6.3 $k - 1$ times starting from $k = 1$ gives

$$T_{\text{mix}}(n, k, \varepsilon) \leq n/v_* + \sqrt{n} \log^2(n) + ck2^k \sqrt{n}.$$

□

Proof of Lemma 6.3. Given $g : \mathcal{M}_{[i_1, \dots, i_k]} \mapsto \mathbb{R}$ such that $\|g\|_\infty \leq 1$, let $g^{(j)}$ be defined recursively as follows:

$$g^{(0)}(\cdot) = g(\cdot), \quad g^{(j)}(\cdot) = g_{\mathcal{I}_{k,j}}^{(j-1)}(\cdot), \quad j \leq N,$$

where $N = n/\Delta$ and $\mathcal{I}_{k,j} = \{i_k\} \times R_j$ with $\Delta = \lfloor \sqrt{n} \rfloor$ and $R_j = ((j-1)\Delta, j\Delta]$. Since $g^{(j)}$ is the π -average of $g^{(j-1)}$ w.r.t. to the entries of the i_k^{th} -column in the interval R_j we get in particular that $g^{(j)}(\cdot) = g_{\{i_k\} \times [1, j\Delta]}(\cdot)$ and $g^{(N)}$ is the π -average of g over the entries on the i_k^{th} -column.

A simple telescopic sum together with the triangle inequality gives

$$\begin{aligned} & |\mathbb{E}_A [g(M(t)) - \pi(g)]| \\ & \leq |\mathbb{E}_M [g^N(M(t)) - \pi(g)]| + \sum_{j=0}^{N-1} |\mathbb{E}_M [g^{(j)}(M(t)) - g^{(j+1)}(M(t))]|. \end{aligned} \quad (6.1)$$

Notice that $g^N(\cdot)$ depends only on the entries in the $i_1^{\text{th}}, \dots, i_{k-1}^{\text{th}}$ -columns so that

$$|\mathbb{E}_M [g^N(M(t)) - \pi(g)]| \leq 2d_{TV}((\mathbf{M}_{i_1}(t), \dots, \mathbf{M}_{i_{k-1}}(t)), \pi),$$

where, with an abuse of notation, π denotes the uniform measure on the chosen k columns of the initial matrix M .

Next we observe that for any $j \leq N - 1$ the function $g^{(j)}$ satisfies the hypothesis of Proposition 6.1 if we choose $a = j\Delta$. Thus, if $t = T_{\text{mix}}(n, \varepsilon', k - 1) + \kappa\Delta$ with κ the constant appearing in Proposition 6.1, we can appeal to Proposition 6.1 and Lemma 6.2 to get that every term in the sum is bounded from above by $8\varepsilon' + 4e^{-c\Delta}$ for some universal constant $c > 0$.

In conclusion

$$\begin{aligned} 2d_{TV}((\mathbf{M}_{i_1}(t), \dots, \mathbf{M}_{i_k}(t)), \pi) &= \max_{\substack{g: \mathcal{M}_{i_1, \dots, i_k} \mapsto \mathbb{R} \\ \|g\|_\infty \leq 1}} |\mathbb{E}_M [g(M(t)) - \pi(g)]| \\ &\leq 2d_{TV}((\mathbf{M}_{i_1}(t), \dots, \mathbf{M}_{i_{k-1}}(t)), \pi) + N(8\varepsilon' + 4e^{-c\Delta}) \\ &\leq 2\varepsilon' + 8\sqrt{n}(\varepsilon' + e^{-c\sqrt{n}}) \leq \varepsilon, \end{aligned}$$

where the last two inequalities follow by the assumption that $t > T_{\text{mix}}(n, \varepsilon', k - 1)$ and our choice of Δ . □

6.3. Local mixing: proofs. Recall the notation defined at the beginning of the section.

Proof of Proposition 6.1. Fix $g : \mathcal{M}_{[i_1, \dots, i_k]} \rightarrow [-1, 1]$ which does not depend on the entries $\{(i_k, j) : 1 \leq j < a\}$ and, for $i \in [a + \Delta, a + 2\Delta]$, let \mathcal{G}_i be the event that i is the largest good index in $[a + \Delta, a + 2\Delta]$ w.r.t. the time interval $[t_1, t_2]$.

Observe that \mathcal{G}_i is measurable with respect to the σ -algebra $\mathcal{F}_{[i,n]}([0, t_2])$ and hence also w.r.t. $\mathcal{F}_{[i,n]}([0, t])$ if $t \geq t_2$. Let

$$t_1 < \tau_1 < \tau_2 < \dots < \tau_\nu < t_2 < \tau_{\nu+1} \quad (6.2)$$

be the set of random times t such that the clock of the $(i-1)^{th}$ -row rings and the i^{th} -entries of the chosen columns are all zero except the last one (i.e. they form the pattern $[0, 0, \dots, 0, 1]$).

Let $\xi_j \equiv \xi_{i-1, \tau_j}$ be the Bernoulli variable associated to τ_j and let $\hat{\mathcal{F}}_t$ be the σ -algebra generated by all the variables generating $\mathcal{F}_{[2, n]}([0, t])$ except the variables $\{\xi_j\}_{j=1}^\nu$. Recalling (4.6), we have

$$\mathbf{M}_{i_k}(t; \omega) = \mathbf{A}_0(t; \omega) + \sum_{j=1}^{\nu} \xi_j(\omega) \mathbf{A}_j(t, \omega),$$

where $\mathbf{A}_j(t; \omega) = \Phi_{I, \Delta_j}(\omega, \mathbf{e}_i)$ with $I = [1, i-1]$ and $\Delta_j = (\tau_j, t]$. Notice that the variables α_j appearing in (4.6) are all equal to one because of the definition of the times τ_1, \dots, τ_ν . The fundamental property of this decomposition is that $\{(\mathbf{A}_0, \mathbf{A}_j)\}_{j=1}^\nu$ as well as all the other columns $(\mathbf{M}_{i_1}, \dots, \mathbf{M}_{i_{k-1}})$ are all measurable with respect to $\hat{\mathcal{F}}$.

Let now \mathcal{P} be the subspace of \mathcal{X}_{i-a} spanned by the restriction of the vectors $\{\mathbf{A}_j(t; \omega)\}_{j=1}^\nu$ to the row interval $[a, a+1, \dots, i-1]$ and let $\mathcal{C}_i = \mathcal{G}_i \cap \{\omega : \mathcal{P} = \mathcal{X}_{i-a}\}$. Clearly the events \mathcal{C}_i are disjoint. Let also $\mathcal{I}_0 = \{i_k\} \times [1, a + \Delta]$.

Claim 6.4.

$$|\mathbb{E}[g(M(t)) - g_{\mathcal{I}_0}(M(t))]| \leq 2 \left[1 - \sum_{i=a+\Delta}^{a+2\Delta} \mathbb{P}(\mathcal{C}_i) \right]. \quad (6.3)$$

Proof of the Claim. Observe that, conditionally on $\hat{\mathcal{F}}_t$, on the event \mathcal{C}_i all the entries $\mathbf{M}_{i_k}(t; x)$, $x \in [a, i-1]$, are i.i.d Bernoulli(1/2) (cf. Lemma A.1). Thus, using the fact that g does not depend on the entries $\mathbf{M}_{i_k}(x)$, $x < a$, and letting $\mathcal{I}_i = \{i_k\} \times [1, i-1]$, we get

$$\begin{aligned} \mathbb{E}[g(M(t)) \mathbf{1}(\mathcal{C}_i) | \hat{\mathcal{F}}_t] &= g_{\mathcal{I}_i}(M(t)) \mathbf{1}(\mathcal{C}_i), \\ \mathbb{E}[g_{\mathcal{I}_0}(M(t)) \mathbf{1}(\mathcal{C}_i) | \hat{\mathcal{F}}_t] &= g_{\mathcal{I}_i}(M(t)) \mathbf{1}(\mathcal{C}_i). \end{aligned} \quad (6.4)$$

In conclusion

$$\begin{aligned} & |\mathbb{E}[g(M(t)) - g_{\mathcal{I}_0}(M(t))]| \\ &= \left| \mathbb{E} \left[[g(M(t)) - g_{\mathcal{I}_0}(M(t))] \left[\sum_{i=a+\Delta}^{a+2\Delta} \mathbf{1}(\mathcal{C}_i) + \left(1 - \sum_{i=a+\Delta}^{a+2\Delta} \mathbf{1}(\mathcal{C}_i) \right) \right] \right] \right| \\ &= \left| \mathbb{E}[g(M(t)) - g_{\mathcal{I}_0}(M(t))] \left[1 - \sum_{i=a+\Delta}^{a+2\Delta} \mathbf{1}(\mathcal{C}_i) \right] \right| \\ &\leq 2 \left(1 - \sum_{i=a+\Delta}^{a+2\Delta} \mathbb{P}(\mathcal{C}_i) \right). \end{aligned} \quad (6.5)$$

□

The last step in the proof of Proposition 6.1 is an upper bound on the r.h.s. of (6.5).

Lemma 6.5. *There exists $n_0 = n_0(k)$ such that, for all $n \geq n_0$ and all $i \in [a + \Delta, a + 2\Delta]$,*

$$\mathbb{P}(\mathcal{C}_i | \mathcal{G}_i) \geq 1 - e^{-c\Delta},$$

where $c > 0$ is a universal constant. In particular

$$1 - \sum_{i=a+\Delta}^{a+2\Delta} \mathbb{P}(\mathcal{C}_i) \leq 1 - (1 - e^{-c\Delta}) \left(\sum_{i=a+\Delta}^{a+2\Delta} \mathbb{P}(\mathcal{G}_i) \right) \leq e^{-c\Delta} + \mathbb{P}(\mathcal{B}_a^c).$$

Clearly the lemma finishes the proof of Proposition 6.1. \square

Proof of Lemma 6.5. Fix $i \in [a + \Delta, a + 2\Delta]$ together with $\omega \in \mathcal{G}_i$ and recall (6.2). Using the *adjoint process argument* described in Section 4.1.1, $\omega \in \mathcal{C}_i$ if for any row vector $\mathbf{Z}^\top \in \mathcal{X}_n^\top$ such that $\mathbf{Z}^\top(x) = 0$ for all $x \notin [a + \Delta, i - 1]$ there exist $j \in [\nu]$ such that the East process on $[n]$, evolving with the adjoint randomness $\omega^* = \omega_i^*$ (cf. the discussion after Remark 4.4) and starting from \mathbf{Z}^\top , at time $t - \tau_j$ is equal to one at the vertex $i - 1$. It is crucial for what follows that, for any given $s \in [t_1, t_2]$, the event $\mathcal{E}_{i-1}(s; \mathbf{Z}^\top)$ that the East process using ω^* and starting from \mathbf{Z}^\top , at time $t - s$ is equal to one at the vertex $i - 1$ involves only the randomness $\xi_{x,k}, t_{x,k}$ with $x \in [1, i - 2]$ and k such that $t_{x,k} \in [s, t]$.

Let also $T_i(\omega) \subset [t_1, t_2]$ be the measurable subset of all times $s \in [t_1, t_2]$ such that the restriction of the i^{th} -row of $M(s; \omega)$ to the columns i_1, \dots, i_k form the pattern $(0, 0, \dots, 0, 1)$. Notice that $T_i(\cdot)$ is measurable w.r.t. $\mathcal{F}_{[i,n]}([0, t_2])$ and that $\tau_j \in T_i \forall j \in [\nu]$. Moreover T_i is the union of intervals and, for $\omega \in \mathcal{G}_i$, its Lebesgue measure $|T_i(\omega)|$ satisfies (cf. Definition 4.7)

$$|T_i(\omega)| \geq (t_2 - t_1)/2^{k+1} \geq \kappa\Delta/2^{k+1}.$$

Finally let $R_{i-1}(\omega) = \{t_{i-1,k}\}_{k=1}^\infty$ be the rings of the Poisson clock of the $(i - 1)^{\text{th}}$ -row. A simple union bound over the at most 2^Δ possible choices of the row vector \mathbf{Z}^\top proves that the lemma follows if we can show that

$$\max_{\mathbf{Z}^\top \text{ as above}} \mathbb{P}(\mathcal{E}_{i-1}(s; \mathbf{Z}^\top) \text{ fails } \forall s \in T_i \cap R_{i-1} \mid \mathcal{G}_i) \leq e^{-c'\Delta},$$

where the constant c' can be made as large as we want (in particular larger than $\log 2$) by choosing the constant κ in $t_2 = t_1 + \kappa\Delta$ large enough.

Using the independence between the event $\mathcal{E}_{i-1}(s; \mathbf{Z}^\top)$ and the σ -algebra $\mathcal{F}_{[i,n]}([t_1, t_2])$, the fact that T_i is measurable w.r.t. $\mathcal{F}_{[i,n]}([t_1, t_2])$ and (4.7) of Theorem 4.3 applied to the East process, we obtain

$$\begin{aligned} & \mathbf{1}(\mathcal{G}_i) \mathbb{P}\left(|\{s \in [t_1, t_2] : \mathcal{E}_{i-1}(s; \mathbf{Z}^\top) \text{ holds}\} \cap T_i| \leq |T_i|/4 \mid \mathcal{F}_{[i,n]}([t_1, t_2])\right) \\ & \leq \mathbf{1}(\mathcal{G}_i) \mathbb{P}\left(\left|\int_{t_1}^{t_2} ds (\mathbf{1}(\mathcal{E}_{i-1}(s; \mathbf{Z}^\top)) - \frac{1}{2}) \mathbf{1}(s \in T_i)\right| \geq |T_i|/2 \mid \mathcal{F}_{[i,n]}([t_1, t_2])\right) \\ & \leq 2^\Delta e^{-c(t_2 - t_1)/2^{k+1}} = 2^\Delta e^{-c\kappa\Delta/2^{k+1}}, \end{aligned}$$

for some numerical constant $c > 0$ related to the spectral gap of the East process.

Finally, using again the independence of R_i from $\mathcal{F}_{[1,i-2]}([0, t])$ and $\mathcal{F}_{[i,n]}([0, t])$ we get that

$$\mathbb{P}(\{s : \mathcal{E}_{i-1}(s; \mathbf{Z}^\top) \text{ holds}\} \cap T_i \cap R_i = \emptyset \mid \mathcal{F}_{[1,i-2] \cup [i,n]}([0, t])) \leq e^{-|\{s : \mathcal{E}_{i-1}(s; \mathbf{Z}^\top) \text{ holds}\} \cap T_i|}.$$

In conclusion, for any \mathbf{Z}^\top as above ,

$$\mathbb{P}(\mathcal{E}_{i-1}(s; \mathbf{Z}^\top) \text{ fails } \forall s \in T_i \cap R_{i-1} \mid \mathcal{G}_i) \leq 2^\Delta e^{-c\kappa\Delta/2^{k+1}} + e^{-\kappa\Delta/2^{k+3}}$$

and the proof is finished by taking κ to be a large enough constant times 2^k . \square

Proof of Lemma 6.2. Using the definition of t_1 we have

$$\begin{aligned} d_{TV} \left((\mathbf{M}_{i_1}(t_1), \dots, \mathbf{M}_{i_{k-1}}(t_1)), \pi \right) &\leq \varepsilon \\ d_{TV} \left((\mathbf{M}_{i_1}(t_1), \dots, \mathbf{M}_{i_{k-2}}(t_1)), \pi \right) &\leq \varepsilon \\ d_{TV} \left((\mathbf{M}_{i_1}(t_1), \dots, \mathbf{M}_{i_{k-2}}(t_1), \mathbf{M}_{i_k}(t_1)), \pi \right) &\leq \varepsilon \\ d_{TV} \left((\mathbf{M}_{i_1}(t_1), \dots, \mathbf{M}_{i_{k-1}}(t_1) + \mathbf{M}_{i_k}(t_1)), \pi \right) &\leq \varepsilon. \end{aligned}$$

Lemma 4.8 and the definition of t_2 now imply that

$$\mathbb{P}(\mathcal{B}^c) \leq e^{-c(t_2-t_1)} + 4\varepsilon \leq e^{-c\Delta} + 4\varepsilon.$$

□

APPENDIX A.

We collect here some result and proofs that were omitted from the main text. We begin with a simple lemma on the distribution of random linear combinations in \mathcal{X}_n that we state here without proof.

Lemma A.1. *For any $n, k \in \mathbb{N}$, given a set of vectors $\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(k)} \in \mathcal{X}_n$, and k independent Bernoulli(1/2) variables $\{\xi_1, \dots, \xi_k\}$, the random vector $\sum_{i=1}^k a_i v_i$ is distributed uniformly in \mathcal{X}_n iff $\mathbf{V}^{(1)}, \dots, \mathbf{V}^{(k)}$ span \mathcal{X}_n .*

1.1. Proof of Lemma 4.4. We first recall one of the main technical results from [8] saying that, for any $\eta \in \Gamma_{\mathbb{F}}$, the increments in the position of the front ((4.9)) (the variables X below) behave asymptotically as a stationary sequence of weakly dependent random variables with exponential moments. In the sequel ν denotes the invariant measure of the process as seen from the front (cf. Section 4.3).

Define $\xi_n := X(\eta(n)) - X(\eta(n-1))$, $n \geq 1$, so that

$$X(\eta(t)) = \sum_{n=1}^{N_t} \xi_n + [X(\eta(t)) - X(\eta(N_t))], \quad N_t = \lfloor t \rfloor. \quad (\text{A.1})$$

Lemma A.2 ([8, Corollary 3.2]). *There exists $\alpha \in (0, 1)$ and $\gamma > 0$ such that the following holds. Let $f : \mathbb{R} \mapsto [0, \infty)$ be such that $e^{-|x|} f^2(x) \in L^1(\mathbb{R})$. Then*

$$C_f \equiv \sup_{\eta \in \Gamma_{\mathbb{F}}} \mathbb{E}_{\eta} [f(\xi_1)^2] < \infty, \quad (\text{A.2})$$

and

$$\sup_{\eta \in \Gamma_{\mathbb{F}}} |\mathbb{E}_{\eta} [f(\xi_n)] - \mathbb{E}_{\nu} [f(\xi_1)]| = O(e^{-\gamma n^{\alpha}}) \quad \forall n \geq 1, \quad (\text{A.3})$$

where the constant in the r.h.s. of (A.3) depend on f only through the constant C_f .

The proof of Lemma 4.4 will follow by constructing the appropriate Doob's martingale and using martingale concentration by bounding the martingale difference. For any probably space $(\Omega, \mathcal{F}, \mathbb{P})$ let $\{X_1, X_2, \dots, X_n\}$, be random variables taking values in a set \mathcal{S} . Consider any function $\phi : \mathcal{S}^n \rightarrow \mathbb{R}$ and a filtration

$$\{\Omega, \emptyset\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \mathcal{F}_k = \mathcal{F}$$

The Doob's martingale is given by $M_i = \mathbb{E}(\phi(X) | \mathcal{F}_i)$, where $X = (X_1, \dots, X_n)$. We define the martingale difference

$$V_i(\phi) = M_i - M_{i-1},$$

and let $D_i(\phi) = \|V_i(\phi)\|_\infty$. Then by the classic Azuma-Hoeffding martingale concentration

$$\mathbb{P}(|\phi(X) - \mathbb{E}(\phi(X))| > r) \leq 2e^{-\frac{r^2}{2\sum_{i=1}^k D_i^2}}.$$

To use the above, in our setting let $k = n$, $\phi(X) = \sum_{i=1}^n X_i$ and let $\mathcal{F}_i = \mathcal{F}_{\mathbb{Z}}([0, i])$ (cf. (4.1)) and $k = n$. Ideally we would want to take $X_i = \xi_i$. However these variables are not quite bounded but, using (A.2), they have exponential tails. Thus we choose

$$X_i = \min(\max(\xi_i, K), -K),$$

(the value of K would be specified later). We now bound the martingale differences. Note that by (A.3) for any $j \geq i + 1$ choosing $f(x) = \min(\max(x, K), -K)$,

$$\begin{aligned} \mathbb{E}(X_j | \mathcal{F}_i) &= O(K e^{-(j-i)\alpha}) + \mathbb{E}_\nu(f(\xi_1)), \\ \mathbb{E}(X_j | \mathcal{F}_{i-1}) &= O(K e^{-(j-i+1)\alpha}) + \mathbb{E}_\nu(f(\xi_1)). \end{aligned}$$

Thus

$$\begin{aligned} |M_i - M_{i-1}| &\leq |X_i| + C K \sum_{i=1}^n e^{-\gamma i \alpha} \\ &\leq CK. \end{aligned}$$

Thus putting everything together we get,

$$\mathbb{P}(|\phi(X) - \mathbb{E}(\phi(X))| > r) \leq 2e^{-\frac{r^2}{(CK)^2}}.$$

All that is left is to take care of the truncation. Let $\xi = (\xi_1, \dots, \xi_n)$. Using (A.2)

$$|\mathbb{E}(\phi(X)) - \mathbb{E}(\phi(\xi))| = O(ne^{-cK}).$$

Thus we get

$$\mathbb{P}(|\phi(\xi) - \mathbb{E}(\phi(\xi))| > r) \leq ne^{-K} + 2e^{-\frac{r'^2}{2K^2n}}$$

where $r' = r - O(nKe^{-cK})$. When $r > C\sqrt{n} \log^{3/2}(n)$ for some large constant C we choose $K^3 = r^2/n$ and see that this choice yields for some constant $c > 0$,

$$\mathbb{P}(|\phi(\xi) - \mathbb{E}(\phi(\xi))| > r) \leq e^{-c(\frac{r^2}{n})^{1/3}}.$$

Thus we see for any n ,

$$\mathbb{P}(|X(w(t_n)) - vt_n| > r) \leq e^{-c(\frac{r^2}{n})^{1/3}}.$$

For a general and large enough, $t_n \leq t < t_{n+1}$ and $r > C\sqrt{n} \log^{3/2}(n)$,

$$\begin{aligned} \mathbb{P}(|X(w(t)) - vt| > r) &\leq \mathbb{P}(|X(w(t_n)) - vt_n| > r/2) + \mathbb{P}(|X(w(t)) - X(w(t_n))| \geq r/4), \\ &\leq e^{-c(\frac{r^2}{n})^{1/3}} + e^{-cr}, \\ &\leq e^{-c(\frac{r^2}{n})^{1/3}}. \end{aligned}$$

□

ACKNOWLEDGEMENTS

A part of this work was completed when both the authors were visiting the Simons Institute for the Theory of Computing during the semester program on Counting Complexity and Phase Transitions. We thank the Institute for its wonderful research environment. SG's research is supported by a Miller Research Fellowship.

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