

# POWERS OF TIGHT HAMILTON CYCLES IN RANDOMLY PERTURBED HYPERGRAPHS

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ABSTRACT. For  $k \geq 2$  and  $r \geq 1$  such that  $k + r \geq 4$ , we prove that, for any  $\alpha > 0$ , there exists  $\varepsilon > 0$  such that the union of an  $n$ -vertex  $k$ -graph with minimum codegree  $\left(1 - \binom{k+r-2}{k-1}^{-1} + \alpha\right)n$  and a binomial random  $k$ -graph  $\mathbb{G}^{(k)}(n, p)$  with  $p \geq n^{-\binom{k+r-2}{k-1}^{-1} - \varepsilon}$  on the same vertex set contains the  $r^{\text{th}}$  power of a tight Hamilton cycle with high probability. This result for  $r = 1$  was first proved by McDowell and Mycroft.

## §1. INTRODUCTION

**1.1. Hamiltonian cycles.** The study of Hamiltonicity (the existence of a cycle as a spanning subgraph) has been a central and fruitful area in graph theory. In particular, by a celebrated result of Karp [19], the decision problem for Hamiltonicity in general graphs is known to be NP-complete. Therefore it is likely that good characterizations of graphs with Hamilton cycles do not exist, and it becomes natural to study sufficient conditions that guarantee Hamiltonicity. Among a large variety of such results, the most famous one is the classical theorem of Dirac from 1952: every  $n$ -vertex graph ( $n \geq 3$ ) with minimum degree at least  $n/2$  is Hamiltonian [10].

Another well-studied object in graph theory is the binomial random graph  $\mathbb{G}(n, p)$ , which contains  $n$  vertices and each pair of vertices forms an edge with probability  $p$  independently from all other pairs. Pósa [33] and Korshunov [21] independently determined the threshold for Hamiltonicity in  $\mathbb{G}(n, p)$ , which is  $(\log n)/n$ . This implies that almost all dense graphs are Hamiltonian. In this sense the degree constraint in Dirac's theorem is very strong. In fact, Bohman, Frieze and Martin [5] studied the random graph model that starts with a given, dense graph and adds  $m$  random edges. In particular, they showed that for every  $\alpha > 0$  there is  $c = c(\alpha)$  such that if we start with a graph with minimum degree at least  $\alpha n$  and we add  $cn$  random edges, then the resulting graph is Hamiltonian a.a.s. (as usual, we say that an event happens *asymptotically almost surely*, or a.a.s., if it happens with probability tending to 1 as  $n \rightarrow \infty$ ). By considering the complete bipartite graph with vertex classes of sizes  $\alpha n$  and  $(1 - \alpha)n$ , one sees that the result above is tight up to the value of  $c$ .

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It is natural to study Hamiltonicity problems in uniform hypergraphs. Given  $k \geq 2$ , a  $k$ -uniform hypergraph (in short,  $k$ -graph)  $H = (V, E)$  consists of a vertex set  $V$  and an edge set  $E \subseteq \binom{V}{k}$ ; thus, every edge of  $H$  is a  $k$ -element subset of  $V$ . Given a  $k$ -graph  $H$  with a set  $S$  of  $d$  vertices (where  $1 \leq d \leq k - 1$ ) we define  $N_H(S)$  to be the collection of  $(k - d)$ -sets  $T$  such that  $S \cup T \in E(H)$ , and let  $\deg_H(S) := |N_H(S)|$  (the subscript  $H$  is omitted whenever  $H$  is clear from the context). The *minimum  $d$ -degree*  $\delta_d(H)$  of  $H$  is the minimum of  $\deg_H(S)$  over all  $d$ -vertex sets  $S$  in  $H$ . We refer to  $\delta_{k-1}(H)$  as the *minimum codegree* of  $H$ .

In the last two decades, there has been growing interest in extending Dirac's theorem to  $k$ -graphs. Among other notions of cycles in  $k$ -graphs (e.g., Berge cycles), the following 'uniform' cycles have attracted much attention. For integers  $1 \leq \ell \leq k - 1$  and  $m \geq 3$ , a  $k$ -graph  $F$  with  $m(k - \ell)$  vertices and  $m$  edges is called an  $\ell$ -cycle if its vertices can be ordered cyclically so that each of its edges consists of  $k$  consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly  $\ell$  vertices. Usually  $(k - 1)$ -cycles are also referred to as *tight* cycles. We say that a  $k$ -graph contains a *Hamilton  $\ell$ -cycle* if it contains an  $\ell$ -cycle as a spanning subgraph. In view of Dirac's theorem, minimum  $d$ -degree conditions that force Hamilton  $\ell$ -cycles (for  $1 \leq d, \ell \leq k - 1$ ) have been studied intensively [2, 3, 8, 9, 13–15, 17, 20, 24, 25, 34, 36–39].

Let  $\mathbb{G}^{(k)}(n, p)$  denote the binomial random  $k$ -graph on  $n$  vertices, where each  $k$ -tuple forms an edge independently with probability  $p$ . The threshold for the existence of Hamilton  $\ell$ -cycles has been studied by Dudek and Frieze [11, 12], who proved that for  $\ell = 1$  the threshold is  $(\log n)/n^{k-1}$ , and for  $\ell \geq 2$  the threshold is  $1/n^{k-\ell}$  (they also determined sharp thresholds for every  $k \geq 4$  and  $\ell = k - 1$ ).

Krivelevich, Kwan and Sudakov [22] considered randomly perturbed  $k$ -graphs, which are  $k$ -graphs obtained by adding random edges to a fixed  $k$ -graph. They proved the following theorem, which mirrors the result of Bohman, Frieze and Martin [5] for randomly perturbed graphs mentioned earlier.

**Theorem 1.1.** [22] *For any  $k \geq 2$  and  $\alpha > 0$ , there is  $c_k = c_k(\alpha)$  for which the following holds. Let  $H$  be a  $k$ -graph on  $n \in (k - 1)\mathbb{N}$  vertices with  $\delta_{k-1}(H) \geq \alpha n$ . If  $p = c_k n^{-(k-1)}$ , then the union  $H \cup \mathbb{G}^{(k)}(n, p)$  a.a.s. contains a Hamilton 1-cycle.*

The authors of [22] also obtained a similar result for perfect matchings. These results are tight up to the value of  $c_k$ , as shown by a simple 'bipartite' construction. McDowell and Mycroft [29] and, subsequently, Han and Zhao [16] extended Theorem 1.1 to Hamilton  $\ell$ -cycles and other degree conditions.

**1.2. Powers of Hamilton cycles.** Powers of cycles are natural generalizations of cycles. Given  $k \geq 2$  and  $r \geq 1$ , we say that a  $k$ -graph with  $m$  vertices is an  $r^{\text{th}}$  *power of a tight cycle* if its vertices can be ordered cyclically so that each consecutive  $k + r - 1$  vertices span a copy of  $K_{k+r-1}^{(k)}$ , the complete  $k$ -graph on  $k + r - 1$  vertices, and there are no other edges than the ones forced by this condition. This extends the notion of (tight) cycles in hypergraphs, which corresponds to the case  $r = 1$ .

The existence of powers of paths and cycles has also been intensively studied. For example, the famous Pósa–Seymour conjecture, which was proved by Komlós, Sárközy and Szemerédi [27, 28] for sufficiently large graphs, states that every  $n$ -vertex graph with minimum degree at least  $rn/(r+1)$  contains the  $r^{\text{th}}$  power of a Hamilton cycle. A general result of Riordan [35] implies that, for  $r \geq 3$ , the threshold for the existence of the  $r^{\text{th}}$  power of a Hamilton cycle in  $\mathbb{G}(n, p)$  is  $n^{-1/r}$ . The case  $r = 2$  was investigated by Kühn and Osthus [26], who proved that  $p \geq n^{-1/2+\varepsilon}$  suffices for the existence of the square of a Hamilton cycle in  $\mathbb{G}(n, p)$ , which is sharp up to the  $n^\varepsilon$  factor. This was further sharpened by Nenadov and Škorić [30]. Moreover, Bennett, Dudek and Frieze [4] proved a result for the square of a Hamilton cycle in randomly perturbed graphs, extending the result of Bohman, Frieze and Martin [5].

**Theorem 1.2.** [4] *For any  $\alpha > 0$  there is  $K > 0$  such that the following holds. Let  $G$  be a graph with  $\delta(G) \geq (1/2 + \alpha)n$  and suppose  $p = p(n) \geq Kn^{-2/3} \log^{1/3} n$ . Then the union  $H \cup \mathbb{G}(n, p)$  a.a.s. contains the square of a Hamilton cycle.*

Note that in Theorem 1.2 the randomness that is required is much weaker than the one needed in the result for the pure random model (which is essentially  $n^{-1/2}$ ). The authors of [4] also asked for similar results for higher powers of Hamilton cycles in randomly perturbed graphs.

Parczyk and Person [31, Theorem 3.7] proved that, for  $k \geq 3$  and  $r \geq 2$ , the threshold for the existence of an  $r^{\text{th}}$  power of a tight Hamilton cycle in the random  $k$ -graph  $\mathbb{G}^{(k)}(n, p)$  is  $n^{-\binom{k+r-2}{k-1}^{-1}}$ . Our main result, Theorem 1.3 below, shows that if we consider randomly perturbed  $k$ -graphs  $H \cup \mathbb{G}^{(k)}(n, p)$  with  $\delta_{k-1}(H)$  reasonably large, then  $p = p(n) \geq n^{-\binom{k+r-2}{k-1}^{-1-\varepsilon}}$  is enough to guarantee the existence of an  $r^{\text{th}}$  power of a tight Hamilton cycle with high probability.

**Theorem 1.3** (Main result). *For all integers  $k \geq 2$  and  $r \geq 1$  such that  $k + r \geq 4$  and  $\alpha > 0$ , there is  $\varepsilon > 0$  such that the following holds. Suppose  $H$  is a  $k$ -graph on  $n$  vertices with*

$$\delta_{k-1}(H) \geq \left(1 - \binom{k+r-2}{k-1}^{-1} + \alpha\right) n \tag{1}$$

*and  $p = p(n) \geq n^{-\binom{k+r-2}{k-1}^{-1-\varepsilon}}$ . Then a.a.s. the union  $H \cup \mathbb{G}^{(k)}(n, p)$  contains the  $r^{\text{th}}$  power of a tight Hamilton cycle.*

We remark that our proof only gives a small  $\varepsilon$ , and it would be interesting to know if one can get a larger gap in  $\alpha$  in comparison with the result in the purely random model, as in Theorem 1.2. We remark that the case  $k \geq 3$  and  $r = 1$  of Theorem 1.3 was first proved by McDowell and Mycroft [29]. Other results in randomly perturbed graphs can be found in [1, 6, 7, 16, 23].

The core of the proof of Theorem 1.3 follows the *Absorbing Method* introduced by Rödl, Ruciński, and Szemerédi in [37], combined with results concerning binomial random hypergraphs.

This paper is organized as follows. In Section 2 we prove some results concerning random hypergraphs. Section 3 contains two essential lemmas in our approach, namely, Lemma 3.1 (Connecting Lemma) and Lemma 3.2 (Absorbing Lemma). In Section 1.3 we prove our main result, Theorem 1.3. Some remarks concerning the hypotheses in Theorem 1.3 are given in Section 5. Throughout the paper, we omit floor and ceiling functions.

## §2. SUBGRAPHS OF RANDOM HYPERGRAPHS

In this section we prove some results related to binomial random  $k$ -graphs. We will apply Chebyshev's inequality and Janson's inequality to prove some concentration results that we shall need. For convenience, we state these two inequalities in the form we need (inequalities (2) and (3) below follow, respectively, from Janson's and Chebyshev's inequalities; see, e.g., [18, Theorem 2.14]).

We first recall Janson's inequality. Let  $\Gamma$  be a finite set and let  $\Gamma_p$  be a random subset of  $\Gamma$  such that each element of  $\Gamma$  is included in  $\Gamma_p$  independently with probability  $p$ . Let  $\mathcal{S}$  be a family of non-empty subsets of  $\Gamma$  and for each  $S \in \mathcal{S}$ , let  $I_S$  be the indicator random variable for the event  $S \subseteq \Gamma_p$ . Thus each  $I_S$  is a Bernoulli random variable  $\text{Be}(p^{|S|})$ . Let  $X := \sum_{S \in \mathcal{S}} I_S$  and  $\lambda := \mathbb{E}(X)$ . Let  $\Delta_X := \sum_{S \cap T \neq \emptyset} \mathbb{E}(I_S I_T)$ , where the sum is over all ordered pairs  $S, T \in \mathcal{S}$  (note that the sum includes the pairs  $(S, S)$  with  $S \in \mathcal{S}$ ). Then Janson's inequality says that, for any  $0 \leq t \leq \lambda$ ,

$$\mathbb{P}(X \leq \lambda - t) \leq \exp\left(-\frac{t^2}{2\Delta_X}\right). \quad (2)$$

Next note that  $\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \leq \Delta_X$ . Then, by Chebyshev's inequality,

$$\mathbb{P}(X \geq 2\lambda) \leq \frac{\text{Var}(X)}{\lambda^2} \leq \frac{\Delta_X}{\lambda^2}. \quad (3)$$

Consider the random  $k$ -graph  $\mathbb{G}^{(k)}(n, p)$  on an  $n$ -vertex set  $V$ . Note that we can view  $\mathbb{G}^{(k)}(n, p)$  as  $\Gamma_p$  with  $\Gamma = \binom{V}{k}$ . For two  $k$ -graphs  $G$  and  $H$ , let  $G \cap H$  (or  $G \cup H$ ) denote the  $k$ -graph with vertex set  $V(G) \cap V(H)$  (or  $V(G) \cup V(H)$ ) and edge set  $E(G) \cap E(H)$  (or  $E(G) \cup E(H)$ ). Finally, let

$$\Phi_F = \Phi_F(n, p) = \min\{n^{v_H} p^{e_H} : H \subseteq F \text{ and } e_H > 0\}.$$

The following simple proposition is useful.

**Proposition 2.1.** *Let  $F$  be a  $k$ -graph with  $s$  vertices and  $f$  edges and let  $G := \mathbb{G}^{(k)}(n, p)$ . Let  $\mathcal{A}$  be a family of ordered  $s$ -subsets of  $V = V(G)$ . For each  $A \in \mathcal{A}$ , let  $I_A$  be the indicator random variable of the event that  $A$  spans a labelled copy of  $F$  in  $G$ . Let  $X = \sum_{A \in \mathcal{A}} I_A$ . Then  $\Delta_X \leq s! 2^{2s} n^{2s} p^{2f} / \Phi_F$ .*

*Proof.* For each ordered  $s$ -subset  $A$  of  $V$ , let  $\alpha_A$  be the bijection from  $V(F)$  to  $A$  following the orders of  $V(F)$  and  $A$ . Let  $F_A$  be the labelled copy of  $F$  spanned on  $A$ . For any  $T \subseteq V(F)$  with  $|F[T]| > 0$ , denote by  $W_T$  the set of all pairs  $A, B \in \mathcal{A}$  such that  $A \cap B = \alpha_A(T)$ . If  $T$  has  $s'$  vertices and  $F[T]$  has  $f'$  edges, then for every  $\{A, B\} \in W_T$ ,  $F_A \cup F_B$  has exactly  $2s - s'$  vertices and at least  $2f - f'$  edges. Therefore, we can bound  $\Delta_X$  by

$$\Delta_X \leq \sum_{T \subseteq V(F)} |W_T| p^{2f - f'}.$$

Given integers  $n$  and  $b$ , let  $(n)_b := n(n-1)(n-2)\cdots(n-b+1) = n!/(n-b)!$ . Note that there are at most  $\binom{n}{2s-s'}$  choices for the vertex set of  $F_A \cup F_B$ , and there are at most

$$(2s - s')_s \cdot \binom{s}{s'} s! \leq (2s - s')! s! 2^s$$

ways to label each  $(2s - s')$ -set to get  $\{A, B\}$ . Thus we have  $|W_T| \leq s!2^s n^{2s-s'}$  and

$$\Delta_X \leq \sum_{T \subseteq V(F)} s!2^s n^{2s-s'} p^{2f-f'} \leq \sum_{T \subseteq V(F)} s!2^s n^{2s} p^{2f} / \Phi_F \leq s!2^{2s} n^{2s} p^{2f} / \Phi_F,$$

because there are at most  $2^s$  choices for  $T$ .  $\square$

The following lemma gives the properties of  $\mathbb{G}^{(k)}(n, p)$  that we will use. Throughout the rest of the paper, we write  $\alpha \ll \beta \ll \gamma$  to mean that ‘we can choose the positive constants  $\alpha, \beta$  and  $\gamma$  from right to left’. More precisely, there are functions  $f$  and  $g$  such that, given  $\gamma$ , whenever  $\beta \leq f(\gamma)$  and  $\alpha \leq g(\beta)$ , the subsequent statement holds. Hierarchies of other lengths are defined similarly.

**Lemma 2.2.** *Let  $F$  be a labelled  $k$ -graph with  $b$  vertices and  $a$  edges. Suppose  $1/n \ll 1/C \ll \gamma, 1/a, 1/b, 1/s$ . Let  $V$  be an  $n$ -vertex set, and let  $\mathcal{F}_1, \dots, \mathcal{F}_t$  be  $t \leq n^s$  families of  $\gamma n^b$  ordered  $b$ -sets on  $V$ . If  $p = p(n)$  is such that  $\Phi_F(n, p) \geq Cn$ , then the following properties hold for the binomial random  $k$ -graph  $G = \mathbb{G}^{(k)}(n, p)$  on  $V$ .*

- (i) *With probability at least  $1 - \exp(-n)$ , every induced subgraph of  $G$  of order  $\gamma n$  contains a copy of  $F$ .*
- (ii) *With probability at least  $1 - \exp(-n)$ , for every  $i \in [t]$ , there are at least  $(\gamma/2)n^b p^a$  ordered  $b$ -sets in  $\mathcal{F}_i$  that span labelled copies of  $F$ .*
- (iii) *With probability at least  $1 - 1/\sqrt{n}$ , there are at most  $2n^b p^a$  ordered  $b$ -sets of vertices of  $G$  that span labelled copies of  $F$ .*
- (iv) *With probability at least  $1 - 1/\sqrt{n}$ , the number of overlapping (i.e., not vertex-disjoint) pairs of copies of  $F$  in  $G$  is at most  $4b^2 n^{2b-1} p^{2a}$ .*

*Proof.* Let  $\mathcal{A}$  be a family of ordered  $b$ -sets of vertices in  $V$ . For each  $A \in \mathcal{A}$ , let  $I_A$  be the indicator random variable of the event that  $A$  spans a labelled copy of  $F$  in  $G$ . Let  $X_{\mathcal{A}} = \sum_{A \in \mathcal{A}} I_A$ . From the hypothesis that  $\Phi_F \geq Cn$  and Proposition 2.1, we have

$$\Delta_X \leq b!2^{2b} n^{2b} p^{2a} / \Phi_F \leq b!2^{2b} n^{2b} p^{2a} / (Cn). \quad (4)$$

Furthermore, let  $\mathcal{S}$  consist of the edge sets of the labelled copies of  $F$  spanned on  $A$  in the complete  $k$ -graph on  $V$  for all  $A \in \mathcal{A}$ . Since we can write  $X_{\mathcal{A}} = \sum_{S \in \mathcal{S}} I_S$ , where  $I_S$  is the indicator variable for the event  $S \subseteq E(G)$ , we can apply (2) to  $X_{\mathcal{A}}$ .

For (i), fix a vertex set  $W$  of  $G$  with  $|W| = \gamma n$ . Let  $\mathcal{A}$  be the family of all labelled  $b$ -sets in  $W$ . Let  $X_{\mathcal{A}}$  be the random variable that counts the number of members of  $\mathcal{A}$  that span a labelled copy of  $F$  and thus  $\mathbb{E}[X_{\mathcal{A}}] = (\gamma n)_b p^a$ . By (4) and (2) and the fact that  $1/C \ll \gamma, 1/b$ , we have  $\mathbb{P}(X_{\mathcal{A}} = 0) \leq \exp(-2n)$ . By the union bound, the probability that there exists a vertex set  $W$  of size  $\gamma n$  such that  $X_{\mathcal{A}} = 0$  is at most  $2^n \exp(-2n) \leq \exp(-n)$ , which proves (i).

For (ii), fix  $i \in [t]$  and let  $X_{\mathcal{F}_i}$  be the random variable that counts the members of  $\mathcal{F}_i$  that span  $F$ . Note that  $\mathbb{E}[X_{\mathcal{F}_i}] = \gamma n^b p^a$ . Thus (2) implies that  $\mathbb{P}(X_{\mathcal{F}_i} \leq (\gamma/2)n^b p^a) \leq \exp(-2n)$ . By the union bound and the fact that  $n^s \exp(-2n) \leq \exp(-n)$ , we see that (ii) holds.

For (iii), let  $X_3$  be the random variable that counts the number of labelled copies of  $F$  in  $G$ . Since  $\mathbb{E}(X_3) = (n)_b p^a$ , by (4) and (3), we obtain

$$\mathbb{P}(X_3 \geq 2p^a n^b) \leq \mathbb{P}(X_3 \geq 2\mathbb{E}[X_3]) \leq \frac{\Delta_{X_3}}{\mathbb{E}[X_3]^2} \leq \frac{b!2^{2b}n^{2b}p^{2a}/(Cn)}{((n)_b p^a)^2} \leq \frac{1}{\sqrt{n}}.$$

For (iv), let  $Y$  be the random variable that denotes the number of overlapping pairs of copies of  $F$  in  $G$ . We first estimate  $\mathbb{E}[Y]$ . We write  $Y = \sum_{A \in \mathcal{Q}} I_A$ , where  $\mathcal{Q}$  is the collection of the edge sets of overlapping pairs of labelled copies of  $F$  in the complete  $k$ -graph on  $n$  vertices. Note that if two overlapping copies of  $F$  do not share any edge, then they induce at most  $2b-1$  vertices and exactly  $2a$  edges. Note that for  $1 \leq i \leq b$ , there are

$$\binom{n}{2b-i} (2b-i)_b \binom{b}{i} b! = (n)_{2b-i} \binom{b}{i} (b)_i \leq (n)_{2b-i} (b)_i^2$$

members of  $\mathcal{Q}$  whose two copies of  $F$  share exactly  $i$  vertices. Thus, the number of choices for the vertex sets of pairs of copies which induce at most  $2b-2$  vertices is at most  $\sum_{2 \leq i \leq b} (n)_{2b-i} (b)_i^2 \leq n^{2b-1}$ . By the definition of  $\Delta_{X_3}$  and (4) we have

$$n^{2b-1} b^2 p^{2a} / 2 \leq \mathbb{E}[Y] \leq (n)_{2b-1} b^2 \cdot p^{2a} + n^{2b-1} \cdot p^{2a} + \Delta_{X_3} \leq 2b^2 n^{2b-1} p^{2a}.$$

We next compute  $\Delta_Y$ . For each  $A \in \mathcal{Q}$ , let  $S_A$  denote the  $k$ -graph induced by  $A$  (thus  $S_A$  is the union of two overlapping copies of  $F$ ). For each  $A, B \in \mathcal{Q}$ , write  $S_A := F_1 \cup F_2$  and  $S_B := F_3 \cup F_4$ , where each  $F_i$  is a copy of  $F$  for  $i \in [4]$  such that  $E(F_1) \cap E(F_3) \neq \emptyset$ . Define  $H_1 := F_1 \cap F_2$ ,  $H_2 := (F_1 \cup F_2) \cap F_3$  and  $H_3 := (F_1 \cup F_2 \cup F_3) \cap F_4$ . Since  $V(F_1) \cap V(F_2) \neq \emptyset$ ,  $V(F_3) \cap V(F_4) \neq \emptyset$ , and  $E(F_1) \cap E(F_3) \neq \emptyset$ , we know that  $v_{H_i} \geq 1$  for  $i = 1, 2, 3$ . We claim that  $n^{v_{H_i}} p^{e_{H_i}} \geq n$  for  $i = 1, 2, 3$ . Indeed, since each  $H_i$  is a subgraph of  $F$ , if  $e_{H_i} \geq 1$ , then  $n^{v_{H_i}} p^{e_{H_i}} \geq \Phi_F \geq Cn$ ; otherwise  $e_{H_i} = 0$  and then we have  $n^{v_{H_i}} p^{e_{H_i}} = n^{v_{H_i}} \geq n^1 = n$ . So we have

$$n^{v_{H_1}} p^{e_{H_1}} \cdot n^{v_{H_2}} p^{e_{H_2}} \cdot n^{v_{H_3}} p^{e_{H_3}} \geq n^3. \quad (5)$$

Now we define  $\Delta_{H_1, H_2, H_3} = \sum_{A, B} \mathbb{E}[I_A I_B]$ , where the sum is over the pairs  $\{A, B\}$  with  $A \cap B \neq \emptyset$  that generate  $H_1, H_2, H_3$ . Observe that the sum contains at most

$$\binom{n}{4b - v_{H_1} - v_{H_2} - v_{H_3}} (4b - v_{H_1} - v_{H_2} - v_{H_3})_b^4 < n^{4b - (v_{H_1} + v_{H_2} + v_{H_3})} (4b)^{3b}$$

terms. Thus, from (5), we obtain

$$\Delta_{H_1, H_2, H_3} = \sum_{A, B} \mathbb{E}[I_A I_B] \leq (4b)^{3b} n^{4b - (v_{H_1} + v_{H_2} + v_{H_3})} p^{4a - (e_{H_1} + e_{H_2} + e_{H_3})} \leq (4b)^{3b} n^{4b-3} p^{4a}.$$

Let  $D = D(b, k, r)$  be the number of choices for  $H_1, H_2, H_3$ , thus

$$\Delta_Y = \sum_{H_1, H_2, H_3} \Delta_{H_1, H_2, H_3} \leq D (4b)^{3b} n^{4b-3} p^{4a}.$$

Therefore, by (3) and the fact that  $n$  is large enough, we get

$$\mathbb{P}(Y \geq 4b^2 n^{2b-1} p^{2a}) \leq \mathbb{P}(Y \geq 2\mathbb{E}[Y]) \leq \frac{\Delta_Y}{\mathbb{E}[Y]^2} \leq \frac{D (4b)^{3b} n^{4b-3} p^{4a}}{(n^{2b-1} p^{2a} / 2)^2} \leq \frac{1}{\sqrt{n}}.$$

This verifies (iv). □

For  $m \geq k + r - 1$ , denote by  $P_m^{k,r}$  the  $r^{\text{th}}$  power of a  $k$ -uniform tight path on  $m$  vertices. Similarly, write  $C_m^{k,r}$  for the  $r^{\text{th}}$  power of a  $k$ -uniform tight cycle on  $m$  vertices. For simplicity we say that  $P_m^{k,r}$  is an  $(r, k)$ -path and  $C_m^{k,r}$  is an  $(r, k)$ -cycle. We write  $P_m^r$  for  $P_m^{k,r}$  whenever  $k$  is clear from the context. Moreover, the ends of  $P_m^r$  are its first and last  $k + r - 1$  vertices (with the order in the  $(r, k)$ -path). We end this section by computing  $\Phi_{P_b^r}$  for the  $p = p(n) \geq n^{-\binom{k+r-2}{k-1}^{-1-\varepsilon}}$  as in Theorem 1.3. For  $b \geq k + r - 1$ , let

$$g(b) := \left( b - \frac{(k-1)(k+r-1)}{k} \right) \binom{k+r-2}{k-1}.$$

Clearly  $g$  is an increasing function. Note that the number of edges in  $P_m^{k,r}$  is given by

$$\begin{aligned} |E(P_m^{k,r})| &= \binom{k+r-1}{k} + (m - (k+r-1)) \binom{k+r-2}{k-1} \\ &= \left( m - \frac{(k-1)(k+r-1)}{k} \right) \binom{k+r-2}{k-1} = g(m). \end{aligned}$$

**Proposition 2.3.** *Suppose  $k \geq 2$ ,  $r \geq 1$ ,  $b \geq k + r - 1$ ,  $k + r \geq 4$  and  $C > 0$ . Let  $\varepsilon$  be such that  $0 < \varepsilon < \min \{ (2g(b))^{-1}, (3\binom{k+r-1}{k})^{-1} \}$ . Suppose  $1/n \ll 1/C, 1/k, 1/r, 1/b$ . If  $p = p(n) \geq n^{-\binom{k+r-2}{k-1}^{-1-\varepsilon}}$ , then  $\Phi_{P_b^r} \geq Cn$ .*

*Proof.* Let  $H$  be a subgraph of  $P_b^r$ . Since for any integer  $k + r - 1 \leq b' \leq b$ , any subgraph of  $P_{b'}^r$  has at most  $g(b')$  edges, we have the following observations.

- (a) If  $e_H > g(b')$  for some  $b' \geq k + r - 1$ , then  $v_H \geq b' + 1$ ;
- (b) if  $e_H > \binom{i}{k}$  for some  $k - 1 \leq i < k + r - 1$ , then  $v_H \geq i + 1$ .

By (a), we have

$$\min_{g(k+r-1) < e_H \leq g(b)} n^{v_H} p^{e_H} = \min_{k+r-1 \leq b' < b} \left( \min_{g(b') < e_H \leq g(b'+1)} n^{v_H} p^{e_H} \right) \geq \min_{k+r-1 \leq b' < b} n^{b'+1} p^{g(b'+1)}.$$

Since  $p \geq n^{-1/\binom{k+r-2}{k-1}^{-1-\varepsilon}}$ , and  $g(b'+1) > 0$ , the following holds for any  $b' < b$ :

$$\begin{aligned} n^{b'+1} p^{g(b'+1)} &\geq n^{b'+1} \left( n^{-1/\binom{k+r-2}{k-1}^{-1-\varepsilon}} \right)^{g(b'+1)} \\ &= n^{-g(b'+1)\varepsilon} n^{(k-1)(k+r-1)/k} \geq n^{-g(b)\varepsilon} n^{(k-1)(k+r-1)/k} \geq Cn, \end{aligned}$$

where we used  $(k-1)(k+r-1)/k \geq 3/2$  and  $g(b)\varepsilon < 1/2$ . Therefore,

$$\min_{g(k+r-1) < e_H \leq g(b)} n^{v_H} p^{e_H} \geq Cn. \tag{6}$$

On the other hand, noting that  $g(k+r-1) = \binom{k+r-1}{k}$ , by (b) we have

$$\min_{0 < e_H \leq g(k+r-1)} n^{v_H} p^{e_H} = \min_{k-1 \leq i < k+r-1} \left( \min_{\binom{i}{k} < e_H \leq \binom{i+1}{k}} n^{v_H} p^{e_H} \right) \geq \min_{k-1 \leq i < k+r-1} n^{i+1} p^{\binom{i+1}{k}}.$$

Since  $p \geq n^{-1/\binom{k+r-2}{k-1}-\varepsilon}$ , and  $\binom{i+1}{k}\varepsilon \leq 1/3$  for any  $k-1 \leq i \leq k+r-2$ , if  $i \geq 2$ , then

$$n^{i+1}p^{\binom{i+1}{k}} \geq n^{i+1}n^{-(1/\binom{k+r-2}{k-1}+\varepsilon)\binom{i+1}{k}} \geq n^{i+1-\frac{i+1}{k}-\binom{i+1}{k}\varepsilon} \geq Cn.$$

Otherwise  $i = 1$  and thus  $k = 2$ , in which case we have  $n^{i+1}p^{\binom{i+1}{k}} = n^2p \geq Cn$ . Therefore,

$$\min_{0 < e_H \leq g(k+r-1)} n^{v_H} p^{e_H} \geq Cn. \quad (7)$$

From (6) and (7), we have  $\Phi_{P_b^r} \geq Cn$ , as desired.  $\square$

### §3. THE CONNECTING AND ABSORBING LEMMAS

For brevity, throughout the rest of this paper, we write

$$h := k + r - 1, \quad t := g(2h), \quad c := \binom{k+r-2}{k-1}^{-1}.$$

Recall that the ends of an  $(r, k)$ -path are ordered  $h$ -sets that span a copy of  $K_h^{(k)}$  in  $H$ .

**3.1. The Connecting Lemma.** Given a  $k$ -graph  $H$  and two ordered  $h$ -sets of vertices  $A$  and  $B$  each spanning a copy of  $K_h^{(k)}$  in  $H$ , we say that an ordered  $2h$ -set of vertices  $C$  *connects*  $A$  and  $B$  if  $C \cap A = C \cap B = \emptyset$  and the concatenation  $ACB$  spans a labelled copy of  $P_{4h}^r$ . We are now ready to state our connecting lemma.

**Lemma 3.1** (Connecting Lemma). *Suppose  $1/n \ll \varepsilon \ll \beta \ll \alpha' \ll 1/k, 1/r$ . Let  $H$  be an  $n$ -vertex  $k$ -graph with  $\delta_{k-1}(H) \geq (1 - c + \alpha')n$  and suppose  $p = p(n) \geq n^{-c-\varepsilon}$ . Then a.a.s.  $H \cup \mathbb{G}^{(k)}(n, p)$  contains a set  $\mathcal{C}$  of vertex-disjoint copies of  $P_{2h}^r$  with  $|\mathcal{C}| \leq \beta n$  such that, for every pair of disjoint ordered  $h$ -sets spanning a copy of  $K_h^{(k)}$  in  $H$ , there are at least  $\beta^2 n / (2h)^2$  ordered copies of  $P_{2h}^r$  in  $\mathcal{C}$  that connect them.*

*Proof.* Let  $\mathcal{S}$  be the set of pairs of disjoint ordered  $h$ -sets that each span a copy of  $K_h^{(k)}$  in  $H$ . Fix  $\{S, S'\} \in \mathcal{S}$  and write  $S := (v_1, \dots, v_h)$  and  $S' := (w_h, \dots, w_1)$ . Since  $\delta_{k-1}(H) \geq (1 - c + \alpha')n$ , we can extend  $S$  to an  $(r, k)$ -path with vertices  $(v_1, \dots, v_{2h})$  such that the vertices of this  $(r, k)$ -path are disjoint with  $\{w_h, \dots, w_1\}$  and there are at least  $(\alpha'n/2)^h$  choices for the ordered set  $(v_{h+1}, \dots, v_{2h})$ . Similarly, we can extend  $S'$  to an  $(r, k)$ -path  $(w_{2h}, \dots, w_1)$  such that the vertices of this  $(r, k)$ -path are disjoint with  $\{v_1, \dots, v_{2h}\}$  and there are at least  $(\alpha'n/2)^h$  choices for the ordered set  $(w_{2h}, \dots, w_{h+1})$ . So there are at least  $(\alpha'n/2)^{2h} \geq 24\beta n^{2h}$  choices for the ordered  $2h$ -sets  $(v_{h+1}, \dots, v_{2h}, w_{2h}, \dots, w_{h+1})$ . Let  $\mathcal{C}_{S, S'}$  be a collection of exactly  $24\beta n^{2h}$  such ordered  $2h$ -sets of vertices. Clearly if an ordered set  $C$  in  $\mathcal{C}_{S, S'}$  spans a copy of  $P_{2h}^r$ , then  $C$  connects  $S$  and  $S'$ .

Now we will use the edges of  $G = \mathbb{G}^{(k)}(n, p)$  to obtain the desired copies of  $P_{2h}^r$  that connect the pairs in  $\mathcal{S}$ . Let  $\mathcal{T}$  be the set of all labelled copies of  $P_{2h}^r$  in  $G$ . We claim that the following properties hold with probability at least  $1 - 3/\sqrt{n}$ :

- (a)  $|\mathcal{T}| \leq 2p^t n^{2h}$ ;
- (b) for every  $\{S, S'\} \in \mathcal{S}$ , at least  $12\beta p^t n^{2h}$  members of  $\mathcal{T}$  connect  $S$  and  $S'$ ;
- (c) the number of overlapping pairs of members of  $\mathcal{T}$  is at most  $4(2h)^2 p^{2t} n^{4h-1}$ .

To see that the claim above holds, note that by Proposition 2.3, we can apply Lemma 2.2 with  $F = P_{2h}^r$ ,  $\gamma = 24\beta$  and  $\mathcal{C}_{S,S'}$  in place of  $\mathcal{F}_i$ . Items (a), (b) and (c) follow, respectively, from Lemma 2.2 (iii), (ii) and (iv).

Next we select a random collection  $\mathcal{C}'$  by including each member of  $\mathcal{T}$  independently with probability  $q := \beta/(2(2h)^2 n^{2h-1} p^t)$ . By using Chernoff's inequality (for (i) and (ii) below) and Markov's inequality (for (iii) below), we know that there is a choice of  $\mathcal{C}'$  that satisfies the following properties:

- (i)  $|\mathcal{C}'| \leq 2q|\mathcal{T}| \leq \beta n$ ;
- (ii) for every  $\{S, S'\} \in \mathcal{S}$ , there are at least  $12\beta(q/2)n^{2h}p^t = 3\beta^2 n/(2h)^2$  members of  $\mathcal{C}'$  that connect  $S$  and  $S'$ ;
- (iii) the number of overlapping pairs of members of  $\mathcal{C}'$  is at most  $8(2h)^2 q^2 n^{4h-1} p^{2t} = 2\beta^2 n/(2h)^2$ .

Deleting one member from each overlapping pair, we obtain a collection  $\mathcal{C}$  of vertex disjoint copies of  $P_{2h}^r$  with  $|\mathcal{C}| \leq \beta n$ , and such that, for every pair of disjoint ordered  $h$ -sets each spanning a  $K_h^{(k)}$  in  $H$ , there are at least  $3\beta^2 n/(2h)^2 - 2\beta^2 n/(2h)^2 = \beta^2 n/(2h)^2$  sets of  $2h$  vertices connecting them.  $\square$

**3.2. The Absorbing Lemma.** In this subsection we prove our absorbing lemma.

**Lemma 3.2** (Absorbing Lemma). *Suppose  $1/n \ll \varepsilon \ll \zeta \ll \alpha \ll 1/k, 1/r$ . Let  $H$  be an  $n$ -vertex  $k$ -graph with  $\delta_{k-1}(H) \geq (1 - c + \alpha)n$  and suppose  $p = p(n) \geq n^{-c-\varepsilon}$ . Then a.a.s.  $H \cup \mathbb{G}^{(k)}(n, p)$  contains an  $(r, k)$ -path  $P_{\text{abs}}$  of order at most  $6h\zeta n$  such that, for every set  $X \subseteq V(H) \setminus V(P_{\text{abs}})$  with  $|X| \leq \zeta^2 n/(2h)^2$ , there is an  $(r, k)$ -path in  $H$  on  $V(P_{\text{abs}}) \cup X$  that has the same ends as  $P_{\text{abs}}$ .*

We call the  $(r, k)$ -paths  $P_{\text{abs}}$  in Lemma 3.2 *absorbing paths*. We now define *absorbers*.

**Definition 3.3.** Let  $v$  be a vertex of a  $k$ -graph. An ordered  $2h$ -set of vertices  $(w_1, \dots, w_{2h})$  is a  $v$ -*absorber* if  $(w_1, \dots, w_{2h})$  spans a labelled copy of  $P_{2h}^r$  and  $(w_1, \dots, w_h, v, w_{h+1}, \dots, w_{2h})$  spans a labelled copy of  $P_{2h+1}^r$ .

*Proof of Lemma 3.2.* Suppose  $1/n \ll \varepsilon \ll \zeta \ll \beta \ll \alpha \ll 1/k, 1/r$ . We split the proof into two parts. We first find a set  $\mathcal{F}$  of absorbers and then connect them to an  $(r, k)$ -path by using Lemma 3.1 (Connecting Lemma). We will expose  $G = \mathbb{G}^{(k)}(n, p)$  in two rounds:  $G = G_1 \cup G_2$  with  $G_1$  and  $G_2$  independent copies of  $\mathbb{G}^{(k)}(n, p')$ , where  $(1 - p')^2 = 1 - p$ .

Fix a vertex  $v$ . By the codegree condition of  $H$ , we can extend  $v$  to a labelled copy of  $P_{2h+1}^r$  in the form  $(w_1, \dots, w_h, v, w_{h+1}, \dots, w_{2h})$  such that there are at least  $(\alpha n/2)^{2h} \geq 24\zeta n^{2h}$  choices for the ordered  $2h$ -set  $(w_1, \dots, w_{2h})$ . Let  $\mathcal{A}_v$  be a collection of exactly  $24\zeta n^{2h}$  such ordered  $2h$ -sets. By definition, if an ordered set  $A$  in  $\mathcal{A}_v$  spans a labelled copy of  $P_{2h}^r$ , then  $A$  is a  $v$ -absorber.

Now consider  $G_1 = \mathbb{G}^{(k)}(n, p')$  and let  $\mathcal{T}$  be the set of all labelled copies of  $P_{2h}^r$  in  $G_1$ . By Proposition 2.3, we can apply Lemma 2.2 with  $F = P_{2h}^r$  and  $\mathcal{A}_v$  in place of  $\mathcal{F}_i$ . Using the union bound we conclude that the following properties hold with probability at least  $1 - 3/\sqrt{n}$ :

- (a)  $|\mathcal{T}| \leq 2p^t n^{2h}$ ;
- (b) for every vertex  $v$  in  $H$ , at least  $12\zeta p^t n^{2h}$  members of  $\mathcal{T}$  are  $v$ -absorbers;
- (c) the number of overlapping pairs of members of  $\mathcal{T}$  is at most  $4(2h)^2 p^{2t} n^{4h-1}$ .

Next we select a random collection  $\mathcal{F}'$  by including each member of  $\mathcal{T}$  independently with probability  $q = \zeta/(2(2h)^2 p^t n^{2h-1})$ . In view of the properties above, by using Chernoff's inequality

(for (i) and (ii) below) and Markov's inequality (for (iii) below), we know that there is a choice of  $\mathcal{F}'$  that satisfies the following properties:

- (i)  $|\mathcal{F}'| \leq \zeta n$ ;
- (ii) for every vertex  $v$ , at least  $12\zeta(q/2)p^t n^{2h} = 3\zeta^2 n / (2h)^2$  members of  $\mathcal{F}'$  are  $v$ -absorbers;
- (iii) there are at most  $8(2h)^2 q^2 n^{4h-1} p^{2t} = 2\zeta^2 n / (2h)^2$  overlapping pairs of members of  $\mathcal{F}'$ .

By deleting from  $\mathcal{F}'$  one member from each overlapping pair and all members that are not in  $\mathcal{T}$ , we obtain a collection  $\mathcal{F}$  of vertex-disjoint copies of  $P_{2h}^r$  such that  $|\mathcal{F}| \leq \zeta n$ , and for every vertex  $v$ , there are at least  $3\zeta^2 n / (2h)^2 - 2\zeta^2 n / (2h)^2 = \zeta^2 n / (2h)^2$   $v$ -absorbers.

Now we connect these absorbers using Lemma 3.1. Let  $V' = V(H) \setminus V(\mathcal{F})$  and  $n' = |V'|$ . In particular,  $n' \geq n/2$  is sufficiently large. Now consider  $H' = H[V']$  and  $G' = G_2[V'] = \mathbb{G}^{(k)}(n', p')$ . Since  $|V(\mathcal{F})| \leq 2h \cdot \zeta n \leq \alpha^2 n$ , we have  $\delta_{k-1}(H') \geq (1 - c + \alpha/2)n$ . We apply Lemma 3.1 on  $H'$  and  $G'$  with  $\alpha' = \alpha/2$  and  $\beta$ , and conclude that a.a.s.  $H' \cup G'$  contains a set  $\mathcal{C}$  of vertex-disjoint copies of  $P_{2h}^r$  such that  $|\mathcal{C}| \leq \beta n$  and for every pair of ordered  $h$ -sets in  $V'$ , there are at least  $\beta^2 n$  members of  $\mathcal{C}$  connecting them.

For each copy of  $P_{2h}^r$  in  $\mathcal{F}$ , we greedily extend its two ends by  $h$  vertices such that all new paths are pairwise vertex disjoint and also vertex disjoint from  $V(\mathcal{C})$ . This is possible because of the codegree condition of  $H_0$  and  $|V(\mathcal{F})| + 2h|\mathcal{F}| + |V(\mathcal{C})| \leq 2h\zeta n + 2h\zeta n + 2h \cdot \beta n < \alpha n/4$ . Note that both ends of these  $(r, k)$ -paths  $P_{4h}^r$  are in  $V' \setminus V(\mathcal{C})$ . Since  $\zeta n \leq \beta^2 n' / (2h)^2$ , we can greedily connect these  $P_{4h}^r$ . Let  $P_{\text{abs}}$  be the resulting  $(r, k)$ -path. By construction,  $|V(P_{\text{abs}})| \leq (4h + 2h) \cdot \zeta n = 6h\zeta n$ . Moreover, for any  $X \subseteq V \setminus V(P_{\text{abs}})$  such that  $|X| \leq \zeta^2 n / (2h)$ , since each vertex  $v$  has at least  $\zeta^2 n / (2h)^2$   $v$ -absorbers in  $\mathcal{F}$ , we can absorb them greedily and conclude that there is an  $(r, k)$ -path on  $V(P_{\text{abs}}) \cup X$  that has the same ends as  $P_{\text{abs}}$ .  $\square$

#### §4. PROOF OF THEOREM 1.3

We now combine Lemmas 3.1 and 3.2 to prove Theorem 1.3.

*Proof of Theorem 1.3.* Suppose  $1/n \ll \varepsilon \ll \beta \ll \zeta \ll \alpha, 1/k, 1/r$ . Furthermore, recall that  $c := \binom{k+r-2}{k-1}^{-1}$  and suppose  $H \cup \mathbb{G}^{(k)}(n, p)$  is an  $n$ -vertex  $k$ -graph with  $\delta_{k-1}(H) \geq (1 - c + \alpha)n$  and  $p = p(n) \geq n^{-c-\varepsilon}$ . We will expose  $G := \mathbb{G}^{(k)}(n, p)$  in three rounds:  $G = G_1 \cup G_2 \cup G_3$  with  $G_1, G_2$  and  $G_3$  three independent copies of  $\mathbb{G}^{(k)}(n, p')$ , where  $(1 - p')^3 = 1 - p$ . Note that  $p' > p/3 > n^{-c-2\varepsilon}$ .

By Lemma 3.2 with  $2\varepsilon$  in place of  $\varepsilon$ , a.a.s. the  $k$ -graph  $H \cup G_1$  contains an absorbing  $(r, k)$ -path  $P_{\text{abs}}$  of order at most  $6h\zeta n$ , that is, for every set  $X \subseteq V(H) \setminus V(P_{\text{abs}})$  such that  $|X| \leq \zeta^2 n / (2h)^2$ , there is an  $(r, k)$ -path in  $H$  on  $V(P_{\text{abs}}) \cup X$  which has the same ends as  $P_{\text{abs}}$ . Let  $V' = V(H) \setminus V(P_{\text{abs}})$  and  $n' = |V'|$ . In particular,  $n' \geq (1 - 6h\zeta)n$  and, since  $\zeta$  is small enough, we have  $(n')^{c+\varepsilon} \geq n^{c+\varepsilon}/2$ . Thus  $p' > p/2 \geq n^{-c-\varepsilon}/2 \geq (n')^{-c-\varepsilon}/4 \geq (n')^{-c-2\varepsilon}$ .

Now consider  $H' = H[V']$  and let  $G'_2 := \mathbb{G}^{(k)}(n', p')$  be the subgraph of  $G_2$  induced by  $V'$ . Note that  $\delta_{k-1}(H') \geq \delta_{k-1}(H) - |V(P_{\text{abs}})| \geq (1 - c + \alpha/2)n'$ . By Lemma 3.1, a.a.s. the  $k$ -graph  $H' \cup G'_2$  contains a set  $\mathcal{C}$  of vertex-disjoint copies of  $P_{2h}^r$  such that  $|\mathcal{C}| \leq \beta n$  and for every pair of disjoint ordered  $h$ -sets in  $V'$  that each spans a copy of  $K_h^{(k)}$ , there are at least  $\beta^2 n' / (2h)^2$  members of  $\mathcal{C}$  connecting them. Since  $|V(\mathcal{C})| + |V(P_{\text{abs}})| \leq 2h \cdot \beta n + 6h\zeta n \leq \alpha n/2$ , we can greedily extend the two ends of  $P_{\text{abs}}$  by  $h$  vertices so that the two new ends  $E_1, E_2$  are in  $V' \setminus V(\mathcal{C})$ .

Let  $m := g^{-1}(1/(2\varepsilon))$ . Note that  $m \geq 1/\sqrt{\varepsilon}$  because  $\varepsilon$  is small enough and  $g$  is linear. By Proposition 2.3, we can apply Lemma 2.2 (i) with  $b = m$  on  $G_3$  and conclude that a.a.s. every induced subgraph of  $G_3$  of order  $\beta n$  contains a copy of  $P_m^r$ . Thus we can greedily find at most  $\sqrt{\varepsilon}n$  vertex-disjoint copies of  $P_m^r$  in  $V \setminus (V(\mathcal{C}) \cup E_1 \cup E_2)$ , which together covers all but at most  $\beta n$  vertices of  $V \setminus V(\mathcal{C})$ . Since  $\sqrt{\varepsilon}n + 1 \leq \beta^2 n' / (2h)^2$ , we can greedily connect these  $(r, k)$ -paths  $P_m^r$  and  $P_{\text{abs}}$  to an  $(r, k)$ -cycle  $Q^r$ . Let  $R := V(H) \setminus V(Q^r)$  and note that  $|R| \leq |V(\mathcal{C})| + \beta n \leq (2h + 1)2\beta n \leq \zeta^2 n / (2h)^2$ . Since  $P_{\text{abs}}$  is an absorber, there is an  $(r, k)$ -path on  $V(P_{\text{abs}}) \cup R$  which has the same ends as  $P_{\text{abs}}$ . So we can replace  $P_{\text{abs}}$  by this  $(r, k)$ -path in  $Q^r$  and obtain the  $r^{\text{th}}$  power of a tight Hamilton cycle.

Moreover, since all previous steps can be achieved a.a.s., by the union bound,  $H \cup G$  a.a.s. contains the desired  $r^{\text{th}}$  power of a tight Hamilton cycle.  $\square$

## §5. CONCLUDING REMARKS

Let us briefly discuss the hypotheses in Theorem 1.3. Note that, for  $r = 1$ , the condition in (1) is simply  $\delta_{k-1}(H) \geq \alpha n$ , with  $\alpha$  any arbitrary positive constant. Thus, in this case, our theorem is in the spirit of the original Bohman, Frieze and Martin [5] set-up, in the sense that we have a similar minimum degree condition on the deterministic graph  $H$ . However, if  $r > 1$ , then our minimum condition (1) is of the form  $\delta_{k-1}(H) \geq (\sigma + \alpha)n$  for some  $\sigma = \sigma(k, r) > 0$  (and arbitrarily small  $\alpha > 0$ ). Thus, for  $r > 1$ , our result is more in line with Theorem 1.2 of Bennett, Dudek and Frieze [4] (in fact, we have  $\sigma(2, 2) = 1/2$  in our result, which matches the minimum degree condition in Theorem 1.2). It is natural to ask whether one can weaken the condition in (1) to  $\delta_{k-1}(H) \geq \alpha n$ , that is, whether one can have  $\sigma = 0$ . This problem was settled positively by Böttcher, Montgomery, Parczyk and Person for graphs [7]. However, the problem remains open for  $k$ -graphs ( $k \geq 3$ ).

**Question 5.1.** *Let integers  $k \geq 3$  and  $r \geq 2$  and  $\alpha > 0$  be given. Is there  $\varepsilon > 0$  such that, if  $H$  is a  $k$ -graph on  $n$  vertices with  $\delta_{k-1}(H) \geq \alpha n$  and  $p = p(n) \geq n^{-\binom{k+r-2}{k-1}^{-1-\varepsilon}}$ , then a.a.s.  $H \cup \mathbb{G}^{(k)}(n, p)$  contains the  $r^{\text{th}}$  power of a tight Hamilton cycle?*

Two remarks on the value of  $\sigma = \sigma(k, r)$  in our degree condition (1) follow. These remarks show that, even though  $\sigma > 0$  if  $r > 1$ , the value of  $\sigma$  is (in the cases considered) below the value that guarantees that  $H$  on its own contains the  $r^{\text{th}}$  power of a tight Hamilton cycle.

Let us first consider the case  $k = 2$ , that is, the case of graphs. In this case,  $\sigma = 1 - 1/r$  and condition (1) is  $\delta(H) \geq (1 - 1/r + \alpha)n$ . We observe that this condition does *not* guarantee that  $H$  contains the  $r^{\text{th}}$  power of a Hamilton cycle; the minimum degree condition that does is  $\delta(H) \geq (1 - 1/(r + 1))n = rn/(r + 1)$ , and this value is optimal.

Let us now consider the case  $k = 3$  and  $4 \mid n$ . In this case, a construction of Pikhurko [32] shows that the condition  $\delta_2(H) \geq 3n/4$  does not guarantee the existence of the square of a tight Hamilton cycle in  $H$  (in fact, his construction is stronger and shows that this condition does not guarantee a  $K_4^{(3)}$ -factor in  $H$ ). Our minimum degree condition for  $k = 3$  and  $r = 2$  is  $\delta_2(H) \geq (2/3 + \alpha)n$ .

Finally, a simple calculation shows that the expected number of  $P_n^r$  in  $\mathbb{G}^{(k)}(n, p)$  is  $o(1)$  if  $p \leq ((1 - \varepsilon)e/n)^{\binom{k+r-2}{k-1}^{-1}}$  and  $\varepsilon > 0$ . Thus, for such a  $p$ , a.a.s.  $\mathbb{G}^{(k)}(n, p)$  does *not* contain the  $r^{\text{th}}$  power of a tight Hamilton cycle.

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