

ON BREADTH-FIRST CONSTRUCTIONS OF SCALING LIMITS OF RANDOM GRAPHS AND RANDOM UNICELLULAR MAPS

GRÉGORIE MIERMONT¹ AND SANCHAYAN SEN²

ABSTRACT. We give alternate constructions of (i) the scaling limit of the uniform connected graphs with given fixed surplus, and (ii) the continuum random unicellular map (CRUM) of a given genus that start with a suitably tilted Brownian continuum random tree and make ‘horizontal’ point identifications, at random heights, using the local time measures. Consequently, this can be seen as a continuum analogue of the breadth-first construction of a finite connected graph. In particular, this yields a breadth-first construction of the scaling limit of the critical Erdős-Rényi random graph which answers a question posed in [2]. As a consequence of this breadth-first construction we obtain descriptions of the radii, the distance profiles, and the two point functions of these spaces in terms of functionals of tilted Brownian excursions.

1. INTRODUCTION AND DEFINITIONS

This paper studies properties of random metric spaces that arise naturally in the study of critical random graphs and random maps, by providing new constructions of these objects. A common feature of these spaces is that they look locally like random trees, which makes it possible to construct them by performing certain gluings in models of continuum random trees. One nice aspect of the particular “breadth-first construction” considered in this paper is that it can be defined canonically for a wide family of (deterministic) metric spaces called \mathbb{R} -graphs. We start by explaining this construction.

Recall that a metric space (X, d) is called geodesic if for every $x, y \in X$, there exists an isometric embedding $\psi : [0, d(x, y)] \rightarrow X$ with $\psi(0) = x$ and $\psi(d(x, y)) = y$. We call ψ a geodesic path between x and y . A compact geodesic metric space is called an \mathbb{R} -tree [26, 36] if it has no embedded cycles. A compact geodesic metric space (X, d) is called an \mathbb{R} -graph [4] if for every $x \in X$, there exists $\varepsilon > 0$ such that

$$B(x, \varepsilon; X) := \{y \in X : d(y, x) \leq \varepsilon\}$$

with the induced metric $d|_{B(x, \varepsilon; X)}$ is an \mathbb{R} -tree. A measured \mathbb{R} -graph is an \mathbb{R} -graph with a probability measure on its Borel σ -algebra. By [4, Theorem 2.7], (X, d) is an \mathbb{R} -graph if and only if there exists a finite connected multigraph $G = (V, E)$ and a collection $\{(T_e, x_e, y_e) :$

¹UNITÉ DE MATHÉMATIQUES PURES ET APPLIQUÉES, ÉCOLE NORMALE SUPÉRIEURE DE LYON, FRANCE

²DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE, BANGALORE, INDIA

E-mail addresses: gregory.miermont@ens-lyon.fr, sanchayan.sen1@gmail.com.

2010 *Mathematics Subject Classification.* Primary: 60C05, 05C80.

Key words and phrases. Erdős-Rényi random graph, critical random graphs, unicellular maps, Gromov-Hausdorff distance, scaling limit, continuum random trees, breadth-first construction, depth-first construction.

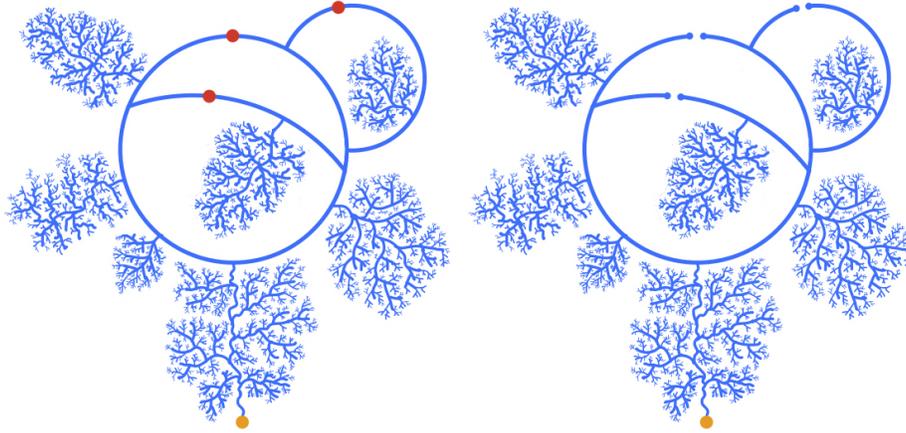


FIGURE 1.1. On the left, an \mathbb{R} -graph X with its root colored orange and points in \mathfrak{X} colored red. On the right, the space X' .

$e \in E$ where T_e is an \mathbb{R} -tree and $x_e, y_e \in T_e$ such that (X, d) is isometric to the space constructed from G by performing a metric gluing of the spaces T_e , replacing each edge $e \in E$ by T_e and identifying x_e with one endpoint of e and y_e with the other endpoint of e . Random measured \mathbb{R} -graphs arise naturally as scaling limits of various random graphs [2, 3, 12, 14–16].

For an \mathbb{R} -graph (X, d) and $x \in X$, choose $\varepsilon > 0$ such that $B(x, \varepsilon; X)$ is an \mathbb{R} -tree, and define the degree of x as

$$\deg(x; X) := \#\{\text{connected components of } B(x, \varepsilon; X) \setminus \{x\}\}.$$

Note that the value of $\deg(x; X)$ is independent of the choice of ε . A triple (X, d, x_*) where (X, d) is an \mathbb{R} -graph and $x_* \in X$ is a distinguished point is called a rooted \mathbb{R} -graph. We can similarly define a measured rooted \mathbb{R} -graph. We define the radius of X to be

$$\text{Rad}(X) := \sup_{x \in X} d(x, x_*).$$

Let (X, d, x_*) be a rooted \mathbb{R} -graph. Let \mathfrak{X} be the cut locus of x_* in X , i.e., the set of all points x in X such that there exist at least two distinct (not necessarily disjoint) geodesic paths from x_* to x . Then $X \setminus \mathfrak{X}$ is connected, since no point on the unique geodesic between a point $x \in X \setminus \mathfrak{X}$ and x_* can be in \mathfrak{X} . Assume further that

$$\deg(x; X) = 2 \text{ for every } x \in \mathfrak{X}. \quad (1.1)$$

Then it is easy to see that \mathfrak{X} is a finite set, for instance from the description of X as a metric gluing of doubly-rooted \mathbb{R} -trees (T_e, x_e, y_e) , $e \in E$, along a finite graph $G = (V, E)$: for every point x of \mathfrak{X} , there exists $e \in E$ such that e belongs to one of the finitely many cycles in G and x is a point on the geodesic path connecting x_e and y_e in T_e , and removing x has the effect of decreasing the total number of cycles. From now on we will only work with \mathbb{R} -graphs that satisfy (1.1).

For a rooted \mathbb{R} -graph (X, d, x_*) satisfying (1.1), define X' to be the completion of the metric space $X \setminus \mathfrak{X}$ endowed with the intrinsic metric inherited from d . Let d' denote the metric on X' , and for any $x \in X \setminus \mathfrak{X}$ let $x' \in X'$ denote the corresponding point. We root X'

at x'_* . Then (X', d', x'_*) is a rooted \mathbb{R} -tree, which we think of as the ‘breadth-first spanning tree’ of (X, d, x_*) . Consider the injective map $x \mapsto x'$ defined on $X \setminus \mathfrak{X}$ and let $p : X' \rightarrow X$ be the unique continuous extension of its inverse to the whole of X' . Then $p^{-1}(x) = \{x'\}$ for $x \in X \setminus \mathfrak{X}$, and $p^{-1}(x)$ has exactly two elements $x^{(1)}, x^{(2)}$ for $x \in \mathfrak{X}$, and both are leaves of X' with $d'(x'_*, x^{(1)}) = d'(x'_*, x^{(2)}) = d(x_*, x)$.

If (X, d, x_*, μ) is a rooted measured \mathbb{R} -graph and μ is non-atomic, then we endow X' with μ' —the natural measure on X' induced by μ , that is, $\mu'(A) = \mu(p(A))$ for a measurable A in (X', d') and p is as described above. This yields a mapping

$$(X, d, x_*, \mu) \mapsto (X', d', x'_*, \{\{x^{(1)}, x^{(2)}\} : x \in \mathfrak{X}\}, \mu') \quad (1.2)$$

from the set of rooted \mathbb{R} -graphs that satisfy (1.1) and are endowed with a non-atomic measure to the set of rooted \mathbb{R} -trees with a non-atomic measure and a finite number of distinguished pairs of leaves that are equidistant to the root. This mapping is inverted simply by identifying the distinguished leaves in pairs and taking the metric quotient. We do not give all details since we are not going to use this correspondence explicitly, but it helps in understanding some parts of our constructions.

The space X' has some nice properties as we describe next. Note that in going from X to X' , distances from the root are preserved: For any $z \in X'$, $d'(z, x'_*) = d(p(z), x_*)$, and in particular,

$$\text{Rad}(X) = \text{Rad}(X'). \quad (1.3)$$

Further, for any $r > 0$,

$$\mu(B(x_*, r; X)) = \mu'(B(x'_*, r; X')). \quad (1.4)$$

For $a < b$, we call a continuous function $e : [a, b] \rightarrow [0, \infty)$ with $e(a) = e(b) = 0$ a continuous excursion on $[a, b]$. For a continuous excursion e on $[0, 1]$, write $(\mathcal{T}_e, d_e, \rho_e, \mu_e)$ for the rooted (at ρ_e) measured \mathbb{R} -tree encoded by e ; see Section 2 for a precise definition. Since X' is an \mathbb{R} -tree (recall that in our definition, \mathbb{R} -trees are always compact), by [25, Corollary 1.2], there exists a continuous excursion e_\bullet on $[0, 1]$ such that (X', d', x'_*) is isometric to $(\mathcal{T}_{e_\bullet}, d_{e_\bullet}, \rho_{e_\bullet})$ as rooted metric spaces. Thus, using (1.3), we get the following description of the radius of X :

$$\text{Rad}(X) = \|e_\bullet\|_\infty := \sup \{e_\bullet(t) : t \in [0, 1]\}. \quad (1.5)$$

If (X', d', x'_*, μ') is isometric to $(\mathcal{T}_{e_\bullet}, d_{e_\bullet}, \rho_{e_\bullet}, \mu_{e_\bullet})$ as rooted metric measure spaces, then using (1.4) and writing λ for the Lebesgue measure on \mathbb{R} , we get

$$\mu(B(x_*, r; X)) = \mu'(B(x'_*, r; X')) = \mu_{e_\bullet}(B(\rho_{e_\bullet}, r; \mathcal{T}_{e_\bullet})) = \lambda(\{t \in [0, 1] : e_\bullet(t) \leq r\}), \quad (1.6)$$

where the last step follows from the definition of the measured \mathbb{R} -tree coded by e_\bullet ; see the beginning of Section 2 for the precise definition.

For any $a > 0$ and any $f \in C[0, a]$ (not necessarily an excursion), we can define the occupation measure of f at time $t \in [0, a]$, which we denote by $\mathfrak{N}(f; t, dy)$, by the formula

$$\int_0^t \phi(f(u)) du = \int_{y \in \mathbb{R}} \phi(y) \mathfrak{N}(f; t, dy)$$

for every bounded continuous $\phi : \mathbb{R} \rightarrow \mathbb{R}$. Now suppose that $\mathfrak{N}(f; t, dy) = \eta(f; t, y) dy$ is absolutely continuous with respect to the Lebesgue measure for every $t \in [0, a]$, and that

we can choose a version of the density $\eta(f; t, y)$ that is jointly continuous in $t \in [0, a]$ and $y \in \mathbb{R}$. Then we call $\eta(f; t, y)$ the *local time* of f at level y and time t , and say that f admits a continuous local time. From this definition it follows that for each fixed $y \in \mathbb{R}$, the function $\eta(f; \cdot, y)$ is non-decreasing on $[0, a]$, and the corresponding Stieltjes measures $\eta(f; dt, y)$ satisfy

$$\int_0^a \phi(t, f(t)) dt = \int_{y \in \mathbb{R}} dy \int_{t=0}^a \phi(t, y) \eta(f; dt, y) \quad (1.7)$$

for every bounded continuous $\phi : [0, a] \times \mathbb{R} \rightarrow \mathbb{R}$.

Now if (X', d', x_*, μ') is isometric to $(\mathcal{T}_{e_\bullet}, d_{e_\bullet}, \rho_{e_\bullet}, \mu_{e_\bullet})$ for a continuous excursion e_\bullet on $[0, 1]$ and e_\bullet admits a continuous local time, then the right side of (1.6) is differentiable w.r.t. r , and further,

$$\frac{d}{dr} \mu(B(x_*, r; X)) = \eta(e_\bullet; 1, r) \quad \text{for all } r > 0. \quad (1.8)$$

The left side of (1.8) gives the distance profile around the root x_* .

Thus, if X is a random rooted measured \mathbb{R} -graph and we can identify the random excursion e_\bullet that encodes X' , and e_\bullet admits a continuous local time, then we can express the radius of X and the distance profile around the root of X in terms of the supremum of e_\bullet and the local time of e_\bullet . For the random \mathbb{R} -graphs considered in this paper, the root x_* will be a μ -distributed point in (X, d, μ) . In this case the two-point function of (X, d, μ) will have the same law as $e_\bullet(U)$, where $U \sim \text{Uniform}[0, 1]$ is independent of e_\bullet .

In this paper we will consider the following two random \mathbb{R} -graphs:

- (i) the scaling limit $\mathcal{H}_{(s)}$ of uniform connected rooted (labeled) graphs with surplus s , and
 - (ii) the scaling limit $\text{CRUM}_{(g)}$ of random unicellular maps of genus g considered in [3].
- The precise definitions of these spaces will be given in Section 2. We will identify the distributions of their breadth-first spanning trees $\mathcal{H}'_{(s)}$ and $\text{CRUM}'_{(g)}$, and we will describe how to glue random points on these \mathbb{R} -trees to recover the spaces $\mathcal{H}_{(s)}$ and $\text{CRUM}_{(g)}$ in distribution.

The spaces $\mathcal{H}_{(s)}$, $s \geq 0$, are the building blocks for the scaling limit of the critical Erdős-Rényi random graph identified in [2]; see [6, Construction 3.10 and Theorem 3.12] for a precise statement. The Erdős-Rényi scaling limit is a universal object in the sense that it arises as the scaling limit of a wide array of standard models of critical random discrete structures exhibiting mean-field behavior. Examples of such models include random regular graphs under critical percolation or more generally critical random graphs with given degree sequence (under finite third moment assumption on the degrees), various models of inhomogeneous random graphs (under appropriate assumptions), bounded-size rules, and the vacant set left by random walks on random regular graphs; see [12, 15, 16]. Further, existing literature suggests that the components of the high-dimensional discrete torus [30, 31, 33] and the hypercube [32] under critical percolation, and the critical quantum random graph model [22] also share the Erdős-Rényi scaling limit. The breadth-first construction of $\mathcal{H}_{(s)}$ (given in Theorem 3.1 below) in particular yields the same for this universal scaling limit.

The construction of the Erdős-Rényi scaling limit given in [2] can be thought of as a ‘depth-first construction’; see Construction 2.1 below. An alternate construction that can

be seen as a ‘core decomposition’ was given in [1]. The advantage of the breadth-first perspective, as explained earlier, is that it directly identifies the radius, the two-point function, and the distance profile in terms functionals of suitably tilted Brownian excursions, which the two constructions above do not. The problem of identifying the breadth-first construction of the Erdős-Rényi scaling limit was asked in [2, Section 6]. This was the main motivation behind this work. Quoting the authors of [2]:

The rescaled breadth-first walk ... converges to the same limit as the rescaled *height profile* (i.e. the number of vertices at each height) of a “breadth-first tree”, which contains less information and, in particular, does not code the structure of that tree. As a result, it seems that it would be much harder to derive a metric space construction of a limiting component using the breadth-first viewpoint.

The breadth-first approach indeed requires more than just the convergence of the rescaled breadth-first walk. This will be discussed in more detail in Section 6.

Comments on the use of maps in our approach: Our proofs will make a thorough use of maps, not only in the study of $\text{CRUM}_{(g)}$, but also in that of the spaces $\mathcal{H}_{(s)}$. This might look a bit surprising at first sight. Indeed, the spaces $\mathcal{H}_{(s)}$ were initially introduced as scaling limits of random labeled graphs. The most natural approach would be to perform a breadth-first search on the same family of graphs, cutting cycles as they appear in this exploration in order to obtain a random labeled tree with a certain law that one can compare to that of a uniform rooted labeled tree $H_{n,0}$ of a given size n . Running the construction backwards, to obtain the random labeled graph starting from this random tree, one has to choose pairs of vertices in the tree at (roughly) the same height and connect them by an edge. A difficulty then appears when trying to understand the scaling limit of this construction: the operation of choosing random pairs of vertices at the same height in a continuum tree coded by some excursion e amounts to sampling them according to measures associated with the local time of e , and proving the convergence then requires a very good understanding of the discrete versions of the local time associated with either the height function or the contour function of the tree $H_{n,0}$ as $n \rightarrow \infty$. Unfortunately, such results are not available in the literature on random trees.

Instead, we take an indirect approach by considering a different model: uniform maps with a fixed surplus. Using Construction 2.1 given below, we will show that the scaling limit of this model is also $\mathcal{H}_{(s)}$. Now, by considering a breadth-first exploration of this model, we circumvent the issue with local times mentioned above by connecting the original problem to a problem about suitably tilted plane trees (as opposed to labeled trees). The contour function of a uniform plane tree is a simple random walk excursion, and this will allow us to apply existing results about local time fields of random walks in our proofs. Owing to this reason, the proof of the breadth-first construction of $\mathcal{H}_{(s)}$ proceeds via a study of such maps.

Organization of this paper: The rest of this paper is organized as follows: In Section 2, we describe the constructions of the random metric spaces involved. In Section 3, we state our main results. In Section 4, we explain the notation and conventions used in this paper; in particular, the notation related to maps will be described. The proofs of our

main results will be given in various subsections in Section 5. Section 6 contains some related questions and further discussions. The proof of a key result used in Section 5 will be outlined in Appendix A.

Convention about metric spaces: We fix a convention here that we will follow throughout the paper. For any metric measure space $X = (X, d, \mu)$ and $\alpha > 0$, $\alpha \cdot X$ or simply αX will denote the metric measure space $(X, \alpha d, \mu)$, i.e, the space where the metric has been multiplied by α and the measure μ has remained unchanged. We can similarly define αX for a rooted metric measure space X by leaving the root unchanged.

When dealing with convergence of rooted metric measure spaces, we will work with the topology induced by the ‘pointed Gromov-Hausdorff-Prokhorov (GHP) distance,’ which we denote by $d_{\text{GHP}}^1(\cdot, \cdot)$. We refer the reader to [4, Section 2.1] for the relevant definitions.

2. CONSTRUCTIONS OF THE SPACES

Let e be a continuous excursion on $[0, 1]$. Let d_e be the pseudo-metric on $[0, 1]$ given by

$$d_e(s, t) := e(s) + e(t) - 2 \inf_{u \in [s, t]} e(u), \text{ for } s, t \in [0, 1]. \quad (2.1)$$

Define the equivalence relation $s \sim t \Leftrightarrow d_e(s, t) = 0$. Let $[0, 1] / \sim$ denote the corresponding quotient space and consider the space $\mathcal{T}_e := [0, 1] / \sim$ endowed with the quotient metric on the equivalence classes induced by d_e . We abuse notation and write d_e for the quotient metric on \mathcal{T}_e as well. Then (\mathcal{T}_e, d_e) is an \mathbb{R} -tree [26, 36]. Let $q_e : [0, 1] \rightarrow \mathcal{T}_e$ be the canonical projection and write μ_e for the push-forward of the Lebesgue measure on $[0, 1]$ onto \mathcal{T}_e via q_e . Further, we let \mathcal{T}_e be rooted at $\rho_e := q_e(0)$. Then $(\mathcal{T}_e, d_e, \rho_e, \mu_e)$ is a rooted measured \mathbb{R} -tree. Note that by construction, for any $x \in \mathcal{T}_e$, the function e is constant on $q_e^{-1}(\{x\})$. Note also that the height of $x \in \mathcal{T}_e$ defined as $\text{ht}(x; \mathcal{T}_e) := d_e(\rho_e, x)$ satisfies $\text{ht}(x; \mathcal{T}_e) = e(u)$ for any $u \in q_e^{-1}(\{x\})$. We define the set of leaves of \mathcal{T}_e to be

$$\mathcal{L}(\mathcal{T}_e) := \{x \in \mathcal{T}_e : \deg(x; \mathcal{T}_e) = 1\}.$$

We will write $(\mathbf{e}(t), t \in [0, 1])$ for a standard Brownian excursion. The \mathbb{R} -tree $2 \cdot \mathcal{T}_{\mathbf{e}}$ is called the Brownian continuum random tree (recall our convention about metric spaces explained at the end of Section 1). It is well-known [7, 8] that the measure $\mu_{\mathbf{e}}$ (also called the mass measure) on $\mathcal{T}_{\mathbf{e}}$ is non-atomic and concentrated on $\mathcal{L}(\mathcal{T}_{\mathbf{e}})$ almost surely.

2.1. Constructions of the scaling limit of uniform connected graphs with fixed surplus.

We will now define the random spaces $\mathcal{H}_{(s)}$, $s \geq 0$, that were introduced in Section 1. For a finite connected graph $G = (V, E)$, let $\text{sp}(G) := |E| - |V| + 1$ denote the number of surplus (also called excess) edges in G . For $n \geq 1$ and $s \geq 0$ define

$$\mathbb{H}_{n,s} = \{G : G \text{ rooted, connected, simple, labeled graph on } [n] \text{ with } \text{sp}(G) = s\}, \quad (2.2)$$

where $[n] := \{1, \dots, n\}$. Let $H_{n,s}$ be uniformly distributed over $\mathbb{H}_{n,s}$. View $H_{n,s}$ as a rooted metric measure space by endowing it with the graph distance and the uniform probability measure on the vertices. Then there exists a random compact, rooted metric measure space $\mathcal{H}_{(s)}$ such that

$$n^{-1/2} \cdot H_{n,s} \xrightarrow{d} \mathcal{H}_{(s)} \quad (2.3)$$

w.r.t. the pointed GHP topology. The space $\mathcal{H}_{(0)}$ is simply $2 \cdot \mathcal{T}_{\mathbf{e}}$ —the Brownian continuum random tree [7, 8]. For $s \geq 1$, (2.3) can be proved by arguments similar to the ones used in [2, 15]; a brief sketch of the proof is given in [6, Section A.1].

The spaces $\mathcal{H}_{(s)}$, $s \geq 0$, are central to describing the scaling limits of many other random discrete structures. As mentioned before, the critical Erdős-Rényi scaling limit identified in [2] can be expressed in terms of the spaces $\mathcal{H}_{(s)}$, $s \geq 0$. The spaces $\mathcal{H}_{(s)}$ also arise as the scaling limit of

- (i) uniform connected graphs with a given degree sequence under some assumptions on the degree sequence [15, Theorem 2.4], and
- (ii) uniform rooted maps with fixed surplus s (without any restriction on the genus); see Remark 4 below.

The scaling limit of the minimal spanning tree of the complete graph identified in [4] can be expressed as the limit, as $s \rightarrow \infty$, of the space obtained by applying a ‘cycle-breaking’ procedure on the space $(12s)^{1/6} \cdot \mathcal{H}_{(s)}$ [6, Theorem 4.8]. We now describe one construction of $\mathcal{H}_{(s)}$.

Construction 2.1 (Depth-first construction of $\mathcal{H}_{(s)}$). *Fix an integer $s \geq 0$.*

- (a) *Sample $\mathbf{e}_{(s)}^{\text{DF}}$ with law given by*

$$\mathbb{E}[\phi(\mathbf{e}_{(s)}^{\text{DF}})] = \frac{\mathbb{E}[\phi(\mathbf{e})(\int_0^1 \mathbf{e}(t) dt)^s]}{\mathbb{E}[(\int_0^1 \mathbf{e}(t) dt)^s]} \quad (2.4)$$

for every bounded continuous $\phi : C[0, 1] \rightarrow \mathbb{R}$.

- (b) *Conditional on $\mathbf{e}_{(s)}^{\text{DF}}$, sample i.i.d. points u_1, \dots, u_s in $[0, 1]$ with density*

$$\left(\frac{\mathbf{e}_{(s)}^{\text{DF}}(u)}{\int_0^1 \mathbf{e}_{(s)}^{\text{DF}}(t) dt} \right) du, \quad u \in [0, 1].$$

- (c) *Conditional on $\mathbf{e}_{(s)}^{\text{DF}}$ and u_1, \dots, u_s , sample independent points z_1, \dots, z_s , where z_i is uniformly distributed in $[0, \mathbf{e}_{(s)}^{\text{DF}}(u_i)]$. For $1 \leq i \leq s$, let*

$$v_i := \inf \{ t \geq u_i : \mathbf{e}_{(s)}^{\text{DF}}(t) = z_i \}.$$

- (d) *Set $\mathcal{H}_{(s)}$ to be the quotient space $2 \cdot (\mathcal{T}_{\mathbf{e}_{(s)}^{\text{DF}}} / \sim)$, where \sim is the smallest equivalence relation under which $q_{\mathbf{e}_{(s)}^{\text{DF}}}(u_i) \sim q_{\mathbf{e}_{(s)}^{\text{DF}}}(v_i)$ for $1 \leq i \leq s$. Here, $\mathcal{H}_{(s)}$ is endowed with the root and the measure inherited from $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{DF}}}$ by the quotient map. That is, $\mathcal{H}_{(s)}$ is the rooted measured \mathbb{R} -graph obtained by identifying the points $q_{\mathbf{e}_{(s)}^{\text{DF}}}(u_i)$ and $q_{\mathbf{e}_{(s)}^{\text{DF}}}(v_i)$ on $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{DF}}}$ and then multiplying the distances by 2.*

The above construction of the space $\mathcal{H}_{(s)}$ is essentially contained in the arguments given in [2]. The reason for using the notation $\mathbf{e}_{(s)}^{\text{DF}}$ is that Construction 2.1 can be thought of as the continuum analogue of the depth-first construction of a finite connected graph. The \mathbb{R} -tree $2 \cdot \mathcal{T}_{\mathbf{e}_{(s)}^{\text{DF}}}$ plays the role of the depth-first spanning tree of $\mathcal{H}_{(s)}$.

One can heuristically argue that the expression for the tilt in (2.4) has to be proportional to $(\int_0^1 \mathbf{e}(t) dt)^s$ as follows: The space $\mathcal{H}_{(s)}$ can be seen as a ‘uniform rooted measured \mathbb{R} -graph having s cycles’ (see (2.3)) just as the Brownian continuum random tree $2 \cdot \mathcal{T}_{\mathbf{e}}$ is, in a

sense, a “uniform random rooted measured \mathbb{R} -tree.” Construction 2.1 gives a construction of $\mathcal{H}_{(s)}$ by identifying s i.i.d. pairs of points on $2 \cdot \mathcal{T}_{\mathbf{e}_{(s)}^{\text{DF}}}$, where each pair is uniformly distributed subject to the constraint that the pair consists of a point and an ancestor of that point. Accordingly, the law of $2 \cdot \mathcal{T}_{\mathbf{e}_{(s)}^{\text{DF}}}$ is the one obtained by tilting the law of $2 \cdot \mathcal{T}_{\mathbf{e}}$ by the total “weight” of possible chosen pairs, which equals

$$\left(\int_{x \in \mathcal{T}_{\mathbf{e}}} 2 \cdot \text{ht}(x; \mathcal{T}_{\mathbf{e}}) \mu_{\mathbf{e}}(dx) \right)^s = \left(\int_0^1 2\mathbf{e}(t) dt \right)^s.$$

Alternately, one can arrive at the expression for the tilt by looking at the analogous problem in the discrete setting; see the proof of Proposition 5.2.

We will now define another space $\mathcal{H}_{(s)}^{\text{BF}}$. Recall, as is well-known, that \mathbf{e} admits a.s. a continuous local time $\eta(\mathbf{e}; \cdot, \cdot)$ as defined around (1.7). The same is true of any random process with a law that is absolutely continuous with respect to that of \mathbf{e} , which justifies the following construction.

Construction 2.2 (The space $\mathcal{H}_{(s)}^{\text{BF}}$). *Fix an integer $s \geq 0$.*

(a) *Sample $\mathbf{e}_{(s)}^{\text{BF}}$ with law given by*

$$\mathbb{E}[\phi(\mathbf{e}_{(s)}^{\text{BF}})] = \frac{\mathbb{E}\left[\phi(\mathbf{e}) \cdot \left(\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy\right)^s\right]}{\mathbb{E}\left[\left(\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy\right)^s\right]} \quad (2.5)$$

for every bounded continuous $\phi : C[0, 1] \rightarrow \mathbb{R}$. Write

$$\eta_{(s)}^{\text{BF}}(\cdot, \cdot) = \eta(\mathbf{e}_{(s)}^{\text{BF}}; \cdot, \cdot). \quad (2.6)$$

(b) *Conditional on $\mathbf{e}_{(s)}^{\text{BF}}$, sample i.i.d. points H_1, \dots, H_s in $[0, \infty)$ with density*

$$\left(\frac{\eta_{(s)}^{\text{BF}}(1, h)^2}{\int_0^\infty \eta_{(s)}^{\text{BF}}(1, y)^2 dy} \right) dh. \quad (2.7)$$

(c) *Conditional on $\mathbf{e}_{(s)}^{\text{BF}}$ and H_1, \dots, H_s , sample $2s$ independent points $u_1, v_1, \dots, u_s, v_s$ in $[0, 1]$, where u_i and v_i are distributed according to the law*

$$\frac{\eta_{(s)}^{\text{BF}}(dt, H_i)}{\eta_{(s)}^{\text{BF}}(1, H_i)}$$

for $1 \leq i \leq s$.

(d) *Set $\mathcal{H}_{(s)}^{\text{BF}}$ to be the quotient space $2 \cdot (\mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}} / \sim)$, where \sim is the smallest equivalence relation under which $q_{\mathbf{e}_{(s)}^{\text{BF}}}(u_i) \sim q_{\mathbf{e}_{(s)}^{\text{BF}}}(v_i)$ for $1 \leq i \leq s$. Here, $\mathcal{H}_{(s)}^{\text{BF}}$ is endowed with the root and the measure inherited from $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$ by the quotient map. That is, $\mathcal{H}_{(s)}^{\text{BF}}$ is the rooted measured \mathbb{R} -graph obtained by identifying the points $q_{\mathbf{e}_{(s)}^{\text{BF}}}(u_i)$ and $q_{\mathbf{e}_{(s)}^{\text{BF}}}(v_i)$ on $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$ and then multiplying the distances by 2.*

Let us describe the above construction in words. We first sample a tilted excursion $\mathbf{e}_{(s)}^{\text{BF}}$. Then we sample s i.i.d. ‘heights’ H_1, \dots, H_s according to density (2.7). For $1 \leq i \leq s$, we sample two points in the tree $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$ independently according to the normalized local time

measure at height H_i , and then we glue these points. Finally, we multiply the metric by 2 in the resulting space.

Almost surely, for every $y > 0$ the measure $\eta(\mathbf{e}; dt, y)$ is concentrated on $q_{\mathbf{e}}^{-1}(\mathcal{L}(\mathcal{T}_{\mathbf{e}}))$. By absolute continuity, the same is true if we replace \mathbf{e} by $\mathbf{e}_{(s)}^{\text{BF}}$. Thus $q_{\mathbf{e}_{(s)}^{\text{BF}}}(u_i)$ and $q_{\mathbf{e}_{(s)}^{\text{BF}}}(v_i)$ are leaves in $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$, $1 \leq i \leq s$. Further, almost surely $\mathbf{e}_{(s)}^{\text{BF}}(u_i) = \mathbf{e}_{(s)}^{\text{BF}}(v_i) = H_i$, $1 \leq i \leq s$, and consequently, $q_{\mathbf{e}_{(s)}^{\text{BF}}}(u_i)$ and $q_{\mathbf{e}_{(s)}^{\text{BF}}}(v_i)$ are equidistant to the root of $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$. So we see that $\mathcal{H}_{(s)}^{\text{BF}}$ is obtained from $2 \cdot \mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$ and the points $\{q_{\mathbf{e}_{(s)}^{\text{BF}}}(u_i), q_{\mathbf{e}_{(s)}^{\text{BF}}}(v_i)\}$, $1 \leq i \leq s$, by applying the inverse of the operation described in (1.2). Hence the breadth-first spanning tree of $\mathcal{H}_{(s)}^{\text{BF}}$ is $(\mathcal{H}_{(s)}^{\text{BF}})' = 2 \cdot \mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$.

Theorem 3.1 given below states that $\mathcal{H}_{(s)} \stackrel{\text{d}}{=} \mathcal{H}_{(s)}^{\text{BF}}$. Thus, Construction 2.2 gives the breadth-first construction of the space $\mathcal{H}_{(s)}$. Similar to Construction 2.1, one can arrive at the expression for the tilt in (2.5) using heuristic arguments. As mentioned before, the space $\mathcal{H}_{(s)}$ can be seen as a uniform rooted measured \mathbb{R} -graph having s cycles, and the aim in Construction 2.2 is to construct a new space $\mathcal{H}_{(s)}^{\text{BF}}$ in a ‘breadth-first fashion’ so that we have $\mathcal{H}_{(s)} \stackrel{\text{d}}{=} \mathcal{H}_{(s)}^{\text{BF}}$. Now, in Construction 2.2 we have constructed $\mathcal{H}_{(s)}^{\text{BF}}$ by identifying s i.i.d. pairs of points on $2 \cdot \mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$, where each pair is uniformly distributed subject to the constraint that the points in the pair are at the same height. Note that the push-forward of the Lebesgue measure on $[0, 1]$ onto $[0, 1] \times \mathbb{R}$ under the mapping $t \mapsto (t, \mathbf{e}(t))$ is $\eta(\mathbf{e}; dt, h)dh$, and accordingly, the law of $2 \cdot \mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$ is the one obtained by a tilting of the law of $2 \cdot \mathcal{T}_{\mathbf{e}}$ by the total ‘weight’ of possible chosen pairs, which equals

$$\left(\int_{h=0}^{\infty} dh \left(\int_{u=0}^1 \int_{v=0}^1 \eta(\mathbf{e}; du, h) \eta(\mathbf{e}; dv, h) \right) \right)^s = \left(\int_0^{\infty} \eta(\mathbf{e}; 1, h)^2 dh \right)^s.$$

Alternately, one can arrive at the expression for the tilt by looking at the analogous problem in the discrete setting; see the proof of Proposition 5.1.

Remark 1. It follows from Jeulin’s local time identity [10, 35] that

$$\int_0^{\infty} \eta(\mathbf{e}; 1, y)^2 dy \stackrel{\text{d}}{=} 2 \int_0^1 \mathbf{e}(t) dt; \quad (2.8)$$

see the argument given around (5.50). However $\mathbf{e}_{(s)}^{\text{DF}}$ and $\mathbf{e}_{(s)}^{\text{BF}}$, as defined in (2.4) and (2.5) respectively, do not have the same distribution. In fact, from the discussion above and (1.5), $\text{Rad}(\mathcal{H}_{(s)}) \stackrel{\text{d}}{=} 2 \cdot \|\mathbf{e}_{(s)}^{\text{BF}}\|_{\infty}$. However, from Construction 2.1, it is clear that $\text{Rad}(\mathcal{H}_{(s)})$ is stochastically dominated by $2 \cdot \|\mathbf{e}_{(s)}^{\text{DF}}\|_{\infty}$.

2.2. Constructions of the continuum random unicellular maps. Fix $g \geq 1$, and let $\mathbb{UM}_{n,g}$ be the set of rooted unicellular maps of genus g having $n + 1$ vertices. Let $UM_{n,g}$ be uniformly distributed over $\mathbb{UM}_{n,g}$. Denote its root edge by e_* . As before, we endow $UM_{n,g}$ with the graph distance and the uniform probability measure on the vertices, and think of it as a rooted metric measure space with the root being $e_*(0)$ —the origin of e_* . (The notation related to maps will be discussed in Section 4.) Then there exists a random compact

rooted metric measure space $\text{CRUM}_{(g)}$ such that

$$n^{-1/2} \cdot \text{UM}_{n,g} \xrightarrow{d} \text{CRUM}_{(g)} \quad (2.9)$$

w.r.t. the pointed GHP topology. As mentioned in [3, Page 940], it seems that a proof of the convergence (2.9) is not written down in the literature. However, the result can be deduced by following the arguments used in [1] or in [2]. A construction of $\text{CRUM}_{(g)}$ that can be viewed as a ‘core decomposition’ is given in [3]¹. We will next describe the breadth-first construction of $\text{CRUM}_{(g)}$. To do so, we first need to set up some notation.

Let $\mathbb{S}_{(g)}$ be the set of permutations on $[4g] = \{1, 2, \dots, 4g\}$ that satisfy the following: $\sigma \in \mathbb{S}_{(g)}$ iff

- (i) the cycle decomposition of σ has $2g$ many transpositions, and
- (ii) the permutation $\rho\sigma$ on $[4g]$ has only one cycle of length $4g$, where $\rho = (1, 2, \dots, 4g)$.

For example, $\mathbb{S}_{(1)} = \{(1, 3)(2, 4)\}$, and $(1, 3)(2, 4)(5, 7)(6, 8)$ and $(1, 7)(2, 5)(3, 8)(4, 6) \in \mathbb{S}_{(2)}$.

Suppose $\sigma = \prod_{j=1}^{2g} (\ell_{2j-1}, \ell_{2j}) \in \mathbb{S}_{(g)}$ with $\ell_1 < \ell_3 < \dots < \ell_{4g-1}$, and $\ell_{2j-1} < \ell_{2j}$ for each $j \in [2g]$. For a continuous excursion e on $[0, 1]$ that admits a continuous local time (as discussed around (1.7)), and $\mathbf{y} := (y_1, \dots, y_{2g}) \in (0, \infty)^{2g}$, define a measure $\nu_{e, \sigma, \mathbf{y}}$ on $[0, 1]^{4g}$ as follows:

$$\nu_{e, \sigma, \mathbf{y}}(dt_1, \dots, dt_{4g}) := \mathbb{1}_{\{t_1 < \dots < t_{4g}\}} \cdot \prod_{j=1}^{2g} \left(\eta(e; dt_{\ell_{2j-1}}, y_j) \eta(e; dt_{\ell_{2j}}, y_j) \right). \quad (2.10)$$

The push-forward of the probability measure $\nu_{e, \sigma, \mathbf{y}}(\cdot) / \nu_{e, \sigma, \mathbf{y}}([0, 1]^{4g})$ onto $(\mathcal{T}_e)^{4g}$ under the $4g$ -fold product of the quotient map q_e can be viewed as the ‘uniform measure’ on the set $\{(q_e(t_1), \dots, q_e(t_{4g})) : 0 < t_1 < \dots < t_{4g} < 1 \text{ and } e(t_{\ell_{2j-1}}) = e(t_{\ell_{2j}}) = y_j \text{ for all } j \in [2g]\}$.

Construction 2.3 (The space $\text{CRUM}_{(g)}^{\text{BF}}$). *Fix an integer $g \geq 1$.*

(a) *Sample $\mathbf{e}_{(g)}^{\text{UM}}$ with law given by*

$$\mathbb{E}[\phi(\mathbf{e}_{(g)}^{\text{UM}})] = 24^g \cdot g! \cdot \mathbb{E} \left[\phi(\mathbf{e}) \cdot \sum_{\sigma \in \mathbb{S}_{(g)}} \left(\int_{(0, \infty)^{2g}} \nu_{e, \sigma, \mathbf{y}}([0, 1]^{4g}) dy_1 \dots dy_{2g} \right) \right] \quad (2.11)$$

for every bounded continuous $\phi : C[0, 1] \rightarrow \mathbb{R}$. (That this indeed gives a valid probability distribution will be shown in (5.70).)

(b) *Conditional on $\mathbf{e}_{(g)}^{\text{UM}}$, sample a permutation Θ with distribution*

$$\mathbb{P}(\Theta = \theta) = \frac{\int_{(0, \infty)^{2g}} \nu_{\mathbf{e}_{(g)}^{\text{UM}}, \theta, \mathbf{y}}([0, 1]^{4g}) dy_1 \dots dy_{2g}}{\sum_{\sigma \in \mathbb{S}_{(g)}} \left(\int_{(0, \infty)^{2g}} \nu_{\mathbf{e}_{(g)}^{\text{UM}}, \sigma, \mathbf{y}}([0, 1]^{4g}) dy_1 \dots dy_{2g} \right)}, \quad \theta \in \mathbb{S}_{(g)}.$$

(c) *Conditional on $\mathbf{e}_{(g)}^{\text{UM}}$ and Θ , sample $\mathbf{H} = (H_1, \dots, H_{2g})$ with density*

$$\frac{\nu_{\mathbf{e}_{(g)}^{\text{UM}}, \Theta, \mathbf{h}}([0, 1]^{4g}) dh_1 \dots dh_{2g}}{\int_{(0, \infty)^{2g}} \nu_{\mathbf{e}_{(g)}^{\text{UM}}, \Theta, \mathbf{y}}([0, 1]^{4g}) dy_1 \dots dy_{2g}}, \quad \mathbf{h} = (h_1, \dots, h_{2g}) \in (0, \infty)^{2g}.$$

¹In fact, the authors consider a more general model in [3].

(d) Conditional on $\mathbf{e}_{(g)}^{\text{UM}}, \Theta$, and \mathbf{H} , sample (u_1, \dots, u_{4g}) according to the law

$$\frac{\nu_{\mathbf{e}_{(g)}^{\text{UM}}, \Theta, \mathbf{H}}(\cdot)}{\nu_{\mathbf{e}_{(g)}^{\text{UM}}, \Theta, \mathbf{H}}([0, 1]^{4g})}.$$

(e) Set $\text{CRUM}_{(g)}^{\text{BF}}$ to be the quotient space $\mathcal{T}_{\mathbf{e}_{(g)}^{\text{UM}}} / \sim$, where \sim is the smallest equivalence relation under which $q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_i) \sim q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_{\sigma(i)})$ for $i \in [4g]$. Thus, $\text{CRUM}_{(g)}^{\text{BF}}$ is the rooted measured \mathbb{R} -graph obtained by identifying the points $q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_i)$ and $q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_{\sigma(i)})$ on $\mathcal{T}_{\mathbf{e}_{(g)}^{\text{UM}}}$ for all i .

The construction of $\text{CRUM}_{(g)}^{\text{BF}}$ has more information than just the metric measure space structure— $\text{CRUM}_{(g)}^{\text{BF}}$ can in fact be viewed as a continuum map; see [3] for a more detailed discussion on continuum random maps. The defining conditions for $\mathbb{S}_{(g)}$ ensure that the cycles resulting from the identifications $q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_i) \sim q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_{\sigma(i)})$ are appropriately ‘entangled’ so that the resulting map is unicellular; see [19] for a detailed account of structure of unicellular maps.

By arguments similar to the ones given below Construction 2.2, $q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_i)$ is a leaf in $\mathcal{T}_{\mathbf{e}_{(g)}^{\text{UM}}}$ for each $i \in [4g]$. Further, $\text{ht}(q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_i); \mathcal{T}_{\mathbf{e}_{(g)}^{\text{UM}}}) = \text{ht}(q_{\mathbf{e}_{(g)}^{\text{UM}}}(u_{\sigma(i)}); \mathcal{T}_{\mathbf{e}_{(g)}^{\text{UM}}})$ for all i . It thus follows that $(\text{CRUM}_{(g)}^{\text{BF}})' = \mathcal{T}_{\mathbf{e}_{(g)}^{\text{UM}}}$. Theorem 3.5 given below states that $\text{CRUM}_{(g)} \stackrel{\text{d}}{=} \text{CRUM}_{(g)}^{\text{BF}}$. Thus, Construction 2.3 gives the breadth-first construction of $\text{CRUM}_{(g)}$.

Remark 2. From Construction 2.2, the space $\mathcal{H}_{(s)}^{\text{BF}}$ can also be viewed as a continuum map. Then it can be shown that the space $\text{CRUM}_{(g)}^{\text{BF}}$ as given in Construction 2.3 has the same law as ‘ $\mathcal{H}_{(2g)}^{\text{BF}}$ conditioned to be unicellular.’ More precisely, let $u_i, v_i, 1 \leq i \leq 2g$, be as in Construction 2.2 (corresponding to $s = 2g$), and order them as $t_1 < \dots < t_{4g}$. Define a permutation ψ on $[4g]$ by letting $\psi(i) = j$ iff $\{t_i, t_j\} = \{u_k, v_k\}$ for some $k \in [2g]$. Then it can be checked that conditional on $\{\psi \in \mathbb{S}_{(g)}\}$, $\mathcal{H}_{(2g)}^{\text{BF}}$ viewed as a continuum map has the same law as $\text{CRUM}_{(g)}^{\text{BF}}$.

3. MAIN RESULTS

The following theorem gives the breadth-first construction of the space $\mathcal{H}_{(s)}$, and consequently, of the critical Erdős-Rényi scaling limit.

Theorem 3.1. Fix $s \geq 0$, and let $\mathcal{H}_{(s)}$ and $\mathcal{H}_{(s)}^{\text{BF}}$ be as in (2.3) and Construction 2.2 respectively. Then $\mathcal{H}_{(s)} \stackrel{\text{d}}{=} \mathcal{H}_{(s)}^{\text{BF}}$.

As explained in Section 1, the following results are immediate from Theorem 3.1.

Corollary 3.2. (i) Radius: We have, $\text{Rad}(\mathcal{H}_{(s)}) \stackrel{\text{d}}{=} 2 \cdot \|\mathbf{e}_{(s)}^{\text{BF}}\|_{\infty}$.

(ii) Two-point function: Denote the metric in $\mathcal{H}_{(s)}$ by $d(\cdot, \cdot)$ and the measure on $\mathcal{H}_{(s)}$ by μ . Let x_1 and x_2 be two i.i.d. μ -distributed points in $\mathcal{H}_{(s)}$. Then

$$d(x_1, x_2) \stackrel{\text{d}}{=} 2 \cdot \mathbf{e}_{(s)}^{\text{BF}}(U),$$

where $U \sim \text{Uniform}[0, 1]$ and is independent of $\mathbf{e}_{(s)}^{\text{BF}}$.

(iii) Distance profile: Denoting the root of $\mathcal{H}_{(s)}$ by x_* , we have,

$$\left(\frac{d}{dr} \mu(B(x_*, r; \mathcal{H}_{(s)})), r > 0 \right) \stackrel{d}{=} \left(\eta(2\mathbf{e}_{(s)}^{\text{BF}}; 1, r), r > 0 \right) = \frac{1}{2} \left(\eta(\mathbf{e}_{(s)}^{\text{BF}}; 1, r/2), r > 0 \right).$$

Recall the definition of $\mathbf{e}_{(s)}^{\text{DF}}$ from (2.4). Using the breadth-first view point we get another representation of $\text{Rad}(\mathcal{H}_{(s)})$ in terms of $\mathbf{e}_{(s)}^{\text{DF}}$ which we state in the next corollary.

Corollary 3.3. *For any $s \geq 1$,*

$$\text{Rad}(\mathcal{H}_{(s)}) \stackrel{d}{=} \int_0^1 \frac{1}{\mathbf{e}_{(s)}^{\text{DF}}(t)} dt.$$

The corresponding result for $s = 0$ is well-known; it says that the height of the Brownian continuum random tree $2\mathcal{T}_e$ has the same distribution as $\int_0^1 dt/\mathbf{e}(t)$. Corollary 3.3 gives the analogue of this result for graphs.

Note that (2.3) together with Corollary 3.2 (i) and (ii) imply that

$$n^{-1/2} \cdot \text{Rad}(H_{n,s}) \xrightarrow{d} 2 \cdot \|\mathbf{e}_{(s)}^{\text{BF}}\|_\infty \quad \text{and} \quad n^{-1/2} \cdot d_n(v_1, v_2) \xrightarrow{d} 2 \cdot \mathbf{e}_{(s)}^{\text{BF}}(U), \quad (3.1)$$

where $d_n(\cdot, \cdot)$ denotes the graph distance in $H_{n,s}$, and v_1, v_2 are independent and uniformly distributed in $[n]$. However, (2.3) and Corollary 3.2 (iii) do not imply the convergence of the (properly rescaled) distance profile in $H_{n,s}$. The following theorem says that this convergence holds as well.

Theorem 3.4. *Fix $s \geq 1$. Let $Z_{n,s}(\ell)$ denote the number of vertices in $H_{n,s}$ at distance ℓ from the root, $\ell = 0, 1, \dots$. Let $\bar{Z}_{n,s}(r) = n^{-1/2} \cdot Z_{n,s}(\lfloor r\sqrt{n} \rfloor)$, $r \geq 0$. Then as $n \rightarrow \infty$,*

$$\left(\bar{Z}_{n,s}(r), r \geq 0 \right) \xrightarrow{d} \left(\eta(2\mathbf{e}_{(s)}^{\text{BF}}; 1, r), r > 0 \right) = \left(\frac{1}{2} \eta(\mathbf{e}_{(s)}^{\text{BF}}; 1, r/2), r \geq 0 \right)$$

w.r.t. Skorohod J_1 topology on $\mathbb{D}([0, \infty) : \mathbb{R})$.

The analogue of Theorem 3.4 for $s = 0$ deals with convergence of the height profiles of uniform rooted labeled trees. This result, in a more general form, was put forward as a conjecture in [9, Conjecture 4] and proved in [24, Theorem 1.1] (see Theorem 5.10 below). Theorem 3.4 gives a generalization of this result for graphs.

The next theorem gives the breadth-first construction of the space $\text{CRUM}_{(g)}$.

Theorem 3.5. *Fix $g \geq 1$, and let $\text{CRUM}_{(g)}$ and $\text{CRUM}_{(g)}^{\text{BF}}$ be as in (2.9) and Construction 2.3 respectively. Then $\text{CRUM}_{(g)} \stackrel{d}{=} \text{CRUM}_{(g)}^{\text{BF}}$.*

The following result is the analogue of Corollary 3.2.

Corollary 3.6. *The conclusions of Corollary 3.2 continue to hold if we replace $\mathcal{H}_{(s)}$ by $\text{CRUM}_{(g)}$ and $2 \cdot \mathbf{e}_{(s)}^{\text{BF}}$ by $\mathbf{e}_{(g)}^{\text{UM}}$.*

4. NOTATION AND CONVENTIONS

We first recall some basic definitions and introduce our notation related to maps. A map is a finite connected multigraph properly embedded in a surface that cuts the surface in a finite collection of simply connected domains, and is viewed up to orientation-preserving homeomorphisms of the surface. We will think of each edge \boldsymbol{e} of a map as being a collection of two directed edges e and \bar{e} that are paths in the underlying surface oriented in opposite directions, and we will write $\boldsymbol{e} = \{e, \bar{e}\}$. We will denote by $e(0)$ and $e(1)$ the origin (or initial vertex) and the target (or terminal vertex) of the directed edge e respectively. Note that $e(0) = \bar{e}(1)$ and $e(1) = \bar{e}(0)$. For a map \mathbf{m} , the set of all directed edges (resp. edges) of \mathbf{m} will be denoted by $\vec{E}(\mathbf{m})$ (resp. $E(\mathbf{m})$). Thus, for $e \in \vec{E}(\mathbf{m})$, \bar{e} will be the corresponding element of $\vec{E}(\mathbf{m})$ that is oriented in the opposite direction, and of course $\bar{\bar{e}} = e$. The image set of e minus the points $e(0), e(1)$ is called the interior of the edge e (or of the corresponding unoriented edge \boldsymbol{e}) and is denoted by $\text{Int}(e) = \text{Int}(\boldsymbol{e})$. A rooted map is a map \mathbf{m} together with a distinguished element $e_* \in \vec{E}(\mathbf{m})$. The latter is called the root edge, and $e_*(0)$ will be called the root vertex of \mathbf{m} . Combinatorially, the data of a map is equivalent to that of a finite connected multigraph where the set of directed edges emanating from each vertex is endowed with a cyclic order (that we think of as being clockwise). For $e \in \vec{E}(\mathbf{m})$, we will write e^- for the corner corresponding to e , that is a small angular sector between f and e , where f is the directed edge emanating from $e(0)$ that comes just before e in the cyclic order around the vertex $e(0)$.

We recall here that there is also an equivalent algebraic description in which a map \mathbf{m} is represented by a triple (α, β, γ) of permutations, where the cycles of α , β , and γ represent the edges, vertices, and faces of \mathbf{m} respectively. We refer the reader to [19, Section 2] for a quick overview of this description. We call γ the face permutation of \mathbf{m} . In this representation of maps, a map is unicellular if and only if its face permutations consists of only one cycle.

We make a note here that we use the notation e, f to denote directed edges of maps, and $\boldsymbol{e}, \boldsymbol{f}$ to denote the corresponding edges. We will also use the notation e for a generic excursion, f for a generic function, and \mathbf{e} for a standard Brownian excursion. The meaning will always be clear from the context, and there should not be any confusion.

For $n \geq 1$ and $s \geq 0$, $\mathbb{M}_{n,s}$ will denote the set of all rooted maps that have $n+1$ vertices and $n+s$ edges, where the root vertex has degree one. In particular, the root edge cannot be a loop. (The condition that the root vertex has degree one is of course artificial. However, it will make certain things simpler.) Thus, $\mathbb{M}_{n,0}$ is the set of plane trees on $n+1$ vertices, where the root vertex has degree one.

For a plane tree or a rooted labeled tree \mathbf{t} , for every vertex v of \mathbf{t} , $\text{ht}(v; \mathbf{t})$ will denote the tree distance between the root and v , and the vertices on the path connecting the root and v (inclusive of both endpoints) will be called ancestors of v . For $k = 0, 1, \dots$, $z(\mathbf{t}; k)$ will denote the number of vertices in \mathbf{t} at height k .

We can explore any $\mathbf{t} \in \mathbb{M}_{n,0}$ in the following fashion: Let e_1 be the root edge of \mathbf{t} , and for $1 \leq i \leq 2n-1$, let e_{i+1} be the directed edge in \mathbf{t} with $e_{i+1}(0) = e_i(1)$ that comes right after \bar{e}_i in the cyclic order on directed edges emanating from $e_i(1)$. We say that the directed edges e_1, \dots, e_{2n} (resp. the corners e_1^-, \dots, e_{2n}^-) appear in the contour order. We refer to e_1^- as the

0-th corner of \mathbf{t} . Let $C_{\mathbf{t}}(0) = 0$ and $C_{\mathbf{t}}(i) = \text{ht}(e_i(1); \mathbf{t})$ for $1 \leq i \leq 2n$. Extend $C_{\mathbf{t}}$ to a function on $[0, 2n]$ by linear interpolation. Then $C_{\mathbf{t}} : [0, 2n] \rightarrow [0, \infty)$ is called the contour function of \mathbf{t} .

For any set A , $\#A$ will denote its cardinality. For two sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ of positive numbers, we will write $a_n \sim b_n$ to mean that $a_n/b_n \rightarrow 1$ as $n \rightarrow \infty$. Throughout this paper c, c' will denote positive universal constants, and their values may change from line to line.

5. PROOFS

The proofs of our main results will be given in this section.

5.1. Exploration of maps. Fix integers $n, s \geq 1$, and $\mathbf{m} \in \mathbb{M}_{n,s}$. Let e_* be the root edge of \mathbf{m} , and let $\mathbf{e}_* = \{e_*, \bar{e}_*\}$.

Depth-first (DF) exploration: We will explore \mathbf{m} and simultaneously grow a plane tree \mathbf{t} . Set \mathbf{t} to be the edge e_* . Root \mathbf{t} at e_* . Set $e_1 = e_*$. Also, set the ‘current’ map $\mathbf{m}_{\text{cur}} = \mathbf{m}$. Set $i = 1$, and iterate as follows:

(a) If $i = 2n$, stop. Otherwise, let f be the directed edge in \mathbf{m}_{cur} with $f(0) = e_i(1)$ that comes right after \bar{e}_i in the cyclic order on directed edges emanating from $e_i(1)$, and let $\mathbf{f} = \{f, \bar{f}\}$. (Note that if \bar{e}_i is the only directed edge emanating from $e_i(1)$ in \mathbf{m}_{cur} , then $f = \bar{e}_i$.) Go to the next step.

(b) If \mathbf{f} is already an edge in \mathbf{t} , set $e_{i+1} = f$, update i to $i + 1$, and go to step (a). Otherwise, go to the next step.

(c) If adding \mathbf{f} to \mathbf{t} results in a tree, then do so while respecting the circular order on directed edges emanating from $e_i(1)$ in \mathbf{m} , set $e_{i+1} = f$, update i to $i + 1$, and go to step (a). Otherwise, go to the next step.

(d) Update \mathbf{m}_{cur} to the map $\mathbf{m}_{\text{cur}} \setminus \text{Int}(\mathbf{f})$, and go to step (a).

After the algorithm terminates, we will get a plane tree $\mathbf{t} \in \mathbb{M}_{n,0}$. Note that $\mathbf{m}_{\text{cur}} = \mathbf{t}$ at this stage. In the contour exploration of \mathbf{t} , its directed edges will appear in the order e_1, \dots, e_{2n} . We can recover \mathbf{m} from \mathbf{t} by adding edges between certain corners in \mathbf{t} .

Motivated by the last fact, we define the ‘depth-first admissible corners’ of a plane tree as follows: Fix a plane tree $\mathbf{t}' \in \mathbb{M}_{n,0}$ and $s \geq 1$. Let f_1, \dots, f_{2n} be the directed edges of \mathbf{t}' in contour order. Let $\text{DFAC}(\mathbf{t}', s)$ be the set of all sequences $(i_1, \dots, i_{2s}, k_1, \dots, k_{2s})$, where $1 \leq i_j \leq 2n - 1$ and $1 \leq k_j \leq 2s$ for $1 \leq j \leq 2s$ such that the following hold:

(A.1) $i_{2j-1} \leq i_{2j}$ and $f_{i_{2j}}(0)$ is an ancestor of $f_{i_{2j-1}}(0)$, $1 \leq j \leq s$.

(A.2) For every maximal subset $\{r_1, \dots, r_j\} \subseteq \{1, \dots, 2s\}$ such that $i_{r_1} = \dots = i_{r_j}$, k_{r_1}, \dots, k_{r_j} is a permutation of $1, \dots, j$.

(A.3) For $1 \leq j < \ell \leq s$, either (a) $i_{2j-1} < i_{2\ell-1}$, or (b) $i_{2j-1} = i_{2\ell-1}$ and $i_{2j} < i_{2\ell}$, or (c) $i_{2j-1} = i_{2\ell-1}$ and $i_{2j} = i_{2\ell}$ and $k_{2j-1} < k_{2\ell-1}$.

The condition $i_{2j-1} \leq i_{2j}$ in (A.1) means that the corner $f_{i_{2j}}^-$ appears after $f_{i_{2j-1}}^-$ in the contour order. We would like to add edges between the corners $f_{i_{2j-1}}^-$ and $f_{i_{2j}}^-$, $1 \leq j \leq s$. Condition (A.2) gives a way of ordering the corresponding directed edges using the integers k_j when multiple i_j -s are the same. Condition (A.3) is needed to pick out one particular representative among different possible permutations of the same sequence.

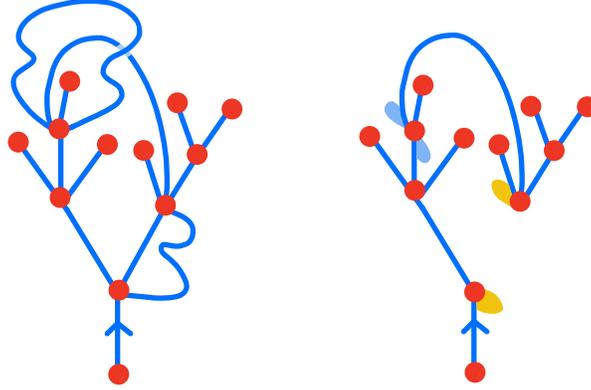


FIGURE 5.1. On the left, a map \mathbf{m} with clockwise order at each vertex. On the right, $\text{DF}(\mathbf{m})$ with the relevant corners colored.

Given $\mathbf{t}' \in \mathbb{M}_{n,0}$ and $\xi' = (i_j, k_j; 1 \leq j \leq 2s) \in \text{DFAC}(\mathbf{t}', s)$, let $\mathcal{S}(\mathbf{t}', \xi')$ to be the map obtained by adding edges between the corners $f_{i_{2j-1}}^-$ and $f_{i_{2j}}^-$, $1 \leq j \leq s$, while using the integers k_j , $1 \leq j \leq 2s$, to order the corresponding directed edges when multiple i_j -s are the same. For example, suppose $i_1 = i_3$ and $k_3 < k_1$. Let $\mathbf{f}_{12} = \{f_{12}, \tilde{f}_{12}\}$ be the edge added between the corners $f_{i_1}^-$ and $f_{i_2}^-$ where f_{12} is directed from $f_{i_1}^-$ towards $f_{i_2}^-$. Similarly define \mathbf{f}_{34} . Then in the resulting map, f_{12} falls between \tilde{f}_{34} and $f_{i_1}^-$ in the circular order on directed edges emanating from $f_{i_1}(0)$. Similarly, if $i_1 < i_3 = i_2$ and $k_3 < k_2$, then \tilde{f}_{12} falls between \tilde{f}_{34} and $f_{i_2}^-$. More generally, suppose $\{r_1, \dots, r_j\}$ is a maximal subset of $\{1, \dots, 2s\}$ such that $i_{r_1} = \dots = i_{r_j} = i_*$, say and $\tilde{r}_1, \dots, \tilde{r}_j$ is a permutation of r_1, \dots, r_j such that $k_{\tilde{r}_\ell} = \ell$ for $1 \leq \ell \leq j$. Then, in the cyclic order at $f_{i_*}(0)$ in the map $\mathcal{S}(\mathbf{t}', \xi')$, the directed edge corresponding to \tilde{r}_ℓ is followed by the directed edge corresponding to $\tilde{r}_{\ell+1}$, $1 \leq \ell \leq j-1$, and the directed edge corresponding to \tilde{r}_j is followed by f_{i_*} .

Now, if \mathbf{t} is the plane tree obtained from the depth-first exploration of $\mathbf{m} \in \mathbb{M}_{n,s}$ as above, then there exists a unique $\xi \in \text{DFAC}(\mathbf{t}, s)$ such that $\mathcal{S}(\mathbf{t}, \xi) = \mathbf{m}$. We set $\text{DF}(\mathbf{m}) = (\mathbf{t}, \xi)$. An illustration is given in Figure 5.1. In this example, an edge is to be added between the two blue corners, and two edges are to be added between the two yellow corners. Here, $i_3 = i_5$, $i_4 = i_6$, $k_1 = k_2 = k_3 = k_6 = 1$, and $k_4 = k_5 = 2$.

Remark 3. In the above depth-first exploration, we follow the contour of the map (similar to the contour exploration of a plane tree), and delete any newly found edge if it completes a cycle in the map. If \mathbf{t} is the plane tree resulting from this exploration, then each deleted edge is of the form (u, v) , where v is an ancestor of u in \mathbf{t} .

There is a variant of the depth-first search where all the neighbors of the vertex being currently explored are immediately ‘discovered.’ If \mathbf{t}_1 is the plane tree resulting from this exploration algorithm, then the edges deleted that are not loops will be of the form (u, v) , where v is a child of some (strict) ancestor of u in \mathbf{t}_1 , and further, v lies to the right of the ancestral line of u . To prove Theorem 3.1 using this latter version of the depth-first search, we need some control over the number of vertices at distance 2 from the ancestral line of a typical vertex in a uniform plane tree. This can be done, but it will make the proof a bit more complicated.

Breadth-first (BF) exploration: As before, we will explore \mathbf{m} and simultaneously grow a plane tree \mathbf{t} . Set \mathbf{t} to be the edge e_* and root \mathbf{t} at e_* . Set $e_1 = e_*$. Set $i = r = 1$ and iterate as follows:

(a) If $r = n$, stop. Otherwise, go to the next step.

(b) Consider the directed edges in \mathbf{m} with origin $e_i(1)$ that come after \bar{e}_i sequentially. Let Δ be the number of edges among them whose addition to \mathbf{t} results in a tree. Update \mathbf{t} by adding these Δ edges to \mathbf{t} while maintaining the cyclic order inherited from the order in \mathbf{m} . For $1 \leq j \leq \Delta$, set e_{r+j} to be the j -th directed edge (after \bar{e}_i) emanating from $e_i(1)$ in \mathbf{t} . Update i to $i + 1$ and r to $r + \Delta$. Go to step (a).

After the algorithm terminates, we will get a plane tree $\mathbf{t} \in \mathbb{M}_{n,0}$, and we can recover \mathbf{m} from \mathbf{t} by adding edges between certain corners in \mathbf{t} . Motivated by this, we define the ‘breadth-first admissible corners’ of a plane tree as follows: Fix a plane tree $\mathbf{t}' \in \mathbb{M}_{n,0}$ and $s \geq 1$. Let f_1, \dots, f_{2n} be the directed edges of \mathbf{t}' in contour order. Let $\text{BFAC}(\mathbf{t}', s)$ be the set of all sequences $(i_1, \dots, i_{2s}, k_1, \dots, k_{2s})$, where $1 \leq i_j \leq 2n - 1$ and $1 \leq k_j \leq 2s$ for $1 \leq j \leq 2s$ such that Conditions (A.2) and (A.3) hold as above, and further the following holds: For $1 \leq j \leq s$, $i_{2j-1} \leq i_{2j}$ and

$$\text{ht}(f_{i_{2j}}(0)) \in \{\text{ht}(f_{i_{2j-1}}(0)), \text{ht}(f_{i_{2j-1}}(0)) - 1\}. \quad (5.1)$$

The only difference from the depth-first case is (5.1). As before, we would like to add edges between the corners $f_{i_{2j-1}}^-$ and $f_{i_{2j}}^-$, $1 \leq j \leq s$. Thus, (5.1) essentially says that edges are to be added between vertices at roughly the same height.

Given $\mathbf{t}' \in \mathbb{M}_{n,0}$ and $\xi' = (i_j, k_j; 1 \leq j \leq 2s) \in \text{BFAC}(\mathbf{t}', s)$, let $\mathcal{S}(\mathbf{t}', \xi')$ to be the map obtained by adding edges between the corners $f_{i_{2j-1}}^-$ and $f_{i_{2j}}^-$, $1 \leq j \leq s$, and as before, we use the integers k_j , $1 \leq j \leq 2s$, to order the corresponding directed edges when multiple i_j -s are the same.

Now, if \mathbf{t} is the plane tree obtained from the breadth-first exploration of $\mathbf{m} \in \mathbb{M}_{n,s}$, then there exists a unique $\xi \in \text{BFAC}(\mathbf{t}, s)$ such that $\mathcal{S}(\mathbf{t}, \xi) = \mathbf{m}$. We set $\text{BF}(\mathbf{m}) = (\mathbf{t}, \xi)$. An illustration is given in Figure 5.2. In this example, an edge is to be added between the green and the blue corner, the green and the yellow corner, and the two purple corners. Here, $i_1 = i_3$ correspond to the green corner, i_2 and i_4 correspond to the blue and yellow corners respectively, i_5 and i_6 correspond to the purple corners, and $k_1 = 1$, $k_3 = 2$, $k_2 = k_4 = k_5 = k_6 = 1$.

We define

$$\text{BFT}(n, s) := \{(\mathbf{t}, \xi) : \mathbf{t} \in \mathbb{M}_{n,0}, \xi \in \text{BFAC}(\mathbf{t}, s)\}, \text{ and} \quad (5.2)$$

$$\text{DFT}(n, s) := \{(\mathbf{t}, \xi) : \mathbf{t} \in \mathbb{M}_{n,0}, \xi \in \text{DFAC}(\mathbf{t}, s)\}. \quad (5.3)$$

5.2. Proof of Theorem 3.1. The proof relies on the following two propositions.

Proposition 5.1. *Fix an integer $s \geq 1$. For each $n \geq 1$, let $(T_n^{\text{BF}}, \Xi_n^{\text{BF}})$ be uniformly distributed over $\text{BFT}(n, s)$. Let $G_{n,s}^{\text{BF}}$ be the rooted metric measure space obtained by endowing $\mathcal{S}(T_n^{\text{BF}}, \Xi_n^{\text{BF}})$ with the graph distance, declaring the root vertex as the root, and finally assigning probability $1/n$ to every vertex of $\mathcal{S}(T_n^{\text{BF}}, \Xi_n^{\text{BF}})$ except the root vertex. Then*

$$\frac{1}{\sqrt{n}} \cdot G_{n,s}^{\text{BF}} \xrightarrow{\text{d}} \frac{1}{\sqrt{2}} \cdot \mathcal{H}_{(s)}^{\text{BF}}, \text{ as } n \rightarrow \infty$$

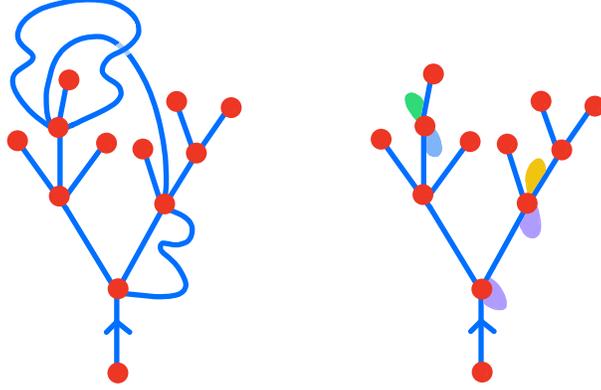


FIGURE 5.2. On the left, a map \mathbf{m} with clockwise order at each vertex. On the right, $\text{BF}(\mathbf{m})$ with the relevant corners colored.

w.r.t. the pointed GHP topology.

Proposition 5.2. Fix an integer $s \geq 1$. For each $n \geq 1$, let $(T_n^{\text{DF}}, \Xi_n^{\text{DF}})$ be uniformly distributed over $\text{DFT}(n, s)$. Let $G_{n,s}^{\text{DF}}$ be the rooted metric measure space obtained by endowing $\mathcal{F}(T_n^{\text{DF}}, \Xi_n^{\text{DF}})$ with the graph distance, declaring the root vertex as the root, and finally assigning probability $1/n$ to every vertex of $\mathcal{F}(T_n^{\text{DF}}, \Xi_n^{\text{DF}})$ except the root vertex. Then

$$\frac{1}{\sqrt{n}} \cdot G_{n,s}^{\text{DF}} \xrightarrow{d} \frac{1}{\sqrt{2}} \cdot \mathcal{H}_{(s)}, \quad \text{as } n \rightarrow \infty$$

w.r.t. the pointed GHP topology.

We first prove Theorem 3.1 using Propositions 5.1 and 5.2. The proofs of these propositions will be given in the Sections 5.3 and 5.4 respectively.

Completing the proof of Theorem 3.1: Fix $s \geq 1$ as the result is trivial for $s = 0$. Let $M_{n,s} \sim \text{Uniform}(\mathbb{M}_{n,s})$. Since $\text{DF} : \mathbb{M}_{n,s} \rightarrow \text{DFT}(n, s)$ and $\text{BF} : \mathbb{M}_{n,s} \rightarrow \text{BFT}(n, s)$ are bijections with inverse \mathcal{F} ,

$$\mathcal{F}(T_n^{\text{DF}}, \Xi_n^{\text{DF}}) \stackrel{d}{=} M_{n,s} \stackrel{d}{=} \mathcal{F}(T_n^{\text{BF}}, \Xi_n^{\text{BF}}). \quad (5.4)$$

View $M_{n,s}$ as a rooted metric measure space by endowing it with the graph distance, declaring the root vertex as the root, and assigning probability $1/n$ to every non-root vertex. Then (5.4) implies that $G_{n,s}^{\text{DF}} \stackrel{d}{=} M_{n,s} \stackrel{d}{=} G_{n,s}^{\text{BF}}$. Thus using Propositions 5.1 and 5.2, we get

$$\sqrt{\frac{2}{n}} \cdot M_{n,s} \xrightarrow{d} \mathcal{H}_{(s)} \stackrel{d}{=} \mathcal{H}_{(s)}^{\text{BF}} \quad (5.5)$$

w.r.t. pointed GHP topology, which completes the proof. \blacksquare

Remark 4. Let $M'_{n,s}$ be uniformly distributed over the set of all rooted maps on n vertices having s surplus edges (i.e., we drop the condition that the degree of the root vertex is one). Then clearly, (5.5) continues to hold if we replace $M_{n,s}$ by $M'_{n,s}$.

Lemma 5.3. *We have,*

$$\frac{1}{\sqrt{n}} \cdot G_{n,s}^\circ \xrightarrow{d} \frac{1}{\sqrt{2}} \cdot \mathcal{H}_{(s)}^{\text{BF}}, \text{ as } n \rightarrow \infty$$

w.r.t. the pointed GHP topology.

Lemma 5.4. *There exists a coupling of $G_{n,s}^\circ$ and $G_{n,s}^{\text{BF}}$ such that*

$$\mathbb{P}(G_{n,s}^\circ \neq G_{n,s}^{\text{BF}}) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

The rest of this section is devoted to the proofs of Lemmas 5.3 and 5.4. To this end, let us define the ‘discrete local time’ of $f \in \mathfrak{C}_n$ as

$$L(f; t, y) := \#\{0 \leq j \leq t : f(j) = y\}, \quad t = 0, 1, \dots, 2n, \quad y \in \mathbb{Z}, \quad (5.9)$$

and extend it to a function on $[0, 2n] \times \mathbb{R}$ via the relation

$$\begin{aligned} L(f; t, y) := & (\langle t \rangle - t)(\langle y \rangle - y)L(f; \lfloor t \rfloor, \lfloor y \rfloor) + (\langle t \rangle - t)(y - \lfloor y \rfloor)L(f; \lfloor t \rfloor, \langle y \rangle) \\ & + (t - \lfloor t \rfloor)(\langle y \rangle - y)L(f; \langle t \rangle, \lfloor y \rfloor) + (t - \lfloor t \rfloor)(y - \lfloor y \rfloor)L(f; \langle t \rangle, \langle y \rangle), \end{aligned} \quad (5.10)$$

where $\lfloor t \rfloor := \max\{j \in \mathbb{Z} : j \leq t\}$ and $\langle t \rangle := \lfloor t \rfloor + 1$. Note that $L(f; \cdot, \cdot)$ is a continuous function on $[0, 2n] \times \mathbb{R}$.

Let C_n be uniformly distributed over \mathfrak{C}_n . Define

$$L_n(\cdot, \cdot) := L(C_n; \cdot, \cdot), \quad \|L_n\|_\infty := \sup_{y \in \mathbb{R}} L_n(2n, y), \quad \text{and} \quad (5.11)$$

$$\bar{L}_n(t, y) := (2n)^{-1/2} L_n(2nt, y\sqrt{2n}), \quad 0 \leq t \leq 1, \quad y \in \mathbb{R}. \quad (5.12)$$

Further, let

$$\bar{C}_n(t) := (2n)^{-1/2} C_n(2nt), \quad 0 \leq t \leq 1. \quad (5.13)$$

We will make use of the following result in our proof.

Proposition 5.5. *The following convergence holds in $C[0, 1] \times C([0, 1] \times \mathbb{R})$:*

$$(\bar{C}_n, \bar{L}_n) \xrightarrow{d} (\mathbf{e}(\cdot), \eta(\mathbf{e}; \cdot, \cdot)). \quad (5.14)$$

Further, there exist constants $c_1, c_2 > 0$ such that for all $n \geq 1$,

$$\mathbb{P}(\|L_n\|_\infty \geq u\sqrt{n}) \leq c_1 \exp(-c_2 u^2), \quad u \geq 0. \quad (5.15)$$

Let T_n be the plane tree whose contour function is C_n , and let e_* be its root edge. Then the tree obtained from T_n by deleting $e_*(0)$ and $\text{Int}(e_*)$ has the same distribution as a uniform plane tree on n vertices. The convergence $\bar{C}_n \xrightarrow{d} \mathbf{e}$ can be deduced from the fact that simple random walk excursions converge, after proper rescaling, to \mathbf{e} , or by using the relation between T_n and uniform plane trees and [8, Theorem 23]. We now state a general result that implies (5.15). This result will also be used in later sections. Recall the notation $z(\mathbf{t}; k)$ from Section 4.

Theorem 5.6 ([5], Theorem 1.1). *Let ξ be a nonnegative integer-valued random variable with $\mathbb{E}\xi = 1$ and $0 < \text{Var}\xi < \infty$. Let T_ξ be a plane Galton-Watson tree with offspring distribution ξ . Let $|T_\xi|$ denote the number of vertices in T_ξ . Then there exist $c_1, c_2 > 0$ such that*

$$\mathbb{P}\left(\max_k z(T_\xi; k) \geq x\sqrt{n} \mid |T_\xi| = n\right) \leq c_1 \exp(-c_2 x^2),$$

for all $x > 0$ and $n \geq 1$ such that $\mathbb{P}(|T_\xi| = n) > 0$.

Using the connection between T_n and uniform plane trees, (5.15) follows if we apply Theorem 5.6 with Geometric(1/2) offspring distribution and the fact that $L_n(2n, k) = z(T_n, k) + z(T_n, k+1)$ for $k \geq 0$. A brief sketch of the proof of the full convergence in (5.14) is given in Appendix A. Let us now prove Lemma 5.3.

Proof of Lemma 5.3: Recall the definition of C_n° from (5.7), and analogous to (5.13), (5.11) and (5.12), define \bar{C}_n° , L_n° , and \bar{L}_n° . Note that

$$\mathbb{E}[\phi(\bar{C}_n^\circ, \bar{L}_n^\circ)] = \frac{\mathbb{E}[\phi(\bar{C}_n, \bar{L}_n)B(C_n)^s]}{\mathbb{E}[B(C_n)^s]} \quad (5.16)$$

for any bounded continuous $\phi : C[0, 1] \times C([0, 1] \times \mathbb{R}) \rightarrow \mathbb{R}$. Now

$$B(C_n) = \sum_{i=1}^{2n-1} [L_n(2n, C_n(i)) - L_n(i-1, C_n(i)) + L_n(2n, C_n(i)-1) - L_n(i-1, C_n(i)-1)]. \quad (5.17)$$

Thus, the following convergence holds jointly with (5.14):

$$\begin{aligned} \frac{B(C_n)}{(2n)^{3/2}} &\xrightarrow{d} 2 \int_0^1 [\eta(\mathbf{e}; 1, \mathbf{e}(t)) - \eta(\mathbf{e}; t, \mathbf{e}(t))] dt \\ &= 2 \int_{y \in \mathbb{R}} dy \int_{t=0}^1 [\eta(\mathbf{e}; 1, y) - \eta(\mathbf{e}; t, y)] \eta(\mathbf{e}; dt, y) = \int_{y \in \mathbb{R}} \eta(\mathbf{e}; 1, y)^2 dy, \end{aligned} \quad (5.18)$$

where the convergence in the first step uses (5.14), the second step uses (1.7), and the last step follows from a direct computation. Now, (5.17) implies that $B(C_n) \leq 4n \|L_n\|_\infty$. Combined with (5.15), this shows that for any $s \geq 1$, the sequence of random variables

$$\{(2n)^{-3s/2} B(C_n)^s\}_{n \geq 1} \text{ is uniformly integrable.} \quad (5.19)$$

Hence, we conclude from (5.16), (5.18), and (5.14) that

$$\mathbb{E}[\phi(\bar{C}_n^\circ, \bar{L}_n^\circ)] \xrightarrow{d} \mathbb{E}[\phi(\mathbf{e}_{(s)}^{\text{BF}}(\cdot), \eta_{(s)}^{\text{BF}}(\cdot, \cdot))],$$

where $\eta_{(s)}^{\text{BF}}$ is as in (2.6). From now on, we work in a space where the following almost sure convergence holds in $C[0, 1] \times C([0, 1] \times \mathbb{R})$:

$$(\bar{C}_n^\circ, \bar{L}_n^\circ) \xrightarrow{\text{a.s.}} (\mathbf{e}_{(s)}^{\text{BF}}, \eta_{(s)}^{\text{BF}}). \quad (5.20)$$

Similar to (5.18), using (5.20), we see that

$$(2n)^{-3/2} B(C_n^\circ) \xrightarrow{\text{a.s.}} \int_0^\infty \eta_{(s)}^{\text{BF}}(1, y)^2 dy. \quad (5.21)$$

From (5.8), the law of $I_{n,1}^\circ$ is given by

$$\mathbb{P}(I_{n,1}^\circ \geq i \mid C_n^\circ) = \frac{1}{B(C_n^\circ)} \sum_{j=i}^{2n} [L_n^\circ(2n, C_n^\circ(j)) - L_n^\circ(j-1, C_n^\circ(j)) + L_n^\circ(2n, C_n^\circ(j)-1) - L_n^\circ(j-1, C_n^\circ(j)-1)]$$

for $i = 0, 1, \dots, 2n$. Using (5.20) and (5.21), we see that $(2n)^{-1}I_{n,1}^\circ$ converges in distribution to a random variable $u_{(1)}$ with law given by

$$\begin{aligned} \mathbb{P}(u_{(1)} \geq u \mid \mathbf{e}_{(s)}^{\text{BF}}) &\cdot \int_0^\infty \eta_{(s)}^{\text{BF}}(1, y)^2 dy = 2 \cdot \int_{t=u}^1 \left[\eta_{(s)}^{\text{BF}}(1, \mathbf{e}_{(s)}^{\text{BF}}(t)) - \eta_{(s)}^{\text{BF}}(t, \mathbf{e}_{(s)}^{\text{BF}}(t)) \right] dt \\ &= 2 \cdot \int_{y=0}^\infty dy \int_{t=u}^1 \left[\eta_{(s)}^{\text{BF}}(1, y) - \eta_{(s)}^{\text{BF}}(t, y) \right] \eta_{(s)}^{\text{BF}}(dt, y) \\ &= \int_{y=0}^\infty \left[\eta_{(s)}^{\text{BF}}(1, y) - \eta_{(s)}^{\text{BF}}(u, y) \right]^2 dy, \quad 0 \leq u \leq 1, \end{aligned} \quad (5.22)$$

where the second equality uses (1.7), and the last step follows from a direct computation. We can assume that we are working in a space where in addition to (5.20),

$$(2n)^{-1}I_{n,1}^\circ \xrightarrow{\text{a.s.}} u_{(1)}. \quad (5.23)$$

Conditional on $I_{n,1}^\circ$, $I_{n,2}^\circ$ is uniformly distributed over $\mathfrak{B}(C_n^\circ; I_{n,1}^\circ)$. Thus, the conditional distribution function of $I_{n,2}^\circ$ is given by

$$\frac{L_n^\circ(j, C_n^\circ(I_{n,1}^\circ)) - L_n^\circ(I_{n,1}^\circ - 1, C_n^\circ(I_{n,1}^\circ)) + L_n^\circ(j, C_n^\circ(I_{n,1}^\circ) - 1) - L_n^\circ(I_{n,1}^\circ, C_n^\circ(I_{n,1}^\circ) - 1)}{L_n^\circ(2n, C_n^\circ(I_{n,1}^\circ)) - L_n^\circ(I_{n,1}^\circ - 1, C_n^\circ(I_{n,1}^\circ)) + L_n^\circ(2n, C_n^\circ(I_{n,1}^\circ) - 1) - L_n^\circ(I_{n,1}^\circ, C_n^\circ(I_{n,1}^\circ) - 1)},$$

for $j = I_{n,1}^\circ, I_{n,1}^\circ + 1, \dots, 2n$, and is a right continuous step function in between. Thus, using (5.20) and (5.23), we see that $(2n)^{-1}I_{n,2}^\circ$ converges in distribution to a random variable $v_{(1)}$ with conditional distribution function (given $\mathbf{e}_{(s)}^{\text{BF}}$ and $u_{(1)}$) given by

$$\frac{\eta_{(s)}^{\text{BF}}(v, \mathbf{e}_{(s)}^{\text{BF}}(u_{(1)})) - \eta_{(s)}^{\text{BF}}(u_{(1)}, \mathbf{e}_{(s)}^{\text{BF}}(u_{(1)}))}{\eta_{(s)}^{\text{BF}}(1, \mathbf{e}_{(s)}^{\text{BF}}(u_{(1)})) - \eta_{(s)}^{\text{BF}}(u_{(1)}, \mathbf{e}_{(s)}^{\text{BF}}(u_{(1)}))}, \quad u_{(1)} \leq v \leq 1. \quad (5.24)$$

Repeating the argument, we can assume that we are working in a space where

$$(2n)^{-1}(I_{n,1}^\circ, I_{n,2}^\circ, \dots, I_{n,2s-1}^\circ, I_{n,2s}^\circ) \xrightarrow{\text{a.s.}} (u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) \quad (5.25)$$

in addition to (5.20). Here, $(u_{(i)}, v_{(i)})$, $1 \leq i \leq s$, are i.i.d. conditionally on $\mathbf{e}_{(s)}^{\text{BF}}$.

It follows from (5.22) and (5.24) that conditional on $\mathbf{e}_{(s)}^{\text{BF}}$,

$$(u_{(i)}, v_{(i)}) \stackrel{\text{d}}{=} (\min\{u_i, v_i\}, \max\{u_i, v_i\}), \quad 1 \leq i \leq s, \quad (5.26)$$

where u_i, v_i , $1 \leq i \leq s$, are as in Construction 2.2. In particular, consider the rooted metric measure space obtained by identifying $q_{\mathbf{e}_{(s)}^{\text{BF}}}(u_{(i)})$ and $q_{\mathbf{e}_{(s)}^{\text{BF}}}(v_{(i)})$, $1 \leq i \leq s$, in $\mathcal{T}_{\mathbf{e}_{(s)}^{\text{BF}}}$. This space has the same distribution as $1/2 \cdot \mathcal{H}_{(s)}^{\text{BF}}$. We denote this space as $1/2 \cdot \bar{\mathcal{H}}_{(s)}^{\text{BF}}$ for the rest of this proof. Let $\psi : [0, 1] \rightarrow 1/2 \cdot \bar{\mathcal{H}}_{(s)}^{\text{BF}}$ denote the quotient map.

Let $\bar{G}_{n,s}$ be the rooted metric measure space obtained from $\mathcal{T}_{\bar{C}_n^\circ}$ by identifying $q_{\bar{C}_n^\circ}(I_{n,2j-1}^\circ/(2n))$ and $q_{\bar{C}_n^\circ}(I_{n,2j}^\circ/(2n))$, $1 \leq j \leq s$. Let $\psi_n : [0, 1] \rightarrow \bar{G}_{n,s}$ denote the quotient map.

Let π_n be the measure on $(1/2 \cdot \bar{\mathcal{H}}_{(s)}^{\text{BF}}) \times \bar{G}_{n,s}$ given by the push-forward of the Lebesgue measure on $[0, 1]$ under $\psi \times \psi_n$, and let \mathcal{R}_n be the correspondence (see, e.g., [4, Section 2.1])

or [18, Section 7.3.3] for the definitions of a correspondence and its distortion) between $1/2 \cdot \mathcal{H}_{(s)}^{\text{BF}}$ and $\bar{G}_{n,s}$ given by

$$\mathcal{R}_n := \{(x, y) : \exists t \in [0, 1] \text{ with } \psi(t) = x \text{ and } \psi_n(t) = y\}.$$

Then using (5.25) and the convergence $\bar{C}_n \xrightarrow{\text{a.s.}} \mathbf{e}_{(s)}^{\text{BF}}$ from (5.20), it is easy to see that the distortion of \mathcal{R}_n satisfies $\text{dis}(\mathcal{R}_n) \xrightarrow{\text{a.s.}} 0$. Further, $\pi_n(\mathcal{R}_n^c) = 0$, and the projections of π_n onto $1/2 \cdot \mathcal{H}_{(s)}^{\text{BF}}$ and $\bar{G}_{n,s}$ are same as the measures on the respective spaces. Combining these observations, it follows that

$$d_{\text{GHP}}^1(\bar{G}_{n,s}, 1/2 \cdot \mathcal{H}_{(s)}^{\text{BF}}) \xrightarrow{\text{a.s.}} 0.$$

There is an obvious coupling between $G_{n,s}^\circ$ and $\bar{G}_{n,s}$, and the proof will be complete if we show that

$$d_{\text{GHP}}^1((2n)^{-1/2} \cdot G_{n,s}^\circ, \bar{G}_{n,s}) \xrightarrow{\text{a.s.}} 0 \quad (5.27)$$

in this coupling. There are two differences between $(2n)^{-1/2} \cdot G_{n,s}^\circ$ and $\bar{G}_{n,s}$: (i) The measure on $\bar{G}_{n,s}$ is the normalized line measure, whereas the measure on $(2n)^{-1/2} \cdot G_{n,s}^\circ$ is the uniform probability measure on all non-root vertices of T_n° . (ii) The points $q_{\bar{C}_n}^\circ(I_{n,2j-1}^\circ/(2n))$ and $q_{\bar{C}_n}^\circ(I_{n,2j}^\circ/(2n))$, $1 \leq j \leq s$, are identified in the construction of $\bar{G}_{n,s}$, whereas an edge of length $(2n)^{-1/2}$ is added between them in the construction of $(2n)^{-1/2} G_{n,s}^\circ$. Using these observations, the proof of (5.27) is routine. We omit the details. \blacksquare

Recall that T_n denotes the plane tree whose contour function is C_n . The proof of Lemma 5.4 relies on the following result:

Lemma 5.7. *We have,*

$$n^{-3s/2} \cdot \mathbb{E}|s! \times \#\text{BFAC}(T_n, s) - B(C_n)^s| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Proof: Note that

$$\begin{aligned} \text{BFAC}(T_n, s) = & \{(i_1, \dots, i_{2s}, 1, \dots, 1) : 1 \leq i_1 < i_3 < \dots < i_{2s-1} \leq 2n-1, \ i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \\ & \text{for } 1 \leq j \leq s, \text{ and } i_\ell\text{-s are all distinct}\} \\ & \cup \{(i_1, \dots, i_{2s}, k_1, \dots, k_{2s}) \in \text{BFAC}(T_n, s) : i_\ell\text{-s are not all distinct}\}. \end{aligned} \quad (5.28)$$

Similarly,

$$\begin{aligned} B(C_n)^s = & \#\{(i_1, \dots, i_{2s}) \in [2n]^{2s} : i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \text{ for } 1 \leq j \leq s, \text{ and } i_\ell\text{-s are all distinct}\} \\ & + \#\{(i_1, \dots, i_{2s}) \in [2n]^{2s} : i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \text{ for } 1 \leq j \leq s, \text{ and } i_\ell\text{-s are not all distinct}\}. \end{aligned}$$

Abbreviating ‘are not all distinct’ as ‘NAD’ and writing (i_1, \dots, i_{2s}) for $(i_1, \dots, i_{2s}) \in [2n]^{2s}$, we see that

$$\begin{aligned} 0 \leq s! \times \#\text{BFAC}(T_n, s) - B(C_n)^s & \leq s! \cdot \#\{(i_1, \dots, i_{2s}, k_1, \dots, k_{2s}) \in \text{BFAC}(T_n, s) : i_\ell\text{-s NAD}\} \\ & \leq s! \times (2s)^{2s} \times \#\{(i_1, \dots, i_{2s}) : i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \text{ for } 1 \leq j \leq s, \text{ and } i_\ell\text{-s NAD}\} \\ & \leq s! \times (2s)^{2s} \times (A_{12} + A_{13} + A_{14} + A_{24}), \end{aligned} \quad (5.29)$$

where

$$A_{12} = \#\{(i_1, \dots, i_{2s}) : i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \text{ for } 1 \leq j \leq s, i_{2\ell-1} = i_{2\ell} \text{ for some } \ell\},$$

$$A_{13} = \#\{(i_1, \dots, i_{2s}) : i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \text{ for } 1 \leq j \leq s, i_{2\ell-1} = i_{2\ell'-1} \text{ for some } \ell \neq \ell'\},$$

and A_{14} and A_{24} are defined similarly with the respective defining conditions being ' $i_{2\ell-1} = i_{2\ell'}$ for some $\ell \neq \ell'$ ' and ' $i_{2\ell} = i_{2\ell'}$ for some $\ell \neq \ell'$.' Now,

$$\begin{aligned} A_{12} &\leq s \cdot \#\{(i_1, \dots, i_{2s}) : i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \text{ for } 1 \leq j \leq s, i_1 = i_2\} \\ &\leq s \cdot (2n) \cdot [2n \cdot \max_i B(C_n; i)]^{s-1} \leq s \cdot (2n) \cdot [2n \cdot 2\|L_n\|_\infty]^{s-1}, \end{aligned}$$

where the last step uses the fact

$$\max_i B(C_n; i) \leq 2\|L_n\|_\infty. \quad (5.30)$$

Combined with (5.15), we get $n^{-3s/2} \cdot \mathbb{E}[A_{12}] = O(n^{-1/2})$. Similarly,

$$\begin{aligned} A_{13} &\leq s^2 \cdot \#\{(i_1, \dots, i_{2s}) : i_{2j} \in \mathfrak{B}(C_n; i_{2j-1}) \text{ for } 1 \leq j \leq s, i_1 = i_3\} \\ &\leq s^2 \cdot \#\{(i, i_2, i, i_4) : i_2, i_4 \in \mathfrak{B}(C_n; i)\} \times [2n \cdot 2\|L_n\|_\infty]^{s-2} \\ &\leq s^2 \cdot (2n) \cdot (2 \cdot \|L_n\|_\infty)^2 \times [2n \cdot 2\|L_n\|_\infty]^{s-2}. \end{aligned} \quad (5.31)$$

Using (5.15) again, we get $n^{-3s/2} \cdot \mathbb{E}[A_{13}] = O(n^{-1})$. We can similarly show that $n^{-3s/2} \cdot \mathbb{E}[A_{14} + A_{24}] = O(n^{-1})$. Combined with (5.29), this yields the desired result. \blacksquare

We record here a useful bound that was used in the previous proof: Recall from Section 4 that $C_{\mathbf{t}}$ denotes the contour function of $\mathbf{t} \in \mathbb{M}_{n,0}$. Then

$$\begin{aligned} &\#\{(i_1, \dots, i_{2s}) : i_{2j} \in \mathfrak{B}(C_{\mathbf{t}}; i_{2j-1}) \text{ for } 1 \leq j \leq s, i_{\ell-s} \text{ NAD}\} \\ &\leq s! \times \#\{(i_1, \dots, i_{2s}, k_1, \dots, k_{2s}) \in \text{BFAC}(\mathbf{t}, s) : i_{\ell-s} \text{ NAD}\} \\ &\leq cn^s \cdot (\max_y L(C_{\mathbf{t}}; 2n, y))^{s-1} =: \alpha_n(\mathbf{t}), \end{aligned} \quad (5.32)$$

where c is a constant that depends only on s . We are now ready for the

Proof of Lemma 5.4: Recall the definition of $(T_n^{\text{BF}}, \Xi_n^{\text{BF}})$ from Proposition 5.1. Clearly,

$$\mathbb{P}(T_n^{\text{BF}} = \mathbf{t}) = \frac{\#\text{BFAC}(\mathbf{t}, s)}{\Sigma \text{BFAC}}, \quad \mathbf{t} \in \mathbb{M}_{n,0},$$

where $\Sigma \text{BFAC} = \sum_{\mathbf{t}' \in \mathbb{M}_{n,0}} \#\text{BFAC}(\mathbf{t}', s)$. Then

$$\begin{aligned} &\sum_{\mathbf{t} \in \mathbb{M}_{n,0}} |\mathbb{P}(T_n^{\circ} = \mathbf{t}) - \mathbb{P}(T_n^{\text{BF}} = \mathbf{t})| = \sum_{\mathbf{t} \in \mathbb{M}_{n,0}} \left| \frac{B(C_{\mathbf{t}})^s}{\sum_{C \in \mathfrak{C}_n} B(C)^s} - \frac{\#\text{BFAC}(\mathbf{t}, s) \times s!}{\Sigma \text{BFAC} \times s!} \right| \\ &\leq \sum_{\mathbf{t} \in \mathbb{M}_{n,0}} \left| \frac{B(C_{\mathbf{t}})^s - \#\text{BFAC}(\mathbf{t}, s) \times s!}{\sum_{C \in \mathfrak{C}_n} B(C)^s} \right| + \sum_{\mathbf{t} \in \mathbb{M}_{n,0}} \#\text{BFAC}(\mathbf{t}, s) \times s! \cdot \left| \frac{1}{\sum_{C \in \mathfrak{C}_n} B(C)^s} - \frac{1}{\Sigma \text{BFAC} \times s!} \right| \\ &\leq 2 \sum_{\mathbf{t} \in \mathbb{M}_{n,0}} \left| \frac{B(C_{\mathbf{t}})^s - \#\text{BFAC}(\mathbf{t}, s) \times s!}{\sum_{C \in \mathfrak{C}_n} B(C)^s} \right| = 2 \cdot \frac{\mathbb{E}[B(C_n)^s - \#\text{BFAC}(T_n, s) \times s!]}{\mathbb{E}[B(C_n)^s]} \rightarrow 0, \end{aligned}$$

where the last step uses Lemma 5.7, (5.18), and (5.19). Let us assume that T_n^{BF} and T_n° are coupled in a way so that

$$\mathbb{P}(T_n^{\text{BF}} \neq T_n^\circ) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (5.33)$$

Let

$$\Xi_n^{\text{BF}} = (I_{n,j}^{\text{BF}}, K_{n,j}^{\text{BF}}; 1 \leq j \leq 2s), \quad \mathbb{P}_{\mathbf{t}}^{\text{BF}}(\cdot) = \mathbb{P}(\cdot \mid T_n^{\text{BF}} = \mathbf{t}), \quad \text{and} \quad \mathbb{P}_{\mathbf{t}}^\circ(\cdot) = \mathbb{P}(\cdot \mid T_n^\circ = \mathbf{t}).$$

Using (5.32), we see that

$$\mathbb{P}_{\mathbf{t}}^{\text{BF}}(I_{n,j}^{\text{BF}}\text{-s NAD}) \leq \frac{\alpha_n(\mathbf{t})}{\#\text{BFAC}(\mathbf{t}, s)} =: \beta_n^{(1)}(\mathbf{t}), \quad \text{and} \quad \mathbb{P}_{\mathbf{t}}^\circ(I_{n,j}^\circ\text{-s NAD}) \leq \frac{\alpha_n(\mathbf{t})}{B(\mathbf{C}_{\mathbf{t}})^s} =: \beta_n^{(2)}(\mathbf{t}). \quad (5.34)$$

Let

$$\Gamma_{n,s}(\mathbf{t}) = \{(i_1, \dots, i_{2s}) \in [2n]^{2s} : \exists k_1, \dots, k_{2s} \text{ such that } (i_j, k_j; 1 \leq j \leq 2s) \in \text{BFAC}(\mathbf{t}, s)\}.$$

Write $\mathbf{I}_n^{\text{BF}} = (I_{n,1}^{\text{BF}}, \dots, I_{n,2s}^{\text{BF}})$ and similarly define \mathbf{I}_n° . Then for any $\mathbf{j} = (j_1, \dots, j_{2s}) \in \Gamma_{n,s}(\mathbf{t})$ with j_ℓ -s all distinct,

$$\left| \mathbb{P}_{\mathbf{t}}^\circ(\text{sort}(\mathbf{I}_n^\circ) = \mathbf{j}) - \mathbb{P}_{\mathbf{t}}^{\text{BF}}(\mathbf{I}_n^{\text{BF}} = \mathbf{j}) \right| = \frac{s!}{B(\mathbf{C}_{\mathbf{t}})^s} - \frac{1}{\#\text{BFAC}(\mathbf{t}, s)},$$

where $\text{sort}(\mathbf{I}_n^\circ) = (I_{n,\pi(1)}^\circ, I_{n,\pi(1)+1}^\circ, I_{n,\pi(3)}^\circ, I_{n,\pi(3)+1}^\circ, \dots, I_{n,\pi(2s-1)}^\circ, I_{n,\pi(2s-1)+1}^\circ)$ for a permutation π of $\{1, 3, \dots, 2s-1\}$ such that $\text{sort}(\mathbf{I}_n^\circ) \in \Gamma_{n,s}(\mathbf{t})$. Hence,

$$\sum_{\substack{\mathbf{j} \in \Gamma_{n,s}(\mathbf{t}) \\ j_\ell\text{-s all distinct}}} \left| \mathbb{P}_{\mathbf{t}}^\circ(\text{sort}(\mathbf{I}_n^\circ) = \mathbf{j}) - \mathbb{P}_{\mathbf{t}}^{\text{BF}}(\mathbf{I}_n^{\text{BF}} = \mathbf{j}) \right| \leq \frac{s! \times \#\text{BFAC}(\mathbf{t}, s) - B(\mathbf{C}_{\mathbf{t}})^s}{B(\mathbf{C}_{\mathbf{t}})^s} =: \beta_n^{(3)}(\mathbf{t}). \quad (5.35)$$

Using (5.34) and (5.35), we see that we can sample \mathbf{I}_n^{BF} and \mathbf{I}_n° in a way so that

$$\mathbb{P}(\text{sort}(\mathbf{I}_n^\circ) \neq \mathbf{I}_n^{\text{BF}} \mid T_n^\circ = T_n^{\text{BF}} = \mathbf{t}) \leq \sum_{j=1}^3 \beta_n^{(j)}(\mathbf{t}) =: \beta_n(\mathbf{t}).$$

Combined with (5.33), we get a coupling of $G_{n,s}^\circ$ and $G_{n,s}^{\text{BF}}$ such that for any $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(G_{n,s}^\circ \neq G_{n,s}^{\text{BF}}) &\leq \mathbb{P}(T_n^\circ \neq T_n^{\text{BF}}) + \sum_{\mathbf{t}: \beta_n(\mathbf{t}) < \varepsilon} \beta_n(\mathbf{t}) \cdot \mathbb{P}(T_n^\circ = T_n^{\text{BF}} = \mathbf{t}) + \mathbb{P}(\beta_n(T_n^\circ) \geq \varepsilon) \\ &\leq o(1) + \varepsilon + \mathbb{P}(\beta_n(T_n^\circ) \geq \varepsilon). \end{aligned} \quad (5.36)$$

To complete the proof, it is enough to show that $\mathbb{P}(\beta_n(T_n^\circ) \geq \varepsilon) \rightarrow 0$. Note that

$$\mathbb{P}(\beta_n(T_n^\circ) \geq \varepsilon) = \frac{\mathbb{E}[\mathbb{1}\{\beta_n(T_n) \geq \varepsilon\} \cdot B(\mathbf{C}_n)^s]}{\mathbb{E}[B(\mathbf{C}_n)^s]} \leq \frac{\mathbb{P}(\beta_n(T_n) \geq \varepsilon)^{1/2} \cdot \mathbb{E}[B(\mathbf{C}_n)^{2s}]^{1/2}}{\mathbb{E}[B(\mathbf{C}_n)^s]}. \quad (5.37)$$

In view of (5.18) and (5.19), it is enough to show that $\mathbb{P}(\beta_n(T_n) \geq \varepsilon) \rightarrow 0$, which is a simple consequence of (5.15), Lemma 5.7, and (5.18). \blacksquare

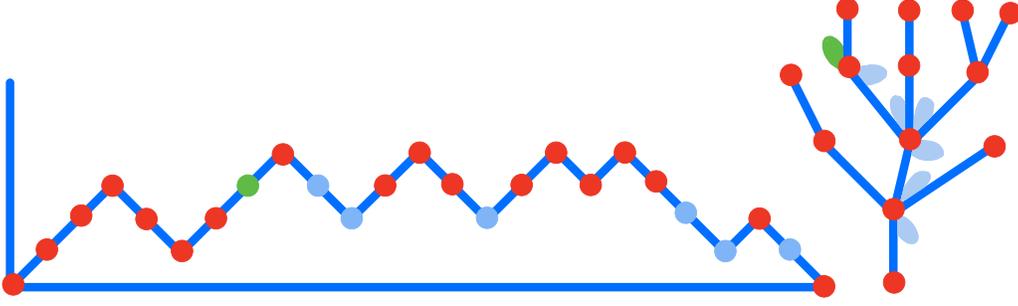


FIGURE 5.4. Left: f with the point on the graph of f corresponding to $i = 7$ colored green, and points that correspond to $\mathcal{D}(f; 7) \setminus \{7\}$ colored light blue. Right: the corresponding plane tree with the relevant corners colored.

5.4. **Proof of Proposition 5.2.** We tailor the argument in the proof of Proposition 5.1 to the depth-first setting. We will work with a fixed $s \geq 1$ throughout this section.

For $a < b$ and a continuous excursion e on $[a, b]$, $t \in [a, b]$, and $y \in [0, e(t)]$, define

$$\mathfrak{R}(e; t, y) := \{u \in [t, b] : e(u) = y, \exists v \in [t, u] \text{ such that } e(v) < y\}.$$

Let \mathfrak{C}_n be as in (5.6). For $f \in \mathfrak{C}_n$ and $i = 0, 1, \dots, 2n$, let

$$\mathcal{D}(f; i) := \bigcup_{j=1}^{f(i)} \mathfrak{R}(f; i, j), \quad D(f; i) := \#\mathcal{D}(f; i), \quad \text{and} \quad D(f) = \sum_{i=0}^{2n} D(f, i).$$

An illustration of $\mathcal{D}(f; i)$ is given in Figure 5.4. In terms of the plane tree whose contour function is f , for $1 \leq i \leq 2n - 1$, $\mathcal{D}(f; i) \setminus \{i\}$ corresponds to the set of corners that (i) are incident to vertices on the ancestral line of the vertex to which the i -th corner is incident, (ii) appear strictly after the i -th corner in the contour order, and (iii) are distinct from the corner associated with the root edge. Note the connection with the set of depth-first admissible corners $\text{DFAC}(\cdot, \cdot)$.

Let C_n^\bullet be distributed as

$$\mathbb{P}(C_n^\bullet = f) = \frac{(D(f))^s}{\sum_{\phi \in \mathfrak{C}_n} (D(\phi))^s}, \quad f \in \mathfrak{C}_n. \quad (5.38)$$

Conditional on C_n^\bullet , sample i.i.d. random variables $(I_{n,2j-1}^\bullet, I_{n,2j}^\bullet)$, $1 \leq j \leq s$, where

$$\mathbb{P}(I_{n,1}^\bullet = i) = D(C_n^\bullet; i) / D(C_n^\bullet), \quad i = 0, 1, \dots, 2n, \quad (5.39)$$

and conditional on $I_{n,1}^\bullet, I_{n,2}^\bullet$ is uniform over $\mathcal{D}(C_n^\bullet; I_{n,1}^\bullet)$.

Let T_n^\bullet be the plane tree whose contour function is C_n^\bullet . Let $G_{n,s}^\bullet$ be the rooted metric measure space obtained by adding an edge in T_n^\bullet between the $I_{n,2j-1}^\bullet$ -th corner and the $I_{n,2j}^\bullet$ -th corner of T_n^\bullet , $1 \leq j \leq s$, endowing the resulting space by the graph distance, rooting it at the root vertex of T_n^\bullet , and assigning probability $1/n$ to every vertex of T_n^\bullet except the root vertex. The next two lemmas combined complete the proof of Proposition 5.2.

Lemma 5.8. *We have,*

$$\frac{1}{\sqrt{n}} \cdot G_{n,s}^\bullet \xrightarrow{d} \frac{1}{\sqrt{2}} \cdot \mathcal{H}_{(s)}, \quad \text{as } n \rightarrow \infty$$

w.r.t. the pointed GHP topology.

Lemma 5.9. *There exists a coupling of $G_{n,s}^\bullet$ and $G_{n,s}^{\text{DF}}$ such that*

$$\mathbb{P}(G_{n,s}^\bullet \neq G_{n,s}^{\text{DF}}) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

We need some control over the functional $D(C_n^\bullet; \cdot)$ in order to prove Lemma 5.8. To this end, for $\mathbf{t} \in \mathbb{M}_{n,0}$, let $(S_{\mathbf{t}}(\ell), 0 \leq \ell \leq n+1)$ be the Łukasiewicz path (see, e.g., [36] for definition) of \mathbf{t} . For $i \in \{0, 1, \dots, 2n-1\}$, consider the vertex of \mathbf{t} to which the i -th corner of \mathbf{t} is incident, and suppose that the Łukasiewicz path of \mathbf{t} explores this vertex at the ℓ_i -th step. Further, denote by $\zeta_{\mathbf{t}}(i)$ the degree of this vertex in \mathbf{t} . Then

$$S_{\mathbf{t}}(\ell_i) - \zeta_{\mathbf{t}}(i) \leq D(C_{\mathbf{t}}; i) - C_{\mathbf{t}}(i) \leq S_{\mathbf{t}}(\ell_i) + 1. \quad (5.40)$$

As in the previous section, let C_n be uniformly distributed over \mathfrak{C}_n , and let T_n be the plane tree whose contour function is C_n . Recall that the tree obtained from T_n by deleting its root edge has the same distribution as a uniform plane tree on n vertices. This observation together with [37, Theorem 3] for the case of Geometric(1/2) offspring distribution gives

$$n^{-1/2} \max_i |S_{T_n}(\ell_i) - C_n(i)| \xrightarrow{d} 0.$$

Combined with (5.40) and the fact $\max_i \zeta_{T_n}(i) = O_P(\log n)$, this yields

$$n^{-1/2} \max_i |D(C_n; i) - 2C_n(i)| \xrightarrow{d} 0. \quad (5.41)$$

Further, it follows from [5, Equation 32 and Theorem 1.2] that² for all $x > 0$,

$$\mathbb{P}\left(\max_i |S_{T_n}(i)| \geq x\sqrt{n}\right) \leq ce^{-c'x^2}, \text{ and } \mathbb{P}\left(\max_i C_n(i) \geq x\sqrt{n}\right) \leq ce^{-c'x^2},$$

which combined with (5.40) yields

$$\mathbb{P}\left(\max_i D(C_n; i) \geq x\sqrt{n}\right) \leq ce^{-c'x^2}, \text{ for all } x > 0. \quad (5.42)$$

Proof of Lemma 5.8: First, using (5.41) and (5.14), we see that

$$\frac{D(C_n)}{(2n)^{3/2}} \xrightarrow{d} 2 \int_0^1 \mathbf{e}(t) dt, \text{ as } n \rightarrow \infty, \quad (5.43)$$

jointly with the convergence $\bar{C}_n \xrightarrow{d} \mathbf{e}$. Also, it follows from (5.42) and the bound $D(C_n) \leq 2n \max_i D(C_n; i)$ that for any $s \geq 1$, the sequence of random variables

$$\{(2n)^{-3s/2} D(C_n)^s\}_{n \geq 1} \text{ is uniformly integrable.} \quad (5.44)$$

Let

$$\bar{C}_n^\bullet(t) := (2n)^{-1/2} C_n^\bullet(2nt), \quad 0 \leq t \leq 1.$$

Then for any bounded continuous $\phi : C[0, 1] \rightarrow \mathbb{R}$,

$$\mathbb{E}[\phi(\bar{C}_n^\bullet)] = \frac{\mathbb{E}[\phi(\bar{C}_n) D(C_n)^s]}{\mathbb{E}[D(C_n)^s]} \xrightarrow{n \rightarrow \infty} \frac{\mathbb{E}[\phi(\mathbf{e}) (\int_0^1 \mathbf{e}(t) dt)^s]}{\mathbb{E}[(\int_0^1 \mathbf{e}(t) dt)^s]} = \mathbb{E}[\phi(\mathbf{e}_{(s)}^{\text{DF}})],$$

²[5, Equation 32] is stated for the maximum of the breadth-first queue, whereas we are dealing with the maximum of the depth-first queue. However, it is well-known that these two have the same distribution.

where the second step uses (5.14), (5.43), and (5.44). Repeating the argument, we can assume that we are working in a space where

$$\bar{C}_n^\bullet \xrightarrow{\text{a.s.}} \mathbf{e}_{(s)}^{\text{DF}}, \quad \frac{1}{\sqrt{n}} \max_i |D(C_n^\bullet; i) - 2C_n^\bullet(i)| \xrightarrow{\text{a.s.}} 0, \quad \text{and} \quad \frac{D(C_n^\bullet)}{(2n)^{3/2}} \xrightarrow{\text{a.s.}} 2 \int_0^1 \mathbf{e}_{(s)}^{\text{DF}}(t) dt. \quad (5.45)$$

Using (5.39), we can further assume that on this space,

$$(2n)^{-1} I_{n,1}^\bullet \xrightarrow{\text{a.s.}} u_1, \quad (5.46)$$

where u_1 is as in Construction 2.1.

Sample $I_{n,2}^\bullet$ in the following two steps: Conditional on $C_n^\bullet, I_{n,1}^\bullet$, first sample J_n^\bullet according to the law

$$\mathbb{P}(J_n^\bullet = j \mid C_n^\bullet, I_{n,1}^\bullet) = \frac{\#\mathfrak{R}(C_n^\bullet; I_{n,1}^\bullet, j)}{D(C_n^\bullet, I_{n,1}^\bullet)}, \quad j = 1, \dots, C_n^\bullet(I_{n,1}^\bullet), \quad (5.47)$$

and then let $I_{n,2}^\bullet \sim \text{Uniform}(\mathfrak{R}(C_n^\bullet; I_{n,1}^\bullet, J_n^\bullet))$. Let $i_j = \min \mathfrak{R}(C_n^\bullet; I_{n,1}^\bullet, j)$ for $j = 1, \dots, C_n^\bullet(I_{n,1}^\bullet)$. Then

$$C_n^\bullet(i_j) = j, \quad \text{and} \quad D(C_n^\bullet; i_j) - D(C_n^\bullet; i_{j-1}) = \#\mathfrak{R}(C_n^\bullet; i_j, j).$$

Consequently, using (5.47) and the second convergence in (5.46), we see that

$$\mathbb{P}(J_n^\bullet \leq j \mid C_n^\bullet, I_{n,1}^\bullet) = \frac{D(C_n^\bullet; i_j)}{D(C_n^\bullet, I_{n,1}^\bullet)} = \frac{2j + o(\sqrt{n})}{2C_n^\bullet(I_{n,1}^\bullet) + o(\sqrt{n})}, \quad j = 1, \dots, C_n^\bullet(I_{n,1}^\bullet),$$

where the $o(\sqrt{n})$ term is uniform over j . Thus, using (5.46) and the first convergence in (5.45), we can assume that

$$(2n)^{-1/2} \cdot J_n^\bullet \xrightarrow{\text{a.s.}} z_1 \quad (5.48)$$

together with (5.45) and (5.46), where z_1 is as in Construction 2.1. Let v_1 be as in Construction 2.1 as well. Then the set $\mathfrak{R}(\mathbf{e}_{(s)}^{\text{DF}}; u_1, z_1)$ is the singleton $\{v_1\}$ almost surely, and hence the first convergence in (5.45), (5.46), and (5.48) imply that

$$(2n)^{-1} \min \mathfrak{R}(C_n^\bullet; I_{n,1}^\bullet, J_n^\bullet) \xrightarrow{\text{a.s.}} v_1, \quad \text{and} \quad (2n)^{-1} \max \mathfrak{R}(C_n^\bullet; I_{n,1}^\bullet, J_n^\bullet) \xrightarrow{\text{a.s.}} v_1.$$

In particular, $(2n)^{-1} I_{n,2}^\bullet \xrightarrow{\text{a.s.}} v_1$. Repeating the same argument, we can assume that

$$(2n)^{-1} (I_{n,2j-1}^\bullet, I_{n,2j}^\bullet) \xrightarrow{\text{a.s.}} (u_j, v_j), \quad 1 \leq j \leq s.$$

Now the proof can be completed by following the argument given after (5.25). ■

Proof of Lemma 5.9: The proof follows the same steps as in the breadth-first case. First note that we can use the argument in the proof of Lemma 5.7 to prove the following analogous result:

$$n^{-3s/2} \cdot \mathbb{E} |s! \times \#\text{DFAC}(T_n, s) - D(C_n)^s| \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty;$$

here the bound in (5.42) plays the role analogous to that of (5.15) in the breadth-first setting. Using the above convergence, we can construct, similar to (5.33), a coupling of T_n^{DF} and T_n^\bullet such that $\mathbb{P}(T_n^{\text{DF}} \neq T_n^\bullet) \rightarrow 0$ as $n \rightarrow \infty$. Then the rest of the argument in the proof of Lemma 5.4 can be mimicked to complete the proof. We omit the details to avoid repetition. ■

5.5. **Proof of Corollary 3.3.** By Jeulin's local time identity [10, 35],

$$(\eta(\mathbf{e}; 1, y/2), y \geq 0) \stackrel{d}{=} (2 \cdot \mathbf{e}(\tau^{-1}(y)), y \geq 0), \text{ where } \tau(t) = \int_0^t \frac{du}{\mathbf{e}(u)}, \quad t \geq 0, \quad (5.49)$$

and $\tau^{-1}(y) = \sup \{t \in [0, 1] : \tau(t) \leq y\}$ for $y \geq 0$. Now,

$$\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy = \frac{1}{2} \int_0^\infty \eta(\mathbf{e}; 1, y/2)^2 dy \stackrel{d}{=} 2 \int_0^\infty (\mathbf{e}(\tau^{-1}(y)))^2 dy = 2 \int_0^1 \mathbf{e}(t) dt, \quad (5.50)$$

where the second step follows from (5.49), and the last step follows if we use the substitution $y = \tau(t)$. Further, since $2 \cdot \|\mathbf{e}\|_\infty = \inf \{y > 0 : \eta(\mathbf{e}; 1, y/2) = 0\}$ almost surely, (5.49) implies

$$2\|\mathbf{e}\|_\infty \stackrel{d}{=} \tau(1) \quad (5.51)$$

jointly with the equality in distribution in (5.50). Consequently, for every bounded continuous $\phi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\phi(\text{Rad}(\mathcal{H}_{(s)}))] &= \mathbb{E}[\phi(2\|\mathbf{e}_{(s)}^{\text{BF}}\|_\infty)] = \frac{\mathbb{E}[\phi(2\|\mathbf{e}\|_\infty)(\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy)^s]}{\mathbb{E}[(\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy)^s]} \\ &= \frac{\mathbb{E}[\phi(\int_0^1 dt/\mathbf{e}(t))(\int_0^1 \mathbf{e}(t) dt)^s]}{\mathbb{E}[(\int_0^1 \mathbf{e}(t) dt)^s]} = \mathbb{E}[\phi(\int_0^1 \frac{dt}{\mathbf{e}_{(s)}^{\text{DF}}(t)})], \end{aligned}$$

where the first step uses Corollary 3.2 (i), and the third step uses (5.50) and (5.51). This completes the proof.

5.6. **Proof of Theorem 3.4.** Let us first state the result concerning convergence of the height profile of $H_{n,0}$.

Theorem 5.10. *Let $Z_{n,0}(\ell)$ denote the number of vertices in $H_{n,0}$ at distance ℓ from the root, $\ell = 0, 1, \dots$. Let $\bar{Z}_{n,0}(r) = n^{-1/2} \cdot Z_{n,0}(\lfloor r\sqrt{n} \rfloor)$, $r \geq 0$. Then*

$$\left(\bar{Z}_{n,0}(r), r \geq 0\right) \xrightarrow{d} \left(\eta(2\mathbf{e}; 1, r), r \geq 0\right) = \left(\frac{1}{2}\eta(\mathbf{e}; 1, r/2), r \geq 0\right), \text{ as } n \rightarrow \infty, \quad (5.52)$$

w.r.t. Skorohod J_1 topology on $\mathbb{D}([0, \infty) : \mathbb{R})$.

As mentioned before, Theorem 5.10 is a special case of [24, Theorem 1.1]. We will make use of this result in our proof.

For $s \geq 1$, we can explore any $G \in \mathbb{H}_{n,s}$ in a breadth-first manner and get a breadth-first spanning tree as follows: Add an extra vertex labeled '0' and connect it to the root of G via an edge, and declare this edge oriented away from 0 to be the root edge. Use the vertex labels to endow the neighbors of every vertex in the resulting graph with a circular order (where our convention is to go from the smallest label to the highest). Then we can explore this map in a breadth-first fashion as in Section 5.1. From the resulting tree we delete the vertex 0 and the edge incident to it, and root this tree at the root vertex of G . Thus, we end up with a tree $\mathbf{t} \in \mathbb{H}_{n,0}$.

Now there exist $(i_k, j_k) \in [n]^2$, $1 \leq k \leq s$, such that

$$0 \leq \text{ht}(i_k; \mathbf{t}) - \text{ht}(j_k; \mathbf{t}) \leq 1, \quad (5.53)$$

and G can be recovered by adding an edge in \mathbf{t} between the vertices i_k and j_k for each $1 \leq k \leq s$. Think of \mathbf{t} as a plane tree by using the vertex labels and consider the plane embedding. Then in this embedding, for each k , the vertex j_k will appear after the vertex i_k in the depth-first order. (These conditions are similar to (5.1) and the condition ' $i_{2j-1} \leq i_{2j}$ ' appearing right above it.)

Thus, there is an asymmetry in the roles of i_k and j_k . This does not cause a problem in the proof of Proposition 5.1 because of the convergence of the second coordinate in (5.14), i.e., the convergence, *as a function of two variables*, of the discrete local time of the contour function to a continuous limit. The analogue of this result for uniform labeled trees is not available in the literature, which poses a problem in working with \mathbf{t} directly. This will be discussed further in Section 6. We, however, do have convergence of the total local time (Theorem 5.10). To go around the above issue by making use of Theorem 5.10, we define a new tree $\bar{\mathbf{t}}$ by applying a kind of symmetrization to \mathbf{t} .

If there exist $1 \leq k \neq q \leq s$ such that $|\text{ht}(i_k; \mathbf{t}) - \text{ht}(i_q; \mathbf{t})| \leq 1$, set $\bar{\mathbf{t}}$ to be the empty tree \emptyset . Otherwise, do the following for $1 \leq k \leq s$: If $\text{ht}(i_k; \mathbf{t}) = \text{ht}(j_k; \mathbf{t})$, do nothing, and if $\text{ht}(i_k; \mathbf{t}) = \text{ht}(j_k; \mathbf{t}) + 1$, then with probability 1/2 do nothing, and with probability 1/2 add an edge in \mathbf{t} between i_k and j_k and delete the edge between i_k and its parent in \mathbf{t} ; call this operation a 'swap.' Denote the resulting tree by $\bar{\mathbf{t}}$. We set

$$\text{bf}(G) = \mathbf{t} \quad \text{and} \quad \bar{\text{bf}}(G) = \bar{\mathbf{t}}.$$

Note that the swapping operations above commute, i.e., it does not matter in what order they are done. Note also that for any $k \geq 1$, the number of vertices in G that are at distance k from the root is same as the number of vertices at height k in $\text{bf}(G)$, and hence the same is true of $\bar{\text{bf}}(G)$ whenever $\bar{\text{bf}}(G) \neq \emptyset$.

To fix ideas, first consider $s = 1$. In this case, $\bar{\text{bf}}(H_{n,1})$ cannot be the empty tree \emptyset . For any $\bar{\mathbf{t}} \in \mathbb{H}_{n,0}$, the event $\{\bar{\text{bf}}(H_{n,1}) = \bar{\mathbf{t}}\}$ holds true iff there exist $i, j \in [n]$ such that $i \neq j$ and j is not the parent of i in $\bar{\mathbf{t}}$, $H_{n,1}$ is the graph obtained by adding an edge in $\bar{\mathbf{t}}$ between i and j , and exactly one of the following happens:

- (i) $\text{ht}(j; \bar{\mathbf{t}}) = \text{ht}(i; \bar{\mathbf{t}})$ and $i < j$. (In this case, $\text{bf}(H_{n,1}) = \bar{\text{bf}}(H_{n,1}) = \bar{\mathbf{t}}.$)
- (ii) Think of $\bar{\mathbf{t}}$ as a plane tree in which the children of a vertex are ordered from left to right in order of increasing label. Then $\text{ht}(j; \bar{\mathbf{t}}) = \text{ht}(i; \bar{\mathbf{t}}) - 1$, and j appears after i in the contour order of $\bar{\mathbf{t}}$, and swapping did not take place in going from $\text{bf}(H_{n,1})$ to $\bar{\text{bf}}(H_{n,1})$. (In this case also, $\text{bf}(H_{n,1}) = \bar{\text{bf}}(H_{n,1}) = \bar{\mathbf{t}}.$)
- (iii) $\text{ht}(j; \bar{\mathbf{t}}) = \text{ht}(i; \bar{\mathbf{t}}) - 1$, j appears before i in the contour order of $\bar{\mathbf{t}}$, and swapping took place in going from $\text{bf}(H_{n,1})$ to $\bar{\text{bf}}(H_{n,1})$. (In this case, $\text{bf}(H_{n,1})$ is the tree obtained from $\bar{\mathbf{t}}$ by adding an edge between i and j and deleting the edge between i and its parent in $\bar{\mathbf{t}}.$)

Then we see from the above discussion that for $\bar{\mathbf{t}} \in \mathbb{H}_{n,0}$,

$$\mathbb{P}(\bar{\text{bf}}(H_{n,1}) = \bar{\mathbf{t}}) = \frac{1}{\#\mathbb{H}_{n,1}} \cdot \sum_{\ell \geq 1} \left[\binom{z(\bar{\mathbf{t}}; \ell)}{2} + \frac{1}{2} \cdot z(\bar{\mathbf{t}}; \ell) \cdot (z(\bar{\mathbf{t}}; \ell - 1) - 1) \right] =: \frac{W_1(\bar{\mathbf{t}})}{\#\mathbb{H}_{n,1}}, \quad (5.54)$$

where $z(\cdot; \cdot)$ is as defined in Section 4. As observed before, $H_{n,1}$ and $\bar{\text{bf}}(H_{n,1})$ have the same distance profile. Thus, using (5.54), we see that for any bounded continuous $\phi: \mathbb{D}[0, \infty) \rightarrow$

\mathbb{R} ,

$$\mathbb{E}[\phi(\bar{Z}_{n,1})] = \frac{\#\mathbb{H}_{n,0}}{\#\mathbb{H}_{n,1}} \cdot \mathbb{E}[\phi(\bar{Z}_{n,0})W_1(H_{n,0})]. \quad (5.55)$$

From Theorem 5.10, it follows that

$$n^{-3/2} \cdot W_1(H_{n,0}) \xrightarrow{d} \frac{1}{2} \int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy \quad (5.56)$$

jointly with (5.52). Further, for any $s \geq 0$,

$$n^{1-n-3s/2} \cdot (\#\mathbb{H}_{n,s}) \xrightarrow{n \rightarrow \infty} \frac{1}{s!} \cdot \mathbb{E} \left[\left(\int_0^1 \mathbf{e}(x) dx \right)^s \right] = \frac{1}{2^s s!} \cdot \mathbb{E} \left[\left(\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy \right)^s \right], \quad (5.57)$$

where the first step follows from the results of [39, 42], and the second step uses (2.8). Finally, using the bound $W_1(\bar{\mathbf{t}}) \leq n \cdot \max_\ell z(\bar{\mathbf{t}}, \ell)$ and applying Theorem 5.6 with Poisson(1) offspring distribution, we see that $n^{-3/2} \cdot W_1(H_{n,0})$ is a sequence of uniformly integrable random variables. Combining this last observation with (5.55), (5.56), and (5.57), and Theorem 5.10, we get

$$\mathbb{E}[\phi(\bar{Z}_{n,1})] \xrightarrow{n \rightarrow \infty} \frac{\mathbb{E}[\phi(\eta(2\mathbf{e}; 1, \cdot)) \int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy]}{\mathbb{E}[\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy]} = \mathbb{E}[\phi(\eta(2\mathbf{e}_{(1)}^{\text{BF}}; 1, \cdot))]. \quad (5.58)$$

This completes the proof for $s = 1$.

For $s \geq 2$, we need some control on $\mathbb{P}(\overline{\text{bf}}(H_{n,s}) = \emptyset)$. To this end, note that for any $\mathbf{t} \in \mathbb{H}_{n,0}$, the number of tuples (i_k, j_k) , $1 \leq k \leq s$, that satisfy (5.53), and $|\text{ht}(i_k; \mathbf{t}) - \text{ht}(j_k; \mathbf{t})| \leq 1$ for some $1 \leq k \neq q \leq s$ is upper bounded by

$$\gamma(\mathbf{t}) := cn^{s-1} \left[\max_\ell z(\mathbf{t}; \ell) \right]^{s+1}, \quad (5.59)$$

where $c > 0$ depends only on s , for instance we can choose $c = 3 \cdot s(s-1) \cdot 2^s$. Consequently,

$$\begin{aligned} \mathbb{P}(\overline{\text{bf}}(H_{n,s}) = \emptyset) &\leq \frac{1}{\#\mathbb{H}_{n,s}} \cdot \sum_{\mathbf{t} \in \mathbb{H}_{n,0}} \gamma(\mathbf{t}) \\ &= cn^{s-1} \cdot (\sqrt{n})^{s+1} \cdot \left(\frac{\#\mathbb{H}_{n,0}}{\#\mathbb{H}_{n,s}} \right) \cdot \mathbb{E} \left[\left(\frac{\max_\ell z(H_{n,0}; \ell)}{\sqrt{n}} \right)^{s+1} \right] \leq \frac{c'}{\sqrt{n}}, \end{aligned} \quad (5.60)$$

where the last step follows if we apply Theorem 5.6 with Poisson(1) offspring distribution, and use (5.57).

Now for any $\bar{\mathbf{t}} \in \mathbb{H}_{n,0}$, the event $\{\overline{\text{bf}}(H_{n,s}) = \bar{\mathbf{t}}\}$ holds true iff there exist $i_k, j_k \in [n]$, $1 \leq k \leq s$, such that the following hold: **(i)** either $\text{ht}(i_k; \bar{\mathbf{t}}) = \text{ht}(j_k; \bar{\mathbf{t}})$ and $i_k < j_k$ or $\text{ht}(i_k; \bar{\mathbf{t}}) = \text{ht}(j_k; \bar{\mathbf{t}}) + 1$ and j_k is not the parent of i_k in $\bar{\mathbf{t}}$, **(ii)** $\text{ht}(i_{k+1}; \bar{\mathbf{t}}) \geq \text{ht}(i_k; \bar{\mathbf{t}}) + 2$ for $k = 1, \dots, s-1$, **(iii)** $H_{n,s}$ is the graph obtained from $\bar{\mathbf{t}}$ by placing an edge between i_k and j_k for $1 \leq k \leq s$, and **(iv)** the unique swaps (if any) needed to go from $\text{bf}(H_{n,s})$ to $\overline{\text{bf}}(H_{n,s})$ were performed.

Writing \sum_1 for sum over all ℓ_1, \dots, ℓ_s such that $\ell_{k+1} \geq \ell_k + 2$, $1 \leq k \leq s-1$, we see that

$$\mathbb{P}(\overline{\text{bf}}(H_{n,s}) = \bar{\mathbf{t}}) = \frac{1}{\#\mathbb{H}_{n,s}} \sum_1 \prod_{k=1}^s \left[\binom{z(\bar{\mathbf{t}}; \ell_k)}{2} + \frac{1}{2} \cdot z(\bar{\mathbf{t}}; \ell_k) \cdot (z(\bar{\mathbf{t}}; \ell_k - 1) - 1) \right] =: \frac{W_s(\bar{\mathbf{t}})}{\#\mathbb{H}_{n,s}}. \quad (5.61)$$

Note that the quantity $W_s(\bar{\mathbf{t}})$ satisfies

$$0 \leq W_1(\bar{\mathbf{t}})^s - s! \cdot W_s(\bar{\mathbf{t}}) \leq c \cdot W_1(\bar{\mathbf{t}})^{s-1} \cdot \max_{\ell} z(\bar{\mathbf{t}}; \ell)^2,$$

which essentially says that $n^{-3s/2} \cdot W_s(H_{n,0})$ can be replaced by $n^{-3s/2} \cdot W_1(H_{n,0})^s / s!$ for obtaining distributional asymptotics. Now the argument given below (5.54) for $s = 1$ can be carried over to complete the proof for a general s .

5.7. Proof of Theorem 3.5. Throughout this section we work with a fixed $g \geq 1$. It is known [19, 29] that

$$\#\mathbb{U}\mathbb{M}_{n,g} \sim \frac{4^g}{3^g \cdot g! \cdot \sqrt{\pi}} n^{3g-3/2} 4^n. \quad (5.62)$$

Further, for any $s \geq 1$, $\text{BF} : \mathbb{M}_{n,s} \rightarrow \text{BFT}(n, s)$ is a bijection, and consequently,

$$\begin{aligned} \#\mathbb{M}_{n,s} &= \sum_{\mathbf{t} \in \mathbb{M}_{n,0}} \#\text{BFAC}(\mathbf{t}, s) = \#\mathbb{M}_{n,0} \cdot \mathbb{E}[\#\text{BFAC}(T_n, s)] \sim \frac{\#\mathbb{M}_{n,0}}{s!} \cdot (2n)^{3s/2} \cdot \frac{\mathbb{E}[B(C_n)^s]}{(2n)^{3s/2}} \\ &\sim \frac{\#\mathbb{M}_{n,0}}{s!} \cdot (2n)^{3s/2} \cdot \mathbb{E}\left[\left(\int_0^\infty \eta(\mathbf{e}; 1, y)^2 dy\right)^s\right] \sim \frac{2^{\frac{5s}{2}-2}}{s! \sqrt{\pi}} \cdot n^{\frac{3}{2}(s-1)} 4^n \cdot \mathbb{E}\left[\left(\int_0^1 \mathbf{e}(t) dt\right)^s\right], \end{aligned} \quad (5.63)$$

where the third step uses Lemma 5.7, the fourth step uses (5.18) and (5.19), and the last step uses (2.8) and the fact that $\#\mathbb{M}_{n,0} = n^{-1} \cdot \binom{2n-2}{n-1} \sim 2^{2n-2} (n\pi)^{-1/2}$.

Remark 5. The enumeration technique in (5.63) is very much in the spirit of [39]. Let us point out here that this can also be deduced from the results of [40]. [40, Formula (5)] states that the generating function $F_s(x) = \sum_{n \geq 0} (\#\mathbb{M}_{n+1,s}) \cdot x^n$ is of the form $F_0(x)(1-4x)^{-s} g_s(y)$. Here, $F_0(x) = (1 - \sqrt{1-4x})/(2x)$ is the generating series for trees³, g_s is a polynomial of degree $(s-1)^+$ with positive integer coefficients, and $y = ((1-4x)^{-1/2} - 1)/2$. (Recall that in our definition of $\mathbb{M}_{n,s}$, we require the root vertex to have degree one. Thus, our notation is slightly different from that in [40], which explains why the coefficient of x^n is $\#\mathbb{M}_{n+1,s}$.) From this and elementary singularity analysis [27, Theorem VI.1], one obtains that as $n \rightarrow \infty$, for any $s \geq 1$ fixed,

$$\#\mathbb{M}_{n,s} \sim \frac{\omega_s^* n^{\frac{3}{2}(s-1)} 4^n}{2^s \cdot \Gamma((3s-1)/2)}, \quad (5.64)$$

where ω_s^* is the leading coefficient of g_s . From [40, Formula (5a)], one immediately deduces that ω_s^* , $s \geq 1$, satisfies the recursion

$$\omega_s^* = \sum_{k=1}^{s-1} \omega_k^* \omega_{s-k}^* + 2(3s-4)\omega_{s-1}^*, \quad (5.65)$$

which is exactly the recursion in [34, Equation (13)] (with the small difference that $\omega_0^* = 1$ here rather than $-1/2$ as in [34], but this does not matter since the initialization of (5.65)

³This corrects a typo in [40], where the x in the denominator of $F_0(x)$ is missing.

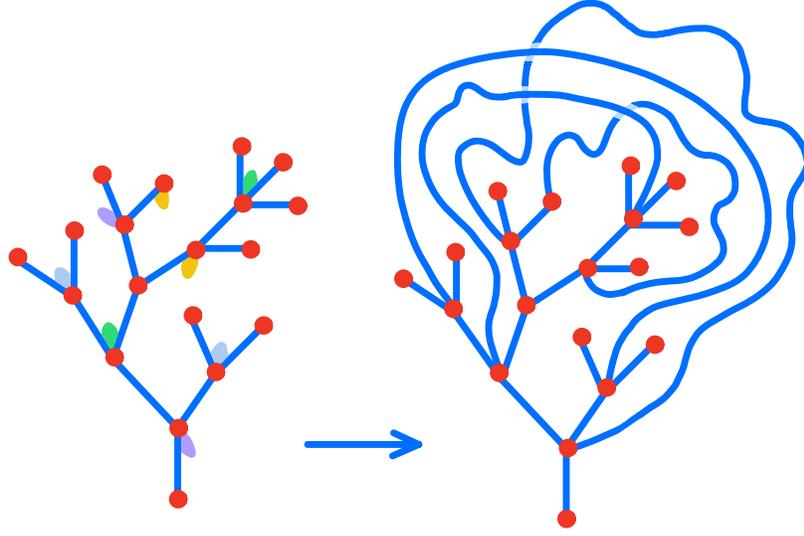


FIGURE 5.5. Left: A plane tree with four pairs of corners corresponding to $\sigma = (1, 7)(2, 5)(3, 8)(4, 6)$ colored. Right: The associated unicellular map.

really starts at $s = 1$). Now [34, Formula (14)] identifies

$$\frac{\omega_s^*}{\Gamma((3s-1)/2)} = \frac{2^{\frac{5s}{2}-2}}{s!\sqrt{\pi}} \cdot \mathbb{E}\left[\left(\int_0^1 2\mathbf{e}(t) dt\right)^s\right],$$

which combined with (5.64) yields the asymptotics for $\#\mathbb{M}_{n,s}$.

Consider $\mathbf{m} \in \mathbb{UM}_{n,g}$, and assume that the root vertex of \mathbf{m} has degree one. Note also that $\text{sp}(\mathbf{m}) = 2g$. Thus, such a map $\mathbf{m} \in \mathbb{UM}_{n,g} \cap \mathbb{M}_{n,2g}$. We can explore \mathbf{m} in a breadth-first way as in Section 5.1. Let $\text{BF}(\mathbf{m}) = (\mathbf{t}, (i_j, k_j; 1 \leq j \leq 4g))$. Then the cardinality of the set of all such \mathbf{m} for which i_1, \dots, i_{4g} are not all distinct is $O(n^{-1/2}) \cdot \#\mathbb{UM}_{n,g}$. Indeed, this follows from the proof of Lemma 5.7 (which shows the analogous result for $\mathbb{M}_{n,s}$) and the fact that $\#\mathbb{UM}_{n,g} = \Theta(\#\mathbb{M}_{n,2g})$ (which is a consequence of (5.62) and (5.63)). Let $\mathbb{UM}_{n,g}^* \subseteq \mathbb{UM}_{n,g} \cap \mathbb{M}_{n,2g}$ be the subset consisting of all \mathbf{m} for which i_1, \dots, i_{4g} are all distinct. Removing the root vertex and the interior of the root edge of $\mathbf{m} \in \mathbb{UM}_{n,g}^*$ gives a map in $\mathbb{UM}_{n-1,g}$. Using this observation and (5.62), it follows that

$$\#\mathbb{UM}_{n,g}^* \sim \#\mathbb{UM}_{n-1,g} \sim \frac{n^{3g-3/2} 4^{g-1}}{3^g \cdot g! \cdot \sqrt{\pi}} 4^n. \quad (5.66)$$

Clearly, $\mathbb{UM}_{n,g}$ has the same scaling limit as a uniform element of $\mathbb{UM}_{n,g}^*$, and so we can restrict our attention to unicellular maps in $\mathbb{UM}_{n,g}^*$.

Suppose σ is a permutation on $[4g]$ whose cycle decomposition consists of $2g$ many transpositions: $\sigma = \prod_{j=1}^{2g} (\ell_{2j-1}, \ell_{2j})$. Suppose further that $\ell_1 < \ell_3 < \dots < \ell_{4g-1}$, and $\ell_{2j-1} < \ell_{2j}$ for each $j \in [2g]$. Recall the definition of the set $\mathbb{S}_{(g)}$ from Section 2.2.

Lemma 5.11. *Suppose $\mathbf{t} \in \mathbb{M}_{n,0}$, σ is a permutation on $[4g]$ as above, and $r_1 < r_2 < \dots < r_{4g}$ are integers in $[2n-1]$. Let $i_j = r_{\ell_j}$ for $j \in [4g]$, and let $\mathbf{m} \in \mathbb{M}_{n,2g}$ be the map obtained by adding an edge in \mathbf{t} between the i_{2j-1} -th corner and the i_{2j} -th corner of \mathbf{t} , $1 \leq j \leq 2g$, and*

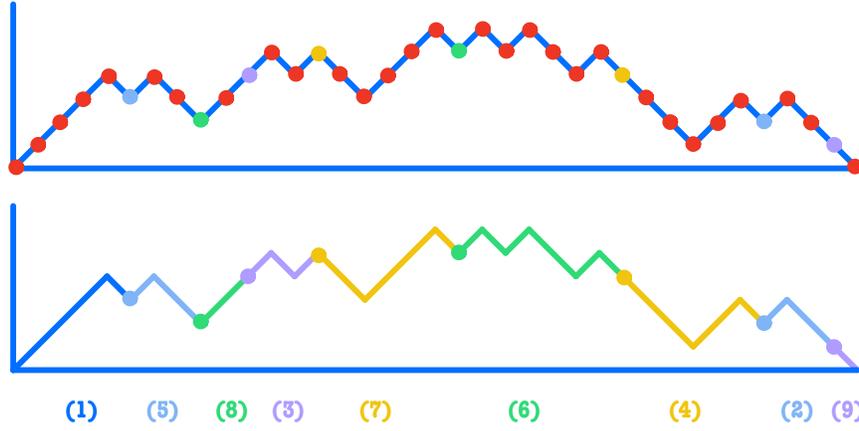


FIGURE 5.6. Top: The contour function of the plane tree in Figure 5.5. Bottom: Different segments of the same function numbered according to the order in which the face exploration of the unicellular map in Figure 5.5 traverses these parts.

endowing the neighbors of each vertex in \mathbf{m} with the natural circular order inherited from \mathbf{t} . Then $\mathbf{m} \in \mathcal{UM}_{n,g} \cap \mathcal{M}_{n,2g}$ iff $\sigma \in \mathcal{S}_{(g)}$.

Consequently, if $\sigma \in \mathcal{S}_{(g)}$ and $0 \leq C_{\mathbf{t}}(i_{2j-1}) - C_{\mathbf{t}}(i_{2j}) \leq 1$ for $j \in [2g]$, then $\mathbf{m} \in \mathcal{UM}_{n,g}^*$ with $\text{BF}(\mathbf{m}) = (\mathbf{t}, (i_1, \dots, i_{4g}, 1, \dots, 1))$.

Before starting the proof, we ask the reader to recall the algebraic description of maps that was mentioned in Section 4. In particular, we will make use of the fact that a map is unicellular if and only if its face permutation consists of only one cycle.

Proof of Lemma 5.11: Let e_1, \dots, e_{2n} be the directed edges of \mathbf{t} in the contour order with e_1 being the root edge. Then $\vec{E}(\mathbf{m})$ consists of e_1, \dots, e_{2n} together with an additional $4g$ directed edges; denote the directed edge going from the r_j -th corner to the $r_{\sigma(j)}$ -th corner by $f_{j,\sigma(j)}$, $j \in [4g]$.

Let us first assume that $\sigma \in \mathcal{S}_{(g)}$. As in Section 2.2, let ρ be the permutation $(1, 2, \dots, 4g)$. Let $r_0 = 0$ and $r_{4g+1} = 2n$. Then the elements of $\vec{E}(\mathbf{m})$ in the contour exploration of \mathbf{m} starting with e_1 appear as follows: For every $j \in \{0, 1, \dots, 4g\}$, $e_{1+r_j}, e_{2+r_j}, \dots, e_{r_{j+1}}$ appear consecutively in this order, and for every $j \in [4g]$, e_{r_j} is followed by the directed edges $f_{j,\sigma(j)}$ and $e_{1+r_{\sigma(j)}}$. (An illustration is given in Figures 5.5 and 5.6.) Using the fact that the permutation $\rho\sigma$ has only one cycle of length $4g$, it follows that the face permutation of \mathbf{m} consists of only one cycle of length $2n + 4g$, and consequently, \mathbf{m} is unicellular.

The converse follows by working the previous argument backwards. If \mathbf{m} is unicellular, then its face permutation has only one cycle of length $2n + 4g$. Observing the order in which the directed edges of \mathbf{m} appear in its contour exploration, it follows that the permutation $\rho\sigma$ must have only one cycle of length $4g$, and hence $\sigma \in \mathcal{S}_{(g)}$.

Now, because of the way ℓ_1, \dots, ℓ_{4g} are arranged among themselves, we have $i_{2j-1} < i_{2j}$ for $j \in [2g]$, and $i_1 < i_3 < \dots < i_{4g-1}$. Hence, if $0 \leq C_{\mathbf{t}}(i_{2j-1}) - C_{\mathbf{t}}(i_{2j}) \leq 1$ for $j \in [2g]$, then $(i_1, \dots, i_{4g}, 1, \dots, 1) \in \text{BFAC}(\mathbf{t}, 2g)$. Since $\mathcal{S}(\mathbf{t}, (i_1, \dots, i_{4g}, 1, \dots, 1)) = \mathbf{m}$, it follows that $\text{BF}(\mathbf{m}) =$

$(\mathbf{t}, (i_1, \dots, i_{4g}, 1, \dots, 1))$. If further $\sigma \in \mathbb{S}_{(g)}$, then $\mathbf{m} \in \mathbb{UM}_{n,g} \cap \mathbb{M}_{n,2g}$, and consequently $\mathbf{m} \in \mathbb{UM}_{n,g}^*$, as i_1, \dots, i_{4g} are distinct. \blacksquare

For the rest of this section, the notation ℓ_1, \dots, ℓ_{2g} will be reserved for integers that are arranged among themselves as described right before the statement of Lemma 5.11, and $\sigma = \prod_{j=1}^{2g} (\ell_{2j-1}, \ell_{2j})$ for a generic $\sigma \in \mathbb{S}_{(g)}$.

In view of Lemma 5.11, $\mathbb{UM}_{n,g}^*$ is in bijective correspondence with the set

$$\{(C, \sigma, (r_1, \dots, r_{4g})) : C \in \mathfrak{C}_n, \sigma \in \mathbb{S}_{(g)}, 1 \leq r_1 < \dots < r_{4g} \leq 2n-1 \quad (5.67)$$

$$\text{such that } 0 \leq C(r_{\ell_{2j-1}}) - C(r_{\ell_{2j}}) \leq 1 \text{ for } j \in [2g]\}.$$

Thus, sampling a uniform element of $\mathbb{UM}_{n,g}^*$ is tantamount to sampling a uniform element in the above set, which can be done in the following steps: For $C \in \mathfrak{C}_n$, $h \in \mathbb{Z}_{>0}$, and $r, r' \in [2n-1]$, let $\psi_{C,h}(r, r') = \mathbb{1}\{C(r) = h = C(r')\} + \mathbb{1}\{C(r) = h, C(r') = h-1\}$.

(a) Sample C_n^\dagger with probability mass function (pmf) proportional to

$$\sum_{\sigma \in \mathbb{S}_{(g)}} \sum_2 \sum_1 \prod_{j=1}^{2g} \psi_{C, h_j}(r_{\ell_{2j-1}}, r_{\ell_{2j}}), \quad C \in \mathfrak{C}_n,$$

where \sum_1 and \sum_2 denote sum over all $1 \leq r_1 < r_2 < \dots < r_{4g} \leq 2n-1$ and $h_1, \dots, h_{2g} \in \mathbb{Z}_{>0}$ respectively.

(b) Conditional on $C_n^\dagger = C$, sample Θ_n with pmf proportional to

$$\sum_2 \sum_1 \prod_{j=1}^{2g} \psi_{C, h_j}(r_{\ell_{2j-1}}, r_{\ell_{2j}}), \quad \sigma \in \mathbb{S}_{(g)}.$$

(c) Conditional on $C_n^\dagger = C$ and $\Theta_n = \sigma$, sample $\mathbf{H}_n = (H_{1,n}, \dots, H_{2g,n})$ with pmf proportional to

$$\sum_1 \prod_{j=1}^{2g} \psi_{C, h_j}(r_{\ell_{2j-1}}, r_{\ell_{2j}}), \quad h_1, \dots, h_{2g} \in \mathbb{Z}_{>0}.$$

(d) Conditional on $C_n^\dagger = C$, $\Theta_n = \sigma$, and $\mathbf{H}_n = (h_1, \dots, h_{2g})$, sample $(v_{1,n}, \dots, v_{4g,n})$ with pmf proportional to

$$\prod_{j=1}^{2g} \psi_{C, h_j}(r_{\ell_{2j-1}}, r_{\ell_{2j}}), \quad 1 \leq r_1 < r_2 < \dots < r_{4g} \leq 2n-1.$$

Let C_n be a uniform element of \mathfrak{C}_n . Then using Proposition 5.5, it follows that for $\sigma \in \mathbb{S}_{(g)}$, as $n \rightarrow \infty$,

$$\frac{1}{(2n)^{3g}} \sum_2 \sum_1 \prod_{j=1}^{2g} \psi_{C_n, h_j}(r_{\ell_{2j-1}}, r_{\ell_{2j}}) \xrightarrow{d} 2^{2g} \int_{(0, \infty)^{2g}} \nu_{\mathbf{e}, \sigma, \mathbf{y}}([0, 1]^{4g}) d\mathbf{y}_1 \dots d\mathbf{y}_{2g}. \quad (5.68)$$

This explains the expression for the tilt in (2.11). Now, using the bijection explained around (5.67),

$$\#\mathbb{UM}_{n,g}^* = \#\mathfrak{C}_n \cdot \mathbb{E} \left[\sum_{\sigma \in \mathbb{S}_{(g)}} \sum_2 \sum_1 \prod_{j=1}^{2g} \psi_{C_n, h_j}(r_{\ell_{2j-1}}, r_{\ell_{2j}}) \right]. \quad (5.69)$$

Note that $\sum_2 \sum_1 \prod_{j=1}^{2g} \psi_{C_n, h_j}(r_{\ell_{2j-1}}, r_{\ell_{2j}}) \leq (2\|L_n\|_\infty)^{2g} \cdot (2n)^{2g}$. Thus, using (5.15), it follows that we also have convergence of expectations in (5.68). Using this observation together with (5.69), (5.66), and the fact that $\#\mathfrak{C}_n = n^{-1} \cdot \binom{2n-2}{n-1}$, a direct calculation shows that

$$\sum_{\sigma \in \mathbb{S}_{(g)}} \mathbb{E} \left[\int_{(0, \infty)^{2g}} v_{\mathbf{e}, \sigma, \mathbf{y}}([0, 1]^{4g}) d\mathbf{y}_1 \dots d\mathbf{y}_{2g} \right] = (24^g \cdot g!)^{-1}. \quad (5.70)$$

This explains the scaling constant in (2.11).

Now using Proposition 5.5 and arguments similar to the ones used in the proof of Proposition 5.1, we can show that the following convergences happen jointly:

$$(2n)^{-1/2} C_n^\dagger(2n \cdot) \xrightarrow{d} \mathbf{e}_{(g)}^{\text{UM}}(\cdot), \quad \Theta_n \xrightarrow{d} \Theta, \quad (2n)^{-1/2} \mathbf{H}_n \xrightarrow{d} \mathbf{H}, \quad \text{and} \quad (5.71)$$

$$(2n)^{-1} (v_{1,n}, \dots, v_{4g,n}) \xrightarrow{d} (u_1, \dots, u_{4g}), \quad (5.72)$$

where the limiting random variables are as in Construction 2.3. The proof of Theorem 3.5 can now be completed using (5.71) and (5.72). We omit the details as no new idea is involved here.

6. DISCUSSION

We will discuss some of the questions related to this work in this section. Firstly, note that the result in Theorem 3.1 concerns only the limiting space $\mathcal{H}_{(g)}$. However, our proof uses discrete approximation techniques. It is natural to ask if this result can be proved directly in the continuum using properties of Brownian excursions.

As mentioned before, the proof of Theorem 3.1 proceeds via a study of maps. The reason is that an analogue of (5.14) for the contour function or the height function of uniform labeled trees, to the best of our knowledge, is not available in the literature. More generally, one may hope for a result of the following form:

Conjecture 6.1. *Let ξ , T_ξ , and $|T_\xi|$ be as in Theorem 5.6. Write $\sigma^2 = \text{Var} \xi$. Let $H_n^\xi(t)$, $t = 0, 1, \dots, n$, (resp. $C_n^\xi(t)$, $t = 0, 1, \dots, 2n$) be the height function (resp. contour function) of T_ξ conditioned to have $(n+1)$ vertices whenever $\mathbb{P}(|T_\xi| = n+1) > 0$. Define a continuous function $L(H_n^\xi; t, y)$ on $[0, n] \times \mathbb{R}$ (resp. a continuous function $L(C_n^\xi; t, y)$ on $[0, 2n] \times \mathbb{R}$) by means of the formulas given in (5.9) and (5.10). Then*

$$\begin{aligned} \left(n^{-1/2} \cdot L(H_n^\xi; nt, y\sqrt{n}); t \in [0, 1], y \in \mathbb{R} \right) &\xrightarrow{d} \left(\eta \left(\frac{2\mathbf{e}}{\sigma}; t, y \right); t \in [0, 1], y \in \mathbb{R} \right), \text{ and} \\ \left(n^{-1/2} \cdot L(C_n^\xi; 2nt, y\sqrt{n}); t \in [0, 1], y \in \mathbb{R} \right) &\xrightarrow{d} \left(2 \cdot \eta \left(\frac{2\mathbf{e}}{\sigma}; t, y \right); t \in [0, 1], y \in \mathbb{R} \right) \end{aligned}$$

in $C([0, 1] \times \mathbb{R})$, as $n \rightarrow \infty$ along the subsequence where $\mathbb{P}(|T_\xi| = n+1) > 0$.

Such a result can be viewed as a generalization of [24, Theorem 1.1]. A more ambitious project would be to identify the breadth-first construction of the stable graphs considered in [28]; see also [13, 23]. [15, Theorem 3.2] gives an algorithm for constructing uniform connected graphs with a given degree sequence. This algorithm can be thought of as a depth-first construction. A similar algorithm can be developed from a breadth-first point

of view. One may try to use such an algorithm to obtain a breadth-first construction of the stable graphs studied in [28] (and thereby identifying its radius, two point function, and distance profile in terms of suitable functionals of a normalized excursion of an α -stable Lévy process). To carry out this program, one needs good control over the local time field of the contour function (or height function) of the corresponding uniform plane tree with given (random) child sequence. Although the question of the full breadth-first construction in the stable setting remains open, some partial results in this direction were obtained in the recent work [20].

Explicit expressions are known for the densities of the two point function and the radius of the space \mathcal{T}_e . The two point function of \mathcal{T}_e follows the Rayleigh distribution [7, 8], whereas $\text{Rad}(\mathcal{T}_e)$ follows the more complicated Theta distribution [17][27, Chapter V.4.3]. In [41], an expression (given in terms of an infinite series) for the joint distribution function of $\text{Rad}(\mathcal{T}_e)$ and the diameter of \mathcal{T}_e is computed using probabilistic arguments. It would be interesting to see if such explicit expressions can be obtained for the laws of the radius and the two point function of the spaces $\mathcal{H}_{(s)}$.

APPENDIX A.

Our aim in this section is to outline a proof of (5.14). For $n \geq 1$, let $(S_n(j), 0 \leq j \leq 2n+1)$ be a simple symmetric random walk of length $2n+1$ started at the origin, i.e., $S_n(0) = 0$ and for $0 \leq j \leq 2n$,

$$\mathbb{P}(S_n(j+1) = y \mid S_n(k), 0 \leq k \leq j) = 1/2 \text{ if } y = S_n(j) \pm 1.$$

Let

$$S_n^{\text{br}} = (S_n \mid S_n(2n+1) = -1) \text{ and } S_n^{\text{ex}} = (S_n^{\text{br}} \mid S_n^{\text{br}}(j) \geq 0 \text{ for } 0 \leq j \leq 2n)$$

be the corresponding bridge and excursion of length $2n+1$ respectively. We extend S_n , S_n^{br} , and S_n^{ex} to continuous functions on $[0, 2n+1]$ by linear interpolation. For $t \in [0, 2n+1]$ and $y \in \mathbb{R}$, define $L(S_n; t, y)$ similar to (5.9) and (5.10). Let

$$\begin{aligned} \ell_n(\cdot, \cdot) &:= L(S_n; \cdot, \cdot), \quad \bar{S}_n(t) := (2n+1)^{-1/2} \cdot S_n((2n+1)t), \quad 0 \leq t \leq 1, \text{ and} \\ \bar{\ell}_n(t, y) &:= (2n+1)^{-1/2} \cdot \ell_n((2n+1)t, y\sqrt{2n+1}), \quad 0 \leq t \leq 1 \quad y \in \mathbb{R}. \end{aligned}$$

Similarly define ℓ_n^{br} , \bar{S}_n^{br} , $\bar{\ell}_n^{\text{br}}$, ℓ_n^{ex} , \bar{S}_n^{ex} , and $\bar{\ell}_n^{\text{ex}}$. Note that the function $(C_n(t+1) - 1, 0 \leq t \leq 2n-1)$ has the same law as S_{n-1}^{ex} . Consequently, (5.14) will follow if we show that

$$(\bar{S}_n^{\text{ex}}, \bar{\ell}_n^{\text{ex}}) \xrightarrow{d} (\mathbf{e}, \eta(\mathbf{e}; \cdot, \cdot)) \tag{A.1}$$

in $C[0, 1] \times C([0, 1] \times \mathbb{R})$.

Let $(B(t), 0 \leq t \leq 1)$ (resp. $(B^{\text{br}}(t), 0 \leq t \leq 1)$) be a standard one dimensional Brownian motion started at the origin (resp. Brownian bridge with $B^{\text{br}}(0) = B^{\text{br}}(1) = 0$). For the rest of the proof, η^{bm} and η^{br} will respectively denote $\eta(B; \cdot, \cdot)$ and $\eta(B^{\text{br}}; \cdot, \cdot)$.

By [21, Theorem F] (see also [11, 38]), it follows that we can construct a standard one dimensional Brownian motion $W(t)$, $t \geq 0$, and S_n for all $n \geq 1$ simultaneously on a single probability space such that for any $\delta > 0$,

$$n^{-1/4-\delta} \left(\max_{0 \leq j \leq 2n+1} \max_{y \in \mathbb{Z}} |\ell_n(j, y) - \eta(W; j, y)| + \max_{0 \leq j \leq 2n+1} |S_n(j) - W(j)| \right) \xrightarrow{\text{a.s.}} 0, \tag{A.2}$$

as $n \rightarrow \infty$. Writing \max_1 for maximum over $t \in \{1/(2n+1), 2/(2n+1), \dots, 1\}$ and $y \in \mathbb{Z}/\sqrt{2n+1}$, it follows from (A.2) that in this space, for any $\delta \in (0, 1/4)$,

$$\begin{aligned} & \max_1 \left| \bar{\ell}_n(t, y) - \eta(\bar{W}; t, y) \right| \\ &= \max_1 \frac{1}{\sqrt{2n+1}} \left| \ell_n((2n+1)t, y\sqrt{2n+1}) - \eta\left(W; (2n+1)t, y\sqrt{2n+1}\right) \right| \\ &= O(n^{-1/4+\delta}), \text{ a.s.}, \end{aligned} \quad (\text{A.3})$$

where $\bar{W}(t) = (2n+1)^{-1/2}W((2n+1)t)$, $0 \leq t \leq 1$. Further, in this space,

$$\max_{t \in \{1/(2n+1), 2/(2n+1), \dots, 1\}} \left| \bar{S}_n(t) - \bar{W}(t) \right| = O(n^{-1/4+\delta}), \text{ a.s.} \quad (\text{A.4})$$

Since $\bar{W} \stackrel{d}{=} B$, (A.3) and (A.4) yield

$$(\bar{S}_n, \bar{\ell}_n) \xrightarrow{d} (B, \eta^{\text{bm}}) \quad (\text{A.5})$$

in $C[0, 1] \times C([0, 1] \times \mathbb{R})$.

Fix $\varepsilon \in (0, 1)$, and let B_ε and $\eta_\varepsilon^{\text{bm}}$ denote the restrictions of B and η^{bm} to $[0, 1 - \varepsilon]$ and $[0, 1 - \varepsilon] \times \mathbb{R}$ respectively. Similarly define $B_\varepsilon^{\text{br}}$ and $\eta_\varepsilon^{\text{br}}$. Then

$$\mathbb{E}[\phi(B_\varepsilon^{\text{br}}, \eta_\varepsilon^{\text{br}})] = \frac{\mathbb{E}[\phi(B_\varepsilon, \eta_\varepsilon^{\text{bm}}) \exp(-B(1-\varepsilon)^2/(2\varepsilon))]}{\mathbb{E}[\exp(-B(1-\varepsilon)^2/(2\varepsilon))]} \quad (\text{A.6})$$

for any bounded continuous ϕ .

Let $2m+1$ be the smallest odd integer bigger than $(1-\varepsilon)(2n+1)$. Let $S_{n,m}^{\text{br}}$ and $\ell_{n,m}^{\text{br}}$ denote the restrictions of S_n^{br} and ℓ_n^{br} to $[0, 2m+1]$ and $[0, 2m+1] \times \mathbb{R}$ respectively. Then

$$\mathbb{E}[\phi(S_{n,m}^{\text{br}}, \ell_{n,m}^{\text{br}})] = \frac{\mathbb{E}[\phi(S_m, \ell_m) \mathbb{P}(X(2n-2m) = S_m(2m+1) + 1 \mid S_m)]}{\mathbb{E}[\mathbb{P}(X(2n-2m) = S_m(2m+1) + 1 \mid S_m)]}, \quad (\text{A.7})$$

where $X(2n-2m)$ is a simple symmetric random walk started at the origin and run up to time $2n-2m$ independent of all other random variables. By the local central limit theorem,

$$\sup_{j \in \mathbb{Z}} \left| \mathbb{P}(X(2n-2m) = 2j) - (4\pi(n-m))^{-1/2} \exp\left(-\frac{4j^2}{4(n-m)}\right) \right| = o(n^{-1/2}). \quad (\text{A.8})$$

It follows from (A.5), (A.6), (A.7), and (A.8) that for every $\varepsilon \in (0, 1)$,

$$(\bar{S}_{n,\varepsilon}^{\text{br}}, \bar{\ell}_{n,\varepsilon}^{\text{br}}) \xrightarrow{d} (B_\varepsilon^{\text{br}}, \eta_\varepsilon^{\text{br}}), \quad (\text{A.9})$$

where $\bar{S}_{n,\varepsilon}^{\text{br}}$ and $\bar{\ell}_{n,\varepsilon}^{\text{br}}$ are restrictions of \bar{S}_n^{br} and $\bar{\ell}_n^{\text{br}}$ to $[0, 1 - \varepsilon]$ and $[0, 1 - \varepsilon] \times \mathbb{R}$ respectively. It would follow from (A.9) that

$$(\bar{S}_n^{\text{br}}, \bar{\ell}_n^{\text{br}}) \xrightarrow{d} (B^{\text{br}}, \eta^{\text{br}}), \quad (\text{A.10})$$

provided we could show that for every $\delta_1, \delta_2 > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$,

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{1-\varepsilon \leq t \leq 1} |\bar{S}_n^{\text{br}}(t)| + \sup_{y \in \mathbb{R}} (\bar{\ell}_n^{\text{br}}(1, y) - \bar{\ell}_n^{\text{br}}(1-\varepsilon, y)) > \delta_1\right) \leq \delta_2. \quad (\text{A.11})$$

However, (A.11) is immediate upon observing that the time reversal of the process $-(1 + S_n^{\text{br}})$ has the same law as S_n^{br} and then using (A.9).

Now the Vervaat transform of B^{br} (resp. S_n^{br}) with respect to its almost surely unique global minima (resp. the first global minima) has the same law as \mathbf{e} (resp. S_n^{ex}). Using this fact together with (A.10), (A.1) follows.

ACKNOWLEDGMENTS

The authors thank Louigi Addario-Berry for many insightful discussions about the results of [3]. The authors also thank two anonymous referees for their careful reading and many comments and suggestions on an earlier version of the paper. Part of the work was done when both authors were attending Oberwolfach workshop 1750-Network Models: Structure and Function. The authors thank the organizers of the workshop and the Oberwolfach Research Institute for Mathematics for a stimulating work environment. GM acknowledges support from Institut Universitaire de France and Fondation Simone et Cino Del Duca. SS was partially supported by MATRICS grant MTR/2019/000745 from SERB, and by the Infosys Foundation, Bangalore.

REFERENCES

- [1] L. Addario-Berry, N. Broutin, and C. Goldschmidt, *Critical random graphs: limiting constructions and distributional properties*, Electron. J. Probab. **15** (2010), no. 25, 741–775. MR2650781 (2011d:60025)
- [2] L. Addario-Berry, N. Broutin, and C. Goldschmidt, *The continuum limit of critical random graphs*, Probab. Theory Related Fields **152** (2012), no. 3-4, 367–406. MR2892951
- [3] L. Addario-Berry, O. Angel, G. Chapuy, É. Fusy, and C. Goldschmidt, *Voronoi tessellations in the CRT and continuum random maps of finite excess*, Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, 2018, pp. 933–946.
- [4] L. Addario-Berry, N. Broutin, C. Goldschmidt, and G. Miermont, *The scaling limit of the minimum spanning tree of the complete graph*, The Annals of Probability **45** (2017), no. 5, 3075–3144.
- [5] L. Addario-Berry, L. Devroye, and S. Janson, *Sub-gaussian tail bounds for the width and height of conditioned galton–watson trees*, The Annals of Probability **41** (2013), no. 2, 1072–1087.
- [6] L. Addario-Berry and S. Sen, *Geometry of the minimal spanning tree of a random 3-regular graph*, arXiv preprint arXiv:1810.03802 (2018).
- [7] D. Aldous, *The continuum random tree. I*, Ann. Probab. **19** (1991), 1–28.
- [8] D. Aldous, *The continuum random tree III*, Ann. Probab. **21** (1993), 248–289.
- [9] D. Aldous, *The continuum random tree II: an overview*, Stochastic analysis **167** (1991), 23–70.
- [10] D. Aldous, G. Miermont, and J. Pitman, *The exploration process of inhomogeneous continuum random trees, and an extension of Jeulin’s local time identity*, Probab. Theory Related Fields **129** (2004), no. 2, 182–218. MR2063375 (2005f:60023)
- [11] R. F Bass and D. Khoshnevisan, *Rates of convergence to Brownian local time*, Stochastic processes and their applications **47** (1993), no. 2, 197–213.
- [12] S. Bhamidi, N. Broutin, S. Sen, and X. Wang, *Scaling limits of random graph models at criticality: Universality and the basin of attraction of the Erdős–Rényi random graph*, arXiv preprint arXiv:1411.3417 (2014).
- [13] S. Bhamidi, S. Dhara, R. van der Hofstad, and S. Sen, *Universality for critical heavy-tailed network models: metric structure of maximal components*, arXiv preprint arXiv:1703.07145 (2017).
- [14] S. Bhamidi, R. v. d. Hofstad, and S. Sen, *The multiplicative coalescent, inhomogeneous continuum random trees, and new universality classes for critical random graphs*, Probability Theory and Related Fields **170** (2018), no. 1-2, 387–474.
- [15] S. Bhamidi and S. Sen, *Geometry of the vacant set left by random walk on random graphs, Wright’s constants, and critical random graphs with prescribed degrees*, To appear in Random Structures & Algorithms (2019+).

- [16] S. Bhamidi, S. Sen, and X. Wang, *Continuum limit of critical inhomogeneous random graphs*, Probability Theory and Related Fields **169** (2017), no. 1-2, 565–641.
- [17] P. Biane, J. Pitman, and M. Yor, *Probability laws related to the jacobi theta and riemann zeta functions, and brownian excursions*, Bulletin of the American Mathematical Society **38** (2001), no. 4, 435–465.
- [18] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, Graduate Studies in Mathematics, vol. 33, American Mathematical Society, Providence, RI, 2001. MR1835418 (2002e:53053)
- [19] G. Chapuy, *The structure of unicellular maps, and a connection between maps of positive genus and planar labelled trees*, Probability Theory and Related Fields **147** (2010), no. 3-4, 415–447.
- [20] D. Clancy Jr, *Epidemics on critical random graphs with heavy-tailed degree distribution*, arXiv preprint arXiv:2104.05826 (2021).
- [21] E. Csáki and P. Révész, *Strong invariance for local times*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **62** (1983), no. 2, 263–278.
- [22] A. Dembo, A. Levit, and S. Vadlamani, *Component sizes for large quantum erdős-rényi graph near criticality*, The Annals of Probability **47** (2019), no. 2, 1185–1219.
- [23] S. Dhara, R. v. d. Hofstad, J. v. Leeuwaarden, and S. Sen, *Critical configuration models with infinite third-moment degrees*, arXiv preprint arXiv:1612.00650 (2016).
- [24] M. Drmota and B. Gittenberger, *On the profile of random trees*, Random Structures & Algorithms **10** (1997), no. 4, 421–451.
- [25] T. Duquesne, *The coding of compact real trees by real valued functions*, arXiv preprint math (2006).
- [26] S. N. Evans, *Probability and real trees*, Lecture Notes in Mathematics, vol. 1920, Springer, Berlin, 2008. Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. MR2351587 (2009d:60014)
- [27] P. Flajolet and R. Sedgewick, *Analytic combinatorics*, Cambridge University press, 2009.
- [28] C. Goldschmidt, B. Haas, and D. Sénizergues, *Stable graphs: distributions and line-breaking construction*, arXiv preprint arXiv:1811.06940 (2018).
- [29] A. Goupil and G. Schaeffer, *Factoring n -cycles and counting maps of given genus*, European Journal of Combinatorics **19** (1998), no. 7, 819–834.
- [30] M. Heydenreich and R. v. d. Hofstad, *Random graph asymptotics on high-dimensional tori*, Comm. Math. Phys. **270** (2007), no. 2, 335–358. MR2276449
- [31] M. Heydenreich and R. v. d. Hofstad, *Random graph asymptotics on high-dimensional tori II: volume, diameter and mixing time*, Probab. Theory Related Fields **149** (2011), no. 3-4, 397–415. MR2776620
- [32] R. v. d. Hofstad and A. Nachmias, *Hypercube percolation*, Preprint (2012). To appear in *Journ. Europ. Math. Soc.*
- [33] R. v. d. Hofstad and A. Sapozhnikov, *Cycle structure of percolation on high-dimensional tori*, Ann. Inst. Henri Poincaré Probab. Stat. **50** (2014), no. 3, 999–1027. MR3224297
- [34] S. Janson, *Brownian excursion area, Wright’s constants in graph enumeration, and other Brownian areas*, Probab. Surv. **4** (2007), 80–145. MR2318402
- [35] Th. Jeulin and M. Yor (eds.), *Grossissements de filtrations: exemples et applications*, Lecture Notes in Mathematics, vol. 1118, Springer-Verlag, Berlin, 1985. Papers from the seminar on stochastic calculus held at the Université de Paris VI, Paris, 1982/1983. MR884713
- [36] J.-F. Le Gall, *Random trees and applications*, Probab. Surv. **2** (2005), 245–311. MR2203728 (2007h:60078)
- [37] J.-F. Marckert and A. Mokkadem, *The depth first processes of Galton-Watson trees converge to the same Brownian excursion*, Annals of probability (2003), 1655–1678.
- [38] P Révész, *Local time and invariance*, Analytical methods in probability theory, 1981, pp. 128–145.
- [39] J. Spencer, *Enumerating graphs and Brownian motion*, Communications on Pure and Applied Mathematics **50** (1997), no. 3, 291–294.
- [40] T. R. S. Walsh and A. B. Lehman, *Counting rooted maps by genus. I*, J. Combinatorial Theory Ser. B **13** (1972), 192–218. MR314686
- [41] M. Wang, *Height and diameter of brownian tree*, Electronic Communications in Probability **20** (2015).
- [42] E. M Wright, *The number of connected sparsely edged graphs*, Journal of Graph Theory **1** (1977), no. 4, 317–330.